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Second Order Rectifiability of Integral Varifolds of Locally Bounded First Variation

Ulrich Menne

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Abstract It is shown that every integral varifold in an open subset of Euclidean space whose first variation with respect to area is representable by integration can be covered by a countable collection of submanifolds of the same dimension of class 2 and that their mean curvature agrees almost everywhere with the variationally defined generalized mean curvature of the varifold.

Keywords Integral varifold · Locally bounded first variation · Second order rectifiability · Second fundamental form

Mathematics Subject Classification (2000) Primary 49Q15 · Secondary 35J60

1 Introduction

Overview In the present paper the existence of an approximate second order structure for integral varifolds in Euclidean space whose first variation with respect to area is representable by integration is established. Such varifolds are called "of locally bounded first variation" in [33]. Moreover, it is proven that the variationally defined generalized mean curvature of the varifold agrees almost everywhere with the mean curvature induced from the approximate second order structure. This problem can be

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U. Menne (✉)
Max-Planck-Institute for Gravitational Physics (Albert-Einstein-Institute), OT Golm,
Am Mühlenberg 1, 14476 Potsdam, Germany
e-mail: Ulrich.Menne@aei.mpg.de

considered a geometric, nonlinear, higher multiplicity version of the following linear one: Prove existence of approximate second order differentials for weakly differentiable functions whose distributional Laplacian is representable by integration (i.e., by a "vector-valued Radon measure") and show that these differentials satisfy the equation Lebesgue almost everywhere. Clearly, the linear case itself is not too hard to solve, and in fact follows immediately from classical results if the distributional Laplacian is integrable with respect to Lebesgue measure to a power larger than 1. Nevertheless, the main objective of the present paper is to develop a method which is based on the study of the nearly linear case and is sufficiently robust to be applied to the present elliptic system of geometric partial differential equations involving higher multiplicity.

Results of the type obtained in the present paper have proven useful, for example, in the context of Brakke's mean curvature flow, or sharp and diffuse interfaces, or image reconstruction, or the Willmore functional; see [7, 9, 26, 28–30, 32] and the references therein.

Result of the Present Paper in the Context of Known Results Fix positive integers m and n with $m < n$. The principal result is as follows; see Sect. 2 for the notation used.

Theorem 1 (see 4.8) *Suppose U is an open subset of $\mathbf{R}^n$, $V \in \mathbf{IV}_m(U)$ and $\|\delta V\|$ is a Radon measure.*

Then there exists a countable collection C of m-dimensional submanifolds of $\mathbf{R}^n$ of class 2 such that $\|V\|(U \sim \bigcup C) = 0$ and each member M of C satisfies

$$\mathbf{h}(V; z) = \mathbf{h}(M; z) \quad \textit{for } \|V\| \textit{ almost all } z \in U \cap M.$$

In the terminology of Anzellotti and Serapioni [8, 3.1] the first part of the conclusion can be expressed equivalently by the condition that $U \cap \{z : 0 < \boldsymbol{\Theta}^m(\|V\|, z) < \infty\}$ meets every compact subset of U in a set which is $(\mathscr{H}^m, m)$ rectifiable of class $\mathscr{C}^2$. The second part of the assertion is sometimes called "locality of the mean curvature"; see Schätzle [32, Sect. 4].

Clearly, if $W \in \mathbf{IV}_m(U)$ then the existence of a countable collection C of m-dimensional submanifolds of $\mathbf{R}^n$ of class 2 with $\|W\|(U \sim \bigcup C) = 0$ is equivalent to the existence of sequences A_i and M_i such that M_i are m-dimensional submanifolds of $\mathbf{R}^n$ of class 2, A_i are $\mathscr{H}^m$ measurable subsets of M_i, and

$$W(f) = \sum_{i=1}^{\infty} \int_{A_i} f(z, \operatorname{Tan}(M_i, z)) \,\mathrm{d}\mathscr{H}^m z \quad \text{for } f \in \mathscr{K}(U \times \mathbf{G}(n, m)).$$

However, in Theorem 1 one cannot require

$$\boldsymbol{\Theta}^m(\|V\|, z) = \operatorname{card}\{M : z \in M \in C\} \quad \text{for } \mathscr{H}^m \text{ almost all } z \in U$$

even if $\operatorname{spt} \|V\|$ is contained in an m-dimensional subspace; see 4.11.

Theorem 1 contains (and re-proves) the fact that $\mathbf{h}(V; z) \in \operatorname{Nor}^m(\|V\|, z)$ for $\|V\|$ almost all z previously obtained by Brakke [9, 5.8]; see 4.9. Moreover, it is worth noting, see 4.10, that if V is a curvature varifold with boundary in U in the sense of Mantegazza [21, Definition 3.1] then V satisfies the hypotheses of Theorem 1 and, taking C as in its conclusion, the second fundamental form of V agrees almost everywhere with the second fundamental form induced by the members M of C.

Evidently, Theorem 1 implies that the function mapping $\|V\|$ almost every z onto the orthogonal projection of $\mathbf{R}^n$ onto the approximate m-dimensional tangent plane of $\|V\|$ at z is $(\|V\|, m)$ approximately differentiable. If the first variation of $\|V\|$ satisfies the integrability condition (H_p) below with sufficiently large exponent p then this map is in fact differentiable in a stronger $\mathbf{L}_2(\|V\|, \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n))$ sense. Whenever U is an open subset of $\mathbf{R}^n$, $V \in \mathbf{IV}_m(U)$ and $1 \leq p \leq \infty$, the varifold V is said to satisfy (H_p) if and only if $\|\delta V\|$ is a Radon measure and, if $p > 1$,

$$\begin{aligned}
&(\delta V)(g) = -\int \mathbf{h}(V; z) \bullet g(z)\, \mathrm{d}\|V\| z \quad \text{for } g \in \mathscr{D}(U, \mathbf{R}^n), \\
&\mathbf{h}(V; \cdot) \in \mathbf{L}_p(\|V\| \llcorner K, \mathbf{R}^n) \quad \text{whenever } K \text{ is a compact subset of } U.
\end{aligned} \tag{H_p}$$

Theorem 2 (see 5.2 and 5.5) *Suppose U is an open subset of $\mathbf{R}^n$, $1 \leq p \leq \infty$, and $V \in \mathbf{IV}_m(U)$ satisfies (H_p).*

If either $m = 1$ or $m = 2$ and $p > 1$ or $m > 2$ and $p \geq 2m/(m+2)$, then for $\|V\|$ almost all a

$$\fint_{\mathbf{B}(a,r)} (|R(z) - R(a) - \langle R(a)(z-a), \operatorname{ap} DR(a)\rangle|/|z-a|)^2 \,\mathrm{d}\|V\| z \to 0$$

as $r \to 0+$ where $R(z) = \operatorname{Tan}^m(\|V\|, z)_\natural \in \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ and the approximate differential is taken with respect to $(\|V\|, m)$.

With the possible exception of the case $m = 2$ this differentiability result is optimal with respect to the assumptions on p, i.e., whenever $m > 2$ and $\frac{mp}{m-p} < 2$ there exists an integral varifold satisfying (H_p) not having the property in question; see 5.4.

In previous work, Schätzle established the following result in codimension one of the existence of submanifolds of class ∞ touching a given varifold; see [31, Proposition 4.1, Theorem 5.1] where it is phrased in terms of upper and lower height functions.

Theorem (Schätzle [31]) *Suppose U is an open subset of $\mathbf{R}^n$, $p > m = n - 1$, $p \geq 2$, and $V \in \mathbf{IV}_m(U)$ satisfies (H_p).*

Then for $\|V\|$ almost all a there exists $0 < r < \infty$ such that

$$\mathbf{U}(a + v, r) \cap \operatorname{spt}\|V\| = \emptyset$$

whenever $v \in \operatorname{Nor}^m(\|V\|, a)$ with $|v| = r$.

This is the key to showing that such a varifold satisfies the conclusion of Theorem 1, see Schätzle [31, Theorem 6.1], and, in combination with previous results of the author in [22, 3.7, 3.9], also that it satisfies the conclusion of Theorem 2. Evidently (see, for example, [22, 1.2]), Schätzle's theorem does not extend to the case

$p < m$. Also, the use of the theory of viscosity solutions for fully nonlinear equations, more precisely the results of Caffarelli [10] and Trudinger [35], leads to the restriction to codimension one, i.e., $m = n - 1$.

Therefore, in order to establish Theorem 1, a different method needs to be developed which is able to deal both with the low integrability of the generalized mean curvature and with higher codimension. The main independent result in this process is the following theorem stated here in the case of Laplace's operator.

Theorem 3 (see 3.11) *Suppose U is an open subset of $\mathbf{R}^m$, $u : U \to \mathbf{R}^{n-m}$ is weakly differentiable, $j \in \{0, 1\}$, $1 \le q < \infty$,*

$$h(a,r) = \inf\left\{ \sum_{i=0}^{j} r^{i-m/q} |\mathbf{D}^i(u-v)|_{q;a,r} : v \in \mathscr{E}(\mathbf{U}(a,r), \mathbf{R}^{n-m}), \operatorname{Lap} v = 0 \right\}$$

whenever $a \in U$, $0 < r < \infty$ with $\mathbf{U}(a,r) \subset U$ and A denotes the set of all $a \in U$ such that

$$\limsup_{r \to 0+} r^{-2} h(a,r) < \infty.$$

Then for $\mathscr{L}^m$ almost all $a \in A$ there exists a polynomial function $Q_a : \mathbf{R}^m \to \mathbf{R}^{n-m}$ of degree at most 2 such that

$$\lim_{r \to 0+} r^{-2} \sum_{i=0}^{j} r^{i-m/q} |\mathbf{D}^i(u - Q_a)|_{q;a,r} = 0.$$

Here the seminorms $|\cdot|_{q;a,r}$ correspond to $\mathbf{L}_q(\mathscr{L}^m \llcorner \mathbf{U}(a,r))$. The weaker statement which results when the condition $\operatorname{Lap} v = 0$ is replaced by $D^2 v = 0$ is contained in Calderón and Zygmund [11, Theorem 5] if $q > 1$. However, the construction of affine comparison functions at a given point from information on the distributional Laplacian of u may—for integral orders of differentiability—fail at individual points; see [24, 10.4]. This corresponds to the well-known fact of the nonexistence of Schauder estimates for the Hölder exponent 1. In this respect the value of the current theorem stems from the fact that harmonic comparison functions are readily constructed independent of the order of differentiability considered, cp. 3.13. In fact, if $j = 1$, $q > 1$, and denoting by $T \in \mathscr{D}'(U, \mathbf{R}^{n-m})$ the distributional Laplacian of u then

$$\Gamma^{-1} h(a,r) \le r^{1-m/q} |T|_{-1,q;a,r} \le \Gamma h(a,r)$$

whenever $a \in U$, $0 < r < \infty$, $\mathbf{U}(a,r) \subset U$, and $u|\mathbf{U}(a,r) \in \mathbf{W}^{1,q}(\mathbf{U}(a,r), \mathbf{R}^{n-m})$ where Γ is a positive, finite number depending only on n and q and $|\cdot|_{-1,q;a,r}$ denotes the seminorm corresponding to $(\mathbf{W}_0^{1,q/(q-1)}(\mathbf{U}(a,r), \mathbf{R}^{n-m}))^*$. In particular, if T is representable by integration and $q < m/(m-1)$ if $m > 1$ then one verifies $\mathscr{L}^m(U \sim A) = 0$. An extensive study of both integral and nonintegral orders of differentiability for solutions of linear elliptic partial differential equations in nondivergence form can be found in Calderón and Zygmund [11].

In passing to divergence form equations, one is naturally lead to consider the related problem for distributions:

Theorem 4 (see 3.13 and A.3) *Suppose U is an open subset of $\mathbf{R}^m$, $1 \le q < \infty$, $T \in \mathscr{D}'(U, \mathbf{R}^{n-m})$ and A denotes the set of all $a \in U$ such that*

$$\limsup_{r \to 0+} r^{-1-m/q} |T|_{-1,q;a,r} < \infty.$$

Then for $\mathscr{L}^m$ almost every $a \in A$ there exists a unique constant distribution $T_a \in \mathscr{D}'(U, \mathbf{R}^{n-m})$ such that

$$\lim_{r \to 0+} r^{-1-m/q} |T - T_a|_{-1,q;a,r} = 0.$$

This may be seen as a Lebesgue point theorem for distributions. In case $q > 1$, it is in fact a corollary to Theorem 3 obtainable by representing T locally as the distributional Laplacian of some function u. In contrast, the case $q = 1$ is independent from the other results of the present paper.

Finally, it should be noted that the proof of Theorem 3 only relies on a priori estimates in Lebesgue spaces, i.e., "$\mathbf{L}_p$ theory", which are known to hold for a much wider class of linear equations; see Agmon, Douglis, and Nirenberg [1, 2].

Outline of the Proofs To prove Theorem 3, one considers the subsets of A_k of A of all $a \in A$ such $h(a,r) \le kr^2$ whenever $0 < r < 1/k$. Denoting by $v_{a,r} : \mathbf{U}(a,r) \to \mathbf{R}^{n-m}$ harmonic functions essentially realizing the infimum in the definition of h, one then uses the partition of unity with estimates from [16, 3.1.13] together with well-known a priori estimates for the Laplace operator to construct functions $v_k : \mathbf{R}^m \to \mathbf{R}^{n-m}$ with the following properties; see 3.9:

(1) There holds

$$\sum_{i=0}^{j} r^{i-m/q} |\mathbf{D}^i (v_k - u)|_{q;a,r} \le \Gamma k r^2$$

for $a \in A_k$ and $0 < r < (36k)^{-1}$ and Γ a positive, finite number depending only on n and q, in particular $v_k(x) = u(x)$ for $\mathscr{L}^m$ almost all $x \in A_k$.

(2) The distributional Laplacian of v_k is represented by a function locally in $\mathbf{L}_\infty(\mathscr{L}^m, \mathbf{R}^{n-m})$.

Then clearly v_k locally belongs to $\mathbf{W}^{2,q}(\mathbf{R}^m, \mathbf{R}^{n-m})$ for $1 \le q < \infty$ and the conclusion of Theorem 3 follows from by now classical differentiability results for functions in Sobolev spaces which were also obtained by Calderón and Zygmund in [11]. An important feature of this proof is that it is readily adapted to the case where the Laplace operator is replaced by the Euler–Lagrange differential operator L_F corresponding to an integrand $F : \mathrm{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m}) \to \mathbf{R}$ of class 2 sufficiently close to the Dirichlet integrand, i.e., $\mathrm{Lip}\, D^2 F < \infty$ and

$$\left| D^2 F(\sigma)(\tau_1, \tau_2) - \tau_1 \bullet \tau_2 \right| \le \varepsilon \quad \text{for } \sigma, \tau_1, \tau_2 \in \mathrm{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m})$$

with a suitable number ε.

Next, it will be explained how this result on a rather restricted class of differential operators can be used to treat the general case. For this purpose let U be an open subset of $\mathbf{R}^n$ and let $V \in \mathbf{IV}_m(U)$ be such that $\|\delta V\|$ is a Radon measure. Comparing the behavior of V near certain "good" points to the behavior of harmonic functions, a procedure developed by De Giorgi in [12] and Almgren in [4], one proves the *tilt decay estimate*

$$\limsup_{r \to 0+} r^{-\tau - m/2} \left(\int_{\mathbf{U}(a,r) \times \mathbf{G}(n,m)} |S_\natural - T_\natural|^2 \, \mathrm{d} V(z, S) \right)^{1/2} < \infty$$

for V almost all (a, T) where $0 < \tau < 1$ if $m \in \{1, 2\}$ and $\tau = \frac{m}{2(m-1)} < 1$ if $m > 2$. This has been done by the author in [24, 10.6] extending results of Brakke [9, 5.7, 5] who proved the case $\tau = 1/2$ with "$< \infty$" replaced by $= 0$, which is sufficient for the proof of all theorems stated in the Introduction. As the order of differentiability considered is nonintegral, i.e., $0 < \tau < 1$, the argument applies, in contrast to those of the present paper, in a direct way to all points satisfying a simple set of conditions; see [24, 10.2].

The principal idea to prove Theorem 1 is now to use the tilt decay estimate to construct a sequence of functions $g_i : \mathbf{R}^m \to \mathbf{R}^{n-m}$, $\mathscr{L}^m$ measurable sets $K_i \subset \mathbf{R}^m$, and distributions $T_i \in \mathscr{D}'(\mathbf{R}^m, \mathbf{R}^{n-m})$ with the following properties:

(1) The varifold is covered by suitably rotated graphs of the $g_i | K_i$.
(2) The distribution T_i corresponds to the Euler–Lagrange differential operator associated with the nonparametric area integrand Φ applied to g_i.
(3) There holds

$$\lim_{r \to 0+} r^{-1-m} \int_{\mathbf{U}(x,r)} |Dg_i(\zeta) - Dg_i(x)|^2 \, \mathrm{d}\mathscr{L}^m \zeta = 0 \quad \text{whenever } x \in K_i.$$

(4) The Lipschitz constant of the g_i is small.
(5) The distributions T_i satisfy the conclusion of Theorem 4 with $q = 1$ and A replaced by K_i with constant distribution given by the generalized mean curvature of the varifold.

Condition (4) is the minimum condition needed to be able to replace Φ with some integrand F of the type discussed before in the definition of T_i without changing it; see 3.21. The basis for the construction of g_i, K_i, and T_i is an approximation by $\mathbf{Q}_Q(\mathbf{R}^{n-m})$-valued functions where the space $\mathbf{Q}_Q(\mathbf{R}^{n-m})$ is isometric to the Q-fold product of $\mathbf{R}^{n-m}$ divided by the action of the group of permutations of $\{1, \dots, Q\}$. Here the version of the author in [24, 5.7] is employed which contains some estimates designed for the current applications and was obtained by combining and extending similar constructions of Almgren in [6, Sect. 3] and Brakke in [9, 5.4]. This yields Lipschitzian functions $f_i : K_i \to \mathbf{Q}_{Q_i}(\mathbf{R}^{n-m})$ with small Lipschitz constant for suitable positive integers Q_i. Denoting the "center" of $S \in \mathbf{Q}_Q(\mathbf{R}^{n-m})$ by $\boldsymbol{\eta}_Q(S) = Q^{-1} \sum_{j=1}^{Q} y_j$ whenever $y_1, \dots, y_Q \in \mathbf{R}^{n-m}$ correspond to S, the functions g_i are then constructed in 4.4 as extensions of $\boldsymbol{\eta}_{Q_i} \circ f_i$. In this process the conditions (3) and (5) are ultimately consequences of the tilt decay estimate.

The final step in the proof of Theorem 1 is now to construct for fixed i and $x \in K_i$ comparison functions $v_r \in \mathbf{W}^{1,2}(\mathbf{U}(x,r), \mathbf{R}^{n-m})$ with $L_F(v_r) = 0$ for $0 < r < \infty$ and estimating $g_i - v_r$ in $\mathbf{U}(x,r)$; see 3.14–3.18. The natural choice is to take v_r as the solution of the Dirichlet problem with boundary values given by g_i. If q in (5) would satisfy $q > 1$ this would immediately yield an estimate of $g_i - v_r$ in $\mathbf{W}^{1,q}(\mathbf{U}(x,r), \mathbf{R}^{n-m})$. In case $q = 1$ the estimate needs to be derived differently, namely, linearizing F and estimating the remaining terms with the help of condition (3), obtaining an estimate in $\mathbf{L}_1(\mathscr{L}^m \llcorner \mathbf{U}(x,r), \mathbf{R}^{n-m})$ instead; see 3.16. Then the extended version of Theorem 3 with L_F replacing Lap, see 3.11, implies the first part of Theorem 1. Recalling condition (5), the second part is derived similarly by using functions $w_r \in \mathbf{W}^{1,2}(\mathbf{U}(x,r), \mathbf{R}^{n-m})$ with $L_F(w_r) = (T_i)_x$ where $(T_i)_x$ is the constant distribution of type $\mathbf{R}^{n-m}$ corresponding to T_i at x as in Theorem 4.

Organization of the Paper Section 2 reviews the notation. Section 3 contains results which can be phrased solely in terms of elliptic partial differential equations and distributions, in particular Theorem 3 and the case $q > 1$ of Theorem 4. Section 4 is devoted to the proof of Theorem 1 whereas Sect. 5 contains Theorem 2. Then, the Appendix gives the proof of the case $q = 1$ of Theorem 4.

Each section starts with a brief overview. Moreover, comments on individual results are supposed to further facilitate the navigation through the paper by explaining the content, the idea, and the role of the result in question without being a prerequisite for its proof. Finally, frequently, references are given also to results certainly known to most experts in order to make the proofs accessible to a wider audience.

2 Notation

Overview The notation from Federer [16] and Allard [3] is used with some modifications and additions described in [24, Sects. 1, 2]. The reader familiar with this notation may directly proceed to the paragraph "Additional notation" at the end of this section. Also, the reader interested only in the results on elliptic partial differential equations and distributions, i.e., Sect. 3 and the Appendix, may skip the paragraphs on varifolds and Almgren's multiple-valued functions.

In order to review the notation cited, suppose m and n are positive integers with $m < n$.

Basic Notation Among the basic symbols used are the following:

$\mathscr{P}$ denotes the positive integers.
$\operatorname{im} f = \{y : (x,y) \in f \text{ for some } x\}$ if f is a relation.
$\operatorname{dmn} f = \{x : (x,y) \in f \text{ for some } y\}$ if f is a relation.
$f|A = \{(x,y) : (x,y) \in f \text{ for some } x \in A\}$ if f is a relation and A is a set.
$f[A] = \operatorname{im} f|A$ if f is a relation and A is a set.
$\mathbf{U}(a,r)$ denotes the open ball with center a and radius r.
$\mathbf{B}(a,r)$ denotes the closed ball with center a and radius r.
$\langle v, f \rangle = f(v)$ whenever f is a linear map and $v \in \operatorname{dmn} f$.
$v \bullet w$ denotes the inner product of v and w.

$\boldsymbol{\alpha}(m) = \mathscr{L}^m\, \mathbf{U}(0,1)$.
$\boldsymbol{\beta}(n)$ is the best constant in Besicovitch's covering theorem in $\mathbf{R}^n$.
$\boldsymbol{\gamma}(m)$ is the best constant in the isoperimetric inequality for m-dimensional rectifiable varifolds.
$\boldsymbol{\mu}_r$ denotes multiplication by r.
$\boldsymbol{\tau}_a$ denotes translation by a.
$\mathbf{O}^*(n,m)$ denotes the set of orthogonal projections of $\mathbf{R}^n$ onto $\mathbf{R}^m$.
$\mathbf{G}(n,m)$ denotes the set of all m-dimensional vector subspaces of $\mathbf{R}^n$.
$T_\natural$ denotes the orthogonal projection of $\mathbf{R}^n$ onto T for $T \in \mathbf{G}(n,m)$.
$T^\perp = \ker T_\natural$ for $T \in \mathbf{G}(n,m)$.
$\operatorname{Tan}(S,a)$ denotes the tangent cone of S at a.
$\operatorname{Tan}^m(\phi,a)$ is the cone of (ϕ,m) approximate tangent vectors.
$\operatorname{Nor}^m(\phi,a)$ is the cone of (ϕ,m) approximate normal vectors.
$\mathscr{K}(X)$ is the space of all real-valued continuous functions with compact support on a locally compact Hausdorff space X.
A function is said to be of class k if and only if it is k times continuously differentiable. A similar usage of the term "of class k" is made concerning submanifolds.
$\fint_A f\,\mathrm{d}\phi = \phi(A)^{-1}\int_A f\,\mathrm{d}\phi$ if ϕ measures X, $0 < \phi(A) < \infty$ and $f \in \mathbf{L}_1(\phi \llcorner A)$.
$\operatorname{Lap} u(a)$ is the Laplacian of a function u of class 2 at a.

Multilinear Algebra Suppose V and W are real vector spaces. Denote by $\bigodot^i(V,W)$ the vector space of all i linear symmetric functions (forms) mapping the i-fold product V^i into W whenever $i \in \mathscr{P}$. One abbreviates $\bigodot^i V = \bigodot^i(V,\mathbf{R})$. Extending the notation for linear maps, the alternate notations

$$\phi(v_1,\dots,v_i) \quad \text{and} \quad \langle v_1 \odot \cdots \odot v_i, \phi\rangle$$

will be used whenever $\phi \in \bigodot^i(V,W)$ and $v_1,\dots,v_i \in V$. In this context the i-fold product $v \odot \cdots \odot v$ will be abbreviated to v^i. (This notation is justified by the fact that $\odot$ is the multiplication in the symmetric graded algebra $\bigodot_* V = \bigoplus_{i=0}^\infty \bigodot_i V$ of V, $v_1 \odot \cdots \odot v_i \in \bigodot_i V$ and ϕ induces a unique linear map from $\bigodot_i V$ into W; see [16, 1.9.1, 1.10.1].) Whenever V and W are inner product spaces and $\dim V = j < \infty$ one may introduce a natural inner product on $\bigodot^i(V,W)$; see [16, 1.10.6]. The induced norm satisfies for every orthonormal base $e_1,\dots,e_j$ of V

$$i!|\phi|^2 = \sum_{s \in \mathscr{S}(j,i)} |\phi(e_{s(1)},\dots,e_{s(i)})|^2 \quad \text{for } \phi \in \bigodot^i(V,W)$$

where $\mathscr{S}(j,i)$ denotes the set of all functions mapping $\{1,\dots,i\}$ into $\{1,\dots,j\}$. In particular, this is the usual norm on $\operatorname{Hom}(V,W) = \bigodot^1(V,W)$ induced by an inner product but may differ by a factor $i!^{-1/2}$ from other definitions if $i > 1$. Additionally, the norm

$$\|\phi\| = \sup\{\phi(v_1,\dots,v_i) : v_k \in V \text{ and } |v_k| \le 1 \text{ for } k = 1,\dots,i\}$$

for $\phi \in \bigodot^i(V,W)$ is employed. These concepts will be mainly used with either $V = \mathbf{R}^m$ and $W = \mathbf{R}^{n-m}$ or $V = \operatorname{Hom}(\mathbf{R}^m,\mathbf{R}^{n-m})$ and $W = \mathbf{R}$.

Weakly Differentiable Functions Whenever U is an open subset of $\mathbf{R}^m$ and u is an $\mathscr{L}^m \llcorner U$ measurable function with values in $\mathbf{R}^{n-m}$, the function u is termed k times weakly differentiable if and only if its distributional derivatives up to order k are representable with respect to $\mathscr{L}^m \llcorner U$ by functions $\mathbf{D}^i u$ belonging to $\mathbf{L}_1(\mathscr{L}^m \llcorner K, \bigodot^i(\mathbf{R}^m, \mathbf{R}^{n-m}))$ whenever K is a compact subset of U and $i = 0, \ldots, k$. The definition of the function $\mathbf{D}^i u$ includes the requirement

$$\mathbf{D}^i u(a) = \lim_{r \to 0+} \fint_{\mathbf{B}(a,r)} \mathbf{D}^i u \, \mathrm{d}\mathscr{L}^m \quad \text{for } a \in U;$$

hence $a \in \operatorname{dmn} \mathbf{D}^i u$ if and only if the limit on the right-hand side exists. In particular, the weak derivative $\mathbf{D}^i u$ will be distinguished notationally from the classical derivative $D^i u$. One abbreviates 1 times weakly differentiable and $\mathbf{D}^1 u$ to weakly differentiable and $\mathbf{D}u$, respectively. Let

$$|f|_{p;a,r} = \left(\int_{\mathbf{U}(a,r)} |f|^p \, \mathrm{d}\mathscr{L}^m \right)^{1/p} \quad \text{if } 1 \leq p < \infty,$$
$$|f|_{\infty;a,r} = \inf\{t : \mathscr{L}^m(\mathbf{U}(a,r) \cap \{x : |f(x)| > t\}) = 0\}$$

whenever $a \in \mathbf{R}^m$, $0 < r < \infty$ with $\mathbf{U}(a,r) \subset U$, and f is an $\mathscr{L}^m \llcorner \mathbf{U}(a,r)$ measurable function with values in a Hilbert space. In particular, this applies with $f = \mathbf{D}^i u$ and $\bigodot^i(\mathbf{R}^m, \mathbf{R}^{n-m})$ equipped with the inner product described in the paragraph on multilinear algebra. The Sobolev space $\mathbf{W}^{k,p}(U, \mathbf{R}^{n-m})$ consists of all k times weakly differentiable functions in U with values in $\mathbf{R}^{n-m}$ such that $\mathbf{D}^i u \in \mathbf{L}_p(\mathscr{L}^m \llcorner U, \bigodot^i(\mathbf{R}^m, \mathbf{R}^{n-m}))$ whenever $i = 0, \ldots, k$ with topology induced by its canonical embedding into $\bigoplus_{i=0}^k \mathbf{L}_p(\mathscr{L}^m \llcorner U, \bigodot^i(\mathbf{R}^m, \mathbf{R}^{n-m}))$. Note that neither in the Sobolev spaces nor in the Lebesgue spaces are functions agreeing $\mathscr{L}^m \llcorner U$ almost everywhere treated as single elements; instead the aforementioned condition on $\mathbf{D}^i u$ is employed.

Distributions Whenever U is an open subset of $\mathbf{R}^m$ and Y is a Banach space, $\mathscr{D}(U, Y)$ denotes the space of functions mapping U into Y of class ∞ having compact support. The space $\mathscr{D}(U, Y)$ is equipped with the usual topology and $\mathscr{D}'(U, Y)$ denotes its topological dual; see [16, 4.1.1]. If $1 \leq p \leq \infty$, $1 \leq q \leq \infty$ with $1/p + 1/q = 1$, i is a negative integer, and $T \in \mathscr{D}'(U, \mathbf{R}^m)$ then

$$|T|_{i,p;a,r} = \sup\{T(\theta) : \theta \in \mathscr{D}(U, \mathbf{R}^{n-m}), \operatorname{spt}\theta \subset \mathbf{U}(a,r) \text{ and } |D^{-i}\theta|_{q;a,r} \leq 1\}.$$

Note, if $U = \mathbf{U}(a,r)$ and $|T|_{-1,p;a,r} < \infty$ then one may use the Hahn–Banach theorem to represent T with the help of either the duality of Lebesgue spaces if $p > 1$ or [16, 2.5.12, 14] if $p = 1$; in fact, if $p > 1$ then there exists $g \in \mathbf{L}_p(\mathscr{L}^m \llcorner \mathbf{U}(a,r), \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m}))$ with $|g|_{p;a,r} = |T|_{-1,p;a,r}$ and

$$T(\theta) = \int D\theta(x) \bullet g(x) \, \mathrm{d}\mathscr{L}^m x \quad \text{for } \theta \in \mathscr{D}(\mathbf{U}(a,r), \mathbf{R}^{n-m}),$$

and if $p = 1$ then there exists a Radon measure ϕ on $\mathbf{U}(a,r)$ and a ϕ measurable function $k : \mathbf{U}(a,r) \to \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m})$ with $|k(x)| = 1$ for ϕ almost all $x \in \mathbf{U}(a,r)$ such that $\phi\, \mathbf{U}(a,r) = |T|_{-1,1;a,r}$ and

$$T(\theta) = \int D\theta(x) \bullet k(x) \,\mathrm{d}\phi\, x \quad \text{for } \theta \in \mathscr{D}(\mathbf{U}(a,r), \mathbf{R}^{n-m}).$$

Moreover, the closure of $\mathscr{D}(U, \mathbf{R}^{n-m})$ in $\mathbf{W}^{k,p}(U, \mathbf{R}^{n-m})$ is $\mathbf{W}_0^{k,p}(U, \mathbf{R}^{n-m})$.

Varifolds The following notation for varifolds based on Allard [3, 3.1, 4.2, 4.3, 3.5] will be used in Sects. 4 and 5.

Suppose U is an open subset of $\mathbf{R}^n$ and $\mathbf{G}(n,m)$ is equipped with the usual topology, e.g., induced by its embedding into $\bigodot_2 \bigwedge_m \mathbf{R}^n$; see [16, 3.2.28 (4)]. An m-dimensional varifold V in U is a Radon measure on $U \times \mathbf{G}(n,m)$. The weight measure $\|V\|$ of V is given by $\|V\|(A) = V(A \times \mathbf{G}(n,m))$ for $A \subset U$. The distributional first variation with respect to area of a varifold V is given by

$$\delta V(\theta) = \int D\theta(z) \bullet S_\natural \,\mathrm{d}V(z,S) \quad \text{whenever } \theta \in \mathscr{D}(U, \mathbf{R}^n)$$

with associated Borel regular measure $\|\delta V\|$ characterized by

$$\|\delta V\|(Z) = \sup\{\delta V(\theta) : \theta \in \mathscr{D}(U, \mathbf{R}^n) \text{ with } \operatorname{spt}\theta \subset Z \text{ and } |\theta(z)| \le 1 \text{ for } z \in U\}$$

whenever Z is an open subset of U. If V is an m-dimensional varifold in U and $\|\delta V\|$ is a Radon measure, the *generalized mean curvature vector of V at z* is the unique $\mathbf{h}(V;z) \in \mathbf{R}^n$ such that

$$\mathbf{h}(V;z) \bullet v = -\lim_{r \to 0+} \frac{(\delta V)(b_{z,r} \cdot v)}{\|V\|\, \mathbf{B}(z,r)} \quad \text{for } v \in \mathbf{R}^n$$

where $b_{z,r}$ is the characteristic function of $\mathbf{B}(z,r)$; hence $z \in \operatorname{dmn} \mathbf{h}(V;\cdot)$ if and only if the above limit exists for every $v \in \mathbf{R}^n$.

An m-dimensional varifold V in U is integral if and only if there exist sequences A_i and M_i such that M_i are m-dimensional submanifolds of $\mathbf{R}^n$ of class 1, A_i are $\mathscr{H}^m$ measurable subsets of M_i, and

$$V(f) = \sum_{i=1}^{\infty} \int_{A_i} f(z, \operatorname{Tan}(M_i, z)) \,\mathrm{d}\mathscr{H}^m z \quad \text{for } f \in \mathscr{K}(U \times \mathbf{G}(n,m)).$$

In this case $\boldsymbol{\Theta}^m(\|V\|, z) \in \mathscr{P}$ and $\operatorname{Tan}^m(\|V\|, z) \in \mathbf{G}(n,m)$ for $\|V\|$ almost all z and

$$V(f) = \int f(z, \operatorname{Tan}^m(\|V\|, z)) \boldsymbol{\Theta}^m(\|V\|, z) \,\mathrm{d}\mathscr{H}^m z \quad \text{for } f \in \mathscr{K}(U \times \mathbf{G}(n,m)).$$

The set of all integral m-dimensional varifolds in U is denoted by $\mathbf{IV}_m(U)$.

Almgren's Multiple-Valued Functions The following notation from Almgren [6, 1.1(1)(3)(9)(10), 2.3(2)] is used in Sect. 4. Suppose $Q \in \mathscr{P}$.

The space $\mathbf{Q}_Q(\mathbf{R}^{n-m})$ equals the set of all 0-dimensional integral currents R such that $R = \sum_{i=1}^{Q} [\![y_i]\!]$ for some $y_1, \ldots, y_Q \in \mathbf{R}^{n-m}$. The metric $\mathscr{G}$ on $\mathbf{Q}_Q(\mathbf{R}^{n-m})$ satisfies

$$\mathscr{G}\left(\sum_{i=1}^{Q} [\![y_i]\!], \sum_{i=1}^{Q} [\![y_i']\!]\right) = \inf\left\{\left(\sum_{i=1}^{Q} |y_i - y_{\pi(i)}'|^2\right)^{1/2} : \pi \in P(Q)\right\}$$

whenever $y_1, \ldots, y_Q, y_1', \ldots, y_Q' \in \mathbf{R}^{n-m}$, where $P(Q)$ denotes the set of permutations of $\{1, \ldots, Q\}$. The function $\boldsymbol{\eta}_Q : \mathbf{Q}_Q(\mathbf{R}^{n-m}) \to \mathbf{R}^{n-m}$ satisfies

$$\boldsymbol{\eta}_Q(R) = Q^{-1} \sum_{i=1}^{Q} y_i \quad \text{whenever } R = \sum_{i=1}^{Q} [\![y_i]\!] \text{ for some } y_1, \ldots, y_Q \in \mathbf{R}^{n-m}$$

and $\operatorname{Lip} \boldsymbol{\eta}_Q = Q^{-1/2}$.

If $f : X \to \mathbf{Q}_Q(\mathbf{R}^{n-m})$ then

$$\operatorname{graph}_Q f = (X \times \mathbf{R}^{n-m}) \cap \{(x, y) : y \in \operatorname{spt} f(x)\}$$

and if additionally $g : X \to \mathbf{R}^{n-m}$ then $f(+)g : X \to \mathbf{Q}_Q(\mathbf{R}^{n-m})$ satisfies

$$(f(+)g)(x) = \sum_{i=1}^{Q} [\![y + y_i]\!] \quad \text{if } f(x) = \sum_{i=1}^{Q} [\![y_i]\!] \text{ and } g(x) = y$$

whenever $x \in X$.

Moreover, a function $f : \mathbf{R}^m \to \mathbf{Q}_Q(\mathbf{R}^{n-m})$ is affine if and only if there exist affine functions $f_i : \mathbf{R}^m \to \mathbf{R}^{n-m}$, $i = 1, \ldots, Q$ such that

$$f(x) = \sum_{i=1}^{Q} [\![f_i(x)]\!] \quad \text{whenever } x \in \mathbf{R}^m.$$

$f_1, \ldots, f_Q$ are uniquely determined up to order. Moreover,

$$|f|^2 = \sum_{i=1}^{Q} |Df_i(0)|^2.$$

Let $a \in A \subset \mathbf{R}^m$ and $f : A \to \mathbf{Q}_Q(\mathbf{R}^{n-m})$. f is affinely approximable at a if and only if $a \in \operatorname{Int} A$ and there exists an affine function $g : \mathbf{R}^m \to \mathbf{Q}_Q(\mathbf{R}^{n-m})$ such that

$$\lim_{x \to a} \mathscr{G}(f(x), g(x))/|x - a| = 0.$$

The function g is unique and denoted by $Af(a)$. f is strongly affinely approximable at a if and only if $Af(a)$ has the following property: If $Af(a)(x) = \sum_{i=1}^{Q} [\![g_i(x)]\!]$

for some affine functions $g_i : \mathbf{R}^m \to \mathbf{R}^{n-m}$ and $g_i(a) = g_j(a)$ for some i and j, then $Dg_i(a) = Dg_j(a)$. The concepts of approximate affine approximability and approximate strong affine approximability are obtained through omission of the condition $a \in \operatorname{Int} A$ and replacement of lim by ap lim. The corresponding affine function is denoted by ap $Af(a)$.

The projections $\mathbf{p} \in \mathbf{O}^*(n, m)$ and $\mathbf{q} \in \mathbf{O}^*(n, n-m)$ satisfy

$$\mathbf{p}(z) = (z_1, \ldots, z_m), \qquad \mathbf{q}(z) = (z_{m+1}, \ldots, z_n)$$

whenever $z = (z_1, \ldots, z_n) \in \mathbf{R}^n$. The closed cuboid $\mathbf{C}(T, a, r, h)$ equals

$$\mathbf{R}^n \cap \{z : |T_\natural(z-a)| \leq r \text{ and } |T_\natural^\perp(z-a)| \leq h\}$$

whenever $T \in \mathbf{G}(n, m)$, $a \in \mathbf{R}^n$, $0 < r < \infty$, and $0 < h \leq \infty$. One abbreviates $\mathbf{C}(T, a, r, \infty) = \mathbf{C}(T, a, r)$. Note that the symbol $\mathbf{C}(T, a, r)$ is used by Allard in [3, 8.10] to denote $\mathbf{R}^n \cap \{z : |T_\natural(z-a)| < r\}$.

References to Constants Each statement asserting the existence of a positive, finite number, small (ε) or large (Γ), will give rise to a function depending on the listed parameters whose "name" is $\varepsilon_{\text{x.y}}$ or $\Gamma_{\text{x.y}}$ where x.y denotes the number of the statement.

Additional Notation If M is a submanifold of $\mathbf{R}^n$ of class 2 and $a \in M$ then the *mean curvature vector of M at a* is the unique $\mathbf{h}(M; a) \in \mathbf{R}^n$ such that

$$\operatorname{Tan}(M; a)_\natural \bullet Dg(a) = -g(a) \bullet \mathbf{h}(M; a)$$

whenever $g : \mathbf{R}^n \to \mathbf{R}^n$ is class 1 and $g(z) \in \operatorname{Nor}(M; z)$ for $z \in M$, cp. Allard [3, 2.5(2)]. And if U is an open subset of $\mathbf{R}^m$ and Y is a Banach space then T is called a *constant distribution in U of type Y* if and only if $T \in \mathscr{D}'(U, Y)$ and for some $\alpha \in Y^*$ there holds $T(\theta) = \int_U \alpha \circ \theta \, d\mathscr{L}^m$ for $\theta \in \mathscr{D}(U, Y)$. Moreover, a subset of a topological space X is called *universally measurable* if and only if it is measurable with respect to every measure ϕ such that all closed subsets of X are ϕ measurable.

3 A Criterion for Second Order Differentiability in Lebesgue Spaces

Overview The purpose of this section is to prove 3.11, which contains Theorem 3 of the Introduction, and to provide the preparations necessary for its application in Sect. 4.

First, in 3.1 the situation studied is described. Then, for the convenience of the reader, in 3.2–3.8 adaptations and applications of standard theory are carried out. The main ingredient in the proof of 3.11 is contained in 3.9. The part $q > 1$ of Theorem 4 is provided in 3.13. Finally, in 3.14–3.18 it is shown how a certain nonintegral differentiability condition on the solution u allows treating the case where estimates for $L_F(u)$, see 3.1, are only available in $|\cdot|_{-1,1;a,r}$.

The following set of definitions will be used frequently in the present section.

3.1 Suppose $m, n \in \mathscr{P}$ with $m < n$. Occasionally, the use of Euclidean coordinates will be useful. For this purpose choose dual orthonormal bases

$$e_1, \dots, e_m \quad \text{and} \quad X_1, \dots, X_m$$

of $\mathbf{R}^m$ and $\bigodot^1 \mathbf{R}^m$ and dual orthonormal bases

$$\upsilon_1, \dots, \upsilon_{n-m} \quad \text{and} \quad Y_1, \dots, Y_{n-m}$$

of $\mathbf{R}^{n-m}$ and $\bigodot^1 \mathbf{R}^{n-m}$. For $\Psi \in \bigodot^2 \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m})$ one then obtains the expression

$$\Psi(\sigma, \tau) = \sum_{i=1}^{m} \sum_{j=1}^{n-m} \sum_{k=1}^{m} \sum_{l=1}^{n-m} \Psi_{i,j;k,l} \langle \sigma(e_i), Y_j \rangle \langle \tau(e_k), Y_l \rangle$$

where $\Psi_{i,j;k,l} = \Psi(X_i \upsilon_j, X_k \upsilon_l)$ and $X\upsilon$ maps $x \in \mathbf{R}^m$ onto $X(x)\upsilon \in \mathbf{R}^{n-m}$ whenever $X \in \bigodot^1 \mathbf{R}^m$ and $\upsilon \in \mathbf{R}^{n-m}$.

Let $\Upsilon \in \bigodot^2 \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m})$ be defined by

$$\Upsilon(\sigma, \tau) = \sigma \bullet \tau \quad \text{for } \sigma, \tau \in \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m}),$$

and suppose $F : \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m}) \to \mathbf{R}$ is of class 2, $0 \le \varepsilon < \infty$, and

$$\| D^2 F(\sigma) - \Upsilon \| \le \varepsilon \quad \text{whenever } \sigma \in \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m}).$$

The quantity $\operatorname{Lip} D^2 F$ will be computed with respect to $|\cdot|$ on $\operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m})$ and $\|\cdot\|$ on $\bigodot^2 \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m})$.

To each such F there corresponds the Euler–Lagrange differential operator L_F which associates with every $u \in \mathbf{W}^{1,1}(U, \mathbf{R}^{n-m})$ for some open subset U of $\mathbf{R}^m$ a distribution $L_F(u)$ in $\mathscr{D}'(U, \mathbf{R}^{n-m})$ defined by

$$L_F(u)(\theta) = -\int_U \langle D\theta(x), DF(\mathbf{D}u(x)) \rangle \, \mathrm{d}\mathscr{L}^m x \quad \text{for } \theta \in \mathscr{D}(U, \mathbf{R}^{n-m}).$$

There also occurs the linear function $C_F(\sigma) : \bigodot^2(\mathbf{R}^m, \mathbf{R}^{n-m}) \to \mathbf{R}^{n-m}$ which for $\sigma \in \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m})$ is given by

$$\langle \phi, C_F(\sigma) \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n-m} \sum_{k=1}^{m} \sum_{l=1}^{n-m} \langle (X_i \upsilon_j, X_k \upsilon_l), D^2 F(\sigma) \rangle \langle \phi(e_i, e_k), Y_j \rangle \upsilon_l$$

whenever $\phi \in \bigodot^2(\mathbf{R}^m, \mathbf{R}^{n-m})$. The function $C_F(\sigma)$ is uniquely determined by $D^2 F(\sigma)$; see [16, 5.2.11]. One obtains by partial integration for $u \in \mathbf{W}^{2,1}(U, \mathbf{R}^{n-m})$, $\theta \in \mathscr{D}(U, \mathbf{R}^{n-m})$

$$L_F(u)(\theta) = \int_U \theta(x) \bullet \langle \mathbf{D}^2 u(x), C_F(\mathbf{D}u(x)) \rangle \, \mathrm{d}\mathscr{L}^m x.$$

Sometimes also $S : \bigodot^2(\mathbf{R}^m, \mathbf{R}^{n-m}) \to \mathbf{R}^{n-m}$ corresponding to the Dirichlet integrand, i.e., $F(\sigma) = |\sigma|^2/2$ for $\sigma \in \mathrm{Hom}(\mathbf{R}^m, \mathbf{R}^m)$, (and therefore to Υ) will be used. Note $\langle \phi, S \rangle = \sum_{i=1}^m \phi(e_i, e_i)$ whenever $\phi \in \bigodot^2(\mathbf{R}^m, \mathbf{R}^{n-m})$. One may check that with $\kappa = 2^{1/2} m(n-m)$

$$|C_F(\sigma)| \le \kappa \|D^2 F(\sigma)\|, \qquad |C_F(\sigma) - S| \le \kappa\varepsilon,$$
$$|C_F(\sigma) - C_F(\tau)| \le \kappa \|D^2 F(\sigma) - D^2 F(\tau)\|$$

for $\sigma, \tau \in \mathrm{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m})$, where $|\cdot|$ denotes the norm associated with the inner product on $\mathrm{Hom}(\bigodot^2(\mathbf{R}^m, \mathbf{R}^{n-m}), \mathbf{R}^{n-m})$; see [16, 1.7.9, 1.10.6].

The development of the present section requires a priori estimates of solutions to linear elliptic systems of second order both in $\mathbf{W}^{1,p}(\mathbf{U}(a,r), \mathbf{R}^{n-m})$ in the case of divergence form and in $\mathbf{W}^{2,p}(\mathbf{U}(a,r), \mathbf{R}^{n-m})$ in case of nondivergence form. The coefficients are possibly neither continuous nor of vanishing mean oscillation. Instead, they are close in an $\mathbf{L}_\infty(\mathscr{L}^m)$ sense to those associated with Laplace's operator. Therefore, the required estimates are obtained in 3.2–3.8 by standard perturbation methods from the case of Laplace's operator.

First, recall the following existence result with corresponding a priori estimates for solutions in $\mathbf{W}_0^{1,p}(\mathbf{U}(a,r), \mathbf{R}^{n-m})$.

Theorem 3.2 *Suppose $n \in \mathscr{P}$ and $1 < p < \infty$.*

Then there exist positive, finite numbers ε and Γ with the following property.

If $n > m \in \mathscr{P}$, Υ is as in 3.1, $a \in \mathbf{R}^m$, $0 < r < \infty$,

$$A : \mathbf{U}(a,r) \to \bigodot^2 \mathrm{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m}) \quad \textit{is } \mathscr{L}^m \llcorner \mathbf{U}(a,r) \textit{ measurable},$$
$$\|A(x) - \Upsilon\| \le \varepsilon \quad \textit{whenever } x \in \mathbf{U}(a,r),$$

then for every $T \in \mathscr{D}'(\mathbf{U}(a,r), \mathbf{R}^{n-m})$ with $|T|_{-1,p;a,r} < \infty$ there exists an $\mathscr{L}^m \llcorner \mathbf{U}(a,r)$ almost unique $u \in \mathbf{W}_0^{1,p}(\mathbf{U}(a,r), \mathbf{R}^{n-m})$ such that

$$-\int_{\mathbf{U}(a,r)} \langle D\theta(x) \odot \mathbf{D}u(x), A(x) \rangle \, \mathrm{d}\mathscr{L}^m x = T(\theta) \quad \textit{for } \theta \in \mathscr{D}(\mathbf{U}(a,r), \mathbf{R}^{n-m}).$$

Moreover, whenever u and T are related as above there holds

$$|\mathbf{D}u|_{p;a,r} \le \Gamma |T|_{-1,p;a,r}.$$

Proof By the Neumann series (cf. [16, 3.1.11]) it is enough to consider the case $\varepsilon = 0$. Note also that there exists $g \in \mathbf{L}_p(\mathscr{L}^m \llcorner \mathbf{U}(a,r), \mathrm{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m}))$ with $T(\theta) = -\int_{\mathbf{U}(a,r)} g \bullet D\theta \, \mathrm{d}\mathscr{L}^m$ for $\theta \in \mathscr{D}(\mathbf{U}(a,r), \mathbf{R}^{n-m})$ and $|T|_{-1,p;a,r} = |g|_{p;a,r}$ by the Hahn–Banach theorem.

The conclusion then follows from [18, Theorem 10.15] in case $p \ge 2$, to which the case $p < 2$ reduces by use of a duality argument. $\square$

Remark 3.3 Representing T via g, the result is contained in Dong and Kim [13, Theorem 8.2 (ii)], where measurable coefficients of nearly vanishing mean oscillation are treated.

In order to express the equation for u in Euclidean coordinates, suppose T is represented by g as in the preceding proof. If u is of class 2, A and g are of class 1 and $A_{i,j;k,l}(x) \in \mathbf{R}$ represent $A(x) \in \bigodot^2 \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m})$ as $\Psi_{i,j;k,l}$ represent Ψ in 3.1, and $g_{i,j}(x) = \langle e_i, g(x)\rangle \bullet \upsilon_j$ both for $x \in \mathbf{U}(a,r)$, the hypothesis relating u to T is equivalent to

$$\sum_{i=1}^{m}\sum_{k=1}^{m}\sum_{l=1}^{n-m} D_i(A_{i,j;k,l}D_k u_l) = \sum_{i=1}^{m} D_i g_{i,j} \quad \text{whenever } j \in \{1,\dots,n-m\}$$

where $\theta_j = Y_j \circ \theta$ and $u_l = Y_l \circ u$; see 3.1.

From the preceding theorem, one deduces as usual local a priori estimates in $\mathbf{W}^{1,p}(\mathbf{U}(a,r), \mathbf{R}^{n-m})$ for weak solutions belonging to $\mathbf{W}^{1,q}(\mathbf{U}(a,r), \mathbf{R}^{n-m})$, possibly with $q < p$.

Theorem 3.4 *Suppose $n \in \mathscr{P}$, $1 < q < \infty$, and $1 < p < \infty$.*

Then there exists a positive, finite number ε with the following property.

If $n > m \in \mathscr{P}$, Υ is as in 3.1, $a \in \mathbf{R}^m$, $0 < r < \infty$,

$$A : \mathbf{U}(a,r) \to \textstyle\bigodot^2 \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m}) \quad \textit{is } \mathscr{L}^m \llcorner \mathbf{U}(a,r) \textit{ measurable},$$
$$\|A(x) - \Upsilon\| \le \varepsilon \quad \textit{whenever } x \in \mathbf{U}(a,r),$$

and $u \in \mathbf{W}^{1,q}(\mathbf{U}(a,r), \mathbf{R}^{n-m})$, $T \in \mathscr{D}'(\mathbf{U}(a,r), \mathbf{R}^{n-m})$ satisfy

$$-\int_{\mathbf{U}(a,r)} \langle D\theta(x) \odot \mathbf{D}u(x), A(x)\rangle \, \mathrm{d}\mathscr{L}^m x = T(\theta) \quad \textit{for } \theta \in \mathscr{D}(\mathbf{U}(a,r), \mathbf{R}^{n-m}),$$

then

$$|\mathbf{D}u|_{p;a,r/2} \le \Gamma\big(r^{-m-1+m/p}|u|_{1;a,r} + |T|_{-1,p;a,r}\big)$$

where Γ is a positive, finite number depending only on n and p.

Proof Let $0 < \delta \le 1$, suppose $n, q, p, m, \Upsilon, a, r, A, u$, and T satisfy the hypotheses in the body of the theorem with ε replaced by δ and assume $q \le p$. It will be shown that u satisfies the estimate in the conclusion of the theorem provided δ is suitably small.

The problem will be reduced.

First, to the case $p = q$ by constructing as solutions of approximating Dirichlet problems by use of 3.2 a sequence of functions $u_i \in \mathbf{W}^{1,p}(\mathbf{U}(a,r), \mathbf{R}^{n-m})$ such that $u_i \to u$ in $\mathbf{W}^{1,q}(\mathbf{U}(a,r), \mathbf{R}^{n-m})$ as $i \to \infty$ and for $i \in \mathscr{P}$

$$-\int_{\mathbf{U}(a,r)} \langle D\theta(x) \odot \mathbf{D}u_i(x), A(x)\rangle \, \mathrm{d}\mathscr{L}^m x = T(\theta) \quad \text{for } \theta \in \mathscr{D}(\mathbf{U}(a,r), \mathbf{R}^{n-m})$$

provided $\delta \le \inf\{\varepsilon_{3.2}(n,p), \varepsilon_{3.2}(n,q)\}$.

Second, to the case $p = q$ and $\delta = 0$ by considering Simon's absorption lemma in [34, p. 398].

Third, to the case $p = q$, $\delta = 0$, and $T = 0$ by use of 3.2 and Poincaré's inequality.

Finally, the remaining case follows by convolution from [17, Theorems 2.8, 2.10]. □

Next, local a priori estimates in $\mathbf{W}^{2,p}(\mathbf{U}(a,r), \mathbf{R}^{n-m})$ for so-called strong solutions of linear elliptic systems in nondivergence form with measurable coefficients close to those of Laplace's operator are stated.

Theorem 3.5 *Suppose $n \in \mathscr{P}$ and $1 < p < \infty$.*

Then there exists a positive, finite number ε with the following property.

If $n > m \in \mathscr{P}$, S is as in 3.1, $a \in \mathbf{R}^m$, $0 < r < \infty$,

$$B : \mathbf{U}(a,r) \to \operatorname{Hom}\big(\textstyle\bigodot^2(\mathbf{R}^m, \mathbf{R}^{n-m}), \mathbf{R}^{n-m}\big) \quad \textit{is } \mathscr{L}^m \llcorner \mathbf{U}(a,r) \textit{ measurable},$$

$$|B(x) - S| \le \varepsilon \quad \textit{whenever } x \in \mathbf{U}(a,r),$$

and $u \in \mathbf{W}^{2,p}(\mathbf{U}(a,r), \mathbf{R}^{n-m})$, $f \in \mathbf{L}_p(\mathscr{L}^m \llcorner \mathbf{U}(a,r), \mathbf{R}^{n-m})$ satisfy

$$\langle \mathbf{D}^2 u(x), B(x)\rangle = f(x) \quad \textit{for } \mathscr{L}^m \textit{ almost all } x \in \mathbf{U}(a,r),$$

then

$$|\mathbf{D}^2 u|_{p;a,r/2} \le \Gamma\big(r^{-2-m+m/p}|u|_{1;a,r} + |f|_{p;a,r}\big)$$

where Γ is a positive, finite number depending only on n and p.

Proof From [17, Theorem 7.26(i)] and Ehring's lemma, see, e.g., [36, Theorem I.7.3], it follows that for every $0 < \kappa < \infty$ there exists a positive, finite number Δ depending only on n, p, and κ such that

$$r^{-2-m/p}|v|_{p;a,r} \le \kappa r^{-m/p}|\mathbf{D}^2 v|_{p;a,r} + \Delta r^{-2-m}|v|_{1;a,r}$$

for $v \in \mathbf{W}^{2,p}(\mathbf{U}(a,r), \mathbf{R}^{n-m})$.

Now, one may readily use [17, Theorem 9.11] in conjunction with the absorption lemma in Simon [34, p. 398] to obtain the conclusion. □

In Euclidean coordinates the equation relating u to f reads

$$\sum_{i=1}^{m}\sum_{k=1}^{m}\sum_{l=1}^{n-m} B_{i,j;k,l}(x)\mathbf{D}_{i,k}u_l(x) = f_j(x) \quad \text{for } \mathscr{L}^m \text{ almost all } x \in \mathbf{U}(a,r)$$

whenever $j \in \{1, \dots, n-m\}$ where, see 3.1, $u_l = Y_l \circ u$, $f_j = Y_j \circ f$ and

$$B_{i,j;k,l}(x) = B((X_i \odot X_k/2)\upsilon_l) \bullet \upsilon_j, \qquad \mathbf{D}_{i,k}u_l(x) = \mathbf{D}^2 u_l(x)(e_i, e_k),$$

$$(\alpha \odot \beta)(v, w) = \alpha(v)\beta(w) + \alpha(w)\beta(v) \quad \text{for } \alpha, \beta \in \textstyle\bigodot^1 \mathbf{R}^m \text{ and } v, w \in \mathbf{R}^m,$$

see [16, 1.10.2].

Proceeding to the Euler–Lagrange differential operator L_F associated with F, one may deduce local a priori estimates in $\mathbf{W}^{2,p}(\mathbf{U}(a,r),\mathbf{R}^{n-m})$ via difference quotients from the local a priori estimates in $\mathbf{W}^{1,p}(\mathbf{U}(a,r),\mathbf{R}^{n-m})$ in 3.4. For the intended use in 3.9 it is important to appropriately subtract an affine function P in the lower order term.

Lemma 3.6 *Suppose $n \in \mathscr{P}$, $1 < q < \infty$, and $1 < p < \infty$.*

Then there exists a positive, finite number ε with the following property.

If F is related to ε as in 3.1, $a \in \mathbf{R}^m$, $0 < r < \infty$, $u \in \mathbf{W}^{1,q}(\mathbf{U}(a,r),\mathbf{R}^{n-m})$, and $f \in \mathbf{L}_p(\mathscr{L}^m \llcorner \mathbf{U}(a,r),\mathbf{R}^{n-m})$ satisfy

$$L_F(u)(\theta) = \int_{\mathbf{U}(a,r)} \theta(x) \bullet f(x) \,\mathrm{d}\mathscr{L}^m x \quad \textit{whenever } \theta \in \mathscr{D}(\mathbf{U}(a,r),\mathbf{R}^{n-m}),$$

then u is twice weakly differentiable and for every affine function $P : \mathbf{R}^m \to \mathbf{R}^{n-m}$ there holds

$$|\mathbf{D}^2 u|_{p;a,r/2} \le \Gamma\big(r^{-2-m+m/p}|u-P|_{1;a,r} + |f|_{p;a,r}\big)$$

where Γ is a positive, finite number depending only on n and p.

Proof Let $\varepsilon = \varepsilon_{3.4}(n,q,p)$ and suppose F, a, r, u, f, and P satisfy the hypotheses in body of the lemma.

Let $v = u - P$, $i \in \{1,\ldots,m\}$ and define for $0 < h < r$, $x \in \mathbf{U}(a,r-h)$

$$u_h(x) = h^{-1}(u(x+he_i) - u(x)), \qquad v_h(x) = h^{-1}(v(x+he_i) - v(x)),$$

$$A_h(x) = \int_0^1 D^2F(t\mathbf{D}u(x+he_i) + (1-t)\mathbf{D}u(x)) \,\mathrm{d}\mathscr{L}^1 t,$$

and let $S_h \in \mathscr{D}'(\mathbf{U}(a,r-h),\mathbf{R}^{n-m})$ be characterized by

$$S_h(\theta|\mathbf{U}(a,r-h)) = h^{-1}\int_{\mathbf{U}(a,r)} (\theta(x-he_i) - \theta(x)) \bullet f(x) \,\mathrm{d}\mathscr{L}^m x$$

whenever $\theta \in \mathscr{D}(\mathbf{R}^m,\mathbf{R}^{n-m})$ with $\operatorname{spt}\theta \subset \mathbf{U}(a,r-h)$. One readily verifies, noting $\mathbf{D}u_h = \mathbf{D}v_h$,

$$-\int_{\mathbf{U}(a,r-h)} \langle D\theta(x) \odot \mathbf{D}v_h(x), A_h(x)\rangle \,\mathrm{d}\mathscr{L}^m x = S_h(\theta)$$

for $\theta \in \mathscr{D}(\mathbf{U}(a,r-h),\mathbf{R}^{n-m})$. Hence, by 3.4,

$$|\mathbf{D}v_h|_{p;a,(r-h)/2} \le \Delta\big((r-h)^{-1-m+m/p}|v_h|_{1;a,r-h} + |S_h|_{-1,p;a,r-h}\big)$$

where $\Delta = \Gamma_{3.4}(n,p)$. Since $|v_h|_{1;a,r-h} \le |\mathbf{D}v|_{1;a,r}$ and $|S_h|_{-1,p;a,r-h} \le |f|_{p;a,r}$, taking the limit $h \to 0+$ one infers that v, hence u, is twice weakly differentiable and satisfies the desired estimate, using Simon's absorption lemma [34, p. 398] as before. □

Remark 3.7 In general, even if $\operatorname{Lip} u \le L < \infty$ and $P = 0$ the condition involving ε cannot be replaced by some uniform strong ellipticity condition on $D^2F(\sigma)$ for $\sigma \in \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m})$ with $\|\sigma\| \le L$, as may be seen from the example of Lawson and Osserman in [20, Theorem 7.1].

Next, differences of solutions to L_F are estimated in $\mathbf{W}^{2,p}(\mathbf{U}(a,r), \mathbf{R}^{n-m})$. Here the nonlinearity of DF enters via $\operatorname{Lip} D^2F$ in the estimate. However, concerning the use of the estimate in 3.9, the factor $r^{-m-1}|u_1 - P|_{1;a,r}$ in the estimate below can be assumed to be uniformly bounded.

Lemma 3.8 *Suppose $n \in \mathscr{P}$, and $1 < q \le p < \infty$.*

Then there exists a positive, finite number ε with the following property.

If $n > m \in \mathscr{P}$, F is related to ε as in 3.1, $\operatorname{Lip} D^2F < \infty$, $a \in \mathbf{R}^m$, $0 < r < \infty$, and $u_i \in \mathbf{W}^{1,q}(\mathbf{U}(a,r), \mathbf{R}^{n-m})$ with $i \in \{1,2\}$ satisfy $L_F(u_i) = 0$, then u_i are twice weakly differentiable and for every affine function $P : \mathbf{R}^m \to \mathbf{R}^{n-m}$ there holds

$$
\begin{aligned}
&r^{-m/p+1}|\mathbf{D}^2(u_2 - u_1)|_{p;a,r/2} \\
&\quad \le \Gamma\big(r^{-m-1}|u_2 - u_1|_{1;a,r} \\
&\qquad + (r^{-m-1}|u_1 - P|_{1;a,r}) \operatorname{Lip}(D^2F)(r^{-m-1}|u_2 - u_1|_{1;a,r})\big)
\end{aligned}
$$

where Γ is a positive, finite number depending only on n and p.

Proof Using an elementary covering argument, it is enough to prove the assertion with $|\mathbf{D}^2(u_2 - u_1)|_{p;a,r/2}$ replaced by $|\mathbf{D}^2(u_2 - u_1)|_{p;a,r/4}$. For this purpose let $\kappa = 2^{1/2}n^2$,

$$
\begin{aligned}
&\varepsilon = \inf\{\varepsilon_{3.6}(n,q,2p), \varepsilon_{3.5}(n,p)/\kappa, \varepsilon_{3.4}(n,q,2p)\}, \qquad \Delta_1 = \Gamma_{3.6}(n,2p), \\
&\Delta_2 = \Gamma_{3.5}(n,p), \qquad \Delta_3 = \Gamma_{3.4}(n,2p), \qquad \Gamma = \Delta_2 \sup\{2^{1+n}, \kappa\Delta_1\Delta_3\}.
\end{aligned}
$$

Suppose F, a, r, and u_i satisfy the hypotheses with ε and that $P : \mathbf{R}^m \to \mathbf{R}^{n-m}$ is an affine function. In order to show that they satisfy the modified conclusions with Γ, it will be assumed $a = 0$ and $r = 1$. Abbreviate $\Lambda = \operatorname{Lip} D^2F$.

By 3.6 the functions u_i are twice weakly differentiable with

$$|\mathbf{D}^2u_i|_{2p;0,1/2} \le \Delta_1|u_i - P|_{1;0,1} \quad \text{for } i \in \{1,2\}$$

and one obtains from 3.1 for $\mathscr{L}^m$ almost all $x \in \mathbf{U}(0,1)$

$$
\begin{aligned}
\langle \mathbf{D}^2u_i(x), C_F(\mathbf{D}u_i(x))\rangle = 0 \quad \text{for } i \in \{1,2\}, \\
\langle \mathbf{D}^2(u_2 - u_1)(x), C_F(\mathbf{D}u_2(x))\rangle = \langle \mathbf{D}^2u_1(x), C_F(\mathbf{D}u_1(x)) - C_F(\mathbf{D}u_2(x))\rangle.
\end{aligned}
$$

Therefore by 3.5, 3.1, and Hölder's inequality

$$
\begin{aligned}
|\mathbf{D}^2(u_2 - u_1)|_{a,1/4;p} &\le \Delta_2\big(2^{2+m-m/p}|u_2 - u_1|_{0,1/2;1} \\
&\qquad + \kappa\Lambda|\mathbf{D}^2u_1|_{2p;0,1/2}|\mathbf{D}(u_2 - u_1)|_{2p;0,1/2}\big).
\end{aligned}
$$

To estimate $|\mathbf{D}(u_2 - u_1)|_{2p;0,1/2}$, one computes for $\theta \in \mathscr{D}(\mathbf{U}(0,1), \mathbf{R}^{n-m})$

$$-\int_{\mathbf{U}(0,1)} \langle D\theta(x) \odot \mathbf{D}(u_2 - u_1)(x), A(x)\rangle \, \mathrm{d}\mathscr{L}^m x = 0,$$

$$\text{where } A(x) = \int_0^1 D^2 F(t\mathbf{D}u_2(x) + (1-t)\mathbf{D}u_1(x)) \, \mathrm{d}\mathscr{L}^1 t,$$

and obtains from 3.4

$$|\mathbf{D}(u_2 - u_1)|_{2p;0,1/2} \le \Delta_3 |u_2 - u_1|_{1;0,1}$$

and the conclusion follows. □

Having gathered the local a priori estimates needed, the patching procedure involved in the proof of the main theorem of this section, 3.11, is carried out. Suppose $v_{a,\varrho}$ are solutions to L_F in balls $\mathbf{U}(a,\varrho)$ with a in some closed set A and $0 < \varrho \le 1$ which are suitably close to some reference function u. Then the following lemma shows how to construct by use of a partition of unity a single function v defined in a neighborhood of A such that v retains from the $v_{a,\varrho}$ both the closeness to u and the a priori estimates for the weak derivatives of second order.

Lemma 3.9 *Suppose $m, n \in \mathscr{P}$, $m < n$, $1 \le p \le r < \infty$, and $1 < q < \infty$.*

Then there exist a positive, finite number ε, a positive, finite number Γ_1 depending only on m and p, and a positive, finite number Γ_2 depending only on m, n, p, and r with the following property.

If F is related to ε as in 3.1, $\operatorname{Lip} D^2F < \infty$, $j \in \{0,1\}$, A is a closed subset of $\mathbf{R}^m$, $u : \mathbf{R}^m \cap \{x : \operatorname{dist}(x,A) < 1\} \to \mathbf{R}^{n-m}$ is j times weakly differentiable, $0 \le \gamma < \infty$, and if for each $a \in A$, $0 < \varrho \le 1$ there are $v_{a,\varrho} \in \mathbf{W}^{1,q}(\mathbf{U}(a,\varrho), \mathbf{R}^{n-m})$ and an affine function $P_{a,\varrho} : \mathbf{R}^m \to \mathbf{R}^{n-m}$ such that

$$L_F(v_{a,\varrho}) = 0,$$

$$\sum_{i=0}^{j} \varrho^{-m/p+i} |\mathbf{D}^i(u - v_{a,\varrho})|_{p;a,\varrho} \le \gamma\varrho^2, \qquad \varrho^{-m/p}|u - P_{a,\varrho}|_{p;a,\varrho} \le \gamma\varrho$$

then there exists a twice weakly differentiable function $v : \mathbf{R}^m \cap \{x : \operatorname{dist}(x,A) < \frac{1}{36}\} \to \mathbf{R}^{n-m}$ with

$$\sum_{i=0}^{j} \varrho^{-m/p+i} |\mathbf{D}^i(u - v)|_{p;a,\varrho} \le \Gamma_1 \gamma \varrho^2,$$

$$\varrho^{-m/r}|\mathbf{D}^2 v|_{r;a,\varrho} \le \Gamma_2\big(\gamma(1 + \operatorname{Lip}(D^2F)\gamma)^2 + \varrho^{-m-2}|u - P_{a,2\varrho}|_{1;a,2\varrho}\big)$$

whenever $a \in A$, $0 < \varrho \le \frac{1}{36}$.

The solutions $v_{a,\varrho}$ approximate u of *second* order in ϱ, and so does the constructed function v. In order to properly treat the nonlinearity of DF the hypothesis concerning approximation of u of *first* order in ϱ by affine functions $P_{a,\varrho}$ is introduced; see also 3.12. Moreover, as a guiding principle for the proof, note that a priori estimates for differences of functions can be controlled via terms of type $|\mathbf{D}^i(u - v_{a,\varrho})|_{p;a,\varrho}$, hence are of second order in ϱ, whereas a priori estimates for single functions additionally involve terms of type $|u - P|_{p;a,\varrho}$, hence are of first order in ϱ.

Proof of 3.9 Assume $r \geq q$ and define

$$\varepsilon = \inf\{1, \varepsilon_{3.6}(n, q, 2r), \varepsilon_{3.8}(n, q, 2r), \varepsilon_{3.6}(n, q, r)\}.$$

Suppose F, j, A, u, γ, $v_{a,\varrho}$, and $P_{a,\varrho}$ are as in the hypotheses in the body of the lemma with ε and abbreviate $\Lambda = \operatorname{Lip} D^2 F$.

By 3.6 and Hölder's inequality

$$\sum_{i=0}^{j} |\mathbf{D}^i v_{a,\varrho}|_{2r;a,1/2} < \infty, \qquad \sum_{i=0}^{j} |\mathbf{D}^i u|_{p;a,1/2} < \infty$$

whenever $a \in A$. Therefore, taking limits (for example, by use of an interpolation inequality similar to [25, Lemma 6.2.2] and weak compactness properties of Sobolev spaces [25, Theorem 3.2.4(e)]) the conclusion can be deduced from the following assertion: *There exist a positive, finite number Γ_1 depending only on m and p, and a positive, finite number Γ_2 depending only on m, n, p, and r such that for every $0 < \delta \leq \frac{1}{18}$ there exists a function $v : \mathbf{R}^m \to \mathbf{R}^{n-m}$ whose restriction to $\mathbf{R}^m \cap \{x : \operatorname{dist}(x, A) < \frac{1}{18}\}$ is twice weakly differentiable satisfying*

$$\sum_{i=0}^{j} \varrho^{-m/p+i} |\mathbf{D}^i(u - v)|_{p;a,\varrho} \leq \Gamma_1 \gamma \varrho^2,$$

$$(\varrho/2)^{-m/r} |\mathbf{D}^2 v|_{r;a,\varrho/2} \leq \Gamma_2 \big(\gamma(1 + \Lambda\gamma)^2 + (\varrho/2)^{-m-2} |u - P_{a,\varrho}|_{1;a,\varrho}\big)$$

whenever $a \in A$, $\delta \leq \varrho \leq \frac{1}{18}$.

Assume $A \neq \emptyset$, let $\Phi = \{\mathbf{R}^m \sim A\} \cup \{\mathbf{U}(a, \delta) : a \in A\}$, note $\bigcup \Phi = \mathbf{R}^m$, define $h : \mathbf{R}^m \to \mathbf{R}$ by

$$h(x) = \frac{1}{20} \sup\{\inf\{1, \operatorname{dist}(x, \mathbf{R}^m \sim U)\} : U \in \Phi\} \quad \text{for } x \in \mathbf{R}^m,$$

and apply [16, 3.1.13] to obtain a countable subset S of $\mathbf{R}^m$ and functions $\varphi_s : \mathbf{R}^m \to \{t : 0 \leq t \leq 1\}$ of class ∞ corresponding to $s \in S$ such that with $S_x = S \cap \{s : \mathbf{B}(x, 10h(x)) \cap \mathbf{B}(s, 10h(s)) \neq \emptyset\}$ for $x \in \mathbf{R}^m$ and a sequence V_i of positive, finite numbers depending only on m there holds

$$\operatorname{card} S_x \leq (129)^m, \qquad \operatorname{spt} \varphi_s \subset \mathbf{B}(s, 10h(s)) \quad \text{for } s \in S,$$

$$1/3 \leq h(x)/h(s) \leq 3 \quad \text{for } s \in S_x, \qquad |D^i \varphi_s(x)| \leq V_i (h(x))^{-i} \quad \text{for } s \in S, i \in \mathscr{P},$$

$$\sum_{s\in S}\varphi_s(y)=\sum_{s\in S_x}\varphi_s(y)=1,\qquad \sum_{s\in S}D^i\varphi_s(y)=\sum_{s\in S_x}D^i\varphi_s(y)=0\quad\text{for } i\in\mathscr{P}$$

whenever $x\in\mathbf{R}^m$, $y\in\mathbf{B}(x,10h(x))$. Note for $x\in\mathbf{R}^m$, $y\in\mathbf{B}(x,10h(x))$, $s\in S$, $i\in\mathscr{P}$

$$|D^i\varphi_s(y)|\le V_i(h(y))^{-i}\le(20)^iV_i(10h(x))^{-i},$$

because $h(x)-h(y)\le\frac{1}{20}|x-y|\le\frac{1}{2}h(x)$. Choose $\xi:S\to A$ such that

$$|\xi(s)-s|=\operatorname{dist}(s,A)\quad\text{whenever } s\in S.$$

Note $20h(x)\le\sup\{\operatorname{dist}(x,A),\delta\}$ for $x\in\mathbf{R}^m$ and observe

$$\mathbf{B}(x,20h(x))\subset\mathbf{B}(\xi(s),120h(s)),\quad 120h(s)\le 1$$

whenever $x\in\mathbf{R}^m$, $\operatorname{dist}(x,A)\le\frac{1}{18}$, $s\in S_x$, because

$$\begin{gathered}
|x-s|\le 10h(x)+10h(s)\le 40h(x)\le 2\sup\{\operatorname{dist}(x,A),\delta\}\le 1/9,\\
|s-\xi(s)|=\operatorname{dist}(s,A)\le|x-s|+\operatorname{dist}(x,A)\le 1/6,\\
|x-\xi(s)|\le|x-s|+|s-\xi(s)|\le 40h(s)+20h(s)=60h(s),\\
|x-\xi(s)|+20h(x)\le 120h(s)\le 360h(x)\le 1.
\end{gathered}$$

Define $R=\bigcup\{S_x : x\in\mathbf{R}^m \text{ and } \operatorname{dist}(x,A)\le\frac{1}{18}\}$,

$$v_s=v_{\xi(s),120h(s)}\quad\text{and}\quad P_s=P_{\xi(s),120h(s)}\quad\text{for } s\in R$$

and, denoting by v_s' the extension of v_s to $\mathbf{R}^m$ by 0, $v:\mathbf{R}^m\to\mathbf{R}^{n-m}$ by

$$v(x)=\sum_{s\in R}\varphi_s(x)v_s'(x)\quad\text{whenever } x\in\mathbf{R}^m.$$

Suppose for the rest of the proof $x\in\mathbf{R}^m$ with $\operatorname{dist}(x,A)\le\frac{1}{18}$ and observe

$$v(y)=\sum_{s\in S_x}\varphi_s(y)v_s(y)\quad\text{whenever } y\in\mathbf{B}(x,10h(x)).$$

The asserted weak differentiability is a consequence of 3.6.

One estimates

$$\begin{aligned}
|\mathbf{D}^i(u-v_s)|_{p;x,20h(x)}&\le|\mathbf{D}^i(u-v_s)|_{p;s,120h(s)}\\
&\le\gamma(120h(s))^{m/p+2-i}\le(18)^{m/p+2}\gamma(20h(x))^{m/p+2-i}
\end{aligned}$$

for $i\in\{0,j\}$, $s\in S_x$, hence by Hölder's inequality

$$(20h(x))^{-m}|u-v_s|_{1;x,20h(x)} \le \boldsymbol{\alpha}(m)^{1-1/p}\sum_{i=0}^{j}(20h(x))^{-m/p+i}|\mathbf{D}^i(u-v_s)|_{p;x,20h(x)} \le 2\Delta_1\gamma(20h(x))^2 \quad \text{(I)}$$

for $s \in S_x$ where $\Delta_1 = \boldsymbol{\alpha}(m)^{1-1/p}(18)^{m/p+2}$. Also

$$\begin{aligned}(20h(x))^{-m}|u-P_s|_{1;x,20h(x)} &\le \boldsymbol{\alpha}(m)^{1-1/p}(20h(x))^{-m/p}|u-P_s|_{p;\xi(s),120h(s)}\\ &\le \Delta_1\gamma(20h(x)),\end{aligned}$$

$$(20h(x))^{-m}|v_s-P_s|_{1;x,20h(x)} \le 3\Delta_1\gamma(20h(x)) \quad \text{(II)}$$

for $s \in S_x$. Using

$$v(y)-u(y) = \sum_{s\in S_x}\varphi_s(y)(v_s(y)-u(y)) \quad \text{whenever } y \in \mathbf{B}(x,10h(x))$$

and the Leibniz formula, one obtains from (I)

$$\sum_{i=0}^{j}(10h(x))^{-m/p+i}|\mathbf{D}^i(u-v)|_{p;x,10h(x)} \le \Delta_2\gamma(10h(x))^2$$

where $\Delta_2 = \boldsymbol{\alpha}(m)^{1/p-1}8\Delta_1 2^{m/p}(1+20V_1)(129)^m$.

In case $x \in \mathbf{B}(a,\varrho)$ for some $a \in A$, $\delta \le \varrho \le \frac{1}{18}$,

$$20h(x) \le \sup\{\operatorname{dist}(x,A),\delta\} \le \varrho, \qquad \mathbf{B}(x,20h(x)) \subset \mathbf{B}(a,2\varrho)$$

and Vitali's covering theorem yields a countable subset T of $\mathbf{B}(a,\varrho)$ such that

$$\{\mathbf{B}(t,2h(t)) : t \in T\} \text{ is disjointed}, \qquad \mathbf{B}(a,\varrho) \subset \bigcup\{\mathbf{B}(t,10h(t)) : t \in T\}$$

and one estimates for $i \in \{0, j\}$

$$\begin{aligned}&|\mathbf{D}^i(u-v)|^p_{p;a,\varrho}\\ &\quad\le \sum_{t\in T}|\mathbf{D}^i(u-v)|^p_{p;t,10h(t)}\\ &\quad\le (\Delta_2\gamma)^p\sum_{t\in T}(10h(t))^{m+(2-i)p}\\ &\quad= (5^{m/p+2-i}\Delta_2\gamma)^p\boldsymbol{\alpha}(m)^{-1-(2-i)p/m}\sum_{t\in T}\mathscr{L}^m(\mathbf{B}(t,2h(t)))^{1+(2-i)p/m}\\ &\quad\le (5^{m/p+2-i}\Delta_2\gamma)^p\boldsymbol{\alpha}(m)^{-1-(2-i)p/m}\mathscr{L}^m(\mathbf{B}(a,2\varrho))^{1+(2-i)p/m}\\ &\quad= \big((10)^{m/p+2-i}\Delta_2\gamma\big)^p\varrho^{m+(2-i)p}.\end{aligned}$$

Therefore, one obtains for $a \in A$, $\delta \le \varrho \le \frac{1}{18}$, $i \in \{0, j\}$

$$\varrho^{-m/p+i} |\mathbf{D}^i (u - v)|_{p;a,\varrho} \le (10)^{m/p+2} \Delta_2 \gamma \varrho^2 \tag{III}$$

and one may take $\Gamma_1 = 2(10)^{m/p+2}\Delta_2$ in the first estimate of the assertion.

According to 3.6 the functions v_s are twice weakly differentiable and satisfy for $s \in S_x$

$$(20h(x))^{-m/(2r)+2} |\mathbf{D}^2 v_s|_{2r;x,10h(x)} \le \Delta_3 (20h(x))^{-m} |v_s - P_s|_{1;x,20h(x)}$$

where $\Delta_3 = \Gamma_{3.6}(n, 2r)$. Combining this with (II) yields

$$(10h(x))^{-m/(2r)+2} |\mathbf{D}^2 v_s|_{2r;x,10h(x)} \le 2^{m/(2r)} 3 \Delta_1 \Delta_3 \gamma (10h(x)) \tag{IV}$$

for $s \in S_x$.

Using 3.8, one obtains for $s, t \in S_x$

$$\begin{aligned}
&(20h(x))^{-m/(2r)+1} |\mathbf{D}^2 (v_s - v_t)|_{2r;x,10h(x)} \\
&\quad \le \Delta_4 \big((20h(x))^{-m-1} |v_s - v_t|_{1;x,20h(x)} \\
&\qquad + \Lambda ((20h(x))^{-m-1} |v_s - P_s|_{1;x,20h(x)}) ((20h(x))^{-m-1} |v_s - v_t|_{1;x,20h(x)}) \big)
\end{aligned}$$

where $\Delta_4 = \Gamma_{3.8}(n, 2r)$. Since

$$(20h(x))^{-m} |v_s - v_t|_{1;x,20h(x)} \le 4 \Delta_1 \gamma (20h(x))^2$$

by (I), one estimates using (II)

$$(10h(x))^{-m/(2r)} |\mathbf{D}^2 (v_s - v_t)|_{2r;x,10h(x)} \le \Delta_5 \gamma (1 + \Lambda \gamma)$$

where $\Delta_5 = 2^{m+2} \Delta_1 \Delta_4 \sup\{3\Delta_1, 1\}$. Using an interpolation inequality (which may be proven similarly to [25, Lemma 6.2.2]), one infers with a positive, finite number Δ_6 depending only n and r

$$\begin{aligned}
&\sum_{i=0}^{2} (10h(x))^{-m/(2r)+i} |\mathbf{D}^i (v_s - v_t)|_{2r;x,10h(x)} \\
&\quad \le \Delta_6 \big((10h(x))^{-m/(2r)+2} |\mathbf{D}^2 (v_s - v_t)|_{2r;x,10h(x)} \\
&\qquad + (10h(x))^{-m} |v_s - v_t|_{1;x,10h(x)} \big) \\
&\quad \le \Delta_6 \big(\Delta_5 (1 + \Lambda \gamma) + 2^{m+4} \Delta_1 \big) \gamma (10h(x))^2.
\end{aligned}$$

This implies for $s, t \in S_x$

$$\sum_{i=0}^{2} (10h(x))^{-m/(2r)+i} |\mathbf{D}^i (v_s - v_t)|_{2r;x,10h(x)} \le \Delta_7 \gamma (1 + \Lambda \gamma) (10h(x))^2$$

where $\Delta_7 = \Delta_6(\Delta_5 + 2^{m+4}\Delta_1)$. Noting $(v - v_s)(y) = \sum_{t \in S_x} \varphi_t(y)(v_t - v_s)(y)$ for $s \in S_x$, $y \in \mathbf{U}(x, 10h(x))$, one infers using the Leibniz formula

$$(10h(x))^{-m/(2r)+i} |\mathbf{D}^i(v - v_s)|_{2r;x,10h(x)} \le \Delta_8 \gamma(1 + \Lambda\gamma)(10h(x))^2 \qquad \text{(V)}$$

for $s \in S_x$, $i \in \{0, 1, 2\}$ where $\Delta_8 = 2(1 + 20V_1 + 400V_2)\Delta_7(129)^m$.

Using 3.1, one defines

$$f(y) = \langle \mathbf{D}^2 v(y), C_F(\mathbf{D}v(y))\rangle$$

whenever $y \in \mathbf{U}(z, 10h(z))$ for some $z \in \mathbf{R}^m$ with $\operatorname{dist}(z, A) \le \frac{1}{18}$ and computes for $s \in S_x$

$$f(y) = \langle \mathbf{D}^2 v_s(y), C_F(\mathbf{D}v(y)) - C_F(\mathbf{D}v_s(y))\rangle + \langle \mathbf{D}^2(v - v_s)(y), C_F(\mathbf{D}v(y))\rangle$$

for $\mathscr{L}^m$ almost all $y \in \mathbf{U}(x, 10h(x))$. Hölder's inequality implies

$$\begin{aligned} |f|_{r;x,10h(x)} \le{}& \kappa\Lambda |\mathbf{D}(v - v_s)|_{2r;x,10h(x)} |\mathbf{D}^2 v_s|_{2r;x,10h(x)} \\ &+ 2\kappa \boldsymbol{\alpha}(m)^{1/(2r)}(10h(x))^{m/(2r)} |\mathbf{D}^2(v - v_s)|_{2r;x,10h(x)}, \end{aligned}$$

hence by (IV) and (V)

$$(10h(x))^{-m/r} |f|_{r;x,10h(x)} \le \Delta_9 \gamma(1 + \Lambda\gamma)^2$$

where $\Delta_9 = \kappa\Delta_8 \sup\{2^{m/(2r)} 3\Delta_1\Delta_3, 2\boldsymbol{\alpha}(m)^{1/(2r)}\}$. Similarly but simpler as in the deduction of (III), one obtains for $\delta \le \varrho \le \frac{1}{18}$, $a \in A$

$$|f|_{r;a,\varrho} \le \Delta_9(10)^{m/r}\gamma(1 + \Lambda\gamma)^2 \varrho^{m/r}$$

and thus, using 3.6 with $\Delta_{10} = \Gamma_{3.6}(n, r)$ and (III),

$$\begin{aligned} \varrho^{-m/r}|\mathbf{D}^2 v|_{r;a,\varrho/2} &\le \Delta_{10}\big(\varrho^{-m-2}(|u - v|_{1;a,\varrho} + |u - P_{a,\varrho}|_{1;a,\varrho}) + \varrho^{-m/r}|f|_{r;a,\varrho}\big) \\ &\le \Delta_{11}\big(\gamma(1 + \Lambda\gamma)^2 + \varrho^{-m-2}|u - P_{a,\varrho}|_{1;a,\varrho}\big) \end{aligned}$$

where $\Delta_{11} = \Delta_{10}(\boldsymbol{\alpha}(m)^{1-1/p}(10)^{m/p+2}\Delta_2 + \Delta_9(10)^{m/r} + 1)$. Therefore, one may take $\Gamma_2 = 2^{m/r}\Delta_{11}$ in the second estimate of the assertion and the proof is completed. □

Remark 3.10 In fact, by Calderón and Zygmund [11, Theorem 10(ii)] (see also [37, Lemma 3.7.2]) or by [22, 3.1]

$$\lim_{\varrho \to 0+} \varrho^{-2} \sum_{i=0}^{j} \varrho^{-m/p+i} |\mathbf{D}^i(u - v)|_{p;a,\varrho} = 0$$

for $\mathscr{L}^m$ almost all $a \in A$. Now, Rešetnyak's result in [27] applied to v yields that for $\mathscr{L}^m$ almost all $a \in A$ there exists a polynomial function $Q_a : \mathbf{R}^m \to \mathbf{R}^{n-m}$ of degree at most 2 such that

$$\limsup_{\varrho \to 0+} \varrho^{-2} \sum_{i=0}^{j} \varrho^{-m/p+i} |\mathbf{D}^i (u - Q_a)|_{p;a,\varrho} = 0.$$

Alternately, this latter fact could have also been deduced by use of Calderón and Zygmund [11, Theorem 12] (see also [37, Theorem 3.4.2]).

The main theorem of the present section, which contains Theorem 3 of the Introduction, now follows by separately considering the subsets where the hypothesized bounds are satisfied uniformly.

Theorem 3.11 *Suppose $m, n \in \mathscr{P}$, $m < n$, $1 \le p < \infty$, and $1 < q < \infty$.*

Then there exists a positive, finite number ε with the following property.

If F is related to ε as in 3.1, $\operatorname{Lip} D^2 F < \infty$, U is an open subset of $\mathbf{R}^m$, $j \in \{0, 1\}$, $u : U \to \mathbf{R}^{n-m}$ is weakly differentiable,

$$h(a, r) = \inf \left\{ \sum_{i=0}^{j} r^{-m/p+i} |\mathbf{D}^i (u - v)|_{p;a,r} : v \in \mathbf{W}^{1,q}(\mathbf{U}(a, r), \mathbf{R}^{n-m}) \text{ and } L_F(v) = 0 \right\}$$

whenever $\mathbf{U}(a, r) \subset U$ for some $a \in \mathbf{R}^m$, $0 < r < \infty$, and if A denotes the set of all $a \in U$ such that

$$\limsup_{r \to 0+} r^{-2} h(a, r) < \infty,$$

then A is a Borel set and for $\mathscr{L}^m$ almost all $a \in A$ there exists a polynomial function $Q_a : \mathbf{R}^m \to \mathbf{R}^{n-m}$ with degree at most 2 such that

$$\lim_{r \to 0+} r^{-2} \sum_{i=0}^{j} r^{-m/p+i} |\mathbf{D}^i (u - Q_a)|_{p;a,r} = 0.$$

Proof In view of 3.6 one may assume $q \ge p$. Let $\varepsilon = \varepsilon_{3.9}(m, n, p, p, q)$. Suppose F, U, j, and u satisfy the hypotheses with ε. Define the open set V by

$$V = U \cap \left\{ x : \sum_{i=0}^{j} |\mathbf{D}^i u|_{p;x,r} < \infty \text{ for some } 0 < r < \operatorname{dist}(x, \mathbf{R}^m \sim U) \right\}$$

and note $A \subset V$. Denote by D the set of all $v \in \mathbf{W}^{1,q}(\mathbf{U}(0, 1), \mathbf{R}^{n-m})$ such that $L_F(v) = 0$ and define

$$W = (V \times \mathbf{R}) \cap \{(a, r) : 0 < r < \operatorname{dist}(a, \mathbf{R}^m \sim V)\}$$

and the continuous map $T : W \to \mathbf{W}^{1,p}(\mathbf{U}(0, 1), \mathbf{R}^{n-m})$ by

$$T(a, r)(x) = r^{-1} u(a + rx) \quad \text{whenever } (a, r) \in W,\ x \in \mathbf{U}(0, 1).$$

Since $D \neq \emptyset$ and

$$h(a,r) = r \inf \left\{ \sum_{i=0}^{j} |\mathbf{D}^i (T(a,r) - v)|_{p;0,1} : v \in D \right\} \quad \text{for } (a,r) \in W,$$

h is continuous. Therefore, A is a Borel set. Similarly, denoting by D' the set of all affine functions mapping $\mathbf{R}^m$ into $\mathbf{R}^{n-m}$ one defines a continuous map $h' : W \to \mathbf{R}$ by

$$h'(a,r) = r \inf\{|T(a,r) - w|_{1;0,1} : w \in D'\} \quad \text{for } (a,r) \in W.$$

By Rešetnyak [27] or [16, 4.5.9(26)(II)(III)] one notes

$$\limsup_{\varrho \to 0+} \varrho^{-1} h'(a,\varrho) < \infty \quad \text{for } \mathscr{L}^m \text{ almost all } a \in U.$$

Define

$$C_k = V \cap \{x : \operatorname{dist}(x, \mathbf{R}^m \sim V) \geq 1/k\},$$

$$A_k = C_k \cap \{a : h(a,r) \leq kr^2 \text{ and } h'(a,r) \leq kr \text{ for } 0 < r < 1/k\}$$

for $k \in \mathscr{P}$ and observe that the sets A_k are closed and

$$\mathscr{L}^m \Big(A \sim \bigcup \{A_k : k \in \mathscr{P}\} \Big) = 0.$$

Finally, the conclusion is obtained by applying (for each $k \in \mathscr{P}$) 3.9 in conjunction with 3.10 to rescaled versions of u, A_k and a suitable number γ. □

Remark 3.12 Instead of using Rešetnyak [27] or [16, 4.5.9(26)(II)(III)], one can also use the functions v occurring in the definition of $h(a,r)$ in a way reminiscent of the familiar harmonic approximation procedure to deduce

$$\limsup_{\varrho \to 0+} \varrho^{-1} h'(a,\varrho) < \infty \quad \text{whenever } a \in A.$$

Therefore, u could have been required to be merely j times weakly differentiable.

An illustrative application of the differentiability criterion is constituted by the following Rademacher type theorem for distributions.

Corollary 3.13 *Suppose $m, n \in \mathscr{P}$, $m < n$, $1 < p < \infty$, U is an open subset of $\mathbf{R}^m$, $T \in \mathscr{D}(U, \mathbf{R}^{n-m})$, and A denotes the set of all $a \in U$ such that*

$$\limsup_{r \to 0+} r^{-1-m/p} |T|_{-1,p;a,r} < \infty.$$

Then A is a Borel set and for $\mathscr{L}^m$ almost all $a \in A$ there exists a unique constant distribution $T_a \in \mathscr{D}'(U, \mathbf{R}^{n-m})$ such that

$$\lim_{r \to 0+} r^{-1-m/p} |T - T_a|_{-1,p;a,r} = 0.$$

Proof The conclusion is local and for each $a \in A$ there exists $0 < r < \infty$ with $|T|_{-1,p;a,r} < \infty$, hence one may assume $\operatorname{spt} T$ to be compact, $U = \mathbf{R}^m$ and $|T|_{-1,p;0,R} < \infty$, $\operatorname{spt} T \subset \mathbf{U}(0,R)$ for some $0 < R < \infty$.

For example, using 3.2, one obtains functions $u \in \mathbf{W}_0^{1,p}(\mathbf{U}(0,R), \mathbf{R}^{n-m})$ and $v_{a,r} \in \mathscr{E}(\mathbf{U}(a,r), \mathbf{R}^{n-m})$ whenever $a \in \mathbf{R}^m$, $0 < r < \infty$ and $\mathbf{U}(a,r) \subset \mathbf{U}(0,R)$ such that

$$-\int_{\mathbf{U}(0,R)} \mathbf{D}u \bullet D\theta \, \mathrm{d}\mathscr{L}^m = T(\theta) \quad \text{for } \theta \in \mathscr{D}(\mathbf{U}(0,R), \mathbf{R}^{n-m}),$$

$$u - v_{a,r} \in \mathbf{W}_0^{1,p}(\mathbf{U}(a,r), \mathbf{R}^{n-m}), \qquad \operatorname{Lap} v_{a,r} = 0.$$

By 3.2 and Poincaré's inequality

$$\sum_{i=0}^{1} r^{i-1} |\mathbf{D}^i(u - v_{a,r})|_{p;a,r} \le \Delta |T|_{-1,p;a,r}$$

for some positive, finite number Δ depending only on n and p, hence the set A agrees with the set "A" defined in 3.11 with $q = p$, F the Dirichlet integrand and $j = 1$. Therefore, applying 3.11, one may take $T_a \in \mathscr{D}'(\mathbf{U}(0,R), \mathbf{R}^{n-m})$ defined by $T_a(\theta) = \int \theta(x) \bullet \operatorname{Lap} Q_a(a) \, \mathrm{d}\mathscr{L}^m x$ for $\theta \in \mathscr{D}(\mathbf{R}^m, \mathbf{R}^{n-m})$.

The uniqueness follows, since every T_a admissible in the conclusion satisfies

$$r^{-m} T_a(\theta \circ \boldsymbol{\mu}_{1/r} \circ \boldsymbol{\tau}_{-a}) = T_a(\theta), \qquad r^{-m} T(\theta \circ \boldsymbol{\mu}_{1/r} \circ \boldsymbol{\tau}_{-a}) \to T_a(\theta) \quad \text{as } r \to 0+$$

whenever $\theta \in \mathscr{D}(\mathbf{R}^m, \mathbf{R}^{n-m})$. □

The remaining part of the present section concerns estimates involving the norm $|\cdot|_{-1,1;a,r}$. As control on the distributional Laplacian of u in this norm does not entail control on the weak derivative $\mathbf{D}u$ in $|\cdot|_{1;a,r}$, a perturbation approach to pass from Laplace's operator to L_F as it was used for the norm $|\cdot|_{-1,p;a,r}$ with $1 < p < \infty$ seems to be impossible. Instead, the analysis is based on the following estimate of u in $|\cdot|_{1;a,r}$, which is readily obtained dualizing global a priori estimates in $\mathbf{W}^{2,p}(\mathbf{U}(a,r), \mathbf{R}^{n-m})$ for $p > m$.

Lemma 3.14 *Suppose $m, n \in \mathscr{P}$, $m < n$, $\Phi \in \bigodot^2 \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m})$, $0 < c \le M < \infty$, $\|\Phi\| \le M$, Φ is strongly elliptic with ellipticity bound c, $a \in \mathbf{R}^m$, $0 < r < \infty$, $u \in \mathbf{W}_0^{1,1}(\mathbf{U}(a,r), \mathbf{R}^{n-m})$, $T \in \mathscr{D}'(\mathbf{U}(a,r), \mathbf{R}^{n-m})$, and*

$$-\int_{\mathbf{U}(a,r)} \langle D\theta(x) \odot \mathbf{D}u(x), \Phi \rangle \, \mathrm{d}\mathscr{L}^m x = T(\theta) \quad \textit{for } \theta \in \mathscr{D}(\mathbf{U}(a,r), \mathbf{R}^{n-m}).$$

Then

$$|u|_{1;a,r} \le \Gamma r |T|_{-1,1;a,r}$$

where Γ is a positive, finite number depending only on n, c, and M.

Proof See [24, 7.8]. □

Introducing an affine function P in the basic $\mathbf{W}^{1,2}(\mathbf{U}(a,r),\mathbf{R}^{n-m})$ estimate for solutions of the Dirichlet problem for L_F yields the following result.

Lemma 3.15 *Suppose $m, n \in \mathscr{P}$, $m < n$, $0 < c \le M < \infty$,*

$$F : \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m}) \to \mathbf{R} \quad \textit{is of class } 2,$$

$$\|D^2F(\sigma)\| \le M, \quad D^2F(\sigma)(\tau,\tau) \ge c|\tau|^2 \quad \textit{for } \sigma, \tau \in \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m}),$$

$a \in \mathbf{R}^m$, $0 < r < \infty$, and $u, v \in \mathbf{W}^{1,2}(\mathbf{U}(a,r),\mathbf{R}^{n-m})$ with

$$u - v \in \mathbf{W}_0^{1,2}(\mathbf{U}(a,r),\mathbf{R}^{n-m}).$$

Then for every affine function $P : \mathbf{R}^m \to \mathbf{R}^{n-m}$

$$|\mathbf{D}(v-u)|_{2;a,r} \le c^{-1}\big(M|\mathbf{D}(u-P)|_{2;a,r} + |L_F(v)|_{-1,2;a,r}\big)$$

where L_F is defined as in 3.1.

Proof Compute for $\theta \in \mathscr{D}(\mathbf{U}(a,r),\mathbf{R}^{n-m})$

$$\begin{aligned} L_F(v)(\theta) &= -\int_{\mathbf{U}(a,r)} \langle D\theta(x), DF(\mathbf{D}v(x)) - DF(DP(x))\rangle \, \mathrm{d}\mathscr{L}^m x \\ &= -\int_{\mathbf{U}(a,r)} \langle D\theta(x) \odot \mathbf{D}(v-P)(x), A(x)\rangle \, \mathrm{d}\mathscr{L}^m x \\ \text{where } A(x) &= \int_0^1 D^2F(t\mathbf{D}v(x) + (1-t)DP(x)) \, \mathrm{d}\mathscr{L}^1 t. \end{aligned}$$

This implies for $\theta \in \mathscr{D}(\mathbf{U}(a,r),\mathbf{R}^{n-m})$

$$\begin{aligned} &\int_{\mathbf{U}(a,r)} \langle D\theta(x) \odot \mathbf{D}(v-u)(x), A(x)\rangle \, \mathrm{d}\mathscr{L}^m x \\ &\quad = -\int_{\mathbf{U}(a,r)} \langle D\theta(x) \odot \mathbf{D}(u-P)(x), A(x)\rangle \, \mathrm{d}\mathscr{L}^m x - L_F(v)(\theta). \end{aligned}$$

Letting θ approximate $v-u$ in $\mathbf{W}^{1,2}(\mathbf{U}(a,r),\mathbf{R}^{n-m})$, one obtains

$$c(|\mathbf{D}(v-u)|_{2;a,r})^2 \le \big(M|\mathbf{D}(u-P)|_{2;a,r} + |L_F(v)|_{-1,2;a,r}\big)|\mathbf{D}(v-u)|_{2;a,r}. \quad \square$$

If DF is linear, one may use 3.14 to estimate $|u-v|_{1;a,r}$ for two functions u and v having the same Dirichlet data in terms of $|L_F(u) - L_F(v)|_{-1,1;a,r}$. In case DF is not linear, the validity of a similar estimate appears to be unclear. Instead, one may—similarly to the familiar harmonic approximation procedure—introduce an additional term $|\mathbf{D}(u-P)|_{2;a,r} + |\mathbf{D}(v-P)|_{2;a,r}$ which enters *quadratically*.

Lemma 3.16 *Suppose* $m, n \in \mathscr{P}$, $m < n$, $\varepsilon = 1/2$ *is related to* F *as in* 3.1, $\operatorname{Lip} D^2 F < \infty$, $a \in \mathbf{R}^m$, $0 < r < \infty$, *and* $u, v \in \mathbf{W}^{1,2}(\mathbf{U}(a,r), \mathbf{R}^{n-m})$ *with* $u - v \in \mathbf{W}_0^{1,2}(\mathbf{U}(a,r), \mathbf{R}^{n-m})$.

Then for every affine function $P : \mathbf{R}^m \to \mathbf{R}^{n-m}$

$$\begin{aligned} r^{-1-m} |v-u|_{1;a,r} \le \Gamma r^{-m} \big(& |L_F(v) - L_F(u)|_{-1,1;a,r} \\ & + \operatorname{Lip}(D^2 F)(|\mathbf{D}(u-P)|_{2;a,r} + |\mathbf{D}(v-P)|_{2;a,r})^2 \big) \end{aligned}$$

where $\Gamma = \Gamma_{3.14}(n, 1/2/, 3/2)$.

Proof Let $\Lambda = \operatorname{Lip} D^2 F$, choose $\sigma \in \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m})$ such that $DP(x) = \sigma$ for $x \in \mathbf{R}^m$, and define $T = L_F(v) - L_F(u)$, the $\mathscr{L}^m \llcorner \mathbf{U}(a,r)$ measurable function $A : \mathbf{U}(a,r) \to \bigodot^2 \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m})$ by

$$A(x) = \int_0^1 D^2 F(t\mathbf{D}v(x) + (1-t)\mathbf{D}u(x)) - D^2 F(\sigma) \, \mathrm{d}\mathscr{L}^1 t$$

whenever $x \in \mathbf{U}(a,r)$, and $S \in \mathscr{D}'(\mathbf{U}(a,r), \mathbf{R}^{n-m})$ by

$$S(\theta) = \int_{\mathbf{U}(a,r)} \langle D\theta(x) \odot \mathbf{D}(v-u)(x), A(x) \rangle \, \mathrm{d}\mathscr{L}^m x + T(\theta)$$

whenever $\theta \in \mathscr{D}(\mathbf{U}(a,r), \mathbf{R}^{n-m})$. One computes

$$\begin{aligned} & DF(\mathbf{D}v(x)) - DF(\mathbf{D}u(x)) \\ & \quad = \left\langle \mathbf{D}(v-u)(x), \int_0^1 DDF(t\mathbf{D}v(x) + (1-t)\mathbf{D}u(x)) \, \mathrm{d}\mathscr{L}^1 t \right\rangle \end{aligned}$$

for $\mathscr{L}^n$ almost all $x \in \mathbf{U}(a,r)$ and infers

$$S(\theta) = -\int_{\mathbf{U}(a,r)} \langle D\theta(x) \odot \mathbf{D}(v-u)(x), D^2 F(\sigma) \rangle \, \mathrm{d}\mathscr{L}^m x$$

whenever $\theta \in \mathscr{D}(\mathbf{U}(a,r), \mathbf{R}^{n-m})$, hence by 3.14 with Φ replaced by $D^2 F(\sigma)$

$$r^{-1-m} |v-u|_{1;a,r} \le \Gamma r^{-m} |S|_{-1,1;a,r}.$$

It remains to estimate $|S|_{-1,1;a,r}$. By use of the definition of S one estimates

$$\begin{aligned} \|A(x)\| & \le \int_0^1 \|D^2 F(t\mathbf{D}v(x) + (1-t)\mathbf{D}u(x)) - D^2 F(t\sigma + (1-t)\sigma)\| \, \mathrm{d}\mathscr{L}^1 t \\ & \le \Lambda \int_0^1 t|\mathbf{D}(v-P)(x)| + (1-t)|\mathbf{D}(u-P)(x)| \, \mathrm{d}\mathscr{L}^1 t \\ & = \Lambda(|\mathbf{D}(v-P)(x)| + |\mathbf{D}(u-P)(x)|)/2 \end{aligned}$$

for $\mathscr{L}^m$ almost all $x \in \mathbf{U}(a, r)$. Finally,

$$|S|_{-1,1;a,r} \leq |T|_{-1,1;a,r} + \Lambda/2 \int_{\mathbf{U}(a,r)} (|\mathbf{D}(u-P)(x)| + |\mathbf{D}(v-P)(x)|)^2 \, \mathrm{d}\mathscr{L}^m x.$$

□

3.17 Whenever $m, n \in \mathscr{P}$, $m < n$, U is an open subset of $\mathbf{R}^m$, $a \in U$, and $T \in \mathscr{D}'(U, \mathbf{R}^{n-m})$ there exists at most one constant distribution $T_a \in \mathscr{D}'(U, \mathbf{R}^{n-m})$ such that

$$\lim_{r \to 0+} r^{-m-1} |T - T_a|_{-1,1;a,r} = 0;$$

see the last paragraph of the proof of 3.13.

Now, second order differentiability of u in $\mathbf{L}_1(\mathscr{L}^m, \mathbf{R}^{n-m})$ spaces is derived from existence of a "value" (in the sense of 3.17) of $L_F(u)$ at a provided a certain *supplementary* $\mathbf{L}_2(\mathscr{L}^m, \mathbf{R}^{n-m})$ *differentiability condition of order* $1/2$ holds for $\mathbf{D}u$.

Lemma 3.18 *Suppose* $m, n \in \mathscr{P}$, $m < n$.

Then there exists a positive, finite number ε *with the following property.*

If F *is related to* ε *as in* 3.1, $\operatorname{Lip} D^2 F < \infty$, U *is an open subset of* $\mathbf{R}^m$, $u : U \to \mathbf{R}^{n-m}$ *is weakly differentiable*, A_1 *denotes the set of all* $a \in U$ *such that*

$$\limsup_{r \to 0+} r^{-m-1} |L_F(u)|_{-1,1;a,r} < \infty,$$

A_2 *denotes the set of all* $a \in U$ *such that there exists a* (*unique, see* 3.17) *constant distribution* $T_a \in \mathscr{D}'(U, \mathbf{R}^{n-m})$ *such that*

$$\lim_{r \to 0+} r^{-m-1} |L_F(u) - T_a|_{-1,1;a,r} = 0,$$

B_1 *denotes the set of all* $b \in \operatorname{dmn} \mathbf{D}u$ *such that*

$$\limsup_{r \to 0+} r^{-m-1} \int_{\mathbf{U}(b,r)} |\mathbf{D}u(x) - \mathbf{D}u(b)|^2 \, \mathrm{d}\mathscr{L}^m x < \infty,$$

and B_2 *denotes the set of all* $b \in \operatorname{dmn} \mathbf{D}u$ *such that*

$$\lim_{r \to 0+} r^{-m-1} \int_{\mathbf{U}(b,r)} |\mathbf{D}u(x) - \mathbf{D}u(b)|^2 \, \mathrm{d}\mathscr{L}^m x = 0,$$

then the following two statements hold:

(1) *For* $\mathscr{L}^m$ *almost all* $a \in A_1 \cap B_1$ *there exists a polynomial function* $Q_a : \mathbf{R}^m \to \mathbf{R}^{n-m}$ *of degree at most* 2 *such that*

$$\lim_{r \to 0+} r^{-2-m} |u - Q_a|_{1;a,r} = 0.$$

(2) *If* $a \in A_2 \cap B_2$ *satisfies the conclusion of* (1) *with* Q_a *then*

$$T_a(\theta) = \int_U \theta(x) \bullet \langle D^2 Q_a(a), C_F(DQ_a(a)) \rangle \, \mathrm{d}\mathscr{L}^m x$$

for $\theta \in \mathscr{D}(U, \mathbf{R}^{n-m})$ *where* C_F *is defined as in* 3.1.

The proof of (1) is readily obtained by constructing comparison functions for use in the differentiability criterion 3.11. The relevant estimates are obtained from 3.15 and 3.16.

In order to link the "value" $y \in \mathbf{R}^{n-m}$ of $T = L_F(u)$ at a to the polynomial function Q_a constructed in (1) one could directly use the equation if the second order differentiability of u would involve the space $\mathbf{W}^{1,1}(\mathbf{R}^m, \mathbf{R}^{n-m})$ rather than $\mathbf{L}_1(\mathscr{L}^m, \mathbf{R}^{n-m})$. However, such control is available, due to the interior a priori estimates 3.6, if u is replaced by solutions w_r of the Dirichlet problem in $\mathbf{U}(a, r)$ with boundary data given by u and the right-hand side given by y.

Proof of 3.18 Let

$$\varepsilon = \inf\{1/2, \varepsilon_{3.11}(m, n, 1, 2), \varepsilon_{3.6}(n, 2, 2)\}.$$

Suppose F and u satisfy the hypotheses with ε. Abbreviate $\Lambda = \operatorname{Lip} D^2 F$ and $T = L_F(u)$. Fix $a \in A_1 \cap B_1$ and $0 < R < \infty$ such that $\mathbf{B}(a, R) \subset U$ and $u|\mathbf{U}(a, R) \in \mathbf{W}^{1,2}(\mathbf{U}(a, R), \mathbf{R}^{n-m})$.

To prove part (1), the criterion 3.11 will be verified with $q = 2$, $j = 0$. Using the direct method of the calculus of variation, see, e.g., [18, Theorems 4.5, 6, Remark 4.1], one constructs for $0 < r < R$ functions $v_r \in \mathbf{W}^{1,2}(\mathbf{U}(a, r), \mathbf{R}^{n-m})$ such that

$$v_r - u \in \mathbf{W}_0^{1,2}(\mathbf{U}(a, r), \mathbf{R}^{n-m}), \qquad L_F(v_r) = 0.$$

By 3.16 one estimates

$$\begin{aligned} &r^{-1-m} |v_r - u|_{1;a,r} \\ &\quad \le \Delta_1 r^{-m} \big(|T|_{-1,1;a,r} + \Lambda(|\mathbf{D}(u - \mathbf{D}u(a))|_{2;a,r} + |\mathbf{D}(v_r - \mathbf{D}u(a))|_{2;a,r})^2 \big) \end{aligned}$$

with $\Delta_1 = \Gamma_{3.16}(n)$. By 3.15 with $c = 1/2$, $M = 2$ one infers

$$|\mathbf{D}(v_r - u)|_{2;a,r} \le 4 |\mathbf{D}(u - \mathbf{D}u(a))|_{2;a,r},$$

hence

$$r^{-1-m} |v_r - u|_{1;a,r} \le \Delta_1 r^{-m} \big(|T|_{-1,1;a,r} + \Lambda(6 |\mathbf{D}(u - \mathbf{D}u(a))|_{2;a,r})^2 \big).$$

Since $a \in A_1 \cap B_1$, this implies

$$\limsup_{r \to 0+} r^{-2-m} |v_r - u|_{1;a,r} < \infty.$$

Therefore, part (1) follows from 3.11.

To prove part (2), assume now additionally that the assumptions of (2) are valid for a, i.e., $a \in A_2 \cap B_2$ and Q_a satisfies the conclusion of (1). Choose $y \in \mathbf{R}^{n-m}$ such that

$$T_a(\theta) = \int_U \theta(x) \bullet y \, \mathrm{d}\mathscr{L}^m x \quad \text{for } \theta \in \mathscr{D}(U, \mathbf{R}^{n-m}).$$

Using the direct method of the calculus of variation as before, one constructs for $0 < r < R$ functions $w_r \in \mathbf{W}^{1,2}(\mathbf{U}(a,r), \mathbf{R}^{n-m})$ such that

$$w_r - u \in \mathbf{W}_0^{1,2}(\mathbf{U}(a,r), \mathbf{R}^{n-m}),$$

$$L_F(w_r)(\theta) = \int_{\mathbf{U}(a,r)} \theta(x) \bullet y \, \mathrm{d}\mathscr{L}^m x \quad \text{whenever } \theta \in \mathscr{D}(\mathbf{U}(a,r), \mathbf{R}^{n-m}).$$

By 3.16 one estimates

$$\begin{aligned} r^{-1-m} |w_r - u|_{1;a,r} \le \Delta_1 r^{-m} \big(&|T - T_a|_{-1,1;a,r} \\ &+ \Lambda (|\mathbf{D}(u - \mathbf{D}u(a))|_{2;a,r} + |\mathbf{D}(w_r - \mathbf{D}u(a))|_{2;a,r})^2 \big) \end{aligned}$$

Since, by Poincaré's inequality,

$$\left| \int_{\mathbf{U}(a,r)} \theta(x) \bullet y \, \mathrm{d}\mathscr{L}^m x \right| \le |y| \Delta_2 r^{1+m/2} |D\theta|_{2;a,r}$$

where Δ_2 is a positive, finite number depending only on n, one infers from 3.15

$$|\mathbf{D}(w_r - u)|_{2;a,r} \le 4 |\mathbf{D}(u - \mathbf{D}u(a))|_{2;a,r} + 2\Delta_2 |y| r^{1+m/2},$$

hence

$$\begin{aligned} &r^{-1-m} |w_r - u|_{1;a,r} \\ &\quad \le \Delta_1 r^{-m} \big(|T - T_a|_{-1,1;a,r} + \Lambda (6 |\mathbf{D}(u - \mathbf{D}u(a))|_{2;a,r} + 2\Delta_2 |y| r^{1+m/2})^2 \big). \end{aligned}$$

Since $a \in A_2 \cap B_2$, this implies

$$\lim_{r \to 0+} r^{-2-m} |w_r - u|_{1;a,r} = 0.$$

Therefore, by the assumption on Q_a

$$\lim_{r \to 0+} r^{-2-m} |w_r - Q_a|_{1;a,r} = 0.$$

In order to estimate derivatives of $w_r - Q_a$, define $P : \mathbf{R}^m \to \mathbf{R}^{n-m}$ by $P(x) = Q_a(a) + \langle x - a, DQ_a(a) \rangle$ for $x \in \mathbf{R}^m$, $R = Q_a - P$, $S : \mathbf{R}^m \to \mathbf{R}^{n-m}$ by $S(x) = \langle x^2/2, D^2 Q_a(a) \rangle$ for $x \in \mathbf{R}^m$ and note $r^{-2} R \circ \boldsymbol{\tau}_a \circ \boldsymbol{\mu}_r = S$ and

$$r^{-2}(w_r - P) \circ \boldsymbol{\tau}_a \circ \boldsymbol{\mu}_r | \mathbf{U}(0,1) \to S | \mathbf{U}(0,1) \quad \text{in } \mathbf{L}_1(\mathbf{U}(0,1), \mathbf{R}^{n-m})$$

as $r \to 0+$. By 3.6

$$r^{-m/2}|\mathbf{D}^2(w_r - P)|_{2;a,r/2} \le \Delta_3(r^{-2-m}|w_r - P|_{1;a,r} + |y|)$$

where $\Delta_3 = \sup\{1, \boldsymbol{\alpha}(m)^{1/2}\}\Gamma_{3.6}(n,2)$, hence

$$\limsup_{r\to 0+} r^{-m/2}|\mathbf{D}^2(w_r - P)|_{2;a,r/2} < \infty.$$

By Rellich's embedding theorem

$$r^{-2}(w_r - P)\circ \boldsymbol{\tau}_a \circ \boldsymbol{\mu}_r|\mathbf{U}(0,1/2) \to S|\mathbf{U}(0,1/2) \quad \text{in } \mathbf{W}^{1,2}(\mathbf{U}(0,1/2), \mathbf{R}^{n-m}),$$
$$r^{-2}(w_r - Q_a)\circ \boldsymbol{\tau}_a \circ \boldsymbol{\mu}_r|\mathbf{U}(0,1/2) \to 0 \quad \text{in } \mathbf{W}^{1,2}(\mathbf{U}(0,1/2), \mathbf{R}^{n-m})$$

as $r \to 0+$. This convergence implies

$$\begin{aligned}&\left| r^{-m-1}\int_{\mathbf{U}(a,r/2)} \langle (D\theta)\circ \boldsymbol{\mu}_{1/r}\circ \boldsymbol{\tau}_{-a}(x), DF(\mathbf{D}w_r(x)) - DF(DQ_a(x))\rangle \, \mathrm{d}\mathscr{L}^m x \right| \\ &\quad \le r^{-m/2-1}(\operatorname{Lip} DF)|D\theta|_{2;0,1}|\mathbf{D}(w_r - Q_a)|_{2;a,r} \to 0 \quad \text{as } r \to 0+\end{aligned}$$

for $\theta \in \mathscr{D}(\mathbf{U}(0,1/2), \mathbf{R}^{n-m})$. Therefore, noting

$$\begin{aligned}&\int_{\mathbf{U}(0,1/2)} \theta(x) \bullet y \, \mathrm{d}\mathscr{L}^m x \\ &\quad = r^{-m}\int_{\mathbf{U}(a,r/2)} (\theta \circ \boldsymbol{\mu}_{1/r}\circ \boldsymbol{\tau}_{-a})(x) \bullet y \, \mathrm{d}\mathscr{L}^m x \\ &\quad = -r^{-m-1}\int_{\mathbf{U}(a,r/2)} \langle (D\theta)\circ \boldsymbol{\mu}_{1/r}\circ \boldsymbol{\tau}_{-a}(x), DF(\mathbf{D}w_r(x))\rangle \, \mathrm{d}\mathscr{L}^m x\end{aligned}$$

for $\theta \in \mathscr{D}(\mathbf{U}(0,1/2), \mathbf{R}^{n-m})$ and

$$\begin{aligned}&- r^{-m-1}\int_{\mathbf{U}(a,r/2)} \langle (D\theta)\circ \boldsymbol{\mu}_{1/r}\circ \boldsymbol{\tau}_{-a}(x), DF(DQ_a(x))\rangle \, \mathrm{d}\mathscr{L}^m x \\ &\quad = r^{-m}\int_{\mathbf{U}(a,r/2)} (\theta \circ \boldsymbol{\mu}_{1/r}\circ \boldsymbol{\tau}_{-a})(x) \bullet \langle D^2Q_a(x), C_F(DQ_a(x))\rangle \, \mathrm{d}\mathscr{L}^m x \\ &\quad \to \int_{\mathbf{U}(0,1/2)} \theta(x) \bullet \langle D^2Q_a(a), C_F(DQ_a(a))\rangle \, \mathrm{d}\mathscr{L}^m x \quad \text{as } r \to 0+,\end{aligned}$$

for $\theta \in \mathscr{D}(\mathbf{U}(0,1/2), \mathbf{R}^{n-m})$, one infers

$$y = \langle D^2Q_a(a), C_F(DQ_a(a))\rangle,$$

as asserted. $\square$

Remark 3.19 Clearly, by Rešetnyak [27] or [16, 4.5.9(26)(II)(III)] for $\mathscr{L}^m$ almost all $a \in A_1 \cap B_1$

$$Q_a(a) = u(a), \qquad DQ_a(a) = \mathbf{D}u(a).$$

Also by Calderón and Zygmund [11, Theorem 9] (see also [37, 3.6–8]), there exists a sequence of functions $u_i : \mathbf{R}^m \to \mathbf{R}^{n-m}$ of class 2 such that

$$\mathscr{L}^m \left(A_1 \cap B_1 \sim \bigcup_{i=1}^{\infty} \left\{ a : D^k u_i(a) = D^k Q_a(a) \text{ for } k \in \{0, 1, 2\} \right\} \right) = 0.$$

Remark 3.20 In A.3 it will be shown that $\mathscr{L}^m(A_1 \sim A_2) = 0$.

The section is concluded by a cut-off lemma which is a consequence of Taylor's formula. It will be used in the proof of the main Theorem 4.8 to replace the nonparametric area integrand by a function F satisfying the conditions in 3.1.

Lemma 3.21 *Suppose H is a Hilbert space with* $\dim H = N < \infty$, $k, l \in \mathscr{P} \cup \{0\}$, $l \geq k$, $\Phi : H \to \mathbf{R}$ *is of class l, $a \in H$,* $0 < \delta < \infty$, *and*

$$s = \sup\{\|D^k\Phi(x) - D^k\Phi(a)\| : x \in \mathbf{B}(a, \delta)\}.$$

Then there exists $F : H \to \mathbf{R}$ of class l such that

$$D^i F(x) = D^i \Phi(x) \quad \textit{for } x \in \mathbf{B}(a, \delta/2), i = 0, \ldots, k,$$

$$\|D^k F(x) - D^k \Phi(a)\| \leq \Gamma s \quad \textit{for } x \in H,$$

$F|H \sim \mathbf{B}(a, \delta)$ *is the restriction of a polynomial function of degree at most k*

where Γ is a positive, finite number depending only on N and k.

Proof Choosing $\varphi \in \mathscr{E}^0(\mathbf{R})$ with $0 \leq \varphi(t) \leq 1$ for $t \in \mathbf{R}$ and

$$\{t : -\infty < t \leq 1/2\} \subset \operatorname{Int}\{t : \varphi(t) = 1\}, \qquad \{t : 1 \leq t < \infty\} \subset \operatorname{Int}\{t : \varphi(t) = 0\}$$

one defines $P : H \to \mathbf{R}$, $F : H \to \mathbf{R}$ by

$$P(x) = \sum_{i=0}^{k} \langle (x-a)^i / i!, D^i\Phi(a) \rangle,$$

$$F(x) = P(x) + \varphi(|x-a|/\delta)(\Phi(x) - P(x))$$

for $x \in H$ and readily estimates $\|D^k F(x) - D^k \Phi(a)\|$ be means of Taylor's formula (cf. [16, 3.1.11]). □

4 An Approximate Second Order Structure for Certain Integral Varifolds

Overview In the present section the main Theorem 4.8, which is Theorem 1 of the Introduction, is proven. In order to do this a general lemma is established which states that the part of a varifold exhibiting a certain decay of its tilt-excess can be covered with some accuracy by suitable rotated graphs of Lipschitzian functions having similar decay properties of their "tilt-excess". This is done by carefully combining the approximation by $\mathbf{Q}_Q(\mathbf{R}^{n-m})$-valued functions of [24, 5.7] with more basic differentiability results in [22]. The "tilt-excess" decay of the Lipschitzian functions is the nonintegral differentiability condition used in Sect. 3 to compensate for the use of the weak norm $|\cdot|_{-1,1;a,s}$ in the estimates, which seems to be unavoidable; see 4.6.

Almgren introduced "multiple-valued", i.e., $\mathbf{Q}_Q(\mathbf{R}^{n-m})$-valued, functions in [6] in order to approximate integral varifolds; see also Almgren [5]. The procedure has been adapted several times, e.g., by Brakke in [9, 5.4], by Schätzle in [31, Appendix D] and by the author in [23, 3.15]. Here, essentially the latter version is used with some simple but crucial modifications carried out in [24, 5.7(1)–(7)(9)].

To explain the basic idea, recall that a *weakly differentiable* function can be approximated by Lipschitzian functions using the fact that points where the maximal function of its weak derivative is bounded are related in a Lipschitzian way; see, e.g., the proof of [15, 6.6.2 Theorem 2]. However, there is no corresponding result for merely *approximately differentiable* functions; see, for example, characteristic functions of $\mathscr{L}^m$ measurable sets (whose graphs obviously correspond to integral varifolds). As such behavior is excluded for stationary varifolds, see Almgren [6, 3.6], one instead considers points satisfying an additional maximal type condition on the first variation. The extension to multiple layers then involves some elementary matching theory.

Lemma 4.1 *Suppose* $n, Q \in \mathscr{P}$, $0 < L < \infty$, $1 \le M < \infty$, $0 < \delta_i \le 1$ *for* $i \in \{1, 2, 3\}$, *and* $0 < \delta_4 \le 1/4$.

Then there exists a positive, finite number ε *with the following property.*

If $m \in \mathscr{P}$, $m < n$, $0 < s < \infty$, $S = \operatorname{im} \mathbf{p}^*$,

$$U = (\mathbf{R}^m \times \mathbf{R}^{n-m}) \cap \{(x, y) : \operatorname{dist}((x, y), \mathbf{C}(S, 0, s, s)) < 2s\},$$

$V \in \mathbf{IV}_m(U)$, $\|\delta V\|$ *is a Radon measure,*

$$(Q - 1 + \delta_1)\boldsymbol{\alpha}(m)s^m \le \|V\|(\mathbf{C}(S, 0, s, s)) \le (Q + 1 - \delta_2)\boldsymbol{\alpha}(m)s^m,$$

$$\|V\|(\mathbf{C}(S, 0, s, s + \delta_4 s) \sim \mathbf{C}(S, 0, s, s - 2\delta_4 s)) \le (1 - \delta_3)\boldsymbol{\alpha}(m)s^m,$$

$$\|V\|(U) \le M\boldsymbol{\alpha}(m)s^m,$$

$0 < \delta \le \varepsilon$, B *denotes the set of all* $z \in \mathbf{C}(S, 0, s, s)$ *with* $\boldsymbol{\Theta}^{*m}(\|V\|, z) > 0$ *such that*

$$\text{either} \quad \|\delta V\|\,\mathbf{B}(z, t) > \delta\|V\|(\mathbf{B}(z, t))^{1-1/m} \quad \text{for some } 0 < t < 2s,$$

$$\text{or} \quad \int_{\mathbf{B}(z,t)\times\mathbf{G}(n,m)} |R_\natural - S_\natural|\,\mathrm{d}V(\xi, R) > \delta\|V\|\,\mathbf{B}(z, t) \quad \text{for some } 0 < t < 2s,$$

$A = \mathbf{C}(S, 0, s, s) \sim B$, $A(x) = A \cap \{z : \mathbf{p}(z) = x\}$ for $x \in \mathbf{R}^m$, X_1 is the set of all $x \in \mathbf{R}^m \cap \mathbf{B}(0, s)$ such that

$$\sum_{z \in A(x)} \boldsymbol{\Theta}^m(\|V\|, z) = Q \quad \textit{and} \quad \boldsymbol{\Theta}^m(\|V\|, z) \in \mathscr{P} \cup \{0\} \quad \textit{for } z \in A(x),$$

X_2 is the set of all $x \in \mathbf{R}^m \cap \mathbf{B}(0, s)$ such that

$$\sum_{z \in A(x)} \boldsymbol{\Theta}^m(\|V\|, z) \leq Q - 1 \quad \textit{and} \quad \boldsymbol{\Theta}^m(\|V\|, z) \in \mathscr{P} \cup \{0\} \quad \textit{for } z \in A(x),$$

$N = \mathbf{R}^m \cap \mathbf{B}(0, s) \sim (X_1 \cup X_2)$, and $f : X_1 \to \mathbf{Q}_Q(\mathbf{R}^{n-m})$ is characterized by the requirement

$$\boldsymbol{\Theta}^m(\|V\|, z) = \boldsymbol{\Theta}^0(\|f(x)\|, \mathbf{q}(z)) \quad \textit{whenever } x \in X_1 \textit{ and } z \in A(x),$$

then the following seven statements hold:

(1) *X_1 and X_2 are universally measurable, and $\mathscr{L}^m(N) = 0$.*
(2) *A and B are Borel sets and*

$$\mathbf{q}[A \cap \operatorname{spt}\|V\|] \subset \mathbf{B}(0, s - \delta_4 s).$$

(3) $\mathbf{p}[A \cap \{z : \boldsymbol{\Theta}^m(\|V\|, z) = Q\}] \subset X_1$.
(4) *The function f is Lipschitzian with* $\operatorname{Lip} f \leq L$.
(5) *For $\mathscr{L}^m$ almost all $x \in X_1$ the following is true*:
   (a) *The function f is approximately strongly affinely approximable at x.*
   (b) *If $(x, y) \in \operatorname{graph}_Q f$ then*

$$\operatorname{Tan}^m(\|V\|, (x, y)) = \operatorname{Tan}\big(\operatorname{graph}_Q \operatorname{ap} Af(x), (x, y)\big) \in \mathbf{G}(n, m).$$

(6) *If $a \in A$, $\boldsymbol{\Theta}^m(\|V\|, a) = Q$, $0 < t \leq s - |\mathbf{p}(a)|$, $|\mathbf{q}(a)| + \delta_4 t \leq s$, and*

$$B_{a,t} = \mathbf{C}(S, a, t, \delta_4 t) \cap B,$$
$$C_{a,t} = \mathbf{B}(\mathbf{p}(a), t) \sim (X_1 \sim \mathbf{p}[B_{a,t}]),$$
$$D_{a,t} = \mathbf{C}(S, a, t, \delta_4 t) \cap \mathbf{p}^{-1}[C_{a,t}],$$

   then $B_{a,t}$ is a Borel set, $C_{a,t}$ and $D_{a,t}$ are universally measurable, and

$$\mathscr{L}^m(C_{a,t}) + \|V\|(D_{a,t}) \leq \Gamma_{(6)} \|V\|(B_{a,t})$$

   with $\Gamma_{(6)} = 3 + 2Q + (12Q + 6)5^m$.
(7) *If a, t, $C_{a,t}$, $D_{a,t}$ are as in (6), $g : \mathbf{R}^m \to \mathbf{R}^{n-m}$, $\operatorname{Lip} g < \infty$, $g|X_1 = \boldsymbol{\eta}_Q \circ f$, $\tau \in \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m})$, $\theta \in \mathscr{D}(\mathbf{R}^m, \mathbf{R}^{n-m})$, $\eta \in \mathscr{D}^0(\mathbf{R}^{n-m})$,*

$$\operatorname{spt}\theta \subset \mathbf{U}(\mathbf{p}(a), t), \qquad 0 \leq \eta(y) \leq 1 \quad \textit{for } y \in \mathbf{R}^{n-m},$$

$$\operatorname{spt}\eta \subset \mathbf{U}(\mathbf{q}(a), \delta_4 t), \qquad \mathbf{B}(\mathbf{q}(a), \delta_4 t/2) \subset \operatorname{Int}(\mathbf{R}^{n-m} \cap \{y : \eta(y) = 1\}),$$

and $\Psi^\S$ *denotes the nonparametric integrand associated with the area integrand* Ψ, *then*

$$\begin{aligned}&\left|Q\int \big\langle D\theta(x), D\Psi_0^\S(Dg(x))\big\rangle \,\mathrm{d}\mathscr{L}^m x - (\delta V)((\eta\circ\mathbf{q})\cdot(\mathbf{q}^*\circ\theta\circ\mathbf{p}))\right|\\ &\quad\le \gamma_1 Q m^{1/2}\operatorname{Lip} g\int_{C_{a,t}} |D\theta|\,\mathrm{d}\mathscr{L}^m\\ &\qquad+\gamma_2\int_{E_{a,t}\sim C_{a,t}} |D\theta(x)||\operatorname{ap} Af(x)(+)(-\tau)|^2\,\mathrm{d}\mathscr{L}^m x\\ &\qquad+m^{1/2}\int_{D_{a,t}} |D((\eta\circ\mathbf{q})\cdot(\mathbf{q}^*\circ\theta\circ\mathbf{p}))|\,\mathrm{d}\|V\|\end{aligned}$$

where

$$\begin{aligned}\gamma_1 &= \sup \|D^2\Psi_0^\S\|[\mathbf{B}(0, m^{1/2}\operatorname{Lip} g)],\\ \gamma_2 &= \operatorname{Lip}\big(D^2\Psi_0^\S|\mathbf{B}(0, m^{1/2}(L+2\|\tau\|))\big),\\ E_{a,t} &= \mathbf{B}(\mathbf{p}(a), t)\cap X_1\cap\{x : \boldsymbol{\Theta}^0(\|f(x)\|, g(x))\neq Q\}.\end{aligned}$$

Proof of 4.1 This follows from [24, 5.7, 8]; in fact the statements (1)–(5) are those in [24, 5.7] with r, h, T replaced by s, s, S, and [24, 5.8] shows that the additional conditions $a\in A$ and $\boldsymbol{\Theta}^m(\|V,\|, a) = Q$ in (6), (7) can be arranged to imply

$$\begin{aligned}\operatorname{graph}_Q f|\mathbf{B}(\mathbf{p}(a), t) &\subset \mathbf{C}(S, a, t, \delta_4 t/2),\\ \|V\|(\mathbf{C}(S, a, t, \delta_4 t)) &\ge (Q - 1/4)\boldsymbol{\alpha}(m)t^m,\end{aligned}$$

hence (6), (7) are consequences of [24, 5.7(6)(7)(9)]. □

Remark 4.2 The nonparametric area integrand at 0, $\Psi_0^\S$, associated with the area integrand Ψ, is given explicitly by

$$\Psi_0^\S(\tau) = \left(\sum_{i=0}^m \Big|\bigwedge\nolimits_i \tau\Big|^2\right)^{1/2}\quad \text{for } \tau\in\operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m}),$$

see [16, 5.1.9].

The main additional feature contained in [24, 5.7(1)–(7)(9)] lies in the fact that the resulting estimates 4.1(6) and 4.1(7) are valid simultaneously for *all* cuboids which are centered in the set $A\cap\{a : \boldsymbol{\Theta}^m(\|V\|, a) = Q\}$ and contained in $\mathbf{C}(S, 0, s, s)$ rather than just for the *single* cuboid $\mathbf{C}(S, 0, s, s)$. Since the results of the preceding section are not and cannot be of pointwise nature, it is important for the purpose of the present paper that the set $A\cap\{a : \boldsymbol{\Theta}^m(\|V\|, a) = Q\}$ will have positive $\|V\|$ measure in the relevant situations.

4.3 The following situation will be studied: $m, n \in \mathscr{P}$, $m < n$, $1 \le p \le \infty$, U is an open subset of $\mathbf{R}^n$, $V \in \mathbf{V}_m(U)$, $\|\delta V\|$ is a Radon measure and, if $p > 1$,

$$(\delta V)(g) = -\int g(z) \bullet \mathbf{h}(V; z) \,\mathrm{d}\|V\|(z) \quad \text{whenever } g \in \mathscr{D}(U, \mathbf{R}^n),$$

$$\mathbf{h}(V; \cdot) \in \mathbf{L}_p(\|V\| \llcorner K, \mathbf{R}^n) \quad \text{whenever } K \text{ is a compact subset of } U.$$

If $p < \infty$ then the measure ψ is defined by

$$\psi = \|\delta V\| \quad \text{if } p = 1, \qquad \psi = |\mathbf{h}(V; \cdot)|^p \|V\| \quad \text{if } p > 1.$$

Next, it is proven that one can cover an integral varifold with locally bounded first variation by a countable family of sets Z such that each Z in a nonempty open set of directions can be expressed as the graph of a Lipschitzian function with certain properties which are discussed below. Exhibiting a single direction for each Z would be sufficient to prove the principal theorem of this paper. The present formulation allows to re-prove rather than use Brakke's perpendicularity of mean curvature in this process.

Lemma 4.4 *Suppose $m, n \in \mathscr{P}$, $m < n$, $1 \le p \le m$, $1 \le q < \infty$, $0 < \alpha \le 1$, $\alpha q(m - p) \le mp$, $0 < L < \infty$, U is an open subset of $\mathbf{R}^n$, $V \in \mathbf{IV}_m(U)$, ψ is related to p and V as in 4.3, and P is the set of all $a \in U$ such that* $\operatorname{Tan}^m(\|V\|, a) \in \mathbf{G}(n, m)$ *and*

$$\limsup_{s \to 0+} s^{-\alpha - m/q} \left(\int_{\mathbf{B}(a,s) \times \mathbf{G}(n,m)} |S_\natural - \operatorname{Tan}^m(\|V\|, a)_\natural|^q \,\mathrm{d}V(z, S) \right)^{1/q} < \infty.$$

Then there exists a countable, disjointed family H of $\|V\|$ measurable subsets of P such that $\|V\|(P \sim \bigcup H) = 0$ and for each $Z \in H$ there exists a nonempty open subset O of $\mathbf{O}^*(n, m)$ such that for each $\pi_1 \in O$ there exist

$$g : \mathbf{R}^m \to \mathbf{R}^{n-m}, \qquad G : \mathbf{R}^m \to \mathbf{R}^n, \qquad K \subset \mathbf{R}^m, \qquad Q \in \mathscr{P},$$
$$\pi_2 \in \mathbf{O}^*(n, n-m), \qquad T \in \mathscr{D}'(\mathbf{R}^m, \mathbf{R}^{n-m})$$

with the following six properties:

(1) $\pi_2 \circ \pi_1^* = 0$, $G = \pi_1^* + \pi_2^* \circ g$, *and* $G[K] = Z$.
(2) $\operatorname{Lip} g \le L$.
(3) *K is an $\mathscr{L}^m$ measurable subset of* $\operatorname{dmn} Dg$.
(4) $\int \langle D\theta(x), D\Psi_0^\S(Dg(x)) \rangle \,\mathrm{d}\mathscr{L}^m x = T(\theta)$ *for $\theta \in \mathscr{D}(\mathbf{R}^m, \mathbf{R}^{n-m})$ where Ψ denotes the area integrand.*
(5) *Whenever $x \in K$ there holds with $z = G(x)$ and $R = \operatorname{Tan}^m(\|V\|, z)$*

$$\mathbf{\Theta}^m(\|V\|, z) = Q, \qquad \operatorname{im} DG(x) = R,$$

$$\limsup_{s \to 0+} s^{-\beta - m/r} \left(\int_{\mathbf{B}(x,s)} |Dg(\zeta) - Dg(x)|^r \,\mathrm{d}\mathscr{L}^m \zeta \right)^{1/r}$$
$$\le 2m^{1/2} \limsup_{s \to 0+} s^{-\beta - m/r} \left(\int_{\mathbf{B}(z,s) \times \mathbf{G}(n,m)} |S_\natural - R_\natural|^r \,\mathrm{d}V(\xi, S) \right)^{1/r}$$

whenever $0 < \beta \le 1$, $1 \le r < \infty$ and $\beta r \le \alpha q$.

(6) *Whenever $x \in K$ there holds*

$$\lim_{s \to 0+} s^{-m-1} |T - T_x|_{-1,1;x,s} = 0$$

where $T_x \in \mathscr{D}'(\mathbf{R}^m, \mathbf{R}^{n-m})$ is defined by

$$T_x(\theta) = -\int \Psi_0^\S(Dg(x))\mathbf{h}(V; G(x)) \bullet (\pi_2^* \circ \theta)(\zeta)\, \mathrm{d}\mathscr{L}^m \zeta$$

whenever $\theta \in \mathscr{D}(\mathbf{R}^m, \mathbf{R}^{n-m})$.

The condition (2) ensures that 3.21 can be applied to replace the nonparametric area integrand by an integrand satisfying the conditions employed in Sect. 3. Condition (5) entails that the tilt-excess decay obtained by Brakke in [9, 5.7,5] or the author in [24, 10.6] carries over to the function g. This is required in order to satisfy the supplementary $\mathbf{L}_2(\mathscr{L}^m, \mathbf{R}^{n-m})$ differentiability condition of order $1/2$ on $\mathbf{D}u$ in 3.18 with $u = g$. Finally, condition (6) guarantees that the "values" (in the sense of 3.17) of T at points $x \in K$ relate to the mean curvature of the varifold so that 3.18 (2) can be used in the proof of the main Theorem 4.8 to link the first and second order $\mathbf{L}_1(\mathscr{L}^m, \mathbf{R}^{n-m})$ derivatives of g to the mean curvature of V.

The proof rests on the fact that the accuracy of the approximation by $\mathbf{Q}_Q(\mathbf{R}^{n-m})$-valued functions is controlled by the $\|V\|$ measure of a set B where either the mean curvature is large at some scale or the tilt-excess is large at some scale. In turn, the $m + \alpha q$ density of B at a generic point can be estimated by use of the local p summability of the mean curvature (see 4.3) and the $\mathbf{L}_q(\|V\|, \mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^n))$ differentiability of order α of the tangent space map $\mathrm{Tan}^m(\|V\|, \cdot)_\natural$ at points in P applying results of the author previously obtained in [22].

Proof of 4.4 First, observe that if some $\|V\|$ measurable set Z has the properties listed in the conclusion so does every $\|V\|$ measurable subset of Z. Therefore, in order to prove the assertion, it is enough to show that for $\|V\|$ almost all $a \in P$ there exists a $\|V\|$ measurable set Z having the stated properties and additionally satisfies $\boldsymbol{\Theta}^{*m}(\|V\| \llcorner Z, a) > 0$; in fact one can then take a maximal, disjointed family H of such Z (hence $\|V\|(Z) > 0$) and note H is countable and $\boldsymbol{\Theta}^m(\|V\| \llcorner \bigcup H, a) = 0$ for $\mathscr{H}^m$ almost all $a \in U \sim \bigcup H$ by [16, 2.10.19(4)] so that $\|V\|(P \sim \bigcup H) > 0$ would contradict the maximality of H.

Define P' to be the set of all $z \in U$ such that $\mathrm{Tan}^m(\|V\|, z) \in \mathbf{G}(n, m)$ and

$$\lim_{t \to 0+} t^{-1/2-m/2} \left(\int_{\mathbf{B}(z,t) \times \mathbf{G}(n,m)} |S_\natural - \mathrm{Tan}^m(\|V\|, z)_\natural|^2 \, \mathrm{d}V(\xi, S) \right)^{1/2} = 0.$$

By Brakke [9, 5.7, 5] or [24, 10.6] there holds $\|V\|(U \sim P') = 0$. Therefore, one may assume $\alpha q \geq 1$ possibly replacing α, q by $1/2, 2$ if $\alpha q < 1$. Assume further $L \leq 1/8$ and suppose $Q \in \mathscr{P}$. The remaining assertion will be shown to hold for $\|V\|$ almost all $a \in P$ with $\boldsymbol{\Theta}^m(\|V\|, a) = Q$. For this purpose define

$$\delta_1 = \delta_2 = \delta_3 = 1/2, \qquad \delta_4 = 1/4, \qquad M = 5^m Q,$$
$$\varepsilon = \inf\left\{\varepsilon_{4.1}(n, Q, L, M, \delta_1, \delta_2, \delta_3, \delta_4), (2\boldsymbol{\gamma}(m))^{-1}\right\},$$

and $R : U \cap \{z : \operatorname{Tan}^m(\|V\|, z) \in \mathbf{G}(n, m)\} \to \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ by

$$R(z) = \operatorname{Tan}^m(\|V\|, z)_\natural \quad \text{whenever } z \in U \text{ with } \operatorname{Tan}^m(\|V\|, z) \in \mathbf{G}(n, m).$$

For $i \in \mathscr{P}$ let C_i denote the set of all $z \in \operatorname{spt}\|V\|$ such that either $\mathbf{B}(z, 1/i) \not\subset U$ or

$$\|\delta V\| \, \mathbf{B}(z, t) > (2\varepsilon/3) \, \|V\|(\mathbf{B}(z, t))^{1-1/m} \quad \text{for some } 0 < t < 1/i,$$

let $D_i(w)$ for $w \in \operatorname{dmn} R$ denote the set of all $z \in U$ such that either $\mathbf{B}(z, 1/i) \not\subset U$ or

$$\int_{\mathbf{B}(z,t)} |R(\xi) - R(w)|^q \, d\|V\|\xi > (\varepsilon/3)^q \, \|V\| \, \mathbf{B}(z, t) \quad \text{for some } 0 < t < 1/i$$

and define X_i for $i \in \mathscr{P}$ by

$$X_i = U \cap \left\{z : \boldsymbol{\Theta}^{m^2/(m-p)}(\|V\| \llcorner C_i, z) = 0\right\} \quad \text{if } p < m,$$
$$X_i = U \sim \operatorname{Clos} C_i \quad \text{if } p = m,$$

as well as Y_i for $i \in \mathscr{P}$ by

$$Y_i = (\operatorname{dmn} R) \cap \left\{w : \boldsymbol{\Theta}^{m+\alpha q}(\|V\| \llcorner D_i(w), w) = 0\right\}.$$

Since $C_{i+1} \subset C_i$ and $D_{i+1}(w) \subset D_i(w)$ for $w \in \operatorname{dmn} R$, one notes $X_i \subset X_{i+1}$ and $Y_i \subset Y_{i+1}$ for $i \in \mathscr{P}$. X_i are Borel sets. Y_i are $\|V\|$ measurable sets by [22, 3.7(ii)]. P is $\|V\|$ measurable by [22, 3.7]. Moreover,

$$\|V\|\left(U \sim \bigcup\{X_i : i \in \mathscr{P}\}\right) = 0, \qquad \|V\|\left(P \sim \bigcup\{Y_i : i \in \mathscr{P}\}\right) = 0$$

by [22, 2.5, 9, 10, 3.7(ii)].

Define a measure μ on U such that $\mu + |\mathbf{h}(V; \cdot)| \|V\| = \|\delta V\|$ and $J = P \cap \{z : \boldsymbol{\Theta}^m(\|V\|, z) = Q\}$. The remaining assertion will be shown at a point a such that for some $i \in \mathscr{P}$

$$a \in X_i \cap Y_i \cap (\operatorname{dmn} R), \quad \mathbf{B}(a, 4/i) \subset U,$$
$$\boldsymbol{\Theta}^m(\|V\|, a) = Q, \qquad \boldsymbol{\Theta}^m(\|V\| \llcorner U \sim (J \cap X_i \cap Y_i), a) = 0,$$
$$R \text{ is approximately continuous at } a \text{ with respect to } \|V\|.$$

These conditions are satisfied by $\|V\|$ almost all $a \in J$ by the preceding remarks and [16, 2.9.11,13]. Fix such a and i, choose $0 < \kappa \le 1/2$ such that $(1 + \kappa)^m Q < Q + 1/2$, and define $\lambda = (1 + \kappa^2)^{-1/2}$ and $\delta = (1 - \lambda)/2$. Noting for $S \in \mathbf{G}(n, m)$ with $|S_\natural - R(a)| < \delta$ and $0 < s < \infty$,

$$\mathbf{R}^n \cap \{z : |S_\natural(z - a)| \le \lambda|z - a|\} \subset \mathbf{R}^n \cap \{z : |R(a)(z - a)| \le (\lambda + \delta)|z - a|\},$$

$$\mathbf{C}(S, a, s) \cap \{z : |S_\natural(z-a)| > \lambda|z-a|\} \subset \mathbf{C}(S, a, s, \kappa s) \subset \mathbf{B}(a, (1+\kappa)s),$$
$$0 < \lambda + \delta < 1, \quad \boldsymbol{\Theta}^m(\|V\| \llcorner \{z : |R(a)(z-a)| \leq (\lambda+\delta)|z-a|\}, a) = 0$$

by [16, 3.2.16], one infers the existence of $0 < s < (2i)^{-1}$ such that

$$(Q - 1/2)\boldsymbol{\alpha}(m)s^m \leq \|V\|(\mathbf{C}(S, a, s, s)) \leq (Q + 1/2)\boldsymbol{\alpha}(m)s^m,$$
$$\|V\|(\mathbf{C}(S, a, s, 5s/4) \sim \mathbf{C}(S, a, s, s/2)) \leq (1/2)\boldsymbol{\alpha}(m)s^m,$$
$$\|V\|(\mathbf{R}^n \cap \{z : \operatorname{dist}(z, \mathbf{C}(S, a, s, s)) < 2s\}) \leq \|V\| \mathbf{B}(a, 4s) \leq M\boldsymbol{\alpha}(m)s^m,$$

whenever $S \in \mathbf{G}(n, m)$ with $|S_\natural - R(a)| < \delta$.

Define A to be the set of all $z \in \mathbf{U}(a, s) \cap \operatorname{spt} \|V\|$ such that

$$\|\delta V\| \mathbf{B}(z, t) \leq (2\varepsilon/3)\|V\|(\mathbf{B}(z, t))^{1-1/m},$$
$$\int_{\mathbf{B}(z,t)} |R(\xi) - R(a)| \, \mathrm{d}\|V\|\xi \leq (2\varepsilon/3)\|V\| \mathbf{B}(z, t)$$

whenever $0 < t < 2s$,

$$O = \mathbf{O}^*(n, m) \cap \{\pi : |\pi^* \circ \pi - R(a)| < \inf\{\delta, \varepsilon/3\}\},$$
$$W = \mathbf{U}(a, s) \cap X_i \cap Y_i \cap \{w : |R(w) - R(a)| \leq \varepsilon/3\}, \qquad Z = W \cap A \cap J \sim N$$

where N is the set of all $w \in W$ such that one of the following three conditions is violated:

$$w \in P', \qquad \boldsymbol{\Theta}^m(\mu, w) = 0, \qquad \lim_{t \to 0+} t^{-m} \int_{\mathbf{B}(w,t)} |\mathbf{h}(V; \xi) - \mathbf{h}(V; w)| \, \mathrm{d}\|V\|\xi = 0.$$

Note $\|V\|(N) = 0$ by [16, 2.9.10, 11].

Now, fix $\pi_1 \in O$, $S = \operatorname{im} \pi_1^*$ and choose $\pi_2 \in \mathbf{O}^*(n, n-m)$ with $\pi_2 \circ \pi_1^* = 0$. The proof will be concluded by showing $\boldsymbol{\Theta}^m(\|V\| \llcorner Z, a) = Q$ and constructing g, G, K, and T with the asserted properties. For this purpose assume $a = 0$ and $\pi_1 = \mathbf{p}$ and $\pi_2 = \mathbf{q}$ using isometries and identifying $\mathbf{R}^n \simeq \mathbf{R}^m \times \mathbf{R}^{n-m}$. Define

$$u(w) = (s - |w-a|)/2 \quad \text{for } w \in W$$

and note $u(w) > 0$. Moreover, define B, f as in 4.1 with δ replaced by ε and whenever $w \in W$ and $0 < t \leq u(w)$ define $B_{w,t}$, $C_{w,t}$, and $D_{w,t}$ as in 4.1(6), (7) with additionally a, s replaced by w, t. Since $|S_\natural - R(a)| \leq \varepsilon/3$ and $Z \subset A \cap \{z : \boldsymbol{\Theta}^m(\|V\|, z) = Q\}$, one infers from 4.1(3) that $Z \subset \operatorname{graph}_Q f$ and

$$\boldsymbol{\Theta}^0(\|f(\mathbf{p}(z))\|, \mathbf{q}(z)) = Q, \qquad (\mathbf{p}^* + \mathbf{q}^* \circ \boldsymbol{\eta}_Q \circ f)(\mathbf{p}(z)) = z$$

whenever $z \in Z$. Using Kirszbraun's theorem (cf. [16, 2.10.43]) one extends $\boldsymbol{\eta}_Q \circ f$ to a function $g : \mathbf{R}^m \to \mathbf{R}^{n-m}$ such that

$$\operatorname{Lip} g = \operatorname{Lip}(\boldsymbol{\eta}_Q \circ f)$$

and defining $G = \mathbf{p}^* + \mathbf{q}^* \circ g$, $K = \mathbf{p}[Z]$ and $T \in \mathscr{D}'(\mathbf{R}^m, \mathbf{R}^{n-m})$ by

$$T(\theta) = \int \left\langle D\theta(x), D\Psi_0^\S(Dg(x))\right\rangle \mathrm{d}\mathscr{L}^m x \quad \text{for } \theta \in \mathscr{D}(\mathbf{R}^m, \mathbf{R}^{n-m}),$$

the properties (1), (2), and (4) are evident noting 4.1(4).

Next, it will be shown

$$B_{w,t} \subset \mathbf{U}(a,s) \cap (\operatorname{spt}\|V\|) \sim A \subset C_i \cup D_i(w)$$

whenever $w \in W$, $0 < t \le u(w)$. The first inclusion is readily verified noting $|S_\natural - R(a)| \le \varepsilon/3$. If $z \in \mathbf{U}(a,s) \cap (\operatorname{spt}\|V\|) \sim A$, then

$$\text{either} \quad \|\delta V\|\,\mathbf{B}(z,t) > (2\varepsilon/3)\|V\|(\mathbf{B}(z,t))^{1-1/m} \quad \text{for some } 0 < t < 2s,$$

$$\text{or} \quad \int_{\mathbf{B}(z,t)} |R(\xi) - R(a)|\,\mathrm{d}\|V\|\xi > (2\varepsilon/3)\|V\|\,\mathbf{B}(z,t) \quad \text{for some } 0 < t < 2s.$$

In the first case, this implies $z \in C_i$, in the second case,

$$\begin{aligned}
(2\varepsilon/3)\|V\|\,\mathbf{B}(z,t) &< \int_{\mathbf{B}(z,t)} |R(\xi) - R(a)|\,\mathrm{d}\|V\|\xi \\
&\le \int_{\mathbf{B}(z,t)} |R(\xi) - R(w)|\,\mathrm{d}\|V\|\xi \\
&\quad + |R(a) - R(w)|\|V\|\,\mathbf{B}(z,t), \\
(\varepsilon/3)\|V\|\,\mathbf{B}(z,t) &< \int_{\mathbf{B}(z,t)} |R(\xi) - R(w)|\,\mathrm{d}\|V\|\xi \\
&\le \|V\|(\mathbf{B}(z,t))^{1-1/q}\left(\int_{\mathbf{B}(z,t)} |R(\xi) - R(w)|^q\,\mathrm{d}\|V\|\xi\right)^{1/q},
\end{aligned}$$

hence $z \in D_i(w)$, and the second inclusion and hence the claim are proven. The inclusions imply the *density estimate*

$$\mathbf{\Theta}^{m+\alpha q}(\|V\| \llcorner B, w) = \mathbf{\Theta}^{m+\alpha q}(\|V\| \llcorner (U \sim A), w) = 0 \quad \text{whenever } w \in W.$$

Noting $a \in W$ and $\mathbf{\Theta}^m(\|V\| \llcorner U \sim (W \cap J), a) = 0$, one infers in particular

$$\mathbf{\Theta}^m(\|V\| \llcorner U \sim Z, a) = 0, \qquad \mathbf{\Theta}^m(\|V\| \llcorner Z, a) = Q$$

and it remains to verify that g, G, K, and T satisfy (3), (5), and (6).

In preparation for this, the following *tilt estimate* will be shown with $\Delta_1 = (1 + L^2)^{1/2}(1 - L^2)^{-1/2}m^{1/2}$

$$\begin{aligned}
&Q^{-1/2}\left(\int_{\mathbf{B}(\mathbf{p}(z),t)\cap \operatorname{dmn} f} |\operatorname{ap} Af(x)(+)(-\tau)|^r\,\mathrm{d}\mathscr{L}^m x\right)^{1/r} \\
&\le \Delta_1\left(\int_{\mathbf{C}(S,z,t,\delta_4 t)} |R(\xi) - \tau_\natural|^r\,\mathrm{d}\|V\|\xi\right)^{1/r}
\end{aligned}$$

whenever $1 \le r < \infty$, $z \in Z$, $0 < t \le u(z)$, $\tau \in \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m})$ with $\|\tau\| \le L$ (here the identification $\tau \subset \mathbf{R}^m \times \mathbf{R}^{n-m} \simeq \mathbf{R}^n$ is used); in fact, recalling $L \le 1/8$ and $z \in \operatorname{graph}_Q f$, one notes

$$\operatorname{graph}_Q f|\mathbf{B}(\mathbf{p}(z), t) \subset \mathbf{C}(S, z, t, \delta_4 t) \subset \mathbf{C}(S, a, s, s),$$

hence for $0 < \gamma < \infty$

$$\mathbf{B}(\mathbf{p}(z), t)) \cap \left\{x : Q^{-1/2}|\operatorname{ap} Af(x)(+)(-\tau)| > \gamma\right\}$$

is $\mathscr{H}^m$ almost contained in

$$\mathbf{p}\big[\mathbf{C}(S, z, t, \delta_4 t) \cap \{\xi : \Delta_1 |R(\xi) - \tau_\natural| > \gamma\}\big]$$

by 4.1(4), (5) and Allard [3, 8.9(5)]. For $x \in K$, taking $z = G(x)$ and τ associated with $\operatorname{im} R(z)$, one infers, noting $\boldsymbol{\Theta}^{m+\alpha q}(\mathscr{L}^m \llcorner \mathbf{R}^m \sim \operatorname{dmn} f, x) = 0$ by the density estimate for B and 4.1(6) and $\Delta_1 \le 2m^{1/2}$,

$$\begin{aligned}
&\limsup_{t\to 0+} t^{-\beta-m/r}\left(\int_{\mathbf{B}(x,t)} |Dg(\zeta) - \tau|^r \, \mathrm{d}\mathscr{L}^m \zeta\right)^{1/r} \\
&\quad \le 2m^{1/2} \limsup_{t\to 0+} t^{-\beta-m/r}\left(\int_{\mathbf{B}(z,t)} |R(\xi) - R(z)|^r \, \mathrm{d}\|V\|\xi\right)^{1/r}
\end{aligned}$$

whenever $x \in K$, $0 < \beta \le 1$, $1 \le r < \infty$, and $\beta r \le \alpha q$, hence in particular, taking $\beta = \alpha \inf\{1, q/r\}$ and noting that the right-hand side in this case is finite by [16, 2.4.17] as $z \in P$,

$$\lim_{t\to 0+}\left(\fint_{\mathbf{B}(x,t)} |Dg(\zeta) - \tau|^r \, \mathrm{d}\mathscr{L}^m \zeta\right)^{1/r} = 0 \quad \text{for } 1 \le r < \infty$$

and g is differentiable at x with $Dg(x) = \tau$ by the argument in [15, Theorem 6.2.1]. Since $Z \subset \operatorname{im} G$, K is $\mathscr{L}^m$ measurable, hence (3) and (5) are now proven and it remains to prove (6).

Choose $\eta \in \mathscr{D}^0(\mathbf{R}^{n-m})$ such that

$$0 \le \eta(y) \le 1 \quad \text{for } y \in \mathbf{R}^{n-m},$$

$$\operatorname{spt} \eta \subset \mathbf{U}(0, 1/4), \qquad \mathbf{B}(0, 1/8) \subset \operatorname{Int}(\mathbf{R}^{n-m} \cap \{y : \eta(y) = 1\})$$

and define T_x for $x \in K$ as in (6). Fix $x \in K$, let $z = G(x)$, note $\mathbf{p}(z) = x$ and abbreviate

$$\theta_t = t^{-m}\theta \circ \boldsymbol{\mu}_{1/t} \circ \boldsymbol{\tau}_{-\mathbf{p}(z)}, \qquad \eta_t = \eta \circ \boldsymbol{\mu}_{1/t} \circ \boldsymbol{\tau}_{-\mathbf{q}(z)}$$

whenever $0 < t \le u(z)$ and $\theta \in \mathscr{D}(\mathbf{R}^m, \mathbf{R}^{n-m})$. The remaining estimate will be carried out by showing that

$$QT_x(\theta_t) - (\delta V)((\eta_t \circ \mathbf{q}) \cdot (\mathbf{q}^* \circ \theta_t \circ \mathbf{p})),$$

$$(\delta V)((\eta_t \circ \mathbf{q}) \cdot (\mathbf{q}^* \circ \theta_t \circ \mathbf{p})) - Q \int \big\langle D\theta_t(\zeta), D\Psi_0^\S(Dg(\zeta))\big\rangle \, \mathrm{d}\mathscr{L}^m \zeta$$

both tend to 0 as $t \to 0+$ uniformly with respect to $\theta \in \mathscr{D}(\mathbf{R}^m, \mathbf{R}^{n-m})$ such that $\operatorname{spt}\theta \subset \mathbf{U}(0,1)$ and $|D\theta|_{\infty;0,1} \le 1$.

To prove the first estimate, one notes that the conditions $\boldsymbol{\Theta}^{m-1}(\|\delta V\|, z) = 0$, $\boldsymbol{\Theta}^m(\|V\|, z) = Q$ and $z \in P$ imply, for example, using Allard [3, 6.4, 5] and [23, 3.1],

$$t^{-m} \int \phi(t^{-1}(\xi - z), \operatorname{im} R(\xi)) \, \mathrm{d}\|V\|\xi \to Q \int_{\operatorname{im} R(z)} \phi(\xi, \operatorname{im} R(z)) \, \mathrm{d}\mathscr{H}^m \xi$$

as $t \to 0+$ whenever $\phi \in \mathscr{K}(\mathbf{R}^n \times \mathbf{G}(n,m))$. Since also, noting

$$(\eta_t \circ \mathbf{q}) \cdot (\mathbf{q}^* \circ \theta_t \circ \mathbf{p}) = t^{-m}\big((\eta \circ \mathbf{q}) \cdot (\mathbf{q}^* \circ \theta \circ \mathbf{p})\big) \circ \boldsymbol{\mu}_{1/t} \circ \boldsymbol{\tau}_{-z},$$

$$\mathbf{C}(T, 0, 1) \cap \operatorname{Tan}^m(\|V\|, z) \subset \mathbf{C}(T, 0, 1, 1/8)$$

as $L \le 1/8$ and $z \in \operatorname{graph}_Q f$, one readily uses the conditions on δV and $\mathbf{h}(V; \cdot)$ imposed by the fact that $z \notin N$ to infer

$$\begin{aligned}
&\lim_{t \to 0+} (\delta V)((\eta_t \circ \mathbf{q}) \cdot (\mathbf{q}^* \circ \theta_t \circ \mathbf{p})) \\
&\quad = -Q \int_{\operatorname{im} R(z)} \mathbf{h}(V; z) \bullet (\eta \circ \mathbf{q})(\xi)(\mathbf{q}^* \circ \theta \circ \mathbf{p})(\xi) \, \mathrm{d}\mathscr{H}^m \xi \\
&\quad = -Q \int \Psi_0^\S(Dg(x))\mathbf{h}(V; z) \bullet (\mathbf{q}^* \circ \theta)(\zeta) \, \mathrm{d}\mathscr{L}^m \zeta = QT_x(\theta_t)
\end{aligned}$$

and the convergence is uniform with respect to $\theta \in \mathscr{D}(\mathbf{R}^m, \mathbf{R}^{n-m})$ such that $\operatorname{spt}\theta \subset \mathbf{U}(0,1)$ and $|D\theta|_{\infty;0,1} \le 1$ as this family of functions is compact with respect to $|\cdot|_{\infty;0,1}$ by [16, 2.10.21] and $\boldsymbol{\Theta}^{*m}(\|\delta V\|, z) < \infty$.

To prove the second estimate, define

$$\gamma_1 = \sup \|D^2\Psi_0^\S\|[\mathbf{B}(0, m^{1/2}L)], \qquad \gamma_2 = \operatorname{Lip}\big(D^2\Psi_0^\S|\mathbf{B}(0, 3m^{1/2}L)\big).$$

Apply 4.1(7) with $\tau = Dg(x)$ and $0 < t \le u(z)$ to obtain

$$\begin{aligned}
&\left| Q \int \big\langle D\theta_t(x), D\Psi_0^\S(Dg(x))\big\rangle \, \mathrm{d}\mathscr{L}^m x - (\delta V)((\eta_t \circ \mathbf{q}) \cdot (\mathbf{q}^* \circ \theta_t \circ \mathbf{p})) \right| \\
&\quad \le \gamma_1 Q m^{1/2} L \int_{C_{z,t}} |D\theta_t| \, \mathrm{d}\mathscr{L}^m \\
&\qquad + \gamma_2 \int_{E_{z,t} \sim C_{z,t}} |D\theta_t(\zeta)| |\operatorname{ap} Af(\zeta)(+)(-Dg(x))|^2 \, \mathrm{d}\mathscr{L}^m \zeta \\
&\qquad + m^{1/2} \int_{D_{z,t}} |D((\eta_t \circ \mathbf{q}) \cdot (\mathbf{q}^* \circ \theta_t \circ \mathbf{p}))| \, \mathrm{d}\|V\|.
\end{aligned}$$

The first and the third summand on the right-hand side may be estimated by use of 4.1(6) as follows:

$$\int_{C_{z,t}} |D\theta_t| \, \mathrm{d}\mathscr{L}^m \leq t^{-m-1} \mathscr{L}^m(C_{z,t}) \leq \Delta_2 t^{-m-1} \|V\|(B_{z,t}),$$

$$\int_{D_{z,t}} |D((\eta_t \circ \mathbf{q}) \cdot (\mathbf{q}^* \circ \theta_t \circ \mathbf{p}))| \, \mathrm{d}\|V\|$$

$$\leq t^{-m-1}(1 + |D\eta|_{\infty;0,1}) \|V\|(D_{z,t}) \leq \Delta_2 t^{-m-1}(1 + |D\eta|_{\infty;0,1}) \|V\|(B_{z,t})$$

where $\Delta_2 = \Gamma_{4.1(6)}(Q, m)$, hence the density estimate for B applies recalling $\alpha q \geq 1$. To estimate the remaining summand, one computes

$$\int_{E_{z,t} \sim C_{z,t}} |D\theta_t(\zeta)| |\operatorname{ap} Af(\zeta)(+)(-Dg(x))|^2 \, \mathrm{d}\mathscr{L}^m \zeta$$

$$\leq t^{-1-m} \int_{\mathbf{B}(x,t) \cap \operatorname{dmn} f} |\operatorname{ap} Af(\zeta)(+)(-Dg(x))|^2 \, \mathrm{d}\mathscr{L}^m \zeta,$$

uses the tilt estimate, and recalls that $z \in P'$. $\square$

Remark 4.5 In 5.2 it will be shown that $\|V\|(U \sim P) = 0$ if $q = 2$ and $(m, p, \alpha) \neq (2, 1, 1)$. The author knows of no m, n, p, q, α, U, and V satisfying the hypotheses of 4.4 such that $\|V\|(U \sim P) > 0$ for the associated set P.

Remark 4.6 It would significantly simplify the treatment in 3.14–3.18 if one could obtain an estimate in $|\cdot|_{-1,r;a,s}$ in (6) for some $r > 1$. However, in this case it seems to be unclear how to control the integral over $D_{z,t}$ in the last paragraph as this set may contain arbitrarily steep parts of the varifold; see Brakke's example in [9, 6.1].

4.7 If $f : \mathbf{R}^m \to \mathbf{R}^{n-m}$ is a linear map, $v \in \mathbf{R}^n$ is orthogonal to $\operatorname{im}(\mathbf{p}^* + \mathbf{q}^* \circ f)$ then $v \in \ker(\mathbf{p}^* + \mathbf{q}^* \circ f)^*$, $\mathbf{p}(v) = -(f^* \circ \mathbf{q})(v)$ and

$$(\mathbf{q}^* - \mathbf{p}^* \circ f^*)(\mathbf{q}(v)) = v.$$

Now, the preceding results are readily combined to obtain the main theorem.

Theorem 4.8 *Suppose $m, n \in \mathscr{P}$, $m \leq n$, U is an open subset of $\mathbf{R}^n$, $V \in \mathbf{IV}_m(U)$, and $\|\delta V\|$ is a Radon measure.*

Then there exists a countable collection C of m-dimensional submanifolds of $\mathbf{R}^n$ of class 2 such that $\|V\|(U \sim \bigcup C) = 0$ and each member M of C satisfies

$$\mathbf{h}(V; z) = \mathbf{h}(M; z) \quad \textit{for } \|V\| \textit{ almost all } z \in U \cap M.$$

Proof Assume $m < n$.

First, note that for $\|V\|$ almost all $z \in U$ there holds $\operatorname{Tan}^m(\|V\|, z) \in \mathbf{G}(n, m)$ and

$$\lim_{r \to 0+} r^{-1/2-m/2} \left(\int_{\mathbf{B}(z,r) \times \mathbf{G}(n,m)} |S_\natural - \operatorname{Tan}^m(\|V\|, z)_\natural|^2 \, \mathrm{d}V(\xi, S) \right)^{1/2} = 0$$

by Brakke [9, 5.7, 5] or [24, 10.6]. Let Ψ denote the area integrand, abbreviate $\Phi = \Psi_0^\S$ and note $D^2\Phi(0) = \Upsilon$ with Υ as in 3.1 by [16, 5.1.9]. Define $\varepsilon = \varepsilon_{3.18}(m, n)$, $\Delta = \Gamma_{3.21}(m(n-m), 2)$, $s = \varepsilon/\Delta$, and choose $0 < \delta < \infty$ such that

$$\|D^2\Phi(\sigma) - D^2\Phi(0)\| \le s \quad \text{whenever } \sigma \in \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m}) \cap \mathbf{B}(0, \delta).$$

Applying 3.21 with H, k, l, a replaced by $\operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m})$, 2, 3, 0, one obtains $F : \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m}) \to \mathbf{R}$ of class 3 such that

$$D^i F(\sigma) = D^i\Phi(\sigma) \quad \text{for } i = \{0, 1, 2\}, \sigma \in \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m}) \cap \mathbf{B}(0, \delta/2),$$

$$\|D^2F(\sigma) - D^2\Phi(0)\| \le \Delta s = \varepsilon \quad \text{whenever } \sigma \in \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m}),$$

$$D^3F \text{ has compact support},$$

hence $\operatorname{Lip} D^2F < \infty$. Define $L = m^{-1/2}\delta/2$ and apply 4.4 with p, q, α replaced by 1, 2, 1/2 to obtain P and H with the properties listed there. Fix $Z \in H$ and take $\pi_1 \in O$ and π_2, g, G, K as in 4.4 to infer from 3.18, 3.19, and 4.4(6), noting 4.4(5) with $\beta = 1/2$ and $r = 2$, the existence a sequence of functions $u_i : \mathbf{R}^m \to \mathbf{R}^{n-m}$ of class 2 such that with $A_i = K \cap \{x : g(x) = u_i(x)\}$ for $i \in \mathscr{P}$

$$\langle D^2u_i(x), C_F(Du_i(x))\rangle = \Phi(Du_i(x))\pi_2(\mathbf{h}(V; G(x)))$$

for $\mathscr{L}^m$ almost all $x \in A_i$. Defining $M_i = \operatorname{im}(\pi_1^* + \pi_2^* \circ u_i)$ and noting

$$\langle D^2u_i(x), C_\Phi(Du_i(x))\rangle = \Phi(Du_i(x))\pi_2(\mathbf{h}(M_i; (\pi_1^* + \pi_2^* \circ u_i)(x)))$$

for $x \in \mathbf{R}^m$ where C_Φ is as in 3.1 and

$$C_\Phi(\sigma) = C_F(\sigma) \quad \text{for } \sigma \in \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m}) \cap \mathbf{B}(0, \delta/2),$$

$$|Du_i(x)| = |Dg(x)| \le Lm^{1/2} = \delta/2 \quad \text{for } \mathscr{L}^m \text{ almost all } x \in A_i$$

by 4.4(2), one concludes

$$\pi_2(\mathbf{h}(V; G(x))) = \pi_2(\mathbf{h}(M_i; G(x))) \quad \text{for } \mathscr{L}^m \text{ almost all } x \in A_i,$$

hence by 4.7, since $\mathbf{h}(V; z) \in \operatorname{Nor}^m(\|V\|, z)$ for $\|V\|$ almost all z by Brakke [9, 5.8],

$$\mathbf{h}(V; G(x)) = \mathbf{h}(M_i; G(x)) \quad \text{for } \mathscr{L}^m \text{ almost all } x \in A_i.$$

Finally, recall $\|V\|(U \sim P) = 0$. □

Remark 4.9 One could also prove Brakke [9, 5.8] instead of using it. Since the proof then still yields a collection C with all properties except of the last one, one can define a $\|V\|$ measurable function h such that for $\|V\|$ almost all $z \in U$ there holds $h(z) = \mathbf{h}(M; z)$ whenever $z \in U \cap M$ and $M \in C$. Following the above proof, one obtains

$$\pi_2(\mathbf{h}(V; G(x))) = \pi_2(h(G(x))) \quad \text{for } \mathscr{L}^m \text{ almost all } x \in A_i$$

whenever $\pi_1 \in O$, $\pi_2 \in \mathbf{O}^*(n, n-m)$ with $\pi_2 \circ \pi_1^* = 0$, and, as O is open and nonempty, this suffices to conclude

$$\mathbf{h}(V; G(x)) = h(G(x)) \in \operatorname{Nor}^m(\|V\|, G(x)) \quad \text{for } \mathscr{L}^m \text{ almost all } x \in A_i.$$

Remark 4.10 Noting [16, 2.10.19(4)], one infers that the function mapping $\|V\|$ almost all z onto $\operatorname{Tan}^m(\|V\|, z)_\natural \in \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ is $(\|V\|, m)$ approximately differentiable at $\|V\|$ almost all z.

Therefore, combining 4.8 with Mantegazza [21, Theorem 5.4], one obtains the following proposition on curvature varifolds with boundary in the sense of Mantegazza [21, Definition 3.1, p. 811]: *If V is a curvature varifold with boundary in an open subset U of $\mathbf{R}^n$ then there exists a countable collection C of m-dimensional submanifolds of $\mathbf{R}^n$ of class 2 such that $\|V\|(U \sim \bigcup C) = 0$ and such that for each member M of C the second fundamental forms of V and M agree at $\|V\|$ almost every $z \in U \cap M$.* Clearly, this includes curvature varifolds in the sense of Hutchinson [19, 5.2.3].

The construction in the following example is included for completeness.

Example 4.11 It will be shown, *if $m > 1$ then there exists an $\mathscr{L}^m$ measurable set A such that $\partial(\mathbf{E}^m \llcorner A)$ is representable by integration and* $\operatorname{spt} \partial(\mathbf{E}^m \llcorner A) = \mathbf{R}^m$; see [16, 4.1.5, 4.1.7]. In particular, since A cannot be $\mathscr{L}^m$ almost equal to an open set, considering $V \in \mathbf{IV}_m(\mathbf{R}^m)$ characterized by $\|V\| = \mathscr{L}^m \llcorner A$ proves that the collection C in 4.8 cannot be required to satisfy

$$\boldsymbol{\Theta}^m(\|V\|, z) = \operatorname{card}\{M : z \in M \in C\} \quad \text{for } \mathscr{H}^m \text{ almost all } z.$$

To construct A, choose a sequence (x_i, s_i) in $\mathbf{R}^m \times \mathbf{R}$ such that $\{x_i : i \in \mathscr{P}\}$ is dense in $\mathbf{R}^m$, with $0 < s_i \le 1$ and $\inf\{s_j : x_i = x_j\} = 0$ for $i \in \mathscr{P}$. Inductively select ε_i, r_i, A_i, S_i, and T_i, satisfying

$$\varepsilon_1 = 1, \qquad r_1 = s_1, \qquad A_1 = \mathbf{B}(x_1, r_1), \qquad S_1 = T_1 = \mathbf{E}^m \llcorner \mathbf{B}(x_1, r_1)$$

and, for $i > 1$, subject to the conditions $0 < r_i \le s_i$, $r_i^{m-1} \le 2^{1-i}$, and

$$\mathbf{B}(x_i, r_i) \subset \operatorname{Int} A_{i-1} \quad \text{and} \quad \varepsilon_i = -1 \quad \text{if } x_i \in \operatorname{Int} A_{i-1},$$
$$\varepsilon_i = 0 \quad \text{if } x_i \in \operatorname{Bdry} A_{i-1},$$
$$\mathbf{B}(x_i, r_i) \subset \operatorname{Int}(\mathbf{R}^m \sim A_{i-1}) \quad \text{and} \quad \varepsilon_i = 1 \quad \text{if } x_i \in \operatorname{Int}(\mathbf{R}^m \sim A_{i-1})$$

and let $S_i = \mathbf{E}^m \llcorner \mathbf{B}(x_i, r_i)$, $T_i = T_{i-1} + \varepsilon_i S_i$, and $A_i = \operatorname{spt} T_i$. Noting

$$T_i = \mathbf{E}^m \llcorner A_i, \qquad \operatorname{Bdry} A_i = \operatorname{spt} \partial T_i,$$
$$\sum_{j=1}^{\infty} \mathbf{M}(S_j) + \mathbf{M}(\partial S_j) < \infty, \qquad T = \lim_{j \to \infty} T_j = \sum_{j=1}^{\infty} \varepsilon_j S_j,$$
$$\mathbf{M}(T - T_j) + \mathbf{M}(\partial T - \partial T_j) \to 0 \quad \text{as } j \to \infty,$$

$$\|\partial T_i\| \le \|\partial T_j\|, \qquad \|\partial T_i\| \le \|\partial (T_j - T)\| + \|\partial T\| \quad \text{for } i \le j \in \mathscr{P},$$
$$\|\partial T_i\| \le \|\partial T\|, \qquad \operatorname{spt} \partial T_i \subset \operatorname{spt} \partial T,$$
$$\operatorname{dist}(x_i, \operatorname{spt} \partial T_i) \le s_i, \qquad \operatorname{spt} \partial T = \mathbf{R}^m,$$

for $i \in \mathscr{P}$, one may take $A = \mathbf{R}^m \cap \{x : \boldsymbol{\Theta}^m(\|T\|, x) \ge 1\}$, since $T = \mathbf{E}^m \llcorner A$.

5 Applications to Decay Rates of Tilt-Excess for Integral Varifolds

Overview The present section discusses some consequences of 4.8 in terms of decay and differentiability of tilt quantities.

These results depend on [24] mainly through the following lemma. It is the result of combining a coercive estimate with an interpolation inequality by use of an approximation by $\mathbf{Q}_Q(\mathbf{R}^{n-m})$-valued functions.

Lemma 5.1 *Suppose $m, n, Q \in \mathscr{P}$, $m < n$, either $p = m = 1$ or $1 < p < m = 2$ or $1 \le p < m > 2$ and $\frac{mp}{m-p} = 2$, $0 < \delta \le 1$, and $1 \le M < \infty$.*

Then there exist positive, finite numbers ε and Γ with the following property.

If $a \in \mathbf{R}^n$, $0 < r < \infty$, $V \in \mathbf{IV}_m(\mathbf{U}(a, 6r))$, ψ and p are related to V as in 4.3, $T \in \mathbf{G}(n, m)$, Z is a $\|V\|$ measurable subset of $\mathbf{C}(T, a, r, 3r)$,

$$(Q - 1/2)\boldsymbol{\alpha}(m) r^m \le \|V\|(\mathbf{C}(T, a, r, 3r)) \le (Q + 1/2)\boldsymbol{\alpha}(m) r^m,$$
$$\|V\|(\mathbf{C}(T, a, r, 4r) \sim \mathbf{C}(T, a, r, r)) \le (1/2)\boldsymbol{\alpha}(m) r^m,$$
$$\|V\|\,\mathbf{U}(a, 6r) \le M\boldsymbol{\alpha}(m) r^m, \qquad \|V\|(\mathbf{C}(T, a, r/2, r/2)) \ge (Q - 1/4)\boldsymbol{\alpha}(m)(r/2)^m,$$
$$\|V\|(\mathbf{C}(T, a, r, 3r) \sim Z) \le \varepsilon \boldsymbol{\alpha}(m) r^m, \qquad \left(\int |S_\natural - T_\natural|^2 \, \mathrm{d}V(z, S) \right)^{1/2} \le \varepsilon r^{m/2},$$

then

$$\begin{aligned} &\left(r^{-m} \int_{\mathbf{C}(T, a, r/4, r/4) \times \mathbf{G}(n, m)} |S_\natural - T_\natural|^2 \, \mathrm{d}V(z, S) \right)^{1/2} \\ &\quad \le \delta \left(r^{-m} \int_{\mathbf{C}(T, a, r, r) \times \mathbf{G}(n, m)} |S_\natural - T_\natural|^2 \, \mathrm{d}V(z, S) \right)^{1/2} \\ &\qquad + \Gamma \left(r^{-m-1} \int_Z \operatorname{dist}(z - a, T) \, \mathrm{d}\|V\| z + r^{1 - m/p} \psi(\mathbf{U}(a, 6r))^{1/p} \right). \end{aligned}$$

Proof See [24, 9.5]. □

Theorem 5.2 *Suppose m, n, p, U, and V are as in 4.3, $V \in \mathbf{IV}_m(U)$ and*

$$\phi(a, r, T) = \left(r^{-m} \int_{\mathbf{U}(a, r) \times \mathbf{G}(n, m)} |S_\natural - T_\natural|^2 \, \mathrm{d}V(z, S) \right)^{1/2}$$

whenever $a \in \mathbf{R}^n$, $0 < r < \infty$, $\mathbf{U}(a, r) \subset U$, and $T \in \mathbf{G}(n, m)$.

Then the following two statements hold:

(1) *If either* $m = 2$ *and* $0 < \tau < 1$ *or* $\sup\{2, p\} < m$ *and* $\tau = \frac{mp}{2(m-p)} < 1$ *then*

$$\lim_{r \to 0+} r^{-\tau} \phi(a, r, T) = 0 \quad \text{for } V \text{ almost all } (a, T) \in U \times \mathbf{G}(n, m).$$

(2) *If either* $m = 1$ *or* $m = 2$ *and* $p > 1$ *or* $m > 2$ *and* $p \geq 2m/(m+2)$ *then*

$$\limsup_{r \to 0+} r^{-1} \phi(a, r, T) < \infty \quad \text{for } V \text{ almost all } (a, T) \in U \times \mathbf{G}(n, m).$$

Proof of (1) From 4.8 one obtains a sequence of maps $R_i : U \to \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ of class 1 such that the sets $A_i = U \cap \{z : R_i(z) = \operatorname{Tan}^m(\|V\|, z)_\natural\}$ cover $\|V\|$ almost all of U. By [24, 10.6] and [22, 3.7(i)] one infers

$$\lim_{r \to 0+} r^{-\tau - m/2} \left(\int_{\mathbf{B}(z,r) \times \mathbf{G}(n,m)} |R_i(z) - S_\natural|^2 \, \mathrm{d}V(\xi, S) \right)^{1/2} = 0$$

for $\|V\|$ almost all $z \in A_i$ and the conclusion follows. □

Proof of (2) Assume that either $p = m = 1$ or $1 < p < m = 2$ or $1 \leq p < m > 2$ and $\frac{mp}{m-p} = 2$. Suppose ψ is related to p and V as in 4.3. Choose C as in 4.8. Then by 4.8 and [16, 2.10.19(4), 2.9.5] for $\|V\|$ almost all $a \in U$ there holds for some $Q \in \mathscr{P}$, $T \in \mathbf{G}(n, m)$ and some $M \in C$

$$T = \operatorname{Tan}(M, a), \qquad \boldsymbol{\Theta}^m(\|V\| \llcorner U \sim M, a) = 0,$$
$$\limsup_{r \to 0+} r^{-m/p} \psi(\mathbf{B}(a, r))^{1/p} < \infty,$$
$$r^{-m} \int \phi(r^{-1}(z - a), S) \, \mathrm{d}V(z, S) \to Q \int_T \phi(z, T) \, \mathrm{d}\mathscr{H}^m z \quad \text{as } r \to 0+$$

whenever $\phi \in \mathscr{K}(\mathbf{R}^n \times \mathbf{G}(n, m))$. Note that

$$\limsup_{r \to 0+} r^{-m-2} \int_{\mathbf{C}(T,a,r,3r) \cap M} \operatorname{dist}(z - a, T) \, \mathrm{d}\|V\| z < \infty$$

as M is a submanifold of class 2. It follows with $\delta = 2^{-m-3}$, $\Delta_1 = 7^m Q$ that there exist $0 < R < \infty$ and $0 \leq \gamma < \infty$ such that $\mathbf{U}(a, 6R) \subset U$,

$$r^{-m-1} \int_{\mathbf{C}(T,a,r,3r) \cap M} \operatorname{dist}(z - a, T) \, \mathrm{d}\|V\| z + r^{1-m/p} \psi(\mathbf{U}(a, 6r))^{1/p} \leq \gamma r$$

for $0 < r \leq R$, and V satisfies the hypotheses of 5.1 for each $0 < r \leq R$ with $\varepsilon = \varepsilon_{5.1}(m, n, Q, p, \delta, \Delta_1)$ and M, Z replaced by Δ_1, $\mathbf{C}(T, a, r, 3r) \cap M$. With $f(r) = r^{-m/2} (\int_{\mathbf{C}(T,a,r,r) \times \mathbf{G}(n,m)} |S_\natural - T_\natural|^2 \, \mathrm{d}V(z, S))^{1/2}$ for $0 < r \leq R$ one defines

$$\Delta_2 = \Gamma_{5.1}(m, n, Q, p, \delta, \Delta_1), \qquad \Delta_3 = \sup\left\{2^{m+3} \Delta_2 \gamma, 2^{m+2} R^{-1} f(R)\right\},$$

one inductively infers from 5.1

$$f(r) \leq \Delta_3 r \quad \text{whenever } 0 < r \leq R;$$

in fact it holds for $R/4 \le r \le R$ and, provided it holds for r,

$$f(r/4) \le 2^m(\delta\Delta_3 r + \Delta_2\gamma r) \le \Delta_3(r/4)$$

by 5.1. The conclusion is now evident. □

Remark 5.3 Having 4.8 at one's disposal, the proof of (2) follows Schätzle in [32, Theorem 3.1] where the case $p \ge 2$ is treated. In extending the result to the present case, the main difference is the use of the coercive estimate in [24, 4.10] in the proof of 5.1 replacing the use of the corresponding estimate in Brakke [9, 5.5] (see also Allard [3, 8.13]).

Remark 5.4 For both parts the family of examples provided in [22, 1.2] shows that if $m > 2$ then p cannot be replaced by any smaller number; see [24, 10.7].

Remark 5.5 In the case of (2) combining this result with [22, 3.9], one obtains

$$\fint_{\mathbf{B}(a,r)} (|R(z) - R(a) - \langle R(a)(z-a), \operatorname{ap} DR(a)\rangle|/|z-a|)^2 \, \mathrm{d}\|V\|z \to 0$$

as $r \to 0+$ for $\|V\|$ almost all a where $R(z) = \operatorname{Tan}^m(\|V\|, z)_\natural$ and the approximate differential is taken with respect to $(\|V\|, m)$.

Remark 5.6 Clearly, one can also obtain decay results for height quantities from this result by use of [23, 4.11].

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Appendix: Lebesgue Points for a Distribution

Overview In this Appendix the part $q = 1$ of Theorem 4 of the Introduction is provided. Its purpose is to clarify the relations of the sets A_1 and A_2 occurring in 3.18.

The result is obtained translating techniques from the differentiation theory of functions and measures to the present setting of distributions.

Lemma A.1 *Suppose* $m, n \in \mathscr{P}$, $m < n$, A *is a closed subset of* $\mathbf{R}^m$, $R \in \mathscr{D}'(\mathbf{R}^m, \mathbf{R}^{n-m})$, $\operatorname{dist}(\operatorname{spt} R, A) > 0$, $0 \le \gamma < \infty$, *and* $0 < r < \infty$ *such that*

$$|R|_{-1,1;x,\varrho} \le \gamma\varrho^{m+1} \quad \text{whenever } 0 < \varrho < 5r, x \in A.$$

Then

$$|R|_{-1,1;a,r} \le \Gamma\gamma r \mathscr{L}^m(\mathbf{B}(a, 4r) \sim A) \quad \text{for } a \in A$$

where Γ *is a positive, finite number depending only on* m.

Proof Assume $r \le \frac{2}{9}$, let $a \in A$, $\theta \in \mathscr{D}(\mathbf{R}^m, \mathbf{R}^{n-m})$ with $\operatorname{spt}\theta \subset \mathbf{U}(a, r)$, choose $0 < \varepsilon \le \inf\{r, \operatorname{dist}(\operatorname{spt} R, A)\}$, define

$$B = \mathbf{R}^m \cap \{x : \operatorname{dist}(x, \operatorname{spt}(R \llcorner \theta)) \le \varepsilon/2\}$$

where $R \llcorner \theta \in \mathscr{E}_0(\mathbf{R}^m)$ is defined by $(R \llcorner \theta)(v) = R(v\theta)$ for $v \in \mathscr{E}^0(\mathbf{R}^m)$, and apply [16, 3.1.13] to obtain S, v_s, and h with $\Phi = \{\mathbf{R}^m \sim A, \mathbf{R}^m \sim B\}$; in particular, S is a countable subset of $\bigcup \Phi$,

$$h(x) = \frac{1}{20} \sup\{\inf\{1, \operatorname{dist}(x, A)\}, \inf\{1, \operatorname{dist}(x, B)\}\} \quad \text{for } x \in \bigcup \Phi$$

and v_s for $s \in S$ form a partition of unity on $\bigcup \Phi$ with $\operatorname{spt} v_s \subset \mathbf{B}(s, 10h(s))$ for $s \in S$. Noting $\bigcup \Phi = \mathbf{R}^m$, one defines $T = S \cap \{s : B \cap \operatorname{spt} v_s \neq \emptyset\}$ and infers

$$\sum_{s \in S \sim T} v_s(x) = 0 \quad \text{for } x \in \mathbf{R}^m \text{ with } \operatorname{dist}(x, \operatorname{spt}(R \llcorner \theta)) < \varepsilon/2,$$

hence $(R \llcorner \theta)(\sum_{s \in S \sim T} v_s) = 0$ and

$$R(\theta) = R\left(\left(\sum_{s \in T} v_s\right)\theta\right) = \sum_{s \in T} R(v_s \theta).$$

Choose $\xi(s) \in A$ for each $s \in T$ such that $|s - \xi(s)| = \operatorname{dist}(s, A)$. If $s \in T$ then there exists $y \in B \cap \operatorname{spt} v_s \subset \mathbf{B}(a, r + \varepsilon/2)$ and one observes

$$\operatorname{dist}(y, A) \le |y - a| \le r + \varepsilon/2 \le (3/2)r \le \frac{1}{3} < 1, \qquad h(y) = \frac{1}{20} \operatorname{dist}(y, A),$$

$$|s - y| \le 10h(s) \le 10h(y) + \frac{1}{2}|s - y|, \qquad |s - y| \le 20h(y) = \operatorname{dist}(y, A) \le |y - a|,$$

$$\operatorname{dist}(s, A) \le |s - y| + \operatorname{dist}(y, A) \le 2 \operatorname{dist}(y, A) \le 3r \le \frac{2}{3} < 1,$$

$$B \cap \mathbf{B}(s, 10h(s)) \neq \emptyset, \qquad \frac{1}{20} \operatorname{dist}(s, B) \le \frac{1}{2} h(s), \qquad 0 < h(s) = \frac{1}{20} \operatorname{dist}(s, A),$$

$$|s - \xi(s)| \le |s - a| \le |s - y| + |y - a| \le 2r + \varepsilon \le 3r \le \frac{2}{3},$$

$$\mathbf{B}(s, h(s)) \subset \mathbf{B}(a, 4r) \sim A.$$

Moreover, for any $x \in \mathbf{B}(s, 10h(s))$, $s \in T$

$$|x - \xi(s)| \le |x - s| + |s - \xi(s)| \le (3/2)|s - \xi(s)| < 5r,$$

$$\operatorname{spt} v_s \subset \mathbf{B}(\xi(s), (3/2)|s - \xi(s)|),$$

$$\operatorname{dist}(s, A) \le \operatorname{dist}(x, A) + |x - s| \le \operatorname{dist}(x, A) + \frac{1}{2} \operatorname{dist}(s, A),$$

$$|s - \xi(s)| = \operatorname{dist}(s, A) \le 2 \operatorname{dist}(x, A),$$

$$\operatorname{dist}(x, A) \le \operatorname{dist}(s, A) + |x - s| \le \frac{3}{2} \operatorname{dist}(s, A) \le 1,$$

$$h(x) \ge \frac{1}{20} \operatorname{dist}(x, A) \ge \frac{1}{40} |s - \xi(s)|.$$

Using the estimates of the preceding paragraph and the estimates of $|Dv_s|$ given in [16, 3.1.13], one infers for $s \in T$, since θ has compact support in $\mathbf{U}(a, r)$,

$$\|(Dv_s)\theta\|_{\infty;a,r} \le 40\Delta |s - \xi(s)|^{-1} r \|D\theta\|_{\infty;a,r},$$

$$\|D(v_s\theta)\|_{\infty;a,r} \le 40\Delta(|s - \xi(s)|^{-1} r + 1) \|D\theta\|_{\infty;a,r}$$

where Δ is a positive, finite number depending only on m with $40\Delta \ge 1$, hence

$$\begin{aligned} |R(v_s\theta)| &\le \gamma (3/2)^{m+1} |s - \xi(s)|^{m+1} 40\Delta(|s - \xi(s)|^{-1} r + 1) \|D\theta\|_{\infty;a,r} \\ &= \gamma (3/2)^{m+1} 40\Delta |s - \xi(s)|^m (r + |s - \xi(s)|) \|D\theta\|_{\infty;a,r} \\ &\le \gamma 160\Delta (3/2)^{m+1} \boldsymbol{\alpha}(m)^{-1} (20)^m r \mathscr{L}^m(\mathbf{B}(s, h(s))) \|D\theta\|_{\infty;a,r}. \end{aligned}$$

Recalling from [16, 3.1.13] that the family $\{\mathbf{B}(s, h(s)) : s \in S\}$ is disjointed, one concludes

$$|R(\theta)| \le \Gamma \gamma r \mathscr{L}^m(\mathbf{B}(a, 4r) \sim A) \|D\theta\|_{\infty;a,r}$$

where $\Gamma = 8(30)^{m+1} \Delta \boldsymbol{\alpha}(m)^{-1}$. □

Remark A.2 Some ideas of the proof were taken from Calderón and Zygmund [11, Theorem 10] and [16, 2.9.17].

Theorem A.3 *Suppose $m, n \in \mathscr{P}$, $m < n$, U is an open subset of $\mathbf{R}^m$, $T \in \mathscr{D}'(U, \mathbf{R}^{n-m})$, and A denotes the set of all $a \in U$ such that*

$$\limsup_{r \to 0+} r^{-1-m} \|T\|_{-1,1;a,r} < \infty.$$

Then A is a Borel set and for $\mathscr{L}^m$ almost all $a \in A$ there exists a unique constant distribution $T_a \in \mathscr{D}'(U, \mathbf{R}^{n-m})$ such that

$$\lim_{r \to 0+} r^{-1-m} \|T - T_a\|_{-1,1;a,r} = 0.$$

Moreover, T_a depends $\mathscr{L}^m \llcorner A$ measurably on a.

Proof The conclusion is local, hence one may assume $\operatorname{spt} T$ to be compact and $U = \mathbf{R}^m$. Since $\|T\|_{-1,1;a,r}$ depends lower semi-continuously on (a, r), the sets

$$A_i = \mathbf{R}^m \cap \{a : \|T\|_{-1,1;a,r} \le i r^{m+1} \text{ for } 0 < r < (10)/i\}$$

defined for $i \in \mathscr{P}$ are closed. Observing $A = \bigcup\{A_i : i \in \mathscr{P}\}$, the conclusion will be shown to hold for $\mathscr{L}^m$ almost all $a \in A_i$.

Let $0 < \varepsilon < 5/i$, choose $\Phi \in \mathscr{D}^0(\mathbf{R}^m)$ with $\int \Phi \, d\mathscr{L}^m = 1$, $\operatorname{spt} \Phi \subset \mathbf{U}(0,1)$ and define $\Phi_\varepsilon(x) = \varepsilon^{-m}\Phi(\varepsilon^{-1}x)$ for $x \in \mathbf{R}^m$,

$$T_\varepsilon(\theta) = T(\Phi_\varepsilon * \theta) = \int f_\varepsilon \bullet \theta \, d\mathscr{L}^m \quad \text{for } \theta \in \mathscr{D}(\mathbf{R}^m, \mathbf{R}^{n-m})$$

with $f_\varepsilon \in \mathscr{E}(\mathbf{R}^m, \mathbf{R}^{n-m})$ given by

$$z \bullet f_\varepsilon(x) = T_y(\Phi_\varepsilon(y-x)z) \quad \text{whenever } x \in \mathbf{R}^m \text{ and } z \in \mathbf{R}^{n-m},$$

see [16, 4.1.2]. Clearly $T_\varepsilon \to T$ as $\varepsilon \to 0+$ and

$$|f_\varepsilon(x)| \leq i2^{m+1} \| D\Phi \|_{\infty;0,1} \quad \text{for } x \in \mathbf{R}^m, a \in A_i \text{ with } |x-a| \leq \varepsilon.$$

One defines a_ε to be the characteristic function of $\mathbf{R}^m \cap \{x : \operatorname{dist}(x, A_i) \leq \varepsilon\}$ and $S_\varepsilon, R_\varepsilon \in \mathscr{D}'(\mathbf{R}^m, \mathbf{R}^{n-m})$ by

$$S_\varepsilon(\theta) = \int a_\varepsilon f_\varepsilon \bullet \theta \, d\mathscr{L}^m \quad \text{for } \theta \in \mathscr{D}(\mathbf{R}^m, \mathbf{R}^{n-m}), \qquad R_\varepsilon = T_\varepsilon - S_\varepsilon.$$

Estimating for $a \in A_i$, $0 < \varrho < 5r < 5/i$, $\theta \in \mathscr{D}(\mathbf{R}^m, \mathbf{R}^{n-m})$ with $\operatorname{spt}\theta \subset \mathbf{U}(a, \varrho)$ and $\| D\theta \|_{\infty;a,\varrho} \leq 1$

$$\begin{gathered}
\operatorname{spt}(\Phi_\varepsilon * \theta) \subset \mathbf{U}(a, \varepsilon + \varrho), \qquad |T_\varepsilon(\theta)| \leq i(\varepsilon+\varrho)^{m+1} \leq i2^{m+1}\varrho^{m+1} \quad \text{if } \varepsilon \leq \varrho, \\
(\operatorname{spt} R_\varepsilon) \cap \{x : \operatorname{dist}(x, A_i) < \varepsilon\} = \emptyset, \qquad R_\varepsilon(\theta) = 0 \quad \text{if } \varepsilon > \varrho, \\
|S_\varepsilon(\theta)| \leq \| a_\varepsilon f_\varepsilon \|_{\infty;a,\varrho} \, \| \theta \|_{1;a,\varrho} \leq i2^{m+1} \| D\Phi \|_{\infty;0,1} \boldsymbol{\alpha}(m) \varrho^{m+1}, \\
\| R_\varepsilon \|_{-1,1;a,\varrho} \leq \gamma \varrho^{m+1} \quad \text{with } \gamma = 2^{m+1} i \big(1 + \| D\Phi \|_{\infty;0,1} \boldsymbol{\alpha}(m)\big),
\end{gathered}$$

Now, A.1 may be applied with A, R replaced by A_i, R_ε to obtain

$$\| R_\varepsilon \|_{-1,1;a,r} \leq \Gamma \gamma r \mathscr{L}^m(\mathbf{B}(a,4r) \sim A_i) \quad \text{for } 0 < r < 1/i.$$

Since $\mathbf{L}_1(\mathscr{L}^m, \mathbf{R}^{n-m})$ is separable, one can use [14, V.4.2, V.5.1, IV.8.3] to infer the existence of $S \in \mathscr{D}'(\mathbf{R}^m, \mathbf{R}^{n-m})$, $f \in \mathbf{L}_\infty(\mathscr{L}^m, \mathbf{R}^{n-m})$, and a sequence ε_j with $\varepsilon_j \downarrow 0$ as $j \to \infty$ such that

$$S(\theta) = \int f \bullet \theta \, d\mathscr{L}^m \quad \text{for } \theta \in \mathscr{D}(\mathbf{R}^m, \mathbf{R}^{n-m}), \qquad S_{\varepsilon_j} \to S \quad \text{as } j \to \infty.$$

Defining $R = T - S$ and noting $R_{\varepsilon_j} \to R$ as $j \to \infty$,

$$\| R \|_{-1,1;a,r} \leq \Gamma \gamma r \mathscr{L}^m(\mathbf{B}(a,4r) \sim A_i) \quad \text{for } 0 < r < 1/i$$

and [16, 2.9.11] implies

$$\lim_{r \to 0+} r^{-1-m} \| R \|_{-1,1;a,r} = 0 \quad \text{for } \mathscr{L}^m \text{ almost all } a \in A_i.$$

Moreover,

$$\left| \int (f(x) - f(a)) \bullet \theta(x) \, \mathrm{d}\mathscr{L}^m x \right| \le \left(\int_{\mathbf{U}(a,r)} |f(x) - f(a)| \, \mathrm{d}\mathscr{L}^m x \right) r |D\theta|_{\infty;a,r}$$

whenever $a \in A$, $0 < r < \infty$, $\theta \in \mathscr{D}(\mathbf{R}^m, \mathbf{R}^{n-m})$ with $\operatorname{spt}\theta \subset \mathbf{U}(a, r)$, and [16, 2.9.9] implies that one can take T_a defined by $T_a(\theta) = \int \theta(x) \bullet f(a) \, \mathrm{d}\mathscr{L}^m x$ for $\theta \in \mathscr{D}(\mathbf{R}^m, \mathbf{R}^{n-m})$ for $\mathscr{L}^m$ almost all $a \in A_i$ in the existence part of the conclusion.

The uniqueness follows from 3.17. □

Remark A.4 The splitting of T into S and R was inspired by a similar procedure for functions used by Calderón and Zygmund in [11, Theorem 7].

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