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Kerr–Schild ansatz in Einstein–Gauss–Bonnet gravity: an exact vacuum solution in five dimensions

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Abstract

As is well known, Kerr–Schild metrics linearize the Einstein tensor. We shall see here that they also simplify the Gauss–Bonnet tensor, which turns out to be only quadratic in the arbitrary Kerr–Schild function f when the seed metric is maximally symmetric. This property allows us to give a simple analytical expression for its trace, when the seed metric is a five-dimensional maximally symmetric spacetime in spheroidal coordinates with arbitrary parameters a and b. We also write in a (fairly) simple form the full Einstein–Gauss–Bonnet tensor (with a cosmological term) when the seed metric is flat and the oblateness parameters are equal, $a = b$. Armed with these results we give in a compact form the solution of the trace of the Einstein–Gauss–Bonnet field equations with a cosmological term and $a \neq b$. We then examine whether this solution for the trace does solve the remaining field equations. We find that it does not in general, unless the Gauss–Bonnet coupling is such that the field equations have a unique maximally symmetric solution.

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1. Introduction

Kerr–Schild metrics [1] are such that there exist coordinate systems x^μ in which the metric coefficients can be written as

$$g_{\mu\nu} = \overline{g}_{\mu\nu} + f h_\mu h_\nu \qquad \text{with} \qquad \overline{g}_{\mu\nu} h^\mu h^\nu = 0 \quad \text{and} \quad h^\mu \overline{D}_\mu h_\rho = 0, \tag{1.1}$$

where $\overline{g}_{\mu\nu}$ are the coefficients of a seed metric in the chosen coordinates x^μ and f is an arbitrary function of the coordinates. The vector $h^\mu = \bar{g}^{\mu\nu} h_\nu$ is null and geodesic. Indices are moved with the seed metric $\overline{g}_{\mu\nu}$ and its inverse $\overline{g}^{\mu\nu}$, and $\overline{D}_\mu$ is its associated covariant derivative.

As is well known, see e.g. [2], the Kerr–Newman black hole solutions of Einstein's equations in four dimensions (with or without a cosmological constant) are of the Kerr–Schild type. In dimensions $D > 4$, the generalization of the Kerr-(A)dS black hole solution is also of the Kerr–Schild type [3]. Note that this is not the case for the black rings [4].

In the Einstein–Gauss–Bonnet (EGB) theory of gravity (see, e.g. [5] for an introduction; for this theory, black holes with horizons of constant curvature have also been found in [6]) the spherically symmetric black hole solution found in [7] which generalizes the Schwarzschild solution is also of the Kerr–Schild type, as we shall see below. However, as already known in the community, despite some claims to the contrary [8], and as we shall see in detail below, the Kerr–Schild ansatz which is used in [3] to obtain the five-dimensional Kerr (AdS) black hole solution of Einstein's equations, does not solve the EGB vacuum field equations. Some numerical results about the existence of five-dimensional rotating black holes with angular momenta of the same magnitude have been presented in [9], and in [10] analytic results have been found up to first order in a single rotation parameter. The problem of finding a rotating black hole solution in the EGB theory of gravity is therefore still open.

In this paper we first study in section 2 the properties of the Gauss–Bonnet tensor when the metric is restricted to be of the Kerr–Schild type. We find that it is quadratic in the arbitrary function f (and not quartic as it could be *a priori*) when the seed metric $\overline{g}_{\mu\nu}$ is maximally symmetric. This property considerably simplifies calculations. Indeed, it allows us to give in section 3 a simple analytical expression for the trace of the Gauss–Bonnet tensor, when the seed metric is a five-dimensional maximally symmetric spacetime in spheroidal coordinates with arbitrary oblateness parameters a and b.

The solution of the trace of the EGB field equations with a cosmological term can hence be given in a compact form, even when the seed metric does not solve the field equation. Furthermore, it is shown in section 4 that when the seed metric is restricted to be flat and when the parameters a and b are equal, the full EGB tensor acquires a (fairly) simple form. Thus we can easily examine in this case whether the solution for the trace obtained in section 3 satisfies the remaining field equations. We find that it does not, unless the Gauss–Bonnet coupling is such that there is a unique maximally symmetric solution of the field equations. In section 5 we generalize this particular solution to the case when the seed metric is no longer flat and the parameters a and b are no longer equal and we comment on a few of its properties (a detailed analysis is left to further work [19]).

2. The Einstein–Gauss–Bonnet tensor for Kerr–Schild metrics

The EGB tensor is [11]:

$$E^\mu_\nu \equiv \Lambda \delta^\mu_\nu + \kappa^{-1} G^\mu_\nu + \alpha H^\mu_\nu, \tag{2.1}$$

with G^μ_ν and H^μ_ν being the Einstein and Gauss–Bonnet tensors, respectively: $G^\mu_\nu \equiv R^\mu_\nu - \frac{1}{2}R\delta^\mu_\nu$, and

$$H^\mu_\nu \equiv 2R^{\mu\alpha}{}_{\beta\gamma}R^{\beta\gamma}{}_{\nu\alpha} - 4R^{\mu\alpha}{}_{\nu\beta}R^\beta_\alpha - 4R^\mu_\alpha R^\alpha_\nu + 2RR^\mu_\nu - \tfrac{1}{2}\delta^\mu_\nu\left(R^{\alpha\beta}{}_{\gamma\delta}R^{\gamma\delta}{}_{\alpha\beta} - 4R^\alpha_\beta R^\beta_\alpha + R^2\right). \tag{2.2}$$

$R^\mu{}_{\nu\rho\sigma} \equiv 2\partial_{[\rho}\Gamma^\mu_{\sigma]\nu} + 2\Gamma^\mu_{\lambda[\rho}\Gamma^\lambda_{\sigma]\nu}$, $R_{\mu\nu} = R^\rho_{\mu\rho\nu}$ and R are the Riemann tensor, Ricci tensor and curvature scalar of the metric $g_{\mu\nu}$. The signature is $(-++\cdots)$. One can rewrite the cosmological constant Λ as

$$\Lambda \equiv -\frac{(D-1)(D-2)}{2\ell^2}\left(\kappa^{-1} - \frac{(D-3)(D-4)\alpha}{\ell^2}\right), \tag{2.3}$$

where D is the dimension of spacetime and where the two roots for ℓ^{-2} are the curvatures of the maximally symmetric solutions of $E^\mu_\nu = 0$. It is worth pointing out that if the Newton constant κ and the Gauss–Bonnet coupling α are related as

$$\kappa = \frac{\ell^2}{2(D-3)(D-4)\alpha} \quad\Longrightarrow\quad \Lambda = -\frac{(D-1)(D-2)(D-3)(D-4)\alpha}{2\ell^4}, \tag{2.4}$$

then the EGB vacuum equations have a unique maximally symmetric vacuum [12], and they admit solutions with a relaxed fall-off as compared with the standard one [13]. This property enlarges the space of allowed solutions, as well as the freedom in the choice of the metric at the boundary [14].

When the metric is of the Kerr–Schild type (1.1) the Ricci tensor R^μ_ν is linear in f, as is well known [1, 3]. As for the Riemann tensors $R^\mu{}_{\nu\rho\sigma}$ and $R^{\mu\nu}{}_{\rho\sigma}$ they are only quadratic in f (see the appendix for their explicit expressions). It also turns out that the contracted products $R^{\mu\alpha}{}_{\beta\gamma}R^{\beta\gamma}{}_{\nu\alpha}$ and $R^{\mu\alpha}{}_{\nu\beta}R^\beta_\alpha$ are also quadratic in f (and not respectively quartic and cubic as they could be *a priori*), at least when the seed metric is maximally symmetric. Hence the result is that the Gauss–Bonnet tensor H^μ_ν is quadratic in f. See the appendix for a justification of these claims.

Let us henceforth restrict our attention to anti-de Sitter seeds with curvature $\mathcal{L}^{-2}$:

$$\overline{R}_{\mu\nu\rho\sigma} = -\frac{1}{\mathcal{L}^2}(\overline{g}_{\mu\rho}\overline{g}_{\nu\sigma} - \overline{g}_{\mu\sigma}\overline{g}_{\nu\rho}). \tag{2.5}$$

(Minkowski spacetime corresponds to $\mathcal{L} \to \infty$ and the de Sitter case is obtained by changing the sign of $\mathcal{L}^2$.) Note that we do not impose the seed metric to solve the field equations, that is, we do not impose $\mathcal{L} = \ell$.

Now, at linear order in f, the Einstein and Gauss–Bonnet tensors are proportional [7] (see also [15]). The remaining, quadratic, part of the Gauss–Bonnet tensor is also easily computed, so that the full EGB tensor reads

$$\begin{aligned} E^\mu_\nu &= \delta^\mu_\nu(\Lambda - \Lambda_{\text{seed}}) + \left(\kappa^{-1} - \frac{2(D-3)(D-4)\alpha}{\mathcal{L}^2}\right)\left[\frac{(D-1)}{\mathcal{L}^2}fh^\mu h_\nu + R^\mu_{(L)\nu} - \frac{1}{2}\delta^\mu_\nu R_{(L)}\right] \\ &\quad + 2\alpha\left(\frac{K}{\mathcal{L}^2}fh^\mu h_\nu + R^{\mu\alpha}_{(L)\beta\gamma}R^{\beta\gamma}_{(L)\nu\alpha} - 2R^{\mu\alpha}_{(L)\nu\beta}R^\beta_{(L)\alpha} - 2R^\mu_{(L)\alpha}R^\alpha_{(L)\nu} + R_{(L)}R^\mu_{(L)\nu}\right) \\ &\quad - \frac{\alpha}{2}\delta^\mu_\nu\left(R^{\alpha\beta}_{(L)\gamma\delta}R^{\gamma\delta}_{(L)\alpha\beta} - 4R^\alpha_{(L)\beta}R^\beta_{(L)\alpha} + R^2_{(L)}\right), \end{aligned} \tag{2.6}$$

with the notation

$$\Lambda_{\text{seed}} \equiv -\frac{(D-1)(D-2)}{2\mathcal{L}^2}\left(\kappa^{-1} - \frac{(D-3)(D-4)\alpha}{\mathcal{L}^2}\right), \tag{2.7}$$

and with the following definitions (see the appendix):

$$R^{\mu\nu}_{(L)\rho\sigma} = \overline{g}^{\nu\alpha}\left(\overline{D}_\rho\Delta^\mu_{\alpha\sigma} - \overline{D}_\sigma\Delta^\mu_{\alpha\rho}\right), \quad R^\mu_{(L)\nu} = \overline{g}^{\mu\sigma}\overline{D}_\rho\Delta^\rho_{\nu\sigma}, \quad R_{(L)} = \overline{D}_\rho[h^\rho\overline{D}_\mu(fh^\mu)] \tag{2.8}$$

with

$$\Delta^{\mu}_{\nu\rho} = \tfrac{1}{2}[\overline{D}_{\nu}(fh^{\mu}h_{\rho}) + \overline{D}_{\rho}(fh^{\mu}h_{\nu}) - \overline{D}^{\mu}(fh_{\nu}h_{\rho})]. \tag{2.9}$$

As for the function K it is given by

$$K \equiv 3(h^{\alpha}\partial_{\alpha}f)\overline{D}_{\beta}h^{\beta} + 2(D-1)f\overline{D}_{\alpha}(h^{\alpha}\overline{D}_{\beta}h^{\beta}) + (4D-7)f\overline{D}_{\alpha}h^{\beta}(\overline{D}_{\beta}h^{\alpha} - \overline{D}^{\alpha}h_{\beta}). \tag{2.10}$$

As the Gauss–Bonnet tensor is quasilinear in the second derivatives [5], all terms quadratic in $\overline{D}_{\lambda}\partial_{\rho}f$ can be ignored, which simplifies the calculations. Moreover, once it has been made clear that the equations of motion are quadratic in f, indices in (2.6) can be moved at will using the seed metric (e.g. $R^{\mu\alpha}_{(L)\beta\gamma}R^{\beta\gamma}_{(L)\nu\alpha}$ can be replaced by $R^{\mu\ \ \beta}_{(L)\alpha\ \gamma}R^{\ \ \alpha\ \gamma}_{(L)\nu\ \beta}$, etc).

3. Solving the trace of the Einstein–Gauss–Bonnet equations in five dimensions

The general D-dimensional (A)dS metric in spheroidal coordinates can be found in [3]. To be specific, the five-dimensional AdS metric in those coordinates $x^{\mu} = (t, r, \theta, \phi, \psi)$ reads

$$\begin{aligned} \mathrm{d}\bar{s}^2 = &-\frac{(1 + r^2/\mathcal{L}^2)\Delta_{\theta}}{\Xi_a\Xi_b}\mathrm{d}t^2 + \frac{r^2\rho^2}{(1 + r^2/\mathcal{L}^2)(r^2 + a^2)(r^2 + b^2)}\mathrm{d}r^2 + \frac{\rho^2}{\Delta_{\theta}}\mathrm{d}\theta^2 \\ &+\frac{r^2 + a^2}{\Xi_a}\sin^2\theta\,\mathrm{d}\phi^2 + \frac{r^2 + b^2}{\Xi_b}\cos^2\theta\,\mathrm{d}\psi^2, \end{aligned} \tag{3.1}$$

where $\Delta_{\theta} \equiv \Xi_a\cos^2\theta + \Xi_b\sin^2\theta$, with

$$\rho^2 \equiv r^2 + a^2\cos^2\theta + b^2\sin^2\theta, \tag{3.2}$$

and where Ξ_a and Ξ_b are related to the parameters a and b by $\Xi_a \equiv 1 - a^2/\mathcal{L}^2$, $\Xi_b \equiv 1 - b^2/\mathcal{L}^2$. As for the null and geodesic vector h_{μ} it is given by

$$h_{\mu}\mathrm{d}x^{\mu} = \frac{\Delta_{\theta}}{\Xi_a\Xi_b}\mathrm{d}t + \frac{r^2\rho^2}{(1 + r^2/\mathcal{L}^2)(r^2 + a^2)(r^2 + b^2)}\,\mathrm{d}r + \frac{a\sin^2\theta}{\Xi_a}\,\mathrm{d}\phi + \frac{b\cos^2\theta}{\Xi_b}\,\mathrm{d}\psi. \tag{3.3}$$

Using the properties listed in the preceding section, and imposing f to depend on r and θ only, the trace of the EGB tensor in these coordinates is easily calculated and turns out to have the following remarkably simple form:

$$E = 5(\Lambda - \Lambda_{\text{seed}}) - \frac{(rQ_t)''}{2r\rho^2} \qquad \text{with} \quad Q_t = \left(\kappa^{-1} - \frac{4\alpha}{\mathcal{L}^2}\right)Q_l + \alpha Q_q \quad \text{and}$$
$$\begin{cases} Q_l = 3\rho^2 f \\ Q_q = 2(4r^2 - \rho^2)\dfrac{f^2}{\rho^2}, \end{cases} \tag{3.4}$$

prime denoting derivation with respect to r.

It is now a simple matter to solve the vacuum EGB equation for the trace, $E = 0$:

$$Q_t = 6m(\theta) + \frac{d(\theta)}{r} + \frac{(\Lambda - \Lambda_{\text{seed}})}{6}r^2(10\rho^2 - 7r^2), \tag{3.5}$$

where $m(\theta)$ and $d(\theta)$ are arbitrary functions of θ. The solutions for f are therefore the roots of the second degree equation

$$2\alpha(4r^2 - \rho^2)\frac{f^2}{\rho^2} + 3\left(\kappa^{-1} - \frac{4\alpha}{\mathcal{L}^2}\right)\rho^2 f = 6m(\theta) + \frac{d(\theta)}{r} + \frac{(\Lambda - \Lambda_{\text{seed}})}{6}r^2(10\rho^2 - 7r^2). \tag{3.6}$$

We have now to check whether this solution for the trace satisfies the remaining field equations. As we shall first see explicitly in the following section, in the particular case when $\mathcal{L} \to \infty$ and $a = b$, the answer is 'no' unless the Gauss–Bonnet coupling is fixed as in equation (2.4).

4. Flat seed metric and equal oblateness parameters

4.1. The EGB tensor in terms of f:

In contrast to the trace, it is a more painstaking task to express the full EGB tensor E^μ_ν in a simple manner. Thus, as a warming up exercise, we shall restrict our attention in this section to the simple case $\mathcal{L} \to \infty$ and $a = b$ (the case $a = -b$ is trivially obtained by a parity transformation). The (flat) seed metric then simply reads

$$d\bar{s}^2 = -dt^2 + \frac{r^2}{r^2+a^2}dr^2 + (r^2+a^2)(d\theta^2 + \sin^2\theta\, d\phi^2 + \cos^2\theta\, d\psi^2), \tag{4.1}$$

and the null and geodesic vector is

$$h_\mu = \left(1, \frac{r^2}{r^2+a^2}0, a\sin^2\theta, a\cos^2\theta\right). \tag{4.2}$$

The trace of the EGB tensor reduces to, see (3.4) and (3.2)

$$E = 5\Lambda - \frac{(rQ_t)''}{2r(r^2+a^2)} \quad \text{with} \quad Q_t = \kappa^{-1}Q_l + \alpha Q_q \quad \text{where} \quad \begin{cases} Q_l = 3(r^2+a^2)f \\ Q_q = \dfrac{2(3r^2-a^2)}{r^2+a^2}f^2, \end{cases} \tag{4.3}$$

and the solution of $E = 0$ is

$$Q_t = 6m + \frac{d}{r} + \frac{\Lambda}{6}r^2(3r^2+10a^2). \tag{4.4}$$

Since Q_t is known in terms of f (see (4.3)), f hence solves the trace equation if it is a root of

$$2\alpha\frac{(3r^2-a^2)}{(r^2+a^2)}f^2 + 3\kappa^{-1}(r^2+a^2)f = 6m + \frac{d}{r} + \frac{\Lambda}{6}r^2(3r^2+10a^2). \tag{4.5}$$

Careful examination then shows that all components of E^μ_ν can be expressed in terms of E^r_r and E^ϕ_ψ as

$$\begin{aligned}
E^t_t &= -\frac{a^2}{3(r^2+a^2)}\left(\frac{a^2+r^2}{r}E^{r\,\prime}_r + \frac{2E^\phi_\psi}{\cos^2\theta}\right) + E^r_r \\
E^t_\phi &= -\frac{a\sin^2\theta}{3}\left(\frac{a^2+r^2}{r}E^{r\,\prime}_r + \frac{2E^\phi_\psi}{\cos^2\theta}\right) \\
E^t_\psi &= -\frac{a\cos^2\theta}{3}\left(\frac{a^2+r^2}{r}E^{r\,\prime}_r + \frac{2E^\phi_\psi}{\cos^2\theta}\right) \\
E^\theta_\theta &= \frac{1}{3}\left(\frac{a^2+r^2}{r}E^{r\,\prime}_r - \frac{E^\phi_\psi}{\cos^2\theta}\right) + E^r_r \\
E^\phi_\phi &= \frac{1}{3}\left(\frac{a^2+r^2}{r}E^{r\,\prime}_r + (2-3\cos^2\theta)\frac{E^\phi_\psi}{\cos^2\theta}\right) + E^r_r \\
E^\psi_\psi &= \frac{1}{3}\left(\frac{a^2+r^2}{r}E^{r\,\prime}_r - (1-3\cos^2\theta)\frac{E^\phi_\psi}{\cos^2\theta}\right) + E^r_r,
\end{aligned} \tag{4.6}$$

all other components being either zero or obtained by raising/lowering indices with the seed metric. As for E^r_r and E^ϕ_ψ they are expressed in terms of the function f or, rather, Q_t and Q_q, see (4.3), as

$$E^r_r = \Lambda + \frac{1}{6r(r^2+a^2)^2}\left[-(3r^2+a^2)Q_t' + 8\alpha a^4\left(\frac{Q_q}{3r^2-a^2}\right)'\right], \tag{4.7}$$

and (an admittedly ugly expression)

$$\frac{E^{\phi}_{\psi}}{\cos^2\theta} = \frac{a^2[(a^2+5r^2)Q_t' - r(r^2+a^2)Q_t'']}{6r^3(r^2+a^2)^2} + \frac{4\alpha a^2(27r^4+42r^2a^2+31a^4)}{(3r^2-a^2)^3(r^2+a^2)^2}Q_q$$
$$- \frac{4\alpha a^2(18r^6+27r^4a^2+16r^2a^4-a^6)}{3r^3(3r^2-a^2)^2(r^2+a^2)^2}Q_q'$$
$$+ \frac{2\alpha a^2(3r^2+2a^2)}{3r^2(3r^2-a^2)(r^2+a^2)}Q_q''. \tag{4.8}$$

4.2. Recovering the Boulware–Deser metric (a = b = 0).

In the case of vanishing oblateness parameters, that is, when the Minkowski seed metric (4.1) is written in standard spherical coordinates, the non-zero components of the five-dimensional EGB tensor (4.6)–(4.8) simplify into

$$E^t_t = E^r_r = -\frac{Q_t'}{2r^3} + \Lambda, \qquad E^{\theta}_{\theta} = E^{\phi}_{\phi} = E^{\psi}_{\psi} = \frac{[r^3 E^r_r]'}{3r^2}. \tag{4.9}$$

Thus, the field equations $E^{\mu}_{\nu} = 0$ are solved by

$$Q_t = 6m + \frac{\Lambda}{2}r^4. \tag{4.10}$$

It then follows from equation (4.3) that the function f is given by

$$f(r) = \frac{r^2}{4\kappa\alpha}\left(-1 \pm \sqrt{1 + \frac{8\kappa^2\alpha}{3r^4}\left(6m + \frac{\Lambda r^4}{2}\right)}\right), \tag{4.11}$$

which is nothing but the EGB solution first found in [7], written here in Kerr–Schild form. (The equivalence of the metrics follows from the generalized Birkhoff theorem [15], and can be explicitly seen from the coordinate transformation, $t = T + \int \frac{f(r)}{1-f(r)}\mathrm{d}r$, which brings it into the Schwarzschild gauge.) The constant of integration m is interpreted as the total mass, see [7, 16–18].

4.3. Switching on the oblateness parameters $(a = b \neq 0)$.

In this case the solutions of the trace equation are the roots of (4.5), that is

$$f(r) = \frac{3(r^2+a^2)^2}{4\kappa\alpha(3r^2-a^2)}\left(-1 \pm \sqrt{1 + \frac{8\kappa^2\alpha(3r^2-a^2)}{9(r^2+a^2)^3}\left(6m + \frac{d}{r} + \frac{\Lambda}{6}r^2(3r^2+10a^2)\right)}\right). \tag{4.12}$$

The Kerr–Schild metric so obtained seems to describe a massive rotating spacetime. However one can show that, with $f(r)$ given by (4.12), the equations of motion $E^{\mu}_{\nu} = 0$ with E^{μ}_{ν} given by (4.6)–(4.8) cannot be satisfied, unless the Gauss–Bonnet coupling is fixed as in equation (2.4).

This can be seen from the behavior of E^r_r when $r \to \infty$, see equation (4.7)

$$E^r_r = \frac{2a^4}{9\kappa^2\alpha r^4}\sqrt{3+4\kappa^2\alpha\Lambda}(\sqrt{3+4\kappa^2\alpha\Lambda} \mp \sqrt{3}) + \frac{d}{2r^5} + \cdots. \tag{4.13}$$

The leading term of equation (4.13) vanishes either for $\Lambda = 0$ taking the upper branch of (4.12), or for $\alpha = -\frac{3}{4\kappa^2\Lambda}$, and the subleading term vanishes for $d = 0$.

In the first case ($\Lambda = 0, d = 0$), the asymptotic behavior of E^r_r becomes

$$E^r_r = -\frac{64a^4\alpha\kappa^2m^2}{r^{12}} + \cdots, \tag{4.14}$$

which means that m also must vanish. Hence $f = 0$ and the solution reduces to the flat seed metric.

In the case now when $\alpha = -\frac{3}{4\kappa^2\Lambda}$ (and $d = 0$), equation (4.7) $E^r_r = 0$ is fulfilled provided $m = \frac{7a^4\Lambda}{36}$, and the solution for f in (4.12) has the remarkably simple form

$$f(r) = -\frac{r^2 + a^2}{\ell^2}, \tag{4.15}$$

where $\ell^{-2} = (4\alpha\kappa)^{-1}$ stands for the (A)dS curvature of the unique maximally symmetric solution of the field equations. It is an exercise to check that E^ϕ_ψ given in (4.8) also vanishes, so that all the components of the EGB equations, $E^\mu_\nu = 0$ with E^μ_ν given by (4.6) are indeed satisfied.

Thus the Kerr–Schild solution with flat seed metric (4.1) and function $f(r)$ given by (4.15) solves the EGB field equations when $\alpha = -\frac{3}{4\kappa^2\Lambda}$ (and hence $\ell^2 = 4\,\alpha\kappa$).

It is worth pointing out that this solution is static, since there is a coordinate transformation that brings it into the Schwarzschild gauge, but where the three-sphere is replaced by a squashed three-sphere, where a parameterizes the squashing [19]. This is not at odds with the Birkhoff theorem [15] since the freedom in the choice of the metric at the boundary is enlarged when the Gauss–Bonnet coupling is fixed as in equation (2.4), [14].

Let us now generalize the previous analysis to the case when the seed metric is no longer flat and b is no longer equal to a.

5. An exact vacuum solution

When the seed metric is no longer flat and b is no longer equal to a, that is when the anti-de Sitter seed metric is given by (3.1), we can no longer give simple expressions for the components of the EGB tensor as we did in the previous section. However we know the general solution for the trace, see (3.6)

$$f(r,\theta) = A(-1 \pm \sqrt{1+B}) \qquad \text{with}$$
$$\begin{cases} A = \dfrac{3(\kappa^{-1} - 4\alpha/\mathcal{L}^2)\rho^4}{4\alpha(4r^2 - \rho^2)}, \qquad B = \dfrac{8\alpha(4r^2 - \rho^2)}{9\rho^6(\kappa^{-1} - 4\alpha/\mathcal{L}^2)^2} Q_t \\ Q_t = \left(6m(\theta) + \dfrac{d(\theta)}{r} + \dfrac{(\Lambda - \Lambda_{\text{seed}})}{6} r^2(10\rho^2 - 7r^2)\right) \end{cases}, \tag{5.1}$$

where we recall that $\rho^2 = r^2 + a^2\cos^2\theta + b^2\sin^2\theta$.

Let us first extend the 'no-go' result of the preceding section: if a and/or b are not zero and if one chooses $\Lambda = \Lambda_{\text{seed}}$, (i.e. the seed metric solves the vacuum field equations) then, due to the properties of the Gauss–Bonnet tensor listed in the appendix, it is possible to show that the solution (5.1) for the trace does not solve the remaining field equations, even in the particular case when $\kappa^{-1} = 4\alpha/\ell^2$ (unless $f = 0$). Therefore the Kerr–Schild ansatz which was used in [3] to obtain the five-dimensional Kerr (AdS) black hole solution of Einstein's equations does not yield a rotating solution of the EGB vacuum field equations.

Let us now generalize the solution obtained in the preceding section. For $\Lambda \neq \Lambda_{\text{seed}}$ and the Gauss–Bonnet coupling fixed as in equation (2.4), the function f is simply given by

$$f(r,\theta) = -\left(\frac{1}{\ell^2} - \frac{1}{\mathcal{L}^2}\right)\rho^2. \tag{5.2}$$

Thus, in five dimensions, the line element

$$ds^2 = d\bar{s}^2 - \left(\frac{1}{\ell^2} - \frac{1}{\mathcal{L}^2}\right)\rho^2(h_\mu\, dx^\mu)^2, \tag{5.3}$$

where $d\bar{s}^2$ is the metric of a seed (anti)-de Sitter spacetime of curvature $\mathcal{L}^{-2}$ in spheroidal coordinates and h_μ is a null and geodesic vector, see equations (3.1) and (3.3), solves the EGB vacuum equations, if the Gauss–Bonnet coupling is fixed in such a way that the field equations have a unique maximally symmetric solution of curvature ℓ^{-2}, see (2.4).

This spacetime is asymptotically locally anti-de Sitter since

$$R^{\alpha\beta}{}_{\gamma\delta} \to -\frac{1}{\ell^2}\left(\delta^\alpha_\gamma\delta^\beta_\delta - \delta^\alpha_\delta\delta^\beta_\gamma\right) \qquad \text{when} \qquad r \to \infty. \tag{5.4}$$

Note that it does not approach the seed metric since $\mathcal{L}$ must be different from ℓ. The solution is parameterized by three constants: $\mathcal{L}^{-2}$, that is, the curvature of the seed (anti)-de Sitter spacetime, and the parameters a and b defining the spheroidal coordinates. Preliminary results show that this solution describes a rotating spacetime [19] if $a \neq b$, but a detailed analysis is left to further work.

6. Conclusions

We considered Kerr–Schild type metrics on maximally symmetric seed spacetimes and showed that the EGB tensor is only quadratic in the Kerr–Schild function f, see equation (2.6). Specializing in a five-dimensional seed metric in spheroidal coordinates we then found a remarkably simple expression for the trace of the Einstein–Gauss–Bonnet tensor, see equation (3.4). Specializing further in a flat seed metric and equal spheroidal parameters we wrote explicitly all the components of the EGB tensor, see equation (4.6)–(4.8). Thanks to those results we were able to show in a transparent manner that the Kerr–Schild ansatz used in [3] to obtain the generalized five-dimensional Kerr solution in Einstein theory does not yield a solution of the EGB vacuum equations, when, as in [3], the (anti)-de Sitter seed metric is chosen so as to solve the field equations. Turning then to Kerr–Schild ansatz whose (anti)-de Sitter seed metric does not solve the field equations, see equation (3.1)–(3.3), we found a new solution given by (5.3) provided the Gauss–Bonnet coupling is fixed as in equation (2.4).

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Appendix A. Some properties of Kerr–Schild metrics

Kerr–Shild metrics read

$$g_{\mu\nu} = \overline{g}_{\mu\nu} + f h_\mu h_\nu \qquad \text{with} \qquad \overline{g}^{\mu\nu} h_\mu h_\nu = 0 \qquad \text{and} \qquad h^\mu \overline{D}_\mu h^\rho = 0.$$

The function f is arbitrary and $h^\nu \equiv \overline{g}^{\mu\nu} h_\mu$. An overlined quantity is built with the seed metric $\overline{g}_{\mu\nu}$.

The Christoffel symbols are given by

$$\Gamma^\mu_{\nu\rho} - \overline{\Gamma}^\mu_{\ \nu\rho} = \Delta^\mu_{\nu\rho} + \delta^\mu_{\nu\rho}$$

with

$$\Delta^\mu_{\nu\rho} = \tfrac{1}{2}[\overline{D}_\nu (f h^\mu h_\rho) + \overline{D}_\rho (f h^\mu h_\nu) - \overline{D}^\mu (f h_\nu h_\rho)] \tag{A.1}$$

and

$$\delta^\mu_{\nu\rho} = \tfrac{1}{2} h^\mu h_\nu h_\rho (f h^\lambda \partial_\lambda f).$$

$\Delta^\mu_{\nu\rho}$ has the following properties:

$$\Delta^\rho_{\nu\rho} = 0, \quad h^\rho \Delta^\mu_{\nu\rho} = \tfrac{1}{2} h^\mu h_\nu (h^\rho \partial_\rho f), \quad h_\mu \Delta^\mu_{\nu\rho} = -\tfrac{1}{2} h_\nu h_\rho (h^\mu \partial_\mu f), \quad \Delta^\alpha_{\nu\beta}\Delta^\beta_{\nu\alpha} \propto h_\mu h_\nu.$$

It follows that the Riemann tensor, which *a priori* could be quartic in f, is in fact only quadratic and reads

$$R^\mu{}_{\nu\rho\sigma} = \overline{R}^\mu{}_{\nu\rho\sigma} + R^\mu_{(lin)\nu\rho\sigma} + R^\mu_{(quad)\nu\rho\sigma}$$

with

$$R^\mu_{(lin)\nu\rho\sigma} = \overline{D}_\rho \Delta^\mu_{\nu\sigma} - \overline{D}_\sigma \Delta^\mu_{\nu\rho}$$

and

$$R^\mu_{(quad)\nu\rho\sigma} = \overline{D}_\rho \delta^\mu_{\nu\sigma} - \overline{D}_\sigma \delta^\mu_{\nu\rho} + \Delta^\mu_{\rho\lambda}\Delta^\lambda_{\nu\sigma} - \Delta^\mu_{\sigma\lambda}\Delta^\lambda_{\nu\rho}.$$

$R^\mu_{(quad)\nu\rho\sigma}$ has the following properties:

$$h^\nu R^\mu_{(quad)\nu\rho\sigma} = 0, \quad h_\mu R^\mu_{(quad)\nu\rho\sigma} = 0, \quad h^\sigma R^\mu_{(quad)\nu\rho\sigma} = -\tfrac{1}{2} h^\mu h_\nu h_\rho (f h^\alpha h^\beta \overline{D}_{\alpha\beta} f).$$

$R^\mu_{(lin)\nu\rho\sigma}$ has the following properties:

$$h^\nu h^\sigma R^\mu_{(lin)\nu\rho\sigma} = -\tfrac{1}{2} h^\mu h_\rho (h^\alpha h^\beta \overline{D}_{\alpha\beta} f), \quad h_\mu h^\sigma R^\mu_{(lin)\nu\rho\sigma} = \tfrac{1}{2} h_\nu h_\rho (h^\alpha h^\beta \overline{D}_{\alpha\beta} f).$$

Another important property of the (contraction) of the Riemann tensor is

$$\overline{g}^{\mu\sigma} R^\lambda_{(quad)\nu\lambda\sigma} = f h^\mu h^\sigma R^\lambda_{(lin)\nu\lambda\sigma}.$$

As it can easily be seen $R^{\mu\nu}{}_{\rho\sigma} \equiv g^{\nu\lambda} R^\mu{}_{\rho\lambda\sigma}$ is also quadratic (and not cubic) in f. More precisely we shall write

$$R^{\mu\nu}{}_{\rho\sigma} = \overline{R}^{\mu\nu}{}_{\rho\sigma} + R^{\mu\nu}_{(lin)\rho\sigma} + R^{\mu\nu}_{(quad)\rho\sigma}$$

with

$$R^{\mu\nu}_{(lin)\rho\sigma} = -f h^\nu h^\alpha \overline{R}^\mu{}_{\alpha\rho\sigma} + R^{\mu\nu}_{(L)\rho\sigma} \quad \text{where} \quad R^{\mu\nu}_{(L)\rho\sigma} = \bar{g}^{\nu\alpha}\left(\overline{D}_\rho \Delta^\mu_{\alpha\sigma} - \overline{D}_\sigma \Delta^\mu_{\alpha\rho}\right) \tag{A.2}$$

and

$$R^{\mu\nu}_{(quad)\rho\sigma} = \overline{g}^{\nu\alpha} R^\mu_{(quad)\alpha\rho\sigma} - f h^\nu h^\alpha R^\mu_{(lin)\alpha\rho\sigma}.$$

$R^{\mu\nu}_{(lin)\rho\sigma}$ and $R^{\mu\nu}_{(quad)\rho\sigma}$ are antisymmetric in their lower and upper two indices.

One then concludes that the Ricci tensor (with indices up-down: $R^{\mu}_{\nu} \equiv g^{\mu\rho} R^{\lambda}{}_{\rho\lambda\nu}$) is linear in f [3] and reads

$$R^{\mu}_{\nu} = \overline{R}^{\mu}_{\nu} + R^{\mu}_{(lin)\nu} \quad \text{with} \quad R^{\mu}_{(lin)\nu} = -f h^{\mu} h^{\sigma} \overline{R}_{\nu\sigma} + R^{\mu}_{(L)\nu} \quad \text{where} \quad R^{\mu}_{(L)\nu} = \overline{g}^{\mu\sigma} \overline{D}_{\rho} \Delta^{\rho}_{\nu\sigma}. \tag{A.3}$$

Finally, the scalar curvature is also linear in f and reads

$$R = \overline{R} + R_{(lin)} \quad \text{with} \quad R_{(lin)} = -f h^{\alpha} h^{\beta} \overline{R}_{\alpha\beta} + R_{(L)} \quad \text{where} \quad R_{(L)} = \overline{D}_{\rho}[h^{\rho} \overline{D}_{\mu}(f h^{\mu})]. \tag{A.4}$$

On can also note for further reference that

$$R_{(L)} = \frac{1}{\sqrt{-g}} \partial_{\rho}[h^{\rho} \partial_{\mu}(\sqrt{-g} f h^{\mu})]$$

and that

$$h_{\alpha} R^{\alpha}_{(L)\mu} = \tfrac{1}{2} f h_{\alpha} h^{\beta} \left(2h^{\gamma} \overline{R}^{\alpha}{}_{\beta\gamma\mu} + \overline{R}^{\alpha}_{\beta} h_{\mu}\right) + \frac{h^{\mu}}{2} \left\{ h^{\alpha} \partial_{\alpha} \left[\frac{1}{\sqrt{-\overline{g}}} \partial_{\beta}(f \sqrt{-\overline{g}} h^{\beta}) \right] - f \overline{D}_{\alpha} h_{\beta} \overline{D}^{\alpha} h^{\beta} \right\}.$$

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