A SOBOLEV POINCARÉ TYPE INEQUALITY FOR INTEGRAL VARIFOLDS

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ABSTRACT. In this work a local inequality is provided which bounds the distance of an integral varifold from a multivalued plane (height) by its tilt and mean curvature. The bounds obtained for the exponents of the Lebesgue spaces involved are shown to be sharp.

CONTENTS

INTRODUCTION

Regularity of integral varifolds is often investigated by use of an approximation by Lipschitzian single or multivalued functions. A basic property of such functions is the Sobolev Poincaré inequality. In this paper a similar inequality is established for the varifold itself. It turns out that this can be done only up to a limiting exponent which is sharp. The initial motivation to examine the validity of a Poincaré type inequality was given by a question arising from [Sch04b], see below.

First, some definitions will be recalled. Suppose throughout the introduction that m, n are as above and U is a nonempty, open subset of $\mathbb{R}^{n+m}$. Using [Sim83, Theorem 11.8] as a definition, μ is a rectifiable [an integral] n varifold in U if and only if μ is a Radon measure on U and for μ almost all $x \in U$ there exists an approximate tangent plane $T_x\mu \in G(n+m,n)$ with multiplicity $\theta^n(\mu,x)$ of μ at x [and $\theta^n(\mu,x) \in \mathbb{N}$], $G(n+m,n)$ denoting the set of n dimensional, unoriented planes in $\mathbb{R}^{n+m}$. The distributional first variation of mass of μ equals

$$(\delta\mu)(\eta) = \textstyle\int \operatorname{div}_\mu \eta \,\mathrm{d}\mu \quad \text{whenever } \eta \in C^1_{\mathrm{c}}(U, \mathbb{R}^{n+m})$$

where $\operatorname{div}_\mu \eta(x)$ is the trace of $D\eta(x)$ with respect to $T_x\mu$. $\|\delta\mu\|$ denotes the total variation measure associated to $\delta\mu$ and μ is said to be of locally bounded first variation if and only if $\|\delta\mu\|$ is a Radon measure. The tilt-excess and the height-excess of μ are defined by

$$\operatorname{tiltex}_\mu(x,\varrho,T) := \varrho^{-n} \textstyle\int_{B_\varrho(x)} |T_\xi\mu - T|^2 \,\mathrm{d}\mu(\xi),$$
$$\operatorname{heightex}_\mu(x,\varrho,T) := \varrho^{-n-2} \textstyle\int_{B_\varrho(x)} \operatorname{dist}(\xi - x, T)^2 \,\mathrm{d}\mu(\xi)$$

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whenever $x \in \mathbb{R}^{n+m}$, $0 < \varrho < \infty$, $B_\varrho(x) \subset U$, $T \in G(n+m,n)$; here $S \in G(n+m,n)$ is identified with the orthogonal projection of $\mathbb{R}^{n+m}$ onto S and $|\cdot|$ denotes the norm induced by the usual inner product on $\operatorname{Hom}(\mathbb{R}^{n+m}, \mathbb{R}^{n+m})$. From the above definition of a rectifiable n varifold μ one obtains that μ almost all of U is covered by a countable collection of n dimensional submanifolds of $\mathbb{R}^{n+m}$ of class $\mathcal{C}^1$. This concept is extended to higher orders of differentiability by adapting a definition of Anzellotti and Serapioni in [AS94] as follows: A rectifiable n varifold μ in U is called countably rectifiable of class $\mathcal{C}^{k,\alpha}$ $[\mathcal{C}^k]$, $k \in \mathbb{N}$, $0 < \alpha \leq 1$, if and only if there exists a countable collection of n dimensional submanifolds of $\mathbb{R}^{n+m}$ of class $\mathcal{C}^{k,\alpha}$ $[\mathcal{C}^k]$ covering μ almost all of U. Throughout the introduction this will be abbreviated to $\mathcal{C}^{k,\alpha}$ $[\mathcal{C}^k]$ rectifiability. Note that $\mathcal{C}^{k,1}$ rectifiability and $\mathcal{C}^{k+1}$ rectifiability agree by [Fed69, 3.1.15].

Decays of tilt-excess or height-excess have been successfully used in [All72, Bra78, Sch04a, Sch04b]. The link to $\mathcal{C}^2$ rectifiability is provided in [Sch04b], see below. In order to explain some of these results, a mean curvature condition is introduced. An integral n varifold in U is said to satisfy (H_p), $1 \leq p \leq \infty$, if and only if either $p > 1$ and for some $\vec{\mathbf{H}}_\mu \in L^p_{\mathrm{loc}}(\mu, \mathbb{R}^{n+m})$, called the generalised mean curvature of μ,

$$(H_p) \qquad (\delta\mu)(\eta) = -\textstyle\int \vec{\mathbf{H}}_\mu \bullet \eta \, \mathrm{d}\mu \quad \text{whenever } \eta \in C^1_{\mathrm{c}}(U, \mathbb{R}^{n+m})$$

or $p = 1$ and

$$(H_1) \qquad \mu \text{ is of locally bounded first variation;}$$

here $\bullet$ denotes the usual inner product on $\mathbb{R}^{n+m}$. Brakke has shown in [Bra78, 5.7] that

$$\operatorname{tiltex}_\mu(x, \varrho, T_x\mu) = o_x(\varrho), \ \operatorname{heightex}_\mu(x, \varrho, T_x\mu) = o_x(\varrho) \quad \text{as } \varrho \downarrow 0$$

for μ almost every $x \in U$ provided μ satisfies (H_1) and

$$\operatorname{tiltex}_\mu(x, \varrho, T_x\mu) = o_x(\varrho^{2-\varepsilon}), \ \operatorname{heightex}_\mu(x, \varrho, T_x) = o_x(\varrho^{2-\varepsilon}) \quad \text{as } \varrho \downarrow 0$$

for every $\varepsilon > 0$ for μ almost every $x \in U$ provided μ satisfies (H_2). In case of codimension 1 and $p > n$ Schätzle has proved the following result yielding optimal decay rates.

Theorem 5.1 in [Sch04a]. *If $m = 1$, $p > n$, $p \geq 2$, and μ is an integral n varifold in U satisfying (H_p), then*

$$\operatorname{tiltex}_\mu(x, \varrho, T_x\mu) = O_x(\varrho^2), \ \operatorname{heightex}_\mu(x, \varrho, T_x\mu) = O_x(\varrho^2) \quad \textit{as } \varrho \downarrow 0$$

for μ almost all $x \in U$.

The importance of the improvement from $2 - \varepsilon$ to 2 stems mainly from the fact that the quadratic decay of tilt-excess can be used to compute the mean curvature vector $\vec{\mathbf{H}}_\mu$ in terms of the local geometry of μ which had already been noted in [Sch01, Lemma 6.3]. In [Sch04b] Schätzle provides the above mentioned link to $\mathcal{C}^2$ rectifiability as follows:

Theorem 3.1 in [Sch04b]. *If μ is an integral n varifold in U satisfying (H_2) then the following two statements are equivalent:*

1. *μ is $\mathcal{C}^2$ rectifiable.*
2. *For μ almost every $x \in U$ there holds*

$$\operatorname{tiltex}_\mu(x, \varrho, T_x\mu) = O_x(\varrho^2), \ \operatorname{heightex}_\mu(x, \varrho, T_x\mu) = O_x(\varrho^2) \quad \textit{as } \varrho \downarrow 0.$$

The quadratic decay of $\operatorname{heightex}_\mu$ implies $\mathcal{C}^2$ rectifiability without the condition (H_2) as may be seen from the proof in [Sch04b]. However, (1) would not imply (2) if μ were merely required to satisfy (H_p) for some p with $1 \leq p < \frac{2n}{n+2}$, an example was be provided in [Men08b, 1.5]. On the other hand, it is evident from the Caccioppoli type inequality relating $\operatorname{tiltex}_\mu$ to $\operatorname{heightex}_\mu$ and mean curvature, see e.g. [Bra78, 5.5], that quadratic decay of $\operatorname{heightex}_\mu$ implies quadratic decay for $\operatorname{tiltex}_\mu$ under the condition (H_2). This leads to the following question:

Problem. Does quadratic decay of $\operatorname{tiltex}_\mu$ imply quadratic decay of $\operatorname{heightex}_\mu$ under the condition (H_2)?

More generally, suppose that μ is an integral n varifold in U satisfying (H_p), $1 \leq p \leq \infty$, and $0 < \alpha \leq 1$, $1 \leq q < \infty$. Does

$$\limsup_{r \downarrow 0} r^{-\alpha - n/q} \big(\textstyle\int_{B_r(x)} |T_\xi \mu - T_x \mu|^q \, \mathrm{d}\mu(\xi) \big)^{1/q} < \infty$$

for μ almost all $x \in U$ imply

$$\limsup_{r \downarrow 0} r^{-1-\alpha - n/q} \big(\textstyle\int_{B_r(x)} \operatorname{dist}(\xi - x, T_x \mu)^q \, \mathrm{d}\mu(\xi) \big)^{1/q} < \infty$$

for μ almost all $x \in U$?

The answer to the second question will be shown in 2.10–2.12 to be in the affirmative if and only if either $p \geq n$ or $p < n$ and $\alpha q \leq \frac{np}{n-p}$, yielding in particular a positive answer to the first question. The main task is to prove the following theorem which in fact provides a quantitative estimate together with the usual embedding in L^q spaces.

Theorem 2.10. *Suppose $Q \in \mathbb{N}$, $0 < \alpha \leq 1$, $1 \leq p \leq n$, and μ is an integral n varifold in U satisfying (H_p).*

Then the following two statements hold:

(1) *If $p < n$, $1 \leq q_1 < n$, $1 \leq q_2 \leq \min\{ \frac{nq_1}{n-q_1}, \frac{1}{\alpha} \cdot \frac{np}{n-p} \}$, then for μ almost all $a \in U$ with $\theta^n(\mu, a) = Q$ there holds*

$$\begin{aligned} &\limsup_{r \downarrow 0} r^{-\alpha - 1 - n/q_2} \| \operatorname{dist}(\cdot - a, T_a \mu) \|_{L^{q_2}(\mu \llcorner B_r(a))} \\ &\leq \Gamma_{(1)} \limsup_{r \downarrow 0} r^{-\alpha - n/q_1} \| T_\mu - T_a \mu \|_{L^{q_1}(\mu \llcorner B_r(a))} \end{aligned}$$

where $\Gamma_{(1)}$ is a positive, finite number depending only on m, n, Q, q_1, and q_2.

(2) *If $p = n$, $n < q \leq \infty$, then for μ almost all $a \in U$ with $\theta^n(\mu, a) = Q$ there holds*

$$\begin{aligned} &\limsup_{r \downarrow 0} r^{-\alpha - 1} \| \operatorname{dist}(\cdot - a, T_a \mu) \|_{L^\infty(\mu \llcorner B_r(a))} \\ &\leq \Gamma_{(2)} \limsup_{r \downarrow 0} r^{-\alpha - n/q} \| T_\mu - T_a \mu \|_{L^{q}(\mu \llcorner B_r(a))} \end{aligned}$$

where $\Gamma_{(2)}$ is a positive, finite number depending only on m, n, Q, and q.

Here T_μ denotes the function mapping x to $T_x \mu$ whenever the latter exists. The connection to higher order rectifiability is provided by the following simple adaption of [Sch04b, Appendix A] by use of [Ste70, VI.2.2.2, VI.2.3.1–3].

Lemma. *Suppose $0 < \alpha \leq 1$, μ is a rectifiable n varifold in U, and A denotes the set of all $x \in U$ such that $T_x \mu$ exists and*

$$\limsup_{\varrho \downarrow 0} \varrho^{-n-1-\alpha} \textstyle\int_{B_\varrho(x)} \operatorname{dist}(\xi - x, T_x \mu) \, \mathrm{d}\mu(\xi) < \infty.$$

Then $\mu \llcorner A$ is $\mathcal{C}^{1,\alpha}$ rectifiable.

The analog of Theorem 2.10 in the case of weakly differentiable functions can be proved simplify by using the Sobolev Poincaré inequality in conjunction with an iteration procedure. In the present case, however, the curvature condition is needed to exclude a behaviour like the one shown by the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \sum_{i=0}^{\infty} (2^{-i}) \chi_{[2^{-i-1}, 2^{-i}[}(x) \quad \text{whenever } x \in \mathbb{R}$$

at 0; in fact an example of this behaviour occurring on a set of positive $\mathcal{L}^1$ measure is provided by $f^{1/2} \circ g$ where g is the distance function from a compact set C such that $\mathcal{L}^1(C) > 0$ and for some $0 < \lambda < 1$

$$\liminf_{r \downarrow 0} r^{-3/2} \mathcal{L}^1([x + \lambda r, x + r[\sim C) > 0 \quad \text{whenever } x \in C.$$

Therefore the strategy to prove Theorem 2.10 is to provide a special Sobolev Poincaré type inequality for integral varifolds involving curvature, see 2.6. In the construction weakly differentiable functions are replaced by Lipschitzian Q valued functions, a Q valued function being a function with values in $Q_Q(\mathbb{R}^m) \cong (\mathbb{R}^m)^Q/\sim$ where $\sim$ is induced by the action of the group of permutations of $\{1, \ldots, Q\}$ on $(\mathbb{R}^m)^Q$.

Roughly speaking, the construction performed in a ball $B_r(a) \subset U$ proceeds as follows. Firstly, a graphical part G of μ in $B_r(a)$ is singled out. The complement of G can be controlled in mass by the curvature, whereas its geometry cannot be controlled in a suitable way as may be seen from the example in [Men08b, 1.2] used to demonstrate the sharpness of the curvature condition. On the graphical part G the varifold μ might not quite correspond to the graph of a Q valued function but still have “holes” or “missing layers”. Nevertheless, it will be shown that μ behaves just enough like a Q valued function to make it possible to reduce the problem to this case. Finally, for Q valued functions Almgren's bi Lipschitzian equivalence of $Q_Q(\mathbb{R}^m)$ to a subset of $\mathbb{R}^{mP}$ for some $P \in \mathbb{N}$ which is a Lipschitz retract of the whole space directly yields a Poincaré inequality. More details about the technical difficulties occurring in the construction and how they are solved will be given at the beginning of Section 1.

The work is organised as follows. In Section 1 the approximation of μ by a Q valued function is constructed. In Section 2 the approximation is used to prove the Sobolev Poincaré type inequality 2.8 and Theorem 2.10.

The notation follows [Sim83] and, concerning Q valued functions, [Alm00, 1.1 (1), (9)–(11)]. Additionally to the symbols already defined, $\operatorname{im} f$ and $\operatorname{dmn} f$ denote the image and the domain of a function f respectively, $T^\perp$ is the orthogonal complement of T for $T \in G(n+m, n)$, γ_n denotes the best constant in the Isoperimetric Inequality as defined in 1.7, and $f(\phi)$ denotes the ordinary push forward of a measure ϕ by a function f, i.e. $f(\phi)(A) := \phi(f^{-1}(A))$ whenever $A \subset Y$, if ϕ is a measure on X and $f : X \to Y$. Definitions are denoted by ‘=’ or, if clarity makes it desirable, by ‘:=’. To simplify verification, in case a statement asserts the existence of a constant, small (ε) or large (Γ), depending on certain parameters this number will be referred to by using the number of the statement as index and what is supposed to replace the parameters in the order of their appearance given in brackets, for example $\varepsilon_{[\text{Men08b, 2.6}]}(m, n, 1 - \delta_3/2)$.

The results have been previously published in the author's PhD thesis, see [Men08a].

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1. Approximation of integral varifolds

In this section an approximation procedure for integral n varifolds μ in $\mathbb{R}^{n+m}$ by Q valued functions is carried out. Similar constructions occur in [Alm00, Chapter 3] and [Bra78, Chapter 5]. Basically, a part of μ which is suitably close to a Q valued plane is approximated "above" a subset Y of $\mathbb{R}^n$ by a Lipschitzian Q valued function. The sets where this approximation fails are estimated in terms of μ and $\mathcal{L}^n$ measure.

In order to obtain an approximation useful for proving the main lemma 2.6 for the Sobolev Poincaré type inequalities 2.8 and 2.10 in the next section, the following three problems had to be solved.

Firstly, in the above mentioned estimate one can only allow for tilt and mean curvature terms and not for a height term as it is present in [Bra78, 5.4]. This is done using a new version of Brakke's multilayer monotonicity which allows for variable offsets, see 1.6.

Secondly, the seemingly most natural way to estimate the height of μ above the complement of Y, namely measure times maximal height h, would not produce sharp enough an estimate. In order to circumvent this difficulty, a "graphical part" G of μ defined mainly in terms of curvature is used which is larger than the part where μ equals the "graph" of the Q valued function. Points in G still satisfy a one sided Lipschitz condition with respect to points above Y, see 1.10 and 1.14 (4). Using this fact in conjunction with a covering argument the actual error in estimating the q height in a ball $\bar{B}_t(\zeta)$ where $\mathcal{L}^n(\bar{B}_t(\zeta) \cap Y)$ and $\mathcal{L}^n(\bar{B}_t(\zeta) \sim Y)$ are comparable, can be estimated by $\mathcal{L}^n(\bar{B}_t(\zeta) \sim Y)^{1/q} \cdot t$ instead of $\mathcal{L}^n(\bar{B}_t(\zeta) \sim Y)^{1/q} \cdot h$; the replacement of h by t being the decisive improvement which allows to estimate the q^* height ($q^* = \frac{nq}{n-q}$, $1 \leq q < n$) instead of the q height in 2.6.

Thirdly, to obtain a sharp result with respect to the assumptions on the mean curvature, all curvature conditions are phrased in terms of isoperimetric ratios in order to allow for the application of the estimates in [Men08b]. In this situation it seems to be impossible to derive monotonicity results from the monotonicity formula, see e.g. [Sim83, (17.3)]. Instead, it is shown that nonintegral bounds for density ratios are preserved provided the varifold is additionally close to a Q valued plane, see 1.3. The latter result appears to be generally useful in deriving sharp estimates involving mean curvature.

1.1. If $m, n \in \mathbb{N}$, $a \in \mathbb{R}^{n+m}$, $0 < r < \infty$, $T \in G(n+m, n)$, and μ is a stationary, integral n varifold in $B_r(a)$ with $T_x\mu = T$ for μ almost all $x \in B_r(a)$, then $T^\perp(\operatorname{spt}\mu)$ is discrete and closed in $T^\perp(B_r(a))$ and for every $x \in \operatorname{spt}\mu$

$$y \in B_r(a),\ y - x \in T \quad \text{implies} \quad \theta^n(\mu, y) = \theta^n(\mu, x) \in \mathbb{N};$$

hence with $S_x = \{y \in B_r(a) : y - x \in T\}$

$$\mu \llcorner S_x = \theta^n(\mu, x)\mathcal{H}^n \llcorner S_x \quad \text{whenever } x \in B_r(a).$$

A similar assertion may be found in [Alm00, 3.6] and is used in [Bra78, 5.3 (16)].

1.2. **Lemma.** *Suppose $0 < M < \infty$, $M \notin \mathbb{N}$, $0 < \lambda_1 < \lambda_2 < 1$, $m, n \in \mathbb{N}$, $T \in G(n+m, n)$, F is the family of all stationary, integral n varifolds in $B_1^{n+m}(0)$ such that*

$$T_x\mu = T \quad \textit{for } \mu \textit{ almost all } x \in B_1^{n+m}(0), \quad \mu(B_1^{n+m}(0)) \leq M\omega_n,$$

and N is the supremum of all numbers

$$(\omega_n r^n)^{-1}\mu(\bar{B}_r^{n+m}(0))$$

corresponding to all $\mu \in F$ and $\lambda_1 \leq r \leq \lambda_2$.

Then for some $\mu \in F$ *and some* $\lambda_1 \leq r \leq \lambda_2$

$$N = (\omega_n r^n)^{-1} \mu(\bar{B}^{n+m}_r(0)) < M.$$

Proof (cf. [Men08a, 1.2]). Noting compactness by [All72, 6.4], the proof reduces to elementary geometry. □

1.3. **Lemma** (Quasi monotonicity)**.** *Suppose* $0 < M < \infty$*,* $M \notin \mathbb{N}$*,* $0 < \lambda < 1$*, and* $m, n \in \mathbb{N}$*.*

Then there exists a positive, finite number ε *with the following property.*

If $a \in \mathbb{R}^{n+m}$*,* $0 < r < \infty$*,* μ *is an integral* n *varifold in* $B_r(a)$ *with locally bounded first variation,*

$$\mu(B_r(a)) \leq M \omega_n r^n,$$

and whenever $0 < \varrho < r$

$$\|\delta\mu\|(\bar{B}_\varrho(a)) \leq \varepsilon\, \mu(\bar{B}_\varrho(a))^{1-1/n},$$

$$\textstyle\int_{\bar{B}_\varrho(a)} |T_x\mu - T| \,\mathrm{d}\mu(x) \leq \varepsilon\, \mu(\bar{B}_\varrho(a)) \quad \textit{for some } T \in G(n+m, n),$$

(here $0^0 := 1$*), then*

$$\mu(\bar{B}_\varrho(a)) \leq M\omega_n \varrho^n \quad \textit{whenever } 0 < \varrho \leq \lambda r.$$

Proof. Using induction, one verifies that it is enough to prove the statement with $\lambda^2 r \leq \varrho \leq \lambda r$ replacing $0 < \varrho \leq \lambda r$ in the last line which is readily accomplished by a contradiction argument using 1.2 and Allard's compactness theorem for integral varifolds [All72, 6.4]. □

1.4. *Remark.* Clearly,

$$(\omega_n \varrho^n)^{-1} \mu(\bar{B}_\varrho(a)) \leq M\lambda^{-n} \quad \text{whenever } 0 < \varrho < r.$$

1.5. **Lemma** (Multilayer monotonicity)**.** *Suppose* $m, n, Q \in \mathbb{N}$*,* $0 < \delta \leq 1$*, and* $0 \leq s < 1$*.*

Then there exists a positive, finite number ε *with the following property.*

If $X \subset \mathbb{R}^{n+m}$*,* $T \in G(n+m, n)$*,* $0 < r < \infty$*,*

$$|T(y-x)| \leq s|y-x| \quad \textit{whenever } x, y \in X,$$

μ *is an integral* n *varifold in* $\bigcup_{x\in X} B_r(x)$ *with locally bounded first variation,*

$$\textstyle\sum_{x\in X} \theta^n_*(\mu, x) \geq Q - 1 + \delta,$$

and whenever $0 < \varrho < r$*,* $x \in X \cap \operatorname{spt}\mu$

$$\|\delta\mu\|(\bar{B}_\varrho(x)) \leq \varepsilon\, \mu(\bar{B}_\varrho(x))^{1-1/n}, \quad \textstyle\int_{\bar{B}_\varrho(x)} |T_\xi\mu - T| \,\mathrm{d}\mu(\xi) \leq \varepsilon\, \mu(\bar{B}_\varrho(x)),$$

then

$$\mu\big(\textstyle\bigcup_{x\in X} B_\varrho(x)\big) \geq (Q-\delta)\omega_n \varrho^n \quad \textit{whenever } 0 < \varrho \leq r.$$

Proof (cf. [Men08a, 1.7]). Noting that lower bounds on $\theta^n_*(\mu, x)$ for $x \in X$ are available, see [Men08b, 2.6] or [Men08a, A.10], the proof is variant of Brakke's (cf. [Bra78, 5.3]). □

1.6. **Lemma** (Multilayer monotonicity with variable offset)**.** *Suppose* $m, n, Q \in \mathbb{N}$*,* $0 \leq M < \infty$*,* $\delta > 0$*, and* $0 \leq s < 1$*.*

Then there exists a positive, finite number ε *with the following property.*

If $X \subset \mathbb{R}^{n+m}$*,* $T \in G(n+m, n)$*,* $0 \leq d < \infty$*,* $0 < r < \infty$*,* $0 < t < \infty$*,* $f : X \to \mathbb{R}^{n+m}$*,*

$$|T(y-x)| \leq s|y-x|, \quad |T(f(y)-f(x))| \leq s|f(y)-f(x)|,$$

$$f(x) - x \in \bar{B}^{n+m}_d(0) \cap T, \quad d \leq Mt, \quad d + t \leq r$$

for $x, y \in X$, μ is an integral n varifold in $\bigcup_{x \in X} B_r(x)$ with locally bounded first variation,

$$\textstyle\sum_{x \in X} \theta^n_*(\mu, x) \geq Q - 1 + \delta, \quad \mu(B_r(x)) \leq M \omega_n r^n \quad \textit{for } x \in X \cap \operatorname{spt} \mu,$$

and whenever $0 < \varrho < r$, $x \in X \cap \operatorname{spt} \mu$

$$\|\delta \mu\|(\bar{B}_\varrho(x)) \leq \varepsilon \, \mu(\bar{B}_\varrho(x))^{1-1/n}, \quad \textstyle\int_{\bar{B}_\varrho(x)} |T_\xi \mu - T| \, \mathrm{d}\mu(\xi) \leq \varepsilon \, \mu(\bar{B}_\varrho(x)),$$

then

$$\mu\big(\textstyle\bigcup_{x \in X} \{ y \in B_t(f(x)) : |T(y - x)| > s|y - x| \}\big) \geq (Q - \delta) \omega_n t^n.$$

Proof. If the lemma were false for some $m, n, Q \in \mathbb{N}$, $0 \leq M < \infty$, $0 < \delta < 1$, and $0 < s < 1$, there would exist a sequence ε_i with $\varepsilon_i \downarrow 0$ as $i \to \infty$ and sequences X_i, T_i, d_i, r_i, t_i, f_i, and μ_i showing that ε_i does not satisfy the conclusion of the lemma.

In view of 1.3, 1.4 one could assume $d_i + t_i = r_i$ for $i \in \mathbb{N}$ by replacing M by $2M$. Using isometries and homotheties one could also assume for some $T \in G(n + m, n)$

$$T_i = T, \quad r_i = 1$$

for $i \in \mathbb{N}$. Finally, one could assume, possibly replacing M by a larger number,

$$X_i \subset \operatorname{spt} \mu_i, \quad \# X_i \leq Q, \quad X_i \subset \bar{B}^{n+m}_M(0)$$

for $i \in \mathbb{N}$.

Therefore passing to a subsequence (cf. [Fed69, 2.10.21]), there would exist a nonempty, closed subset X of $\bar{B}^{n+m}_M(0)$, $0 \leq d < \infty$, $0 \leq t < \infty$, and a nonempty, closed subset f of $\mathbb{R}^{n+m} \times \mathbb{R}^{n+m}$ such that $\# X \leq Q$,

$$d_i \to d \text{ and } t_i \to t \text{ as } i \to \infty,$$
$$X_i \to X \text{ and } f_i \to f \text{ in Hausdorff distance as } i \to \infty.$$

There would hold

$$s^{-1} |T(y - x)| \leq |y - x| \quad \text{for } x, y \in X, \quad d \leq Mt, \quad d + t = 1, \quad t > 0.$$

Moreover, since

$$(1 - s^2)^{1/2} |y_i - x_i| \leq \big| T^\perp(y_i - x_i) \big| = \big| T^\perp(f_i(y_i) - f_i(x_i)) \big| \leq |f_i(y_i) - f_i(x_i)|$$

for $x_i, y_i \in X_i$, and $i \in \mathbb{N}$, f were a function and one could readily verify $\operatorname{dmn} f = X$, and

$$f(x) - x \in \bar{B}^{n+m}_d(0) \cap T \quad \text{for } x \in X,$$
$$s^{-1} |T(f(y) - f(x))| \leq |f(y) - f(x)| \quad \text{for } x, y \in X.$$

Possibly passing to another subsequence, one could construct (cf. [All72, 6.4]) a stationary, integral n varifold μ in $U := \bigcup_{x \in X} B_1(x)$ with

$$T_x \mu = T \quad \text{for } \mu \text{ almost all } x \in U$$

such that

$$\textstyle\int \varphi \, \mathrm{d}\mu_i \to \int \varphi \, \mathrm{d}\mu \quad \text{as } i \to \infty \text{ for } \varphi \in C^0_{\mathrm{c}}(\mathbb{R}^{n+m}) \text{ with } \operatorname{spt} \varphi \subset U.$$

According to 1.5 one would estimate for large i

$$\mu_i\big(\textstyle\bigcup_{x \in X_i} B_\varrho(x)\big) \geq (Q - \delta) \omega_n \varrho^n \quad \text{whenever } 0 < \varrho \leq 1,$$

hence

$$\mu\big(\textstyle\bigcup_{x \in X} B_\varrho(x)\big) \geq (Q - \delta) \omega_n \varrho^n \quad \text{whenever } 0 < \varrho \leq 1.$$

Therefore, passing to the limit $\varrho \downarrow 0$, one would infer the lower bound (noting 1.1)

$$\textstyle\sum_{x \in X} \theta^n(\mu, x) \geq Q - \delta.$$

For $y, z \in \mathbb{R}^{n+m}$, $0 < \varrho < \infty$ define $V(y, z, \varrho)$ to be the set of all $x \in B_\varrho(z)$ such that

$$s^{-1}|T(y-x)| > |y-x|,$$

and note that every compact subset K of $\bigcup_{x\in X} V(x, f(x), t)$ would satisfy

$$K \subset \bigcup_{x\in X_i} V(x, f_i(x), t_i) \quad \text{for large } i;$$

hence

$$\mu\big(\bigcup_{x\in X} V(x, f(x), t)\big) \le \liminf_{i\to\infty} \mu_i\big(\bigcup_{x\in X_i} V(x, f_i(x), t_i)\big) \le (Q-\delta)\omega_n t^n.$$

On the other hand 1.1 would imply in conjunction with the fact

$$\{x \in \mathbb{R}^{n+m} : x - y \in T\} \cap \{x \in \mathbb{R}^{n+m} : x - z \in T\} = \emptyset$$

for $y, z \in X$ with $y \ne z$ and the lower bound previously derived

$$\mu\big(\bigcup_{x\in X} V(x, f(x), t)\big) \ge \big(\textstyle\sum_{x\in X} \theta^n(\mu, x)\big)\omega_n t^n \ge (Q-\delta)\omega_n t^n,$$

hence $\sum_{x\in X} \theta^n(\mu, x) = Q - \delta$ which is incompatible with $Q - \delta \notin \mathbb{N}$. □

1.7. **Definition.** Whenever $n \in \mathbb{N}$ the symbol γ_n will denote the smallest number with the following property:

If $m \in \mathbb{N}_0$, μ is a rectifiable n varifold in $\mathbb{R}^{n+m}$ with $\mu(\mathbb{R}^{n+m}) < \infty$ and $\|\delta\mu\|(\mathbb{R}^{n+m}) < \infty$, then

$$\mu\big(\big\{x \in \mathbb{R}^{n+m} : \theta^n(\mu, x) \ge 1\big\}\big) \le \gamma_n\, \mu(\mathbb{R}^{n+m})^{1/n} \|\delta\mu\|(\mathbb{R}^{n+m}).$$

1.8. *Remark.* $\gamma_n < \infty$ by the Isoperimetric Inequality of Micheal and Simon. Further properties of this number are given in [Men08b, Section 2].

1.9. **Lemma.** *Suppose $m, n \in \mathbb{N}$, $0 < \delta < 1$, $0 \le s < 1$, and $0 \le M < \infty$.*

Then there exists a positive, finite number ε with the following property.

If $a \in \mathbb{R}^{n+m}$, $0 < r < \infty$, $T \in G(n+m, n)$, $0 \le d < \infty$, $0 < t < \infty$, $\zeta \in \mathbb{R}^{n+m}$,

$$\max\{d, r\} \le Mt, \quad \zeta \in \bar{B}^{n+m}_d(0) \cap T, \quad d + t \le r,$$

μ is an integral n varifold in $B_r(a)$ with locally bounded first variation, $a \in \operatorname{spt}\mu$,

$$\|\delta\mu\|(B_r(a)) \le \varepsilon\, \mu(B_r(a))^{1-1/n}, \quad \mu(B_r(a)) \le M\omega_n r^n,$$
$$\textstyle\int_{B_r(a)} |T_\xi\mu - T|\, d\mu(\xi) \le \varepsilon\, \mu(B_r(a))$$

and for $0 < \varrho < r$

$$\|\delta\mu\|(\bar{B}_\varrho(a)) \le (2\gamma_n)^{-1}\mu(\bar{B}_\varrho(a))^{1-1/n}$$

(see 1.7), then

$$\mu(\{x \in B_t(a+\zeta) : |T(x-a)| > s|x-a|\}) \ge (1-\delta)\omega_n t^n.$$

Proof. A contradiction argument using [Men08b, 2.5], 1.1, and [All72, 6.4] yields the result. □

1.10. **Lemma.** *Suppose $m, n, Q \in \mathbb{N}$, $0 < \delta_1 \le 1$, $0 < \delta_2 \le 1$, $0 \le s < 1$, $0 \le s_0 < 1$, $0 \le M < \infty$, and $0 < \lambda < 1$ is uniquely defined by the requirement*

$$(1-\lambda^2)^{n/2} = (1-\delta_2) + \Big(\frac{(s_0)^2}{1-(s_0)^2}\Big)^{n/2}\lambda^n.$$

Then there exists a positive, finite number ε with the following property.

If $X \subset \mathbb{R}^{n+m}$, $T \in G(n+m, n)$, $0 \le d < \infty$, $0 < r < \infty$, $0 < t < \infty$, $\zeta \in \mathbb{R}^{n+m}$,

$$\#T(X) = 1, \quad \zeta \in \bar{B}^{n+m}_d(0) \cap T, \quad d \le Mt, \quad d + t \le r,$$

μ is an integral n varifold in $\bigcup_{x\in X} B_r(x)$ with locally bounded first variation, $\theta^n(\mu,x)\in\mathbb{N}$ for $x\in X$,

$$\textstyle\sum_{x\in X}\theta^n(\mu,x)=Q,\quad \mu(B_r(x))\le M\omega_n r^n \quad \textit{for } x\in X,$$

and whenever $0<\varrho<r$, $x\in X$

$$\|\delta\mu\|(\bar B_\varrho(x))\le\varepsilon\,\mu(\bar B_\varrho(x))^{1-1/n},\quad \textstyle\int_{\bar B_\varrho(x)}|T_\xi\mu-T|\,\mathrm{d}\mu(\xi)\le\varepsilon\,\mu(\bar B_\varrho(x))$$

satisfying

$$\mu\big(\textstyle\bigcup_{x\in X}\{y\in B_t(x+\zeta):|T(y-x)|>s_0|y-x|\}\big)\le(Q+1-\delta_2)\omega_n t^n,$$

then the following two statements hold:

(1) *If $0<\tau\le\lambda t$, then*
$$\mu\big(\textstyle\bigcup_{x\in X}\bar B_\tau(x)\big)\le(Q+\delta_1)\omega_n\tau^n.$$
(2) *If $y\in\operatorname{spt}\mu$ with $\operatorname{dist}(y,X)\le\lambda t/2$ and*
$$\|\delta\mu\|(\bar B_\varrho(y))\le(2\gamma_n)^{-1}\mu(\bar B_\varrho(y))^{1-1/n}\quad \textit{for } 0<\varrho<\delta_1\operatorname{dist}(y,X),$$
then for some $x\in X$
$$|T(y-x)|\ge s|y-x|.$$

Proof of (1). One may first assume $\max\{\delta_1,\delta_2\}\le 1/2$ and then $\lambda^2\le\tau/t\le\lambda$ by iteration of the result observing that the remaining assertion implies inductively

$$\mu\big(\textstyle\bigcup_{x\in X}\bar B_{\lambda^{-i}\tau}(x)\big)\le(Q+\delta_1)\omega_n(\lambda^{-i}\tau)^n$$

whenever $i\in\mathbb{N}$, $\lambda^{-i}\tau\le\lambda t$. Moreover, in view of 1.3, 1.4, only the case $r=d+t$ needs to be considered.

The remaining assertion will be proved by contradiction. If it were false for some $m,n,Q\in\mathbb{N}$, $0<\delta_1\le 1/2$, $0<\delta_2\le 1/2$, $0<s_0<1$, and $0\le M<\infty$, there would exist a sequence ε_i with $\varepsilon_i\downarrow 0$ as $i\to\infty$ and sequences X_i, T_i, d_i, r_i, t_i ζ_i, μ_i, and τ_i with $i\in\mathbb{N}$ showing that ε_i does not satisfy the assertion.

The argument follows the pattern of 1.6. First, one could assume for some $T\in G(n+m,n)$

$$T_i=T,\quad r_i=1$$

for $i\in\mathbb{N}$ and then noting $\#X_i\le Q$ that $X_i\subset\bar B_M^{n+m}(0)$ and hence, possibly passing to a subsequence, the existence of real numbers d, t, τ, of a nonempty, closed subset X of $\bar B_M^{n+m}(0)$, of $\zeta,\in\mathbb{R}^{n+m}$, and of a stationary, integral n varifold μ in $U:=\bigcup_{x\in X}B_1(x)$ such that $\#X\le Q$, and, as $i\to\infty$,

$$d_i\to d,\quad t_i\to t,\quad \tau_i\to\tau,\quad \zeta_i\to\zeta,$$
$$X_i\to X\quad\text{in Hausdorff distance,}$$
$$\textstyle\int\varphi\,\mathrm{d}\mu_i\to\int\varphi\,\mathrm{d}\mu\quad\text{for }\varphi\in C^0_{\mathrm{c}}(\mathbb{R}^{n+m})\text{ with }\operatorname{spt}\varphi\subset U,$$

and additionally

$$T_x\mu=T\quad\text{for }\mu\text{ almost all }x\in U.$$

Clearly,

$$d\le Mt,\quad d+t=1,\quad t>0,\quad \lambda^2\le\tau/t\le\lambda,$$
$$\#T(X)=1,\quad \zeta\in\bar B_d^{n+m}(0)\cap T,$$

and one would readily verify

$$\mu\big(\textstyle\bigcup_{x\in X}\{y\in B_t(x+\zeta):|T(y-x)|>s_0|y-x|\}\big)\le(Q+1-\delta_2)\omega_n t^n,$$
$$\mu\big(\textstyle\bigcup_{x\in X}\bar B_\tau(x)\big)\ge(Q+\delta_1)\omega_n\tau^n.$$

Moreover, 1.5 would imply with $S_x := \{z \in \mathbb{R}^{n+m} : T^\perp(z-x) = 0\}$ for $x \in \mathbb{R}^{n+m}$

$$\begin{gathered}\mu\big(\textstyle\bigcup_{x\in X} B_\varrho(x)\big) \geq (Q-\delta_1)\omega_n \varrho^n \quad \text{for } 0 < \varrho \leq 1,\\ \textstyle\sum_{x\in X} \theta^n(\mu, x) \geq Q,\\ \textstyle\sum_{x\in X} \theta^n(\mu,x)\big(\mathcal{H}^n \llcorner S_x\big)(A) \leq \mu(A) \quad \text{for } A \subset U.\end{gathered}$$

Therefore if $x \in X$, $y \in \operatorname{spt}\mu$, $T^\perp(y) \notin T^\perp(X)$, $0 < |T^\perp(y-x)| = h < t$, then one would find

$$\begin{aligned}&\{z \in S_y : |T(z-x)| \leq s_0|z-x|\} = S_y \cap \bar{B}_{(s_0^{-2}-1)^{-1/2}h}(x + T^\perp(y-x)),\\ &\big((1-(h/t)^2)^{n/2} - (s_0^{-2}-1)^{-n/2}(h/t)^n\big)\omega_n t^n\\ &= \big(\mathcal{H}^n \llcorner S_y\big)(B_t(x+\zeta)) - \big(\mathcal{H}^n \llcorner S_y\big)(\{z \in \mathbb{R}^{n+m} : |T(z-x)| \leq s_0|z-x|\})\\ &\leq \big(\mathcal{H}^n \llcorner S_y\big)(\{z \in B_t(x+\zeta) : |T(z-x)| > s_0|z-x|\})\\ &\leq (1-\delta_2)\omega_n t^n,\end{aligned}$$

hence $h \geq \lambda t$, in particular, since $\lambda t \geq \tau$ and $\# T(X) = 1$,

$$\begin{gathered}(\operatorname{spt}\mu) \cap \textstyle\bigcup_{x\in X} B_\tau(x) = \bigcup_{x\in X} S_x \cap B_\tau(x),\\ \mu\big(\textstyle\bigcup_{x\in X} \bar{B}_\tau(x)\big) = Q\omega_n\tau^n\end{gathered}$$

contradicting the previously derived lower bound because $\tau > 0$. □

Proof of (2) *(cf. [Men08a, 1.10 (2)]).* Having part (1) at one's disposal, the proof can be carried out using an argument similar to 1.6 and part (1). □

1.11 (cf. [Men08a, D.11])**.** The following proposition links approximate affine approximability of Q valued functions to approximate differentiability of Lipschitzian functions.

If $n, m, Q \in \mathbb{N}$, A is $\mathcal{L}^n$ measurable, $f : A \to Q_Q(\mathbb{R}^m)$ is Lipschitzian, I is countable, and to each $i \in I$ there corresponds a function $f_i \subset \operatorname{graph}_Q f$ with $\mathcal{L}^n$ measurable domain and $\operatorname{Lip} f_i \leq \operatorname{Lip} f$ such that

$$\#\{i : (x,y) \in f_i\} = \theta^0(\|f(x)\|, y) \quad \textit{whenever } (x,y) \in A \times \mathbb{R}^m,$$

then f is approximately strongly affinely approximable with

$$\operatorname{ap} Af(a)(v) = \textstyle\sum_{i \in I(a)} [\![f_i(x) + \langle v, \operatorname{ap} Df_i(x)\rangle]\!] \quad \textit{whenever } v \in \mathbb{R}^n$$

at $\mathcal{L}^n$ almost all $a \in A$ where $I(a) = \{i \in I : a \in \operatorname{dmn} f_i\}$. Moreover, for any n, m, Q, A, and f as above such functions f_i do exist.

In fact, the existence is proved using [Fed69, 3.3.5] and the relation to the Q valued function is established adapting [Fed69, 3.1.5, 3.1.9] and using [Fed69, 2.9.11, 3.1.2, 3.1.7].

1.12. **Definition.** Suppose $m, n, Q \in \mathbb{N}$, and $T \in G(n+m, n)$.

Then P is called a *Q valued plane parallel to T* if and only if for some $S \in Q_Q(T^\perp)$ (see [Alm00, 1.1 (1)])

$$P = \big(\theta^0(\|S\|, \cdot) \circ T^\perp\big)\mathcal{H}^n.$$

S is uniquely determined by P. For any two Q valued planes P_1 and P_2 parallel to T associated to $S_1, S_2 \in Q_Q(T^\perp)$ one defines (see [Alm00, 1.1 (1)])

$$\mathcal{G}(P_1, P_2) := \mathcal{G}(S_1, S_2).$$

In case $S = \sum_{i=1}^Q [\![z_i]\!]$ for some $z_1, \ldots, z_Q \in T^\perp$, then

$$\|S\| = \sum_{i=1}^Q \delta_{z_i}, \quad P = \sum_{i=1}^Q \mathcal{H}^n \llcorner \{x \in \mathbb{R}^{n+m} : T^\perp(x) = z_i\}$$

where δ_x denotes the Dirac measure at the point x.

1.13. In studying approximations of integral varifolds the following notation will be convenient. Suppose $m, n \in \mathbb{N}$, and $T \in G(n+m, n)$. Then there exist orthogonal projections $\pi : \mathbb{R}^{n+m} \to \mathbb{R}^n$, $\sigma : \mathbb{R}^{n+m} \to \mathbb{R}^m$ such that $T = \operatorname{im} \pi^*$ and $\pi \circ \sigma^* = 0$, hence

$$T = \pi^* \circ \pi, \quad T^\perp = \sigma^* \circ \sigma, \quad \mathbb{1}_{\mathbb{R}^{n+m}} = \pi^* \circ \pi + \sigma^* \circ \sigma.$$

Whenever $a \in \mathbb{R}^{n+m}$, $0 < r < \infty$, $0 < h \leq \infty$ the *closed cylinder* $C(T, a, r, h)$ is defined by

$$\begin{aligned} C(T, a, r, h) &= \{x \in \mathbb{R}^{n+m} : |T(x-a)| \leq r \text{ and } |T^\perp(x-a)| \leq h\} \\ &= \{x \in \mathbb{R}^{n+m} : |\pi(x-a)| \leq r \text{ and } |\sigma(x-a)| \leq h\}. \end{aligned}$$

This definition extends Allard's definition in [All72, 8.10] where $h = \infty$.

1.14. **Lemma** (Approximation by Q valued functions)**.** *Suppose $m, n, Q \in \mathbb{N}$, $0 < L < \infty$, $1 \leq M < \infty$, and $0 < \delta_i \leq 1$ for $i \in \{1, 2, 3, 4\}$.*

Then there exists a positive, finite number ε with the following property.

If a, r, h, T, π, and σ are as in 1.13, $h > 2\delta_4 r$,

$$U = \{x \in \mathbb{R}^{n+m} : \operatorname{dist}(x, C(T, a, r, h)) < 2r\},$$

μ is an integral n varifold in U with locally bounded first variation,

$$\begin{gathered} (Q - 1 + \delta_1)\omega_n r^n \leq \mu(C(T, a, r, h)) \leq (Q + 1 - \delta_2)\omega_n r^n, \\ \mu(C(T, a, r, h + \delta_4 r) \sim C(T, a, r, h - 2\delta_4 r)) \leq (1 - \delta_3)\omega_n r^n, \\ \mu(U) \leq M\omega_n r^n, \end{gathered}$$

$0 < \varepsilon_1 \leq \varepsilon$, B denotes the set of all $x \in C(T, a, r, h)$ with $\theta^{*n}(\mu, x) > 0$ such that

$$\begin{array}{lll} \textit{either} & \|\delta\mu\|(\bar{B}_\varrho(x)) > \varepsilon_1\, \mu(\bar{B}_\varrho(x))^{1-1/n} & \textit{for some } 0 < \varrho < 2r, \\ \textit{or} & \int_{\bar{B}_\varrho(x)} |T_\xi \mu - T| \,\mathrm{d}\mu(\xi) > \varepsilon_1\, \mu(\bar{B}_\varrho(x)) & \textit{for some } 0 < \varrho < 2r, \end{array}$$

and G denotes the set of all $x \in C(T, a, r, h) \cap \operatorname{spt} \mu$ such that

$$\begin{gathered} \|\delta\mu\|(B_{2r}(x)) \leq \varepsilon\, \mu(B_{2r}(x))^{1-1/n}, \\ \int_{B_{2r}(x)} |T_\xi \mu - T| \,\mathrm{d}\mu(\xi) \leq \varepsilon\, \mu(B_{2r}(x)), \\ \|\delta\mu\|(\bar{B}_\varrho(x)) \leq (2\gamma_n)^{-1} \mu(\bar{B}_\varrho(x))^{1-1/n} \quad \textit{for } 0 < \varrho < 2r, \end{gathered}$$

then there exist an $\mathcal{L}^n$ measurable subset Y of $\mathbb{R}^n$ and a function $f : Y \to Q_Q(\mathbb{R}^m)$ with the following seven properties:

(1) *$Y \subset \bar{B}_r(\pi(a))$ and f is Lipschitzian with $\operatorname{Lip} f \leq L$.*
(2) *Defining $A = C(T, a, r, h) \sim B$ and $A(y) = \{x \in A : \pi(x) = y\}$ for $y \in \mathbb{R}^n$, the sets A and B are Borel sets and there holds (see [Alm00, 1.1 (1)])*

$$\begin{gathered} \sigma(A \cap \operatorname{spt} \mu) \subset \bar{B}_{h - \delta_4 r}(\sigma(a)), \quad \operatorname{spt} f(y) \subset \sigma(A(y)), \\ \|f(y)\| = \sigma\big(\theta^n(\mu, \cdot)\mathcal{H}^0 \llcorner A(y)\big) \end{gathered}$$

whenever $y \in Y$.
(3) *Defining the sets*

$$C = \bar{B}_r(\pi(a)) \sim (Y \sim \pi(B)), \quad D = C(T, a, r, h) \cap \pi^{-1}(C),$$

there holds

$$\mathcal{L}^n(C) + \mu(D) \leq \Gamma_{(3)}\, \mu(B).$$

with $\Gamma_{(3)} = \max\{3 + 2Q + (12Q + 6)5^n, 4(Q + 2)/\delta_1\}$.

(4) *If* $x_1 \in G$, *then*

$$|\sigma(x_1 - a)| \le h - \delta_4 r$$

and for $y \in Y \cap \bar{B}_{\lambda_{(4)}}(\pi(x_1))$ *there exists* $x_2 \in A(y)$ *with* $\theta^n(\mu, x_2) \in \mathbb{N}$ *and*

$$\left|T^\perp(x_2 - x_1)\right| \le L\,|T(x_2 - x_1)|,$$

where $0 < \lambda_{(4)} < 1$ *depends only on* n, δ_2, *and* δ_4. *Moreover,* $G \supset A \cap \operatorname{spt}\mu$ *and*[1]

$$(\pi \bowtie \sigma)\big(G \cap \pi^{-1}(Y)\big) = \operatorname{graph}_Q f.$$

(5) $\overline{Y} \sim Y$ *has measure* 0 *with respect to* $\mathcal{L}^n$ *and* $\pi(\mu \llcorner G)$.

(6) *If* $\mathcal{L}^n(\bar{B}_r(\pi(a)) \sim Y) \le \frac{1}{2}\omega_n(\lambda_{(4)} r/6)^n$, $1 \le q < \infty$, $S \in Q_Q(\mathbb{R}^m)$, $P = (\theta^0(\|S\|, \cdot) \circ \sigma)\mathcal{H}^n$ *is the* Q *valued plane associated to* S *via* σ, *and* $g : Y \to \mathbb{R}$ *is defined by* $g(y) = \mathcal{G}(f(y), S)$ *for* $y \in Y$, *then*

$$\begin{aligned}\|\operatorname{dist}(\cdot, \operatorname{spt} P)\|_{L^q(\mu \llcorner G)} \\ \le (12)^{n+1} Q\big(\|g\|_{L^q(\mathcal{L}^n \llcorner Y)} + \Gamma_{(6)} \mathcal{L}^n(\bar{B}_r(\pi(a)) \sim Y)^{1/q + 1/n}\big),\end{aligned}$$

where $\Gamma_{(6)}$ *is a positive, finite number depending only on* q, *and* n, *and*

$$\begin{aligned}\sup\{\operatorname{dist}(x, \operatorname{spt} P) : x \in G\} \\ \le \|g\|_{L^\infty(\mathcal{L}^n \llcorner Y)} + 2\big(\mathcal{L}^n(\bar{B}_r(\pi(a)) \sim Y)/\omega_n\big)^{1/n}.\end{aligned}$$

(7) *For* $\mathcal{L}^n$ *almost all* $y \in Y$ *the following is true:*
 (a) f *is approximately strongly affinely approximable at* y.
 (b) *Whenever* $x \in G$ *with* $\pi(x) = y$

$$(\pi \bowtie \sigma)(T_x\mu) = \operatorname{Tan}\big(\operatorname{graph}_Q \operatorname{ap} Af(y), (y, \sigma(x))\big)$$

 where $\operatorname{Tan}(S, a)$ *denotes the classical tangent cone of* S *at* a *in the sense of [Fed69, 3.1.21].*
 (c) $\|T_x\mu - T\| \le \|\operatorname{ap} Af(y)\|$ *for* $x \in G$ *with* $\pi(x) = y$.
 (d) $\|\operatorname{ap} Af(y)\|^2 \le Q(1 + (\operatorname{Lip} f)^2) \max\{\|T_x\mu - T\|^2 : x \in \pi^{-1}(\{y\}) \cap G\}$.

Choice of constants. One can assume $3L \le \delta_4$.

Choose $0 < s_0 < 1$ close to 1 such that $2(s_0^{-2} - 1)^{1/2} \le \delta_4$, define $\lambda = \lambda_{1.10}(n, \delta_2, s_0)/4$, choose $s_0 \le s < 1$ close to 1 satisfying

$$(s^{-2} - 1)^{1/2} \le \lambda/4, \quad Q^{1/2}(s^{-2} - 1)^{1/2} \le L,$$

and define $\varepsilon > 0$ so small that

$$\begin{gathered}\varepsilon \le (2\gamma_n)^{-1}, \quad Q - 1 + \delta_1/2 \le (1 - n\varepsilon^2)(Q - 1 + \delta_1), \\ Q - 1/2 \le (1 - n\varepsilon^2)(Q - 1/4), \quad 1 - n\varepsilon^2 \ge 1/2,\end{gathered}$$

and not larger than the minimum of the following seven numbers

$$\begin{gathered}\varepsilon_{\text{[Men08b, 2.6]}}(m, n, 1 - \delta_3/2), \quad \varepsilon_{1.6}(m, n, 1, M, \delta_3/2, s), \\ \varepsilon_{1.6}(m, n, Q + 1, M, \delta_2/2, s), \quad \varepsilon_{1.6}(m, n, Q, M, 1/4, s), \\ \varepsilon_{1.9}(m, n, \min\{\delta_2/3, \delta_3/2\}, s, \max\{M, 2\}), \quad \varepsilon_{1.6}(m, n, Q, M, \delta_2/3, s), \\ \varepsilon_{1.10}(m, n, Q, 1, \delta_2, s, s_0, M).\end{gathered}$$

Clearly, ε_1 satisfies the same inequalities as ε and one can assume $a = 0$, and $r = 1$. □

[1] Recall from [Alm00, T.1 (23)] that $(\pi \bowtie \sigma)(x) = (\pi(x), \sigma(x))$ for $x \in \mathbb{R}^m$.

Proof of (1) *and* (2). Since $\theta^{*n}(\mu, \cdot)$ is a Borel function, one may verify that A and B are Borel sets (cp. [Fed69, 2.9.14]).

First, the following *basic properties of* A are proved: For $x \in A \cap \operatorname{spt} \mu$

$$\theta_*^n(\mu, x) \geq \delta_3/2,$$
$$\{\xi \in \pi^{-1}(\bar{B}_1^n(0)) : |T(\xi - x)| > s|\xi - x|\} \subset \sigma^{-1}(\bar{B}_{\min\{\lambda/2, \delta_4\}}(\sigma(x))),$$
$$\sigma(A \cap \operatorname{spt} \mu) \subset \bar{B}_{h-\delta_4}^m(0).$$

The first is implied by [Men08b, 2.6]. The second is a consequence of the fact that for $\xi \in \pi^{-1}(\bar{B}_1^n(0))$ with $|T(\xi - x)| > s|\xi - x|$

$$|\sigma(\xi) - \sigma(x)| < (s^{-2} - 1)^{1/2} |\pi(\xi) - \pi(x)| \leq 2(s^{-2} - 1)^{1/2} \leq \min\{\lambda/2, \delta_4\}.$$

To prove the third, note that 1.6 applied with

Q, δ, X, d, r, t, and f replaced by

1, $\delta_3/2$, $\{x\}$, 1, 2, 1, and $T^\perp | \{x\}$

yields

$$\mu\big(\pi^{-1}(\bar{B}_1^n(0)) \cap \sigma^{-1}(\bar{B}_{\delta_4}(\sigma(x)))\big) \geq (1 - \delta_3/2)\omega_n,$$

so that $h - \delta_4 < |\sigma(x)| \leq h$ would be incompatible with

$$\mu(C(T, 0, 1, h + \delta_4) \sim C(T, 0, 1, h - 2\delta_4)) \leq (1 - \delta_3)\omega_n.$$

Next, it will be shown *if* $X \subset A \cap \operatorname{spt} \mu$, $\theta^n(\mu, x) \in \mathbb{N}_0$ *for* $x \in X$,

$$s^{-1}|T(x_2 - x_1)| \leq |x_2 - x_1| \quad \textit{whenever } x_1, x_2 \in X,$$

then $\sum_{x \in X} \theta^n(\mu, x) \leq Q$. Using the basic properties of A to verify

$$\begin{aligned}\{\xi \in B_1(T^\perp(x)) : |T(\xi - x)| > s|\xi - x|\} &\subset \pi^{-1}(\bar{B}_1^n(0)) \cap \sigma^{-1}(\bar{B}_{\delta_4}(\sigma(x))) \\ &\subset C(T, 0, 1, h)\end{aligned}$$

there holds

$$\begin{aligned}\mu\big(\textstyle\bigcup_{x \in X} \{\xi \in B_1(T^\perp(x)) : |T(\xi - x)| > s|\xi - x|\}\big) &\leq \mu(C(T, 0, 1, h)) \\ &\leq (Q + 1 - \delta_2)\omega_n\end{aligned}$$

and 1.6 applied with

Q, δ, d, r, t, and f replaced by

$Q + 1$, $\delta_2/2$, 1, 2, 1, and $T^\perp | X$

yields

$$\textstyle\sum_{x \in X} \theta^n(\mu, x) < Q + \delta_2/2 < Q + 1,$$

hence $\sum_{x \in X} \theta^n(\mu, x) \leq Q$. In particular, $\sum_{x \in A(y)} \theta^n(\mu, x) \leq Q$ whenever $y \in \bar{B}_1^n(0)$ and $\theta^n(\mu, x) \in \mathbb{N}_0$ for each $x \in A(y)$.

Let Y be the set of all $y \in \bar{B}_1^n(0)$ such that

$$\textstyle\sum_{x \in A(y)} \theta^n(\mu, x) = Q \quad \text{and} \quad \theta^n(\mu, x) \in \mathbb{N}_0 \text{ for } x \in A(y),$$

Z be the set of all $z \in \bar{B}_1^n(0)$ such that

$$\textstyle\sum_{x \in A(z)} \theta^n(\mu, x) \leq Q - 1 \quad \text{and} \quad \theta^n(\mu, x) \in \mathbb{N}_0 \text{ for } x \in A(z),$$

and $N = \bar{B}_1^n(0) \sim (Y \cup Z)$. Clearly, $Y \cap Z = \emptyset$. Note by the concluding remark of the preceding paragraph $\mathcal{L}^n(N) = 0$ because $\theta^n(\mu, x) \in \mathbb{N}_0$ for $\mathcal{H}^n$ almost all $x \in U$. Since $\theta^n(\mu, \cdot)$ is a Borel function whose domain is a Borel set and A is a

Borel set, Y and Z are $\mathcal{L}^n$ measurable by [Fed69, 3.2.22 (3)]. Let $f : Y \to Q_Q(\mathbb{R}^m)$ be defined by

$$f(y) = \sigma_\#\big(\textstyle\sum_{x \in A(y)} \theta^n(\mu, x)[\![x]\!]\big) \quad \text{whenever } y \in Y.$$

One infers from the assertion of the preceding paragraph and [LP86, Theorem 1.1.3] (cp. e.g. [Men08a, D.12])

$$\mathcal{G}(f(y_2), f(y_1)) \le Q^{1/2}(s^{-2} - 1)^{1/2}|y_2 - y_1| \quad \text{for } y_1, y_2 \in Y.$$

(1) and (2) are now evident. □

Proof of (3). For the estimate some preparations are needed. Let ν denote the Radon measure defined by the requirement

$$\nu(X) = \textstyle\int_X J^\mu T \,\mathrm{d}\mu \quad \text{for every Borel subset } X \text{ of } U$$

where J^μ denotes the Jacobian with respect μ. Note

$$|T_x\mu - T| \le \varepsilon \quad \text{for } \mu \text{ almost all } x \in A,$$

hence $1 - J^\mu T(x) \le 1 - (J^\mu T)(x)^2 \le n\varepsilon^2$. Therefore

$$(1 - n\varepsilon^2)\, \mu \llcorner A \le \nu \llcorner A.$$

This implies the *coarea estimate*

$$\begin{aligned} &(1 - n\varepsilon^2)\, \mu\big(C(T, 0, 1, h) \cap \pi^{-1}(W)\big) \\ &\le \mu\big(B \cap \pi^{-1}(W)\big) + Q\mathcal{L}^n(Y \cap W) + (Q - 1)\mathcal{L}^n(Z \cap W) \end{aligned}$$

for every subset W of $\mathbb{R}^n$; in fact the estimate holds for every Borel set by [Fed69, 3.2.22 (3)] and $\pi(\mu \llcorner B)$ is a Radon measure by [Fed69, 2.2.17]. Also note that in view of the choice of $\Gamma_{(3)}$ one can assume

$$\mu(B) \le (\delta_1/4)\omega_n,$$

which implies $\mathcal{L}^n(Y) > 0$ because it follows from the coarea estimate applied with $W = \bar{B}^n_1(0)$

$$\begin{aligned} (Q - 1 + \delta_1/2)\omega_n &\le (1 - n\varepsilon^2)\mu(C(T, 0, 1, h)) \\ &\le \mu(B) + Q\mathcal{L}^n(Y) + (Q - 1)\mathcal{L}^n(Z) \\ &\le (\delta_1/4)\omega_n + (Q - 1 + \delta_1/4)\omega_n + \mathcal{L}^n(Y) - (\delta_1/4)\mathcal{L}^n(Z), \end{aligned}$$

hence $\mathcal{L}^n(Z) \le (4/\delta_1)\mathcal{L}^n(Y)$.

In order to derive an upper bound for the $\mathcal{L}^n$ measure of Z, the following assertion will be proved. *If $z \in Z$ with $\theta^n(\mathcal{L}^n \llcorner \mathbb{R}^n \sim Z, a) = 0$, then there exist $\zeta \in \mathbb{R}^n$ and $0 < t < \infty$ with*

$$z \in \bar{B}_t(\zeta) \subset \bar{B}^n_1(0), \quad \mathcal{L}^n(\bar{B}_{5t}(\zeta)) \le 6 \cdot 5^n\, \mu\big(B \cap \pi^{-1}(\bar{B}_t(\zeta))\big).$$

Since $\mathcal{L}^n(Y) > 0$, some element $\bar{B}_t(\zeta)$ of the family of balls

$$\{\bar{B}_\theta((1 - \theta)z) : 0 < \theta \le 1\}$$

will satisfy

$$z \in \bar{B}_t(\zeta) \subset \bar{B}^n_1(0), \quad 0 < \mathcal{L}^n(Y \cap \bar{B}_t(\zeta)) \le \tfrac{1}{2}\mathcal{L}^n(Z \cap \bar{B}_t(\zeta)).$$

Hence there exists $y \in Y \cap B_t(\zeta)$. Noting for $\xi \in A(y)$ with $\theta^n(\mu, \xi) > 0$, and $\eta \in \mathbb{R}^{n+m}$ with $|\eta_{-\pi^*(\zeta-y),1}(\xi) - \eta| < t$,[2]

$$t \leq 1, \quad \pi(\xi) = y,$$
$$|\pi(\eta) - \zeta| = |\pi(\xi + \pi^*(\zeta - y) - \eta)| \leq |\eta_{-\pi^*(\zeta-y),1}(\xi) - \eta| < t,$$
$$B_t(\eta_{-\pi^*(\zeta-y),1}(\xi)) \subset \pi^{-1}(\bar{B}_t(\zeta)),$$

and, recalling the basic properties of A,

$$\{\kappa \in B_t(\eta_{-\pi^*(\zeta-y),1}(\xi)) : |T(\kappa - \xi)| > s|\kappa - \xi|\} \subset C(T, 0, 1, h) \cap \pi^{-1}(\bar{B}_t(\zeta)),$$

one can apply 1.6 with

$$\delta,\ X,\ d,\ r, \text{ and } f \text{ replaced by}$$
$$1/4,\ \{\xi \in A(y) : \theta^n(\mu, \xi) > 0\},\ t,\ 2, \text{ and}$$
$$\eta_{-\pi^*(\zeta-y),1} | \{\xi \in A(y) : \theta^n(\mu, \xi) > 0\}$$

to obtain

$$(Q - 1/4)\omega_n t^n \leq \mu\big(C(T, 0, 1, h) \cap \pi^{-1}(\bar{B}_t(\zeta))\big).$$

The coarea estimate with $W = \bar{B}_t(\zeta)$ now implies

$$\begin{aligned}
&(Q - 1/2)\omega_n t^n \\
&\leq \mu\big(B \cap \pi^{-1}(\bar{B}_t(\zeta))\big) + Q\mathcal{L}^n(Y \cap \bar{B}_t(\zeta)) + (Q-1)\mathcal{L}^n(Z \cap \bar{B}_t(\zeta)) \\
&= \mu\big(B \cap \pi^{-1}(\bar{B}_t(\zeta))\big) + (Q - 1/2)\omega_n t^n \\
&\quad + \tfrac{1}{2}\mathcal{L}^n(Y \cap \bar{B}_t(\zeta)) - \tfrac{1}{2}\mathcal{L}^n(Z \cap \bar{B}_t(\zeta)),
\end{aligned}$$

hence

$$\tfrac{2}{3}\mathcal{L}^n(\bar{B}_t(\zeta)) \leq \mathcal{L}^n(Z \cap \bar{B}_t(\zeta)) \leq 4\,\mu\big(B \cap \pi^{-1}(\bar{B}_t(\zeta))\big)$$

and the assertion follows.

$\mathcal{L}^n$ almost all $z \in Z$ satisfy the assumptions of the last assertion (cf. [Fed69, 2.9.11]) and Vitali's covering theorem (cf. [Fed69, 2.8.5]) implies

$$\mathcal{L}^n(Z) \leq 6 \cdot 5^n\, \mu(B).$$

Clearly,

$$\mathcal{L}^n(\pi(B)) \leq \mathcal{H}^n(B) \leq \mu(B).$$

Since $C \sim N \subset Z \cup \pi(B)$, it follows

$$\mathcal{L}^n(C) \leq (1 + 6 \cdot 5^n)\, \mu(B).$$

Finally, applying the coarea estimate with $W = C$ yields

$$(1 - n\varepsilon^2)\, \mu(D) \leq \mu(B) + Q\mathcal{L}^n(C) \leq (1 + Q + 6Q \cdot 5^n)\, \mu(B). \qquad \square$$

Proof of (4). Assuming now that x_1 and y satisfy the conditions of (4), it will be shown that one can take $\lambda_{(4)} = \lambda$. Verifying

$$\{\xi \in \pi^{-1}(\bar{B}^n_1(0)) : |T(\xi - x_1)| > s|\xi - x_1|\} \subset \sigma^{-1}(\bar{B}_{\min\{\lambda/2, \delta_4\}}(\sigma(x_1))),$$

defining $\delta_5 = \min\{\delta_2/3, \delta_3/2\}$ and applying 1.9 with

$$\delta,\ M,\ a,\ r,\ d,\ t, \text{ and } \zeta \text{ replaced by}$$
$$\delta_5,\ \max\{M, 2\},\ x_1,\ 2,\ 1,\ 1, \text{ and } -T(x_1)$$

yields

$$\mu\big(\pi^{-1}(\bar{B}^n_1(0)) \cap \sigma^{-1}(\bar{B}_{\min\{\lambda/2,\delta_4\}}(\sigma(x_1)))\big) \geq (1 - \delta_5)\omega_n$$

[2]Recall from [Sim83] that the functions $\eta_{a,r} : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ are given by $\eta_{a,r}(x) = r^{-1}(x-a)$ for $a, x \in \mathbb{R}^{n+m}$, $0 < r < \infty$.

so that $h - \delta_4 < |\sigma(x_1)| \leq h$ would be incompatible with

$$\mu\big(C(T,0,1,h+\delta_4) \sim C(T,0,1,h-2\delta_4)\big) \leq (1-\delta_3)\omega_n$$

and the first part of (4) follows.

To prove the second part, one defines $X = \{\xi \in A(y) : \theta^n(\mu,\xi) \in \mathbb{N}\}$ and first observes that 1.6 applied with

$$\delta,\ d,\ r,\ t, \text{ and } f \text{ replaced by,}$$
$$\delta_2/3,\ 1,\ 2,\ 1, \text{ and } \eta_{\pi^*(y),1}|X$$

yields

$$\mu\big(\textstyle\bigcup_{x\in X}\{\xi \in B_1(x-\pi^*(y)) : |T(\xi-x)| > s|\xi - x|\}\big) \geq \big(Q - \frac{\delta_2}{3}\big)\omega_n.$$

On the other hand

$$\mu(C(T,0,1,h)) \leq (Q+1-\delta_2)\omega_n.$$

Therefore, using the basic properties of A, for some $x \in X$

$$C(T,0,1,h) \cap \sigma^{-1}(\bar{B}_{\lambda/2}(\sigma(x_1))) \cap \sigma^{-1}(\bar{B}_{\lambda/2}(\sigma(x))) \neq \emptyset,$$

hence $|\sigma(x_1 - x)| \leq \lambda$ and

$$\operatorname{dist}(x_1, X) \leq |\pi(x_1 - x)| + |\sigma(x_1 - x)| \leq 2\lambda = \lambda_{1.10}(n,\delta_2,s_0)/2 \leq 1.$$

Finally, the point $x_2 \in X$ may be constructed by applying 1.10 (2) with

$$\delta_1,\ \lambda,\ d,\ r,\ t,\ \zeta, \text{ and } y \text{ replaced by}$$
$$1,\ \lambda_{1.10}(n,\delta_2,s_0),\ 1,\ 2,\ 1,\ -\pi^*(y), \text{ and } x_1$$

noting

$$\{\xi \in B_1(x-\pi^*(y)) : |T(\xi - x)| > s_0|\xi - x|\} \subset C(T,0,1,h)$$

for $x \in X$.

The postscript follows readily from the second part and $\varepsilon_1 \leq \varepsilon \leq (2\gamma_n)^{-1}$. □

Proof of (5). Recalling $(\mu \llcorner A)/2 \leq \nu \llcorner A$ and $\mathscr{L}^n(N) = 0$, it is enough to prove

$$\overline{Y} \subset N \cup Y, \quad \pi^{-1}(\overline{Y}) \cap G \subset A \cap \operatorname{spt}\mu$$

in view of the coarea formula [Fed69, 3.2.22 (3)].

Suppose for this purpose $y \in \overline{Y}$. Since f is Lipschitzian, there exists a unique $S \in Q_Q(\mathbb{R}^m)$ such that

$$(y,S) \in \overline{\operatorname{graph} f}.$$

Let $R = \pi^{-1}(\{y\}) \cap \sigma^{-1}(\operatorname{spt} S)$. Since $A \cap \operatorname{spt}\mu$ is closed (cp. [Fed69, 2.9.14]),

$$R \subset A \cap \operatorname{spt}\mu$$

and (4) implies $G \cap \pi^{-1}(\{y\}) \subset R$, the second inclusion follows.

Choose a sequence $y_i \in Y$ with $y_i \to y$ as $i \to \infty$ and abbreviate $X_i = \{\xi \in A(y_i) : \theta^n(\mu,\xi) \in \mathbb{N}\}$ for $i \in \mathbb{N}$. 1.6 applied with

$$\delta,\ X,\ d,\ r, \text{ and } f \text{ replaced by}$$
$$1/4,\ X_i,\ 0,\ 2, \text{ and } \mathbb{1}_{X_i}$$

yields for $i \in \mathbb{N}$

$$\mu\big(\textstyle\bigcup_{x\in X_i}\bar{B}_t(x)\big) \geq (Q - 1/4)\omega_n t^n \quad \text{whenever } 0 < t < 2.$$

Since $f(y_i) \to S$ in Hausdorff distance as $i \to \infty$ the same estimate holds with X_i replaced by R and

$$Q - 1/4 \le \limsup_{t \downarrow 0} \frac{\mu\big(\bigcup_{x \in R} \bar{B}_t(x)\big)}{\omega_n t^n} \le \sum_{x \in R} \theta^{*n}(\mu, x)$$

implies $y \notin Z$, hence the first inclusion. □

Proof of (6). Let $\psi := \mu \llcorner G$. Using $(\pi(\psi)) \llcorner Y \le 2(\pi(\nu \llcorner G)) \llcorner Y \le 2Q\mathcal{L}^n \llcorner Y$,

$$\{x \in G \cap \pi^{-1}(Y) : \operatorname{dist}(x, \operatorname{spt} P) > \gamma\} \subset G \cap \pi^{-1}(\{y \in Y : g(y) > \gamma\})$$

for $0 < \gamma < \infty$, one infers

$$\| \operatorname{dist}(\cdot, \operatorname{spt} P)\|_{L^q(\mu \llcorner G \cap \pi^{-1}(Y))} \le 2Q \|g\|_{L^q(\mathcal{L}^n \llcorner Y)}.$$

Hence only $\| \operatorname{dist}(\cdot, \operatorname{spt} P)\|_{L^q(\mu \llcorner G \sim \pi^{-1}(Y))}$ needs to be estimated in the first part of (6).

Whenever $z \in \bar{B}^n_1(0) \sim \overline{Y}$ there exist $\zeta \in \mathbb{R}^n$ and $0 < t \le \lambda/6$ such that

$$z \in \bar{B}_t(\zeta) \subset \bar{B}^n_1(0), \quad \mathcal{L}^n(\bar{B}_t(\zeta) \cap Y) = \mathcal{L}^n(\bar{B}_t(\zeta) \sim Y)$$

as may be verified by consideration of the family of closed balls

$$\{\bar{B}_\theta((1-\theta)z) : 0 < \theta \le \lambda\}.$$

Therefore [Fed69, 2.8.5] yields a countable set I and $\zeta_i \in \mathbb{R}^n$, $0 < t_i \le \lambda/6$ and $y_i \in Y \cap \bar{B}_{t_i}(\zeta_i)$ for each $i \in I$ such that

$$\bar{B}_{t_i}(\zeta_i) \subset \bar{B}^n_1(0), \quad \mathcal{L}^n(\bar{B}_{t_i}(\zeta_i) \cap Y) = \mathcal{L}^n(\bar{B}_{t_i}(\zeta_i) \sim Y),$$
$$\bar{B}_{t_i}(\zeta_i) \cap \bar{B}_{t_j}(\zeta_j) = \emptyset \quad \text{whenever } i, j \in I \text{ with } i \ne j,$$
$$\bar{B}^n_1(0) \sim \overline{Y} \subset \textstyle\bigcup_{i \in I} E_i \subset \bar{B}^n_1(0)$$

where $E_i = \bar{B}_{5t_i}(\zeta_i) \cap \bar{B}^n_1(0)$ for $i \in I$. Let

$$h_i := \mathcal{G}(f(y_i), S), \quad X_i := \{\xi \in A(y_i) : \theta^n(\mu, \xi) \in \mathbb{N}\}$$

for $i \in I$, $J := \{i \in I : h_i \ge 18t_i\}$, and $K := I \sim J$.

In view of (5) there holds

$$\|d\|_{L^q(\mu \llcorner G \sim \pi^{-1}(Y))} \le \|d\|_{L^q(\psi \llcorner \pi^{-1}(\bigcup_{j \in J} E_j))} + \|d\|_{L^q(\psi \llcorner \pi^{-1}(\bigcup_{i \in K} E_i))}$$

for every ψ measurable function $d : \mathbb{R}^{n+m} \to [0, \infty[$. In order to estimate the terms on the right hand side for $d = \operatorname{dist}(\cdot, \operatorname{spt} P)$, two observations will be useful. *If $i \in I$, $x_1 \in G \cap \pi^{-1}(E_i)$, then*

$$\operatorname{dist}(x_1, \operatorname{spt} P) \le 6t_i + h_i;$$

in fact $|\pi(x_1) - y_i| \le 6t_i \le \lambda$ and (4) yields a point $x_2 \in X_i$ and

$$\big|T^\perp(x_2 - x_1)\big| \le L\,|T(x_2 - x_1)| = L\,|\pi(x_1) - y_i| \le 6t_i,$$

implying

$$\operatorname{dist}(x_1, \operatorname{spt} P) \le \big|T^\perp(x_2 - x_1)\big| + \operatorname{dist}(x_2, \operatorname{spt} P) \le 6t_i + h_i.$$

Moreover,

$$|x_2 - x_1| \le |T(x_2 - x_1)| + \big|T^\perp(x_2 - x_1)\big| \le 12t_i, \quad x_1 \in \bar{B}_{12t_i}(x_2),$$

hence

$$G \cap \pi^{-1}(E_i) \subset \textstyle\bigcup_{x \in X_i} \bar{B}_{12t_i}(x)$$

and 1.10 (1) applied with

$$\delta_1,\ s,\ \lambda,\ X,\ d,\ r,\ t,\ \zeta, \text{ and } \tau \text{ replaced by}$$
$$1,\ 0,\ \lambda_{1.10}(n, \delta_2, s_0),\ X_i,\ 1,\ 2,\ 1,\ -\pi^*(y_i), \text{ and } 12t_i$$

yields

$$\psi\big(\pi^{-1}(E_i)\big) \le (Q+1)\omega_n(12t_i)^n \quad \textit{whenever } i \in I.$$

Now, the first term will be estimated. Note, if $j \in J$, then

$$\operatorname{dist}(x, \operatorname{spt} P) \le \tfrac{4}{3}h_j \quad \text{whenever } x \in G \cap \pi^{-1}(E_j),$$
$$\tfrac{4}{3}h_j \le 2\mathcal{G}(f(y), S) \quad \text{whenever } y \in Y \cap \bar{B}_{t_j}(\zeta_j),$$

because

$$\mathcal{G}(f(y), S) \ge \mathcal{G}(f(y_j), S) - L|y - y_j| \ge h_j - 2Lt_j \ge \tfrac{2}{3}h_j.$$

Using this fact and the preceding observations, one estimates with $J(\gamma) := \{j \in J : \frac{4}{3}h_j > \gamma\}$ for $0 < \gamma < \infty$

$$\begin{aligned}
\psi\big(\pi^{-1}(\textstyle\bigcup_{j\in J} E_j) \cap \{x \in \mathbb{R}^{n+m} : \operatorname{dist}(x, \operatorname{spt} P) > \gamma\}\big) &\le \textstyle\sum_{j \in J(\gamma)} \psi\big(\pi^{-1}(E_j)\big) \\
\le \textstyle\sum_{j\in J(\gamma)} (Q+1)\omega_n(12t_j)^n &\le (Q+1)(12)^n \mathcal{L}^n\big(\textstyle\bigcup_{j\in J(\gamma)} \bar{B}_{t_j}(\zeta_j)\big) \\
&\le 2(Q+1)(12)^n \mathcal{L}^n\big(\textstyle\bigcup_{j\in J(\gamma)} \bar{B}_{t_j}(\zeta_j) \cap Y\big) \\
&\le 2(Q+1)(12)^n\, \mathcal{L}^n(\{y \in Y : \mathcal{G}(f(y), S) > \gamma/2\}),
\end{aligned}$$

hence

$$\|\operatorname{dist}(\cdot, \operatorname{spt} P)\|_{L^q(\psi \llcorner \pi^{-1}(\bigcup_{j\in J} E_j))} \le (2(Q+1)(12)^n)2\,\|g\|_{L^q(\mathcal{L}^n \llcorner Y)}.$$

To estimate the second term, one notes, if $i \in K$, $x \in G \cap \pi^{-1}(E_i)$, then

$$\operatorname{dist}(x, \operatorname{spt} P) < 24t_i.$$

Therefore one estimates with $K(\gamma) := \{i \in K : 24t_i > \gamma\}$ for $0 < \gamma < \infty$ and $u : \mathbb{R}^n \to \mathbb{R}$ defined by $u = \sum_{i\in I} 2t_i\, \chi_{\bar{B}_{t_i}(\zeta_i)}$

$$\begin{aligned}
\psi\big(\pi^{-1}(\textstyle\bigcup_{i\in K} E_i) \cap \{x \in \mathbb{R}^{n+m} : \operatorname{dist}(x, \operatorname{spt} P) > \gamma\}\big) &\le \textstyle\sum_{i\in K(\gamma)} \psi\big(\pi^{-1}(E_i)\big) \\
\le \textstyle\sum_{i\in K(\gamma)} (Q+1)\omega_n(12t_i)^n &\le (Q+1)(12)^n \mathcal{L}^n\big(\textstyle\bigcup_{i\in K(\gamma)} \bar{B}_{t_i}(\zeta_i)\big) \\
&\le (Q+1)(12)^n \mathcal{L}^n\big(\{y \in \mathbb{R}^n : u(y) > \gamma/(12)\}\big),
\end{aligned}$$

hence

$$\|\operatorname{dist}(\cdot, \operatorname{spt} P)\|_{L^q(\psi \llcorner \pi^{-1}(\bigcup_{i\in K} E_i))} \le (Q+1)(12)^{n+1}\|u\|_{L^q(\mathcal{L}^n)}.$$

Combining these two estimates and

$$\mathcal{L}^n\big(\textstyle\bigcup_{i\in I} \bar{B}_{t_i}(\zeta_i)\big) \le 2\mathcal{L}^n(\bar{B}^n_1(0) \sim Y),$$
$$\begin{aligned}
\textstyle\int |u|^q \, \mathrm{d}\mathcal{L}^n = \textstyle\sum_{i\in I} (2t_i)^q \omega_n (t_i)^n &\le 2^q \omega_n^{-q/n} \big(\textstyle\sum_{i\in I} \mathcal{L}^n(\bar{B}_{t_i}(\zeta_i))\big)^{1+q/n} \\
&\le 2^{q+1+q/n}\omega_n^{-q/n}\big(\mathcal{L}^n(\bar{B}^n_1(0) \sim Y)\big)^{1+q/n},
\end{aligned}$$

one obtains the first part of the conclusion of (6).

To prove the second part, suppose $x_1 \in G$. Since

$$\bar{B}_\theta((1-\theta)\pi(x_1)) \subset \bar{B}^n_1(0), \quad \mathcal{L}^n(\bar{B}_\theta((1-\theta)\pi(x_1)) \cap Y) > 0$$

for $(\mathcal{L}^n(\bar{B}^n_1(0) \sim Y)/\omega_n)^{1/n} < \theta < 1$, there exists for any $\delta > 0$ a $y \in Y$ with

$$\mathcal{G}(f(y), S) \le \|g\|_{L^\infty(\mathcal{L}^n \llcorner Y)},$$
$$|\pi(x_1) - y| \le 2\big(\mathcal{L}^n(\bar{B}^n_1(0) \sim Y)/\omega_n\big)^{1/n} + \delta,$$

in particular $|\pi(x_1) - y| \le \lambda$ for small δ. Therefore (4) may be applied to construct a point $x_2 \in A(y)$ with $\theta^n(\mu, x_2) \in \mathbb{N}$ and

$$\big|T^\perp(x_2 - x_1)\big| \le L\,|T(x_2 - x_1)| \le |\pi(x_1) - y|.$$

Finally,

$$\begin{aligned} \operatorname{dist}(x_1, \operatorname{spt} P) &\le \operatorname{dist}(x_2, \operatorname{spt} P) + \left| T^\perp(x_2 - x_1) \right| \\ &\le \mathcal{G}(f(y), S) + 2\big(\mathcal{L}^n(\bar{B}^n_1(0) \sim Y)/\omega_n\big)^{1/n} + \delta \end{aligned}$$

and δ can be chosen arbitrarily small. ☐

Proof of (7). Combine (1), (4), [Alm00, 1.1 (9)–(11)], 1.11, and estimates for orthogonal projections, see e.g. [All72, 8.9 (5)]. ☐

1.15. *Remark.* The idea to prove (4) was taken from [Alm00, 3.8 (4)].

2. A Sobolev Poincaré type inequality for integral varifolds

In this section the two main theorems, 2.8 and 2.10, are proved, the first being a Sobolev Poincaré type inequality at some fixed scale r but involving of necessity mean curvature, the second considering the limit r tends to 0. For this purpose the distance of an integral n varifold from a Q valued plane is introduced. One cannot use ordinary planes in 2.8 (without additional assumptions) as may be seen from the fact that any Q valued plane is stationary with vanishing tilt. In 2.10–2.12 an answer to the Problem posed in the introduction is provided.

2.1. **Definition.** Suppose $m, n, Q \in \mathbb{N}$, $1 \le q \le \infty$, $a \in \mathbb{R}^{n+m}$, $0 < r < \infty$, $0 < h \le \infty$, $T \in G(n+m, n)$, P is a Q valued plane parallel to T (see 1.12), μ is an integral n varifold in an open superset of $C(T, a, r, h)$, A is the $\mathcal{H}^n$ measurable set of all $x \in T \cap \bar{B}_r(T(a))$ such that for some $R(x), S(x) \in Q_Q(\mathbb{R}^{n+m})$

$$\begin{aligned} \|R(x)\| &= \theta^n(P \llcorner C(T, a, r, h), \cdot)\mathcal{H}^0 \llcorner T^{-1}(\{x\}), \\ \|S(x)\| &= \theta^n(\mu \llcorner C(T, a, r, h), \cdot)\mathcal{H}^0 \llcorner T^{-1}(\{x\}) \end{aligned}$$

and $g : A \to \mathbb{R}$ is the $\mathcal{H}^n$ measurable function defined by $g(x) = \mathcal{G}(R(x), S(x))$ for $x \in A$.[3]

Then the *q height of μ with respect to P in* $C(T, a, r, h)$, denoted by

$$H_q(\mu, a, r, h, P),$$

is defined to be the sum of

$$r^{-1-n/q} \| \operatorname{dist}(\cdot, \operatorname{spt} P) \|_{L^q(\mu \llcorner C(T,a,r,h))}$$

and the infimum of the numbers

$$r^{-1-n/q} \|g\|_{L^q(\mathcal{H}^n \llcorner Y)} + r^{-1-n/q} \mathcal{H}^n(T \cap \bar{B}_r(T(a)) \sim Y)^{1/q+1/n}$$

corresponding to all $\mathcal{H}^n$ measurable subsets Y of A. The *q tilt of μ with respect to T in* $C(T, a, r, h)$ is defined by

$$T_q(\mu, a, r, h, T) = r^{-n/q} \|T_\mu - T\|_{L^q(\mu \llcorner C(T,a,r,h))}.$$

Moreover,

$$H_q(\mu, a, r, h, Q, T)$$

is defined to be the infimum of all numbers $H_q(\mu, a, r, h, P)$ corresponding to all Q valued planes P parallel to T.

[3]The asserted measurabilities may be shown by use of the coarea formula (cf. [Fed69, 3.2.22 (3)]).

2.2. *Remark.* $T_q(\mu, a, r, h, T)$ generalises $\operatorname{tiltex}_\mu$ in an obvious way.

$H_q(\mu, a, r, h, P)$ measures the distance of μ in $C(T, a, r, h)$ from the Q valued plane P. To obtain a reasonable definition of distance, neither the first nor the second summand would be sufficient. The first summand is 0 if $\mu = P \llcorner B$ for some $\mathcal{H}^n$ measurable set B. The second summand is 0 if $\mu = P + \mathcal{H}^n \llcorner B$ for some $\mathcal{H}^n$ measurable subset B of $C(T, a, r, h)$ with $\mathcal{H}^n(B) < \infty$ and $\mathcal{H}^n(T(B)) = 0$. From a more technical point of view, the second summand is added because it is useful in the iteration procedure occurring in 2.10 where the distance of Q valued planes corresponding to different radii r has to be estimated.

2.3. *Remark.* One readily checks that $H_q(\mu, a, r, h, P) = 0$ implies

$$\mu \llcorner C(T, a, r, h) = P \llcorner C(T, a, r, h)$$

and $H_q(\mu, a, r, h, Q, T) = 0$, $h < \infty$ implies $H_q(\mu, a, r, h, P) = 0$ for some Q valued plane P parallel to T.

More generally, the infima occurring in the definition of $H_q(\mu, a, r, h, P)$ and $H_q(\mu, a, r, h, Q, T)$ are attained. However, this latter fact will neither be used nor proved in this work.

2.4. **Definition.** Suppose $m, n, Q \in \mathbb{N}$, $S \in Q_Q(\mathbb{R}^m)$, $1 \leq q \leq \infty$, A is $\mathcal{L}^n$ measurable, and $f : A \to Q_Q(\mathbb{R}^m)$ is an $\mathcal{L}^n \llcorner A$ measurable function.

Then the *q height of f with respect to S* is defined to be the $L^q(\mathcal{L}^n \llcorner A)$ (semi) norm of the function mapping $x \in A$ to $\mathcal{G}(f(x), S)$, denoted by $h_q(f, S)$, and, if f is additionally Lipschitzian, then the *q tilt of f* is defined to be the $L^q(\mathcal{L}^n \llcorner A)$ (semi) norm of the function mapping $x \in A$ to $|\operatorname{ap} Af(x)|$, denoted by $t_q(f)$. Moreover, the *q height of f* is defined to be the infimum of the numbers $h_q(f, S)$ corresponding to all $S \in Q_Q(\mathbb{R}^m)$ and denoted by $h_q(f)$.

2.5. **Theorem.** *Suppose $m, n, Q \in \mathbb{N}$, $f : \bar{B}^n_1(0) \to Q_Q(\mathbb{R}^m)$, and $\operatorname{Lip} f < \infty$.*

Then the following two statements hold:

(1) *If $1 \leq q < n$, $q^* = \frac{qn}{n-q}$, then there exists a positive, finite number $\Gamma_{(1)}$ depending only on m, n, Q, and q such that*
$$h_{q^*}(f) \leq \Gamma_{(1)}\, t_q(f).$$

(2) *If $q < n \leq \infty$, then there exists a positive, finite number $\Gamma_{(2)}$ depending only on m, n, Q, and q such that*
$$h_\infty(f) \leq \Gamma_{(2)}\, t_q(f).$$

Proof. Using Almgren's functions $\boldsymbol{\xi}$ and $\boldsymbol{\rho}$ [Alm00, 1.2 (3), 1.3 (1), 1.4 (3) (5)], the assertion is readily deduced from classical embedding results. ☐

2.6. **Lemma.** *Suppose $m, n, Q \in \mathbb{N}$, $1 \leq M < \infty$, and $0 < \delta \leq 1$.*

Then there exists a positive, finite number ε with the following property.

If $a \in \mathbb{R}^{n+m}$, $0 < r < \infty$, $0 < h \leq \infty$, $T \in G(n + m, n)$, $\delta r < h$, μ is an integral n varifold in an open superset of $C(T, a, 3r, h + 2r)$ with locally bounded first variation satisfying

$$(Q - 1 + \delta)\omega_n r^n \leq \mu(C(T, a, r, h)) \leq (Q + 1 - \delta)\omega_n r^n,$$
$$\mu(C(T, a, r, h + \delta r) \sim C(T, a, r, h - \delta r)) \leq (1 - \delta)\omega_n r^n,$$
$$\mu(C(T, a, 3r, h + 2r)) \leq M\omega_n r^n,$$
$$\|\delta\mu\|(C(T, a, 3r, h + 2r)) \leq \varepsilon r^{n-1}, \quad T_1(\mu, a, 3r, h + 2r, T) \leq \varepsilon,$$

G is the set of all $x \in C(T, a, r, h) \cap \operatorname{spt} \mu$ such that

$$\|\delta\mu\|(\bar{B}_\varrho(x)) \leq (2\gamma_n)^{-1}\, \mu(\bar{B}_\varrho(x))^{1-1/n} \quad \textit{whenever } 0 < \varrho < 2r,$$

and A is the set defined as G with ε replacing $(2\gamma_n)^{-1}$, then the following two statements hold:

(1) *If $1 \leq q < n$, $q^* = \frac{nq}{n-q}$, then*

$$H_{q^*}(\mu \llcorner G, a, r, h, Q, T) \leq \Gamma_{(1)} \big(T_q(\mu, a, 3r, h+2r, T) + (r^{-n} \mu(C(T,a,r,h) \sim A))^{1/q} \big)$$

where $\Gamma_{(1)}$ is a positive, finite number depending only on m, n, Q, M, δ, and q.

(2) *If $n < q \leq \infty$, then*

$$H_\infty(\mu \llcorner G, a, r, h, Q, T) \leq \Gamma_{(2)} \big(T_q(\mu, a, 3r, h+2r, T) + (r^{-n} \mu(C(T,a,r,h) \sim A))^{1/q} \big).$$

where $\Gamma_{(2)}$ is a positive, finite number depending only on m, n, Q, M, δ, and q.

Proof. Let $\Gamma_0 := \operatorname{Lip}(\boldsymbol{\xi}^{-1}) \operatorname{Lip}(\boldsymbol{\varrho}) \operatorname{Lip}(\boldsymbol{\xi})$ with $\boldsymbol{\xi}$, $\boldsymbol{\varrho}$ as in [Alm00, 1.3 (2)], hence Γ_0 is a positive, finite number depending only on m and Q, and let

$$\Gamma_1 := \Gamma_{1.14(3)}(Q, n, \delta/2), \quad L := 1, \quad \varepsilon_0 := \varepsilon_{1.14}(m, n, Q, 1, M, \delta/2, \delta/2, \delta/2, \delta/2),$$
$$\varepsilon_1 := \varepsilon_0, \quad \lambda := \lambda_{1.14(4)}(n, \delta/2, \delta/2)$$

and choose $0 < \varepsilon \leq \varepsilon_0$ such that

$$\varepsilon \leq \varepsilon_0 (n\gamma_n)^{1-n}, \quad 3^n \varepsilon \leq \varepsilon_0 (n\gamma_n)^{-n},$$
$$\Gamma_1 N(n+m) 3^n \varepsilon \leq \tfrac{1}{2} \omega_1(\lambda/6) \quad \text{if } n = 1,$$
$$\Gamma_1 N(n+m) \big(3^n \varepsilon + \varepsilon^{n/(n-1)} \big) \leq \tfrac{1}{2} \omega_n (\lambda/6)^n \quad \text{if } n > 1;$$

recall that $N(n+m)$ denotes the best constant in Besicovitch's covering theorem in $\mathbb{R}^{n+m}$, see [Sim83, Lemma 4.6].

Assume $a = 0$ and $r = 1$. Choose orthogonal projections $\pi : \mathbb{R}^{n+m} \to \mathbb{R}^n$, $\sigma : \mathbb{R}^{n+m} \to \mathbb{R}^m$ with $\pi \circ \sigma^* = 0$ and $\operatorname{im} \pi^* = T$. Applying 1.14, one obtains sets Y, B and a Lipschitzian function $f : Y \to Q_Q(\mathbb{R}^m)$ with the properties listed there. Using 1.14 (1) (2) and [Alm00, 1.3 (2)] and noting the existence of a retraction of $\mathbb{R}^m$ to $\bar{B}^m_h(0)$ with Lipschitz constant 1 (cf. [Fed69, 4.1.16]), one constructs an extension $g : \bar{B}^n_1(0) \to Q_Q(\mathbb{R}^m)$ of f with $\operatorname{Lip} g \leq \Gamma_0$ and $\operatorname{spt} g(x) \subset \bar{B}^m_h(0)$ for $x \in \bar{B}^n_1(0)$.

Next, it will be verified that the set G agrees with the set G defined in 1.14; in fact for $x \in G$ using [Men08b, 2.5] yields

$$\|\delta\mu\|(B_2(x)) \leq \|\delta\mu\|(C(T,0,3,h+2)) \leq \varepsilon \leq \varepsilon_0\, \mu(B_2(x))^{1-1/n},$$
$$\textstyle\int_{B_2(x)} |T_\xi \mu - T| \, \mathrm{d}\mu(\xi) \leq \int_{C(T,0,3,h+2)} |T_\xi \mu - T| \, \mathrm{d}\mu(\xi) \leq 3^n \varepsilon \leq \varepsilon_0\, \mu(B_2(x)).$$

In order to be able to apply 1.14 (6), it will be shown

$$\mathscr{L}^n(\bar{B}^n_1(0) \sim Y) \leq \tfrac{1}{2} \omega_n (\lambda/6)^n.$$

Let B_1 be the set of all $x \in B$ such that

$$\|\delta\mu\|(\bar{B}_\varrho(x)) > \varepsilon_0\, \mu(\bar{B}_\varrho(x))^{1-1/n} \quad \text{for some } 0 < \varrho < 2,$$

and let B_2 be the set of all $x \in B$ such that

$$\textstyle\int_{\bar{B}_\varrho(x)} |T_\xi \mu - T| \, \mathrm{d}\mu(\xi) > \varepsilon_0\, \mu(\bar{B}_\varrho(x)) \quad \text{for some } 0 < \varrho < 2.$$

Clearly, Besicovitch's covering theorem implies

$$\mu(B_2) \leq N(n+m) 3^n T_1(\mu, 0, 3, h+2, T) \leq N(n+m) 3^n \varepsilon.$$

Moreover, $B_1 = \emptyset$ if $n = 1$, and Besicovitch's covering theorem implies in case $n > 1$

$$\mu(B_1) \leq N(n+m) \, \|\delta\mu\|(C(T,0,3,h+2))^{n/(n-1)} \leq N(n+m)\varepsilon^{n/(n-1)}.$$

Therefore the desired estimate is implied by 1.14 (3) and the choice of ε.

To prove part (1), let $1 \leq q < n$, $q^* = \frac{nq}{n-q}$, define

$$\Gamma_2 = 1 + (12)^{n+1} Q \max\{1, \Gamma_{1.14(6)}(q^*, n)\}, \quad \Gamma_3 = 2\Gamma_{2.5(1)}(m,n,Q,q),$$
$$\Gamma_4 = N(n+m)^{1/q}\varepsilon^{-1}3^{n/q}, \quad \Gamma_5 = 2^{1/2}Qm^{1/2}, \quad \Gamma_6 = \Gamma_0 m^{1/2}Q^{1/2},$$

choose $S \in Q_Q(\mathbb{R}^m)$ such that (see 2.4)

$$h_{q^*}(g,S) \leq \Gamma_3 \, t_q(g), \quad \operatorname{spt} S \subset \bar{B}^m_h(0)$$

with the help of 2.5 (1) noting again [Fed69, 4.1.16] and denote by

$$P := (\theta^0(\|S\|,\cdot) \circ \sigma)\mathcal{H}^n$$

the Q valued plane associated to S via σ. The estimate for $H_{q^*}(\mu \llcorner G, 0, 1, h, P)$ is obtained by combining the following six inequalities:

(I) $$H_{q^*}(\mu \llcorner G, 0, 1, h, P) \leq \Gamma_2\big(h_{q^*}(g,S) + \mathcal{L}^n(\bar{B}^n_1(0) \sim Y)^{1/q}\big),$$

(II) $$h_{q^*}(g,S) \leq \Gamma_3 \, t_q(g),$$

(III) $$\mathcal{L}^n(\bar{B}^n_1(0) \sim Y)^{1/q} \leq (\Gamma_1)^{1/q} \, \mu(B)^{1/q},$$

(IV) $$\mu(B \cap A)^{1/q} \leq \Gamma_4 \, T_q(\mu, 0, 3, h+2, T),$$

(V) $$t_q(g|Y) \leq \Gamma_5 \, T_q(\mu, 0, 1, h, T),$$

(VI) $$t_q(g|\bar{B}^n_1(0) \sim Y) \leq \Gamma_6 \, \mathcal{L}^n(\bar{B}^n_1(0) \sim Y)^{1/q}.$$

(I) is implied by 1.14 (2) (4) (6) and $\operatorname{spt} S \subset \bar{B}^m_h(0)$, (II) is implied by the choice of S, (III) is implied by 1.14 (3), (VI) is elementary (cf. [Alm00, 1.1 (9)–(11)]). To prove (IV), note that for every $x \in B \cap A$ there exists $0 < \varrho < 2$ such that

$$\varepsilon_0 \, \mu(\bar{B}_\varrho(x)) < \textstyle\int_{\bar{B}_\varrho(x)} |T_\xi\mu - T| \, d\mu(\xi),$$

hence by Hölder's inequality

$$(\varepsilon_0)^q \, \mu(\bar{B}_\varrho(x)) < \textstyle\int_{\bar{B}_\varrho(x)} |T_\xi\mu - T|^q \, d\mu(\xi)$$

and Besicovitch's covering theorem implies (IV). Observing that

$$\{y \in Y : |\operatorname{ap} Ag(y)| > \gamma\} \sim \pi\big(\{\xi \in G \cap \pi^{-1}(Y) : |T_\xi\mu - T| > \gamma/\Gamma_5\}\big)$$

has $\mathcal{L}^n$ measure 0 by 1.14 (7d) and [Alm00, 1.1 (9)–(11)], inequality (V) is a consequence of

$$\begin{aligned}&\mathcal{L}^n(\{y \in Y : |\operatorname{ap} Ag(y)| > \gamma\})\\ &\leq \mathcal{H}^n(\{\xi \in G \cap \pi^{-1}(Y) : |T_\xi\mu - T| > \gamma/\Gamma_5\})\\ &\leq \mu(\{\xi \in G \cap \pi^{-1}(Y) : |T_\xi\mu - T| > \gamma/\Gamma_5\}).\end{aligned}$$

The proof of part (2) exactly parallels the proof of part (1) with ∞, q, and 2.5 (2) replacing q^*, q, and 2.5 (1). □

2.7. *Remark.* Part (2) can be sharpened using Lorentz spaces to

$$\begin{aligned}&H_\infty(\mu \llcorner G, a, r, h, Q, T)\\ &\leq \Gamma\big(T_{n,1}(\mu, a, 3r, h+2r, T) + (r^{-n}\mu(C(T,a,r,h) \sim A))^{1/n}\big)\end{aligned}$$

with a positive, finite number Γ depending only on m, n, Q, M, and δ. Here $T_{n,1}$ is the obvious generalisation of T_q to Lorenz spaces.

A similar improvement is possible for part (1) using Peetre's theorem.

2.8. **Theorem.** *Suppose* $m, n, Q \in \mathbb{N}$, $1 \leq M < \infty$, $0 < \delta \leq 1$, $a \in \mathbb{R}^{n+m}$, $0 < r < \infty$, $T \in G(n+m, n)$, $1 \leq p \leq n$, μ *is an integral* n *varifold in an open superset of* $C(T, a, 3r, 3r)$ *satisfying* (H_p) *and*

$$\psi = \|\delta\mu\| \quad \text{if } p = 1, \quad \psi = |\vec{\mathbf{H}}_\mu|^p \mu \quad \text{if } p > 1,$$
$$(Q - 1 + \delta)\omega_n r^n \leq \mu(C(T, a, r, r)) \leq (Q + 1 - \delta)\omega_n r^n,$$
$$\mu(C(T, a, r, (1+\delta)r) \sim C(T, a, r, (1-\delta)r)) \leq (1 - \delta)\omega_n r^n,$$
$$\mu(C(T, a, 3r, 3r)) \leq M\omega_n r^n.$$

Then the following two statements hold:

(1) *If* $p < n$, $1 \leq q < n$, *then*

$$H_{\frac{nq}{n-q}}(\mu, a, r, r, Q, T) \leq \Gamma_{(1)}\big(T_q(\mu, a, 3r, 3r, T) + (r^{p-n}\psi(C(T, a, 3r, 3r)))^{\frac{n-q}{q(n-p)}}\big)$$

where $\Gamma_{(1)}$ *is a positive, finite number depending only on* m, n, Q, M, δ, p, *and* q.

(2) *If* $p = n$ *and* $\psi(C(T, a, 3r, 3r)) \leq \varepsilon_{(2)}$ *where* $\varepsilon_{(2)}$ *is a positive, finite number depending only on* m, n, Q, M, *and* δ, *then*
 (a) $H_{\frac{nq}{n-q}}(\mu, a, r, r, Q, T) \leq \Gamma_{(2a)}\, T_q(\mu, a, 3r, 3r, T)$ *whenever* $1 \leq q < n$,
 (b) $H_\infty(\mu, a, r, r, Q, T) \leq \Gamma_{(2b)}\, T_q(\mu, a, 3r, 3r, T)$ *whenever* $n < q \leq \infty$
 where $\Gamma_{(2a)}$, $\Gamma_{(2b)}$ *are positive, finite numbers depending only on* m, n, Q, M, δ, *and* q.

Proof. To prove part (1), assume $a = 0$, $r = 1$, define $q^* = \frac{nq}{n-q}$, and suppose that ε, A, and G are as in 2.6. Observing

$$\begin{aligned}
H_{q^*}(\mu, 0, 1, 1, Q, T) - H_{q^*}(\mu \llcorner G, 0, 1, 1, Q, T) &\leq 2\mu(C(T, 0, 1, 1) \sim G)^{1/q^*} \\
&\quad + \mathcal{H}^n(T(\{\xi \in C(T, 0, 1, 1) : \theta^{*n}(\mu, \xi) > 0\} \sim G))^{1/q} \\
&\leq (2 + \omega_n^{1/n})\mu(C(T, 0, 1, 1) \sim G)^{1/q^*},
\end{aligned}$$
$$\mu(C(T, 0, 1, 1) \sim G) \leq N(n+m)(2\gamma_n)^{\frac{np}{n-p}}\psi(C(T, 0, 3, 3))^{\frac{n}{n-p}},$$
$$\mu(C(T, 0, 1, 1) \sim A) \leq N(n+m)\varepsilon^{-\frac{np}{n-p}}\psi(C(T, 0, 3, 3))^{\frac{n}{n-p}},$$
$$\begin{aligned}
\|\delta\mu\|(C(T, 0, 3, 3)) &\leq \mu(C(T, 0, 3, 3))^{1-1/p}\psi(C(T, 0, 3, 3))^{1/p} \\
&\leq (M\omega_n)^{1-1/p}\psi(C(T, 0, 3, 3))^{1/p},
\end{aligned}$$
$$T_1(\mu, 0, 3, 3, T) \leq 3^{-n+n/q}(M\omega_n)^{1-1/q}T_q(\mu, 0, 3, 3, T),$$
$$H_{q^*}(\mu, 0, 1, 1, Q, T) \leq \mu(C(T, 0, 1, 1))^{1/q^*} + \omega_n^{1/q} \leq M^{1/q^*}\omega_n^{1/q^*} + \omega_n^{1/q},$$

a suitable number $\Gamma_{(1)}$ is readily constructed using 2.6 (1).

Part (2) is proved similarly using 2.6 (2). □

2.9. *Remark.* In case μ additionally satisfies

$$\mu(\{x \in C(T, a, r, r) : \theta^n(\mu, x) = Q\}) \geq \delta\omega_n r^n,$$

there exists $z \in T^\perp$ such that for $P := Q\mathcal{H}^m \llcorner \{x \in \mathbb{R}^{n+m} : T^\perp(x) = z\}$

$$H_{\frac{nq}{n-q}}(\mu, a, r, r, P) \leq \Gamma\big(T_q(\mu, a, 3r, 3r, T) + (r^{p-n}\psi(C(T, a, 3r, 3r)))^{\frac{n-q}{q(n-p)}}\big)$$

provided $p < n$, $1 \leq q < n$ where Γ is a positive, finite number depending only on m, n, Q, M, δ, p, and q.

In fact from 1.14 (2) (3) and the coarea formula [Fed69, 3.2.22 (3)] one obtains for the set Y_0 of all $y \in T \cap B_r(T(a))$ such that for some $x_0 \in C(T, a, r, r)$ with

$T(x_0) = y$

$$\theta^n(\mu, x_0) = Q, \qquad \theta^n(\mu, x) = 0 \quad \text{for } x \in T^{-1}(\{y\}) \cap C(T, a, r, r) \sim \{x_0\}$$

the estimate

$$\mathcal{L}^1(Y_0) \geq (2\delta/3)\omega_n r^n$$

provided the right hand side of the inequality in question is suitably small (depending only on m, n, Q, M, δ, p, and q), hence for any Q valued plane P' parallel to T such that

$$(2H_{\frac{nq}{n-q}}(\mu, a, r, r, P'))^q \leq (\delta/3)\omega_n$$

there holds

$$((\delta/3)\omega_n)^{1/q-1/n} \frac{\operatorname{diam} T^\perp(\operatorname{spt} P')}{2r} \leq 2H_{\frac{nq}{n-q}}(\mu, a, r, r, P')$$

and suitable z and Γ are readily constructed.

A similar remark holds for the second part.

2.10. **Theorem.** *Suppose $m, n, Q \in \mathbb{N}$, $0 < \alpha \leq 1$, $1 \leq p \leq n$, U is an open subset of $\mathbb{R}^{n+m}$, and μ is an integral n varifold in U satisfying (H_p).*

Then the following two statements hold:

(1) *If $p < n$, $1 \leq q_1 < n$, $1 \leq q_2 \leq \min\{\frac{nq_1}{n-q_1}, \frac{1}{\alpha} \cdot \frac{np}{n-p}\}$, then for μ almost all $a \in U$ with $\theta^n(\mu, a) = Q$ there holds*

$$\begin{aligned} &\limsup_{r \downarrow 0} r^{-\alpha-1-n/q_2} \| \operatorname{dist}(\cdot - a, T_a\mu) \|_{L^{q_2}(\mu \llcorner B_r(a))} \\ &\leq \Gamma_{(1)} \limsup_{r \downarrow 0} r^{-\alpha-n/q_1} \| T_\mu - T_a\mu \|_{L^{q_1}(\mu \llcorner B_r(a))} \end{aligned}$$

where $\Gamma_{(1)}$ is a positive, finite number depending only on m, n, Q, q_1, and q_2.

(2) *If $p = n$, $n < q \leq \infty$, then for μ almost all $a \in U$ with $\theta^n(\mu, a) = Q$ there holds*

$$\begin{aligned} &\limsup_{r \downarrow 0} r^{-\alpha-1} \| \operatorname{dist}(\cdot - a, T_a\mu) \|_{L^\infty(\mu \llcorner B_r(a))} \\ &\leq \Gamma_{(2)} \limsup_{r \downarrow 0} r^{-\alpha-n/q} \| T_\mu - T_a\mu \|_{L^q(\mu \llcorner B_r(a))} \end{aligned}$$

where $\Gamma_{(2)}$ is a positive, finite number depending only on m, n, Q, and q.

Proof. For $a \in \mathbb{R}^{n+m}$, $0 < r < \infty$ such that $B_{7r}(a) \subset U$ denote by $G_r(a)$ the set of all $x \in \bar{B}_{5r}(a) \cap \operatorname{spt} \mu$ satisfying

$$\|\delta\mu\|(\bar{B}_\varrho(x)) \leq (2\gamma_n)^{-1} \mu(\bar{B}_\varrho(x))^{1-1/n} \quad \text{whenever } 0 < \varrho < 2r.$$

To prove (1), one may assume first that $q_2 \geq \frac{n}{n-1}$ possibly replacing q_2 by a larger number since $\min\{\frac{nq_1}{n-q_1}, \frac{1}{\alpha} \cdot \frac{np}{n-p}\} \geq \frac{n}{n-1}$, and thus also that $q_2 = \frac{nq_1}{n-q_1}$ possibly replacing q_1 by a smaller number. Define $M = 6^n Q$, $\delta = 1/2$, $q = q_1$, $q^* = q_2$,

$$\varepsilon = \min\{\varepsilon_{2.6}(m, n, Q, M, \delta), (2\gamma_n)^{-1}\}, \quad \Gamma = \Gamma_{2.6(1)}(m, n, Q, M, \delta, q).$$

Denote by C_i for $i \in \mathbb{N}$ the set of all $x \in \operatorname{spt} \mu$ such that $B_{1/i}(x) \subset U$ and

$$\|\delta\mu\|(\bar{B}_\varrho(x)) \leq \varepsilon \, \mu(\bar{B}_\varrho(x))^{1-1/n} \quad \text{whenever } 0 < \varrho < 1/i.$$

The conclusion will be shown for $a \in \operatorname{dmn} T_\mu$ such that

$$\theta^n(\mu, a) = Q, \quad \theta^{n-1}(\|\delta\mu\|, a) = 0,$$

$$\lim_{r \downarrow 0} r^{-n^2/(n-p)} \mu(\bar{B}_r(x) \sim C_i) = 0 \quad \text{for some } i \in \mathbb{N}.$$

Note that according to [Fed69, 2.9.5] and [Men08b, 2.9, 2.10] with s replaced by n this is true for μ almost all $a \in U$ with $\theta^n(\mu, a) = Q$, fix such a, i, and abbreviate $T := T_a\mu$.

For a there holds

$$\lim_{r \downarrow 0} \frac{\mu(C(T,a,r,r))}{\omega_n r^n} = Q,$$
$$\lim_{r \downarrow 0} \frac{\mu(C(T,a,r,3r/2) \sim C(T,a,r,r/2))}{\omega_n r^n} = 0$$

and one can assume for some $0 < \gamma < \infty$

$$\limsup_{r \downarrow 0} r^{-\alpha} T_q(\mu, a, r, r, T) < \gamma.$$

Choose $0 < s < \min\{(2i)^{-1}, \operatorname{dist}(a, \mathbb{R}^{n+m} \sim U)/7\}$ so small that for $0 < \varrho < s$

$$(Q - 1/2)\omega_n \varrho^n \leq \mu(C(T,a,\varrho,\varrho)) \leq (Q+1/2)\omega_n\varrho^n,$$
$$\mu(C(T,a,\varrho,3\varrho/2) \sim C(T,a,\varrho,\varrho/2)) \leq (1/2)\omega_n\varrho^n,$$
$$\mu(C(T,a,3\varrho,3\varrho)) \leq \mu(\bar{B}_{5\varrho}(a)) \leq \omega_n 6^n Q \varrho^n,$$
$$\|\delta\mu\|(C(T,a,3\varrho,3\varrho)) \leq \varepsilon\varrho^{n-1}, \quad T_1(\mu,a,3\varrho,3\varrho,T) \leq \varepsilon,$$
$$T_q(\mu,a,3\varrho,3\varrho,T) + (\varrho^{-n}\mu(C(T,a,\varrho,\varrho) \sim C_i))^{1/q} \leq 4\gamma\varrho^\alpha;$$

in particular 2.6 (1) can be applied to any such ϱ with (r,h) replaced by (ϱ,ϱ). For each $0 < \varrho < s$ use 2.3 to choose Q valued planes P_ϱ parallel to T such that

$$H_{q^*}(\mu \llcorner G_\varrho(a), a, \varrho, \varrho, P_\varrho) \leq 2H_{q^*}(\mu \llcorner G_\varrho(a), a, \varrho, \varrho, Q, T),$$

denote by A_ϱ the $\mathcal{H}^n$ measurable sets of all $x \in T \cap \bar{B}_\varrho(T(a))$ such that for some $R_\varrho(x), S_\varrho(x) \in Q_Q(\mathbb{R}^{n+m})$

$$\|R_\varrho(x)\| = \theta^n(P_\varrho \llcorner C(T,a,\varrho,\varrho), \cdot)\,\mathcal{H}^0 \llcorner T^{-1}(\{x\}),$$
$$\|S_\varrho(x)\| = \theta^n(\mu \llcorner G_\varrho(a) \cap C(T,a,\varrho,\varrho), \cdot)\,\mathcal{H}^0 \llcorner T^{-1}(\{x\}),$$

and by $g_\varrho : A_\varrho \to \mathbb{R}$ the $\mathcal{H}^n$ measurable functions defined by

$$g_\varrho(x) = \mathcal{G}(R_\varrho(x), S_\varrho(x)) \quad \text{for } x \in A_\varrho.$$

By 2.3 there exist $\mathcal{H}^n$ measurable subset Y_ϱ of A_ϱ such that

$$2H_{q^*}(\mu \llcorner G_\varrho(a), a, \varrho, \varrho, P_\varrho) \geq \varrho^{-n/q}\|\operatorname{dist}(\cdot, \operatorname{spt} P_\varrho)\|_{L^{q^*}(\mu \llcorner G_\varrho(a) \cap C(T,a,\varrho,\varrho))}$$
$$+ \varrho^{-n/q}\|g_\varrho\|_{L^{q^*}(\mathcal{H}^n \llcorner Y_\varrho)} + \varrho^{-n/q}\mathcal{H}^n(T \cap \bar{B}_\varrho(T(a)) \sim Y_\varrho)^{1/q}.$$

Possibly replacing s by a smaller number, one may assume for $0 < \varrho < s$ that

$$(2H_{q^*}(\mu \llcorner G_\varrho(a), a, \varrho, \varrho, P_\varrho))^q \leq 2^{-n-2}\omega_n$$

by 2.6 (1) and also that

$$\mu(C(T,a,\varrho,\varrho) \sim C_i) \leq 2^{-n-2}\omega_n\varrho^n.$$

Noting $C_i \cap C(T,a,\varrho/2,\varrho) \subset G_\varrho(a) \cap G_{\varrho/2}(a)$, one obtains directly from the additional assumptions on s that

$$\mathcal{H}^n(T \cap \bar{B}_\varrho(T(a)) \sim Y_\varrho) \leq 2^{-n-2}\omega_n\varrho^n,$$
$$\mathcal{H}^n(T \cap \bar{B}_{\varrho/2}(T(a)) \sim Y_{\varrho/2})\} \leq 2^{-n-2}\omega_n\varrho^n,$$
$$\mathcal{H}^n(\{x \in Y_{\varrho/2} \cap Y_\varrho : S_\varrho(x) \neq S_{\varrho/2}(x)\})$$
$$\leq \mathcal{H}^n\big(T(\{x \in C(T,a,\varrho/2,\varrho) : \theta^{*n}(\mu,x) \geq 1\} \sim C_i)\big)$$
$$\leq \mu(C(T,a,\varrho,\varrho) \sim C_i) \leq 2^{-n-2}\omega_n\varrho^n,$$

hence for $B_\varrho := Y_\varrho \cap Y_{\varrho/2} \cap \{x : S_\varrho(x) = S_{\varrho/2}(x)\}$

$$\mathcal{H}^n(B_\varrho) \geq \tfrac{1}{4}\omega_n(\varrho/2)^n \quad \text{for } 0 < \varrho < s,$$

in particular

$$\operatorname{dmn} R_\varrho = A_\varrho \supset Y_\varrho \supset B_\varrho \neq \emptyset, \quad \mathcal{G}(P_\varrho, Q\mathcal{H}^n \llcorner T) \leq Q^{1/2}\varrho.$$

By integration over the set B_ϱ one obtains

$$\begin{aligned}
&(\tfrac{1}{4}\omega_n(\varrho/2)^n)^{1/q-1/n}\mathcal{G}(P_\varrho, P_{\varrho/2}) \\
&\leq \|g_\varrho\|_{L^{q^*}(\mathcal{H}^n \llcorner Y_\varrho)} + \|g_{\varrho/2}\|_{L^{q^*}(\mathcal{H}^n \llcorner Y_{\varrho/2})} \\
&\leq \varrho^{n/q} 4\big(H_{q^*}(\mu \llcorner G_\varrho(a), a, \varrho, \varrho, Q, T) + H_{q^*}(\mu \llcorner G_{\varrho/2}(a), \varrho/2, \varrho/2, Q, T)\big)
\end{aligned}$$

for $0 < \varrho < s$. Therefore 2.6 (1) implies

$$\mathcal{G}(P_\varrho, P_{\varrho/2}) \leq \Gamma_1 \gamma \varrho^{1+\alpha}$$

where $\Gamma_1 = 2^{3+n/q+2/q-2/n}\omega_n^{1/n-1/q}\Gamma$, hence

$$\mathcal{G}(Q\mathcal{H}^n \llcorner T, P_\varrho) \leq \textstyle\sum_{i=0}^\infty \mathcal{G}(P_{2^{-i}\varrho}, P_{2^{-i-1}\varrho}) \leq 2\Gamma_1\gamma\varrho^{1+\alpha}$$

because $\mathcal{G}(P_\varrho, Q\mathcal{H}^n \llcorner T) \to 0$ as $\varrho \downarrow 0$. From the definition of the q^* height of μ in $C(T, a, \varrho, \varrho)$ one obtains

$$\begin{aligned}
&H_{q^*}(\mu \llcorner G_\varrho(a), a, \varrho, \varrho, Q\mathcal{H}^n \llcorner T) - H_{q^*}(\mu \llcorner G_\varrho(a), a, \varrho, \varrho, P_\varrho) \\
&\leq \varrho^{-n/q}\big(\mu(C(T, a, \varrho, \varrho))^{1/q^*} + \mathcal{H}^n(Y_\varrho)^{1/q^*}\big)\mathcal{G}(Q\mathcal{H}^n \llcorner T, P_\varrho) \leq \Gamma_2\gamma\varrho^\alpha
\end{aligned}$$

for $0 < \varrho < s$ where $\Gamma_2 = \omega_n^{1/q^*} 2(Q+1)^{1/q^*} 2\Gamma_1$, hence

$$\begin{aligned}
&\limsup_{\varrho \downarrow 0} \varrho^{-\alpha} H_{q^*}(\mu \llcorner G_\varrho(a), a, \varrho, \varrho, Q\mathcal{H}^n \llcorner T) \\
&\qquad\qquad \leq (8\Gamma + \Gamma_2) \limsup_{\varrho \downarrow 0} \varrho^{-\alpha} T_q(\mu, a, \varrho, \varrho, T)
\end{aligned}$$

by 2.6 (1). Combining this with the fact that

$$\lim_{\varrho \downarrow 0} \varrho^{-\alpha-1-n/q^*} \|\operatorname{dist}(\cdot - a, T_a\mu)\|_{L^{q^*}(\mu \llcorner B_\varrho(a) \sim G_\varrho(a))} = 0,$$

the conclusion follows.

(2) may be proved by a similar argument using 2.6 (2) and [Men08b, 2.5] instead of 2.6 (1) and [Men08b, 2.9, 2.10]. □

2.11. *Remark.* As in 2.7, in (2) the L^q norm can be replaced by $L^{n,1}$, in particular $n = q = 1$ is admissible. The latter fact can be derived without the use of Lorentz spaces, of course.

2.12. *Remark.* If $1 \leq p < n$, $1 \leq q_1 \leq q_2 < \infty$, $\frac{1}{\alpha} \cdot \frac{np}{n-p} < q_2$, then the conclusion of (1) fails for some μ; in fact one can assume $q_1 = q_2$ possibly enlarging q_1 and then take $\alpha_2 = \alpha$ and α_1 slightly larger than α_2 in [Men08b, 1.2]. Clearly, also in (2) the assumption $p = n$ cannot be weakened.

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