Four-dimensional deformed special relativity from group field theories

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We derive a scalar field theory of the deformed special relativity type, living on noncommutative κ-Minkowski space-time and with a κ-deformed Poincaré symmetry, from the SO(4,1) group field theory defining the transition amplitudes for topological BF theory in 4 space-time dimensions. This is done at a nonperturbative level of the spin foam formalism working directly with the group field theory (GFT). We show that matter fields emerge from the fundamental model as perturbations around a specific phase of the GFT, corresponding to a solution of the fundamental equations of motion, and that the noncommutative field theory governs their effective dynamics.

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I. INTRODUCTION

The progress toward a quantum theory of gravity, in the past 20 years or so, has been substantial. On the theory side, many different approaches, the most notable being probably string theory, have been developed and achieved considerable successes [1].

Group field theories (GFTs) [2,3] are quantum field theories over group manifolds, characterized by a nonlocal pairing of field arguments in the action, which can be seen as a generalization of matrix models [4]. They can be interpreted as a simplicial "third quantization" of gravity [5], in which a discrete space-time emerges as a Feynman diagram of the theory in perturbative expansion. The field arguments assign group-theoretic data to these cellular complexes, and the GFT perturbative expansion in Feynman amplitudes define uniquely and completely a so-called spin foam model [6]. Spin foam models [7], in turn, can be understood as a covariant formulation of the dynamics of loop quantum gravity [8] and as a new algebraic implementation of discrete quantum gravity approaches, such as Regge calculus [9] and dynamical triangulations [10]. This makes GFTs a very useful tool, and suggests that they may provide the fundamental definition of a dynamical theory of spin networks, and be of great help in investigating nonperturbative and collective properties of their quantum dynamics [2,3,11].

In recent years, moreover, the possibility of testing experimentally Planck scale effects using astrophysical or cosmological observations has been investigated to a great extent and led to a whole set of approaches to possible quantum gravity phenomenology [12]. The general idea is that there exist several physical amplifying mechanisms, e.g., in gamma-ray bursts, cosmic rays, or gravitational wave physics, that could bring quantum gravity effects, even if suppressed by (negative) powers of the Planck energy or by (positive) powers of the Planck length, within reach of near future (if not current, e.g., the on-going GLAST experiment) experiments. The most studied effects are that of a breaking (e.g., Einstein-Aether theory) or of a deformation (e.g., deformed special relativity) of fundamental space-time symmetries, like the Lorentz or Poincaré invariance [12]. This last case is implemented in the context of noncommutative models of space-time, with symmetry groups implemented by means of appropriate Hopf algebras [13]. In many of the interesting cases, in particular, those we are concerned with in this work, space-time coordinates, turned into operators, have Lie algebra-type commutation relations, with a corresponding momentum space given instead by a group manifold, following the general principle [13] that noncommutativity in configuration space is related to curvature in momentum space, a sort of "cogravity" [13]. One class of models that have attracted much attention in this context is given by so-called deformed (or doubly) special relativity (DSR) [12,14], based on the idea of introducing a second invariant scale, given by the Planck length (or energy) and assumed to encode quantum gravity effects in a semiclassical and flat space-time, on top of the velocity scale of usual special relativity, while maintaining the relativity principle, and thus a 10-dimensional transformation group relating the observations made by inertial observers. In one particular incarnation of DSR, space-time is noncommutative and its structure is of κ-Minkowski type [15]. This is the space-time we are concerned with here. In fact, it is now clear that these effective models of quantum gravity can in principle be falsified. Unfortunately, we are still lacking any fundamental formulation of quantum gravity that, on top of

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being clearly defined at the Planck scale, can produce unambiguously any of the effective models that have been proposed, thus producing falsifiable predictions.

Very interesting results have been obtained in the 3D context [16–18] where it has been shown that effective models with quantum group symmetries and a noncommutative space-time structure (although different from the DSR one) arise very naturally when considering the coupling of point particles to a spin foam model for 3D quantum gravity, in the Riemannian setting, with the physics of these particles being that of noncommutative field theories on Lie algebra spaces. While no similarly solid links between spin foam models and noncommutative field theories have been discovered in the 4D context, several arguments have been put forward suggesting that these links should exist and that the relevant effective models in 4D should indeed be of the DSR type [19,20].

For reasons that should become apparent in the following, group field theories are a natural framework for establishing such links, and for actually *deriving* effective noncommutative models of quantum gravity from more fundamental (if tentative) descriptions of quantum spacetime. Once more, in 3D this is technically easier to do, and it has been shown recently [21] that one can indeed derive the same effective field theory obtained in [17] directly from GFT model corresponding to the spin foam model on which that earlier work was based. The procedure used, moreover, appears not to depend too much in the details of the 3D model considered, but only on general properties of the GFT formalism.

What we do in this paper is to apply the same procedure to the more technically challenging case of four space-time dimensions, and Lorentzian signature, and derive from a group field theory model related to 4-dimensional quantum gravity an effective noncommutative field theory of the DSR type and living on κ-Minkowski space-time.

As said, not only is this the first example of a derivation of a DSR model for matter from a more fundamental quantum gravity model, and one further example of the link between noncommutative geometry and quantum gravity formulated in terms of spin foam/loop quantum gravity ideas, but it is of great interest from the point of view of quantum gravity phenomenology. It is also interesting, more generally, as another possible way of bridging the gap between quantum gravity at the Planck scale and effective physics at low energies and macroscopic distances. For a possible interpretative framework of our results relating them to the issue of the continuum approximation of group field theory, and for a connection with the analogue gravity models, we refer to [22].

II. 4D GROUP FIELD THEORY AND PERTURBATIONS

We present in this section the generalization of the 3D framework of [21] to the 4-dimensional case. Given the GFT action given by Ooguri [23], which is related to the quantization of the BF theory in four dimension, we identify a specific type of fluctuations of the group field ϕ, around some classical solution, as matter degrees of freedom propagating on some effective flat noncommutative background.

We show here the general form of the class of solutions we deal with, the type of perturbations we study which lead to the emergent matter fields, and the general form of the effective actions that result from the expansion. We will see that this part of the construction, which works for any group G, is straightforward. The real task, which we tackle in the rest of the paper, will be to identify the specific classical solutions and perturbations whose effective actions are defined on the specific momentum space characterizing DSR theories (i.e., the group AN_3) and possess the right kinetic term, (i.e., the one characterized by the appropriate symmetries in DSR).

Let us consider the 4D GFT related to topological BF quantum field theories [23], i.e., whose Feynman expansion leads to amplitudes that can be interpreted as discrete BF path integrals, for a compact semisimple gauge group $\mathcal{G}$. This is given by the following action:

$$\begin{aligned} S_{4d} = \frac{1}{2}\int [dg]^4 \phi(g_1, g_2, g_3, g_4)\phi(g_4, g_3, g_2, g_1) \\ - \frac{\lambda}{5!}\int [dg]^{10} \phi(g_1, g_2, g_3, g_4)\phi(g_4, g_5, g_6, g_7) \\ \times \phi(g_7, g_3, g_8, g_9)\phi(g_9, g_6, g_2, g_{10}) \\ \times \phi(g_{10}, g_8, g_5, g_1), \end{aligned} \tag{1}$$

where the field is required to be gauge invariant, $\phi(g_1, g_2, g_3, g_4) = \phi(g_1 g, g_2 g, g_3 g, g_4 g)$ for all group elements $g \in \mathcal{G}$. The relevant groups for 4D quantum gravity are $\mathcal{G} = \mathrm{Spin}(4)$ [and SO(5)] in the Riemannian case and $\mathcal{G} = \mathrm{SL}(2, \mathbb{C})$ [and SO(4,1)] in the Lorentzian case. In this section, we focus on the compact group case. We will deal with the noncompact group case relevant to Lorentzian gravity in the next section. It will require proper and careful regularization to avoid divergencies due to the noncompact nature of the group.

We generalize the "flat solution" ansatz of the 3D group field theory to the 4-dimensional case [21]:

$$\phi^{(0)}(g_i) \equiv \sqrt[3]{\frac{4!}{\lambda}} \int dg \delta(g_1 g) F(g_2 g) \tilde{F}(g_3 g) \delta(g_4 g). \tag{2}$$

It is straightforward to check that this provides a solution to the classical equations of motion as soon as $(\int F\tilde{F})^3 = 1$. We let aside for a moment this normalization condition, and we compute the effective action for 2-dimensional variations around such background configurations for arbitrary functions F and $\tilde{F}$:

$$S_{\rm eff}[\psi] \equiv S_{4d}[\phi^{(0)} + \psi(g_1 g_4^{-1})] - S_{4d}[\phi^{(0)}].$$

We obtain an effective action with a linear term proportional to $\psi(\mathbb{I})$, a nontrivial quadratic kinetic term, and interaction vertices of order 3 to 5:

$$S_{\rm eff}[\psi] = \sqrt[3]{\frac{4!}{\lambda}}\psi(\mathbb{I})\int F\int\tilde{F}\Big[1-\Big(\int F\tilde{F}\Big)^3\Big] + \frac{1}{2}\int\psi(g)\psi(g^{-1})\mathcal{K}(g) - \sqrt[3]{\frac{\lambda}{4!}}\int F\int\tilde{F}\int\psi(g_1)\ldots\psi(g_3)\delta(g_1\ldots g_3)$$
$$\times\Big[\int F\int\tilde{F} + \int dhF(hg_3)\tilde{F}(h)\Big] - \Big(\sqrt[3]{\frac{\lambda}{4!}}\Big)^2\int F\int\tilde{F}\int\psi(g_1)\ldots\psi(g_4)\delta(g_1\ldots g_4)$$
$$-\frac{\lambda}{5!}\int\psi(g_1)\ldots\psi(g_5)\delta(g_1\ldots g_5), \tag{3}$$

with the new kinetic operator given by:

$$\mathcal{K}(g) = \Big[1 - 2\Big(\int F\int\tilde{F}\Big)^2\int F\tilde{F} - 2\int F\int\tilde{F}\int dhF(hg)\tilde{F}(h) \times\int dhF(h)\tilde{F}(hg)\Big]. \tag{4}$$

Taking into account the normalization condition $(\int F\tilde{F})^3 = 1$ and thus working with an exact solution ϕ_0 of the equations of motion, we see that the linear term vanishes exactly due to this condition. We also notice that, if we were to relax this normalization condition and work with a "partial solution requirement" as in the 3D case, the linear term could still be made to vanish and with the same condition $\int F = 0$ (or with $\int\tilde{F} = 0$). However, in this 4D case, this other condition also makes all new terms (among which the nontrivial kinetic term) vanish. Another possibility could be to renormalize the coupling constant λ by reabsorbing in it the factors $\int F\int\tilde{F}$, and then impose the same condition of vanishing integral in some limiting procedure. The interest and consequences of doing this, however, are not clear at the present stage.

At any rate, we obtain an effective field theory for the field ψ defined on two copies of the initial group manifold, but reduced by means of the symmetry requirement to a function of a single group element, with a nontrivial quadratic propagator. The group $\mathcal{G}$ is now interpreted again as the momentum space for the quanta corresponding to this field, with the $\delta(g_1\ldots g_n)$ factors in the action imposing momentum conservation in the field interactions. And again, after introducing a suitable Fourier transform, such effective group field theory appears as the dual of a noncommutative field theory. This same duality implies that position space field theory is defined in terms of functions on $\mathbb{R}^d$, with d the dimension of the group G, endowed with a suitable star product structure, or, equivalently, by elements of the enveloping algebra for the Lie algebra of the same group G, i.e., noncommutative fields living on a noncommutative space-time given by the same Lie algebra. The noncommutativity reflects the curvature of the group manifold and the non-Abelian group multiplication leads to a deformation of the addition of momenta. We will show how this works in detail in the next section for the noncompact group $G = \mathrm{SO}(4,1)$ and for a group field theory more closely related to 4D quantum gravity.

We conclude this section by considering the special case when the function $\tilde{F}$ is fixed to be the δ distribution while F is kept arbitrary as long as $F(\mathbb{I}) = 1$. This ansatz clearly satisfies the normalization condition $\int F\tilde{F} = 1$ and thus provides a solution to the classical field equations. Calling $c \equiv \int F$, the effective action takes has a simpler expression:

$$S_{\rm eff}[\psi] = \frac{1}{2}\int\psi(g)\psi(g^{-1})[1-2c^2-2cF(g)F(g^{-1})] - c\Big(\sqrt[3]{\frac{\lambda}{4!}}\Big)\int\psi(g_1)\ldots\psi(g_3)\delta(g_1\ldots g_3)[c+F(g_3)]$$
$$- c\Big(\sqrt[3]{\frac{\lambda}{4!}}\Big)^2\int\psi(g_1)\ldots\psi(g_4)\delta(g_1\ldots g_4) - \frac{\lambda}{5!}\int\psi(g_1)\ldots\psi(g_5)\delta(g_1\ldots g_5). \tag{5}$$

III. DEFORMED SPECIAL RELATIVITY AS A GROUP FIELD THEORY

The term DSR has been used to describe many different theories. Here we are interested in the original construction which described a noncommutative space-time, of the Lie algebra type (κ Minkowski) together with some deformed Poincaré symmetries. In particular the latter are consistent with the existence of another universal scale (the Planck mass/momentum) than the speed of light.

When dealing with such theory, the literature has often emphasized its noncommutative geometry aspect. Moreover it is also known since some time [13] that a Fourier transform from a noncommutative space-time of the Lie algebra type leads to a (curved) momentum space with a (non-Abelian) group structure. From this perspective, it is clear that a scalar field theory over κ Minkowski can also be interpreted as a group field theory, where the group is the momentum space (contrary to the usual GFT approach for quantum gravity models where the group is usually considered as the configuration space). This aspect of DSR was certainly known but never exploited before from the group field theory perspective. In fact having this in mind will allow us to derive a DSR scalar field theory from a group field theory describing the BF quantum amplitudes in the next section.

Before doing so, we recall the definition of the κ-Minkowski space and its associated momentum space, the AN group. The construction can be done in any dimension. This means that we can also obtain, in principle, an effective field theory on κ-Minkowski space-time in any dimension from a group field theory, using our procedure. However, we focus on the 4D case which is directly relevant for quantum gravity. We then review the construction of scalar field theory on κ Minkowski, emphasizing the group field theory aspect.

A. κ-Minkowski and the AN momentum space

As a vector space, the κ-Minkowski space-time is isomorphic to $\mathbb{R}^n$ and is defined as the Lie algebra $\mathfrak{an}_{n-2}$, which is a subalgebra of the Lorentz algebra $\mathfrak{so}(n-1,1)$. In the following, we work with the signature $(-,+,\ldots,+)$. The $n-1$ generators of $\mathfrak{an}_{n-2}$ are given by

$$X_0 = \frac{1}{\kappa}J_{n0}, \qquad X_k = \frac{1}{\kappa}(J_{nk} + J_{0k}), \qquad k = 1,\ldots,n-2, \tag{6}$$

where the $J_{\mu\nu}$ are the generators of the Lorentz algebra $\mathfrak{so}(n-1,1)$. It is easy to see that $\mathfrak{an}_{n-2}$ is therefore encoded by the following commutation relations:

$$[X_0, X_k] = -\frac{i}{\kappa}X_k, \qquad [X_k, X_l] = 0, \qquad k,l = 1,\ldots,n. \tag{7}$$

Their explicit matrix elements in the fundamental (n-dimensional) representation of $\mathfrak{so}(n-1,1)$ are [24]

$$X_0 = \frac{i}{\kappa}\begin{pmatrix} 0 & \mathbf{0} & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 1 & \mathbf{0} & 0 \end{pmatrix}, \qquad X_k = \frac{i}{\kappa}\begin{pmatrix} 0 & {}^t\mathbf{x} & 0 \\ \mathbf{x} & \mathbf{0} & \mathbf{x} \\ 0 & -{}^t\mathbf{x} & 0 \end{pmatrix}, \tag{8}$$

where ${}^t\mathbf{x}$ are the $(n-2)$-dimensional basis vectors (1,0,. . .,0), (0,1,0,. . .), and so on. For explicit calculations, it is convenient to notice that the matrices X_k are nilpotent with $(X_k)^3 = 0$. There are indeed $n-2$ Abelian and nilpotent generators, hence the name AN_{n-2}. The corresponding exponentiated group elements are

$$e^{ik_0X_0} = \begin{pmatrix} \cosh\frac{k_0}{\kappa} & \mathbf{0} & -\sinh\frac{k_0}{\kappa} \\ \mathbf{0} & \mathbb{1} & \mathbf{0}, \\ -\sinh\frac{k_0}{\kappa} & \mathbf{0} & \cosh\frac{k_0}{\kappa} \end{pmatrix}$$
$$e^{ik_iX_i} = \begin{pmatrix} 1+\frac{\mathbf{k}^2}{2\kappa^2} & -\frac{{}^t\mathbf{k}}{\kappa} & \frac{\mathbf{k}^2}{2\kappa^2} \\ -\frac{\mathbf{k}}{\kappa} & \mathbb{1} & -\frac{\mathbf{k}}{\kappa} \\ -\frac{\mathbf{k}^2}{2\kappa^2} & \frac{{}^t\mathbf{k}}{\kappa} & 1-\frac{\mathbf{k}^2}{2\kappa^2} \end{pmatrix}, \tag{9}$$

where $\mathbb{1}$ is the $(n-2)\times(n-2)$ identity matrix. We parametrize generic AN_{n-2} group elements as

$$h(k_\mu) = h(k_0, k_i) \equiv e^{ik_0X_0}e^{ik_iX_i}. \tag{10}$$

As we will see in the next subsection, this group element can be interpreted as the noncommutative plane wave and the coordinates on the group k_μ as the wave vector (and therefore related to the momentum). To multiply group elements in this parametrization, we check that

$$e^{ik_0X_0}e^{ik_iX_i} = e^{i(e^{k_0/\kappa})k_iX_i}e^{ik_0X_0}.$$

This is the exponentiated version of the commutation relation between X_0 and the X_i's. This allows one to derive the multiplication law for AN_{n-2} group elements:

$$h(k_0,k_i)h(q_0,q_i) = h(k_0+q_0, e^{-q_0/\kappa}k_i + q_i), \tag{11}$$

which defines a deformed noncommutative addition of the wave vectors:

$$(k\oplus q)_0 \equiv k_0 + q_0, \qquad (k\oplus q)_i \equiv e^{-q_0/\kappa}k_i + q_i. \tag{12}$$

This also gives the inverse group elements

$$h(k_0,k_i)^{-1} = h(-k_0, -e^{k_0/\kappa}k_i), \tag{13}$$

which defines the opposite momentum $S(k_\mu)$ for the noncommutative addition

$$S(k_0) = -k_0, \qquad S(k_i) = -e^{k_0/\kappa}k_i. \tag{14}$$

The relation between the $\mathrm{SO}(n-1,1)$ group and AN_{n-2} is given by the Iwasawa decomposition (see, e.g., [24,25]):

$$\mathrm{SO}(n-1,1) = \mathrm{AN}_{n-2}\mathrm{SO}(n-2,1) \cup \mathrm{AN}_{n-2}\mathcal{M}\mathrm{SO}(n-2,1), \tag{15}$$

where the two sets are disjoint and $\mathcal{M}$ is the following diagonal matrix,

$$\mathcal{M} = \begin{pmatrix} -1 & & \\ & \mathbb{1} & \\ & & -1 \end{pmatrix}.$$

To understand the geometric meaning of this decomposition, we look at the map between AN_{n-2} and the de Sitter space-time dS_{n-1} defined as the coset $\mathrm{SO}(n-1,1)/\mathrm{SO}(n-2,1)$. We introduce a reference spacelike vector $v^{(0)} \equiv (0,\ldots,0,1) \in \mathbb{R}^n$. The little group of this vector is the Lorentz group $\mathrm{SO}(n-2,1)$ and the action of $\mathrm{SO}(n-1,1)$ on it sweeps the whole de Sitter space.

Looking at the action of AN_{n-2} on $v^{(0)}$, using the plane wave parametrization (10), we define the vector $v \equiv h(k_\mu).v^{(0)}$ with explicit coordinates:

$$v_0 = -\sinh\frac{k_0}{\kappa} + \frac{\mathbf{k}^2}{2\kappa^2}e^{k_0/\kappa} \qquad v_i = -\frac{k_i}{\kappa}$$
$$v_n = \cosh\frac{k_0}{\kappa} - \frac{\mathbf{k}^2}{2\kappa^2}e^{k_0/\kappa}. \tag{16}$$

We easily check that $v_A v^A = -v_0^2 + \vec{v}^2 + v_n^2 = 1$. However, since $v_0 + v_n = \exp(-k_0/\kappa)$, this action of AN_{n-2} on $v^{(0)}$ sweeps only half of the de Sitter space defined by the condition $v_+ = v_0 + v_n > 0$. Assuming this condition, we can reverse the previous relation and express the AN_{n-2} group element in terms of the n vector v:

$$h(k_\mu) = \begin{pmatrix} v_n + \frac{\mathbf{v}^2}{v_0+v_n} & \frac{{}^t\mathbf{v}}{v_0+v_n} & v_0 \\ \mathbf{v} & \mathbb{1} & \mathbf{v} \\ v_0 - \frac{\mathbf{v}^2}{v_0+v_n} & \frac{-{}^t\mathbf{v}}{v_0+v_n} & v_n \end{pmatrix}, \quad \text{with} \quad h(k_\mu)^{-1} = \begin{pmatrix} v_n + \frac{\mathbf{v}^2}{v_0+v_n} & -{}^t\mathbf{v} & -v_0 + \frac{\mathbf{v}^2}{v_0+v_n} \\ \frac{-\mathbf{v}}{v_0+v_n} & \mathbb{1} & \frac{-\mathbf{v}}{v_0+v_n} \\ -v_0 & {}^t\mathbf{v} & v_n \end{pmatrix}. \tag{17}$$

To recover the full de Sitter space, we need to use the other part of the Iwasawa decomposition. Considering the action of $\mathcal{M}$, we obtain

$$h(k_\mu)\mathcal{M}.v^{(0)} = -v$$
$$h(k_\mu) = \begin{pmatrix} -v_n - \frac{\mathbf{v}^2}{v_0+v_n} & \frac{{}^t\mathbf{v}}{v_0+v_n} & -v_0 \\ -\mathbf{v} & \mathbb{1} & -\mathbf{v} \\ -v_0 + \frac{\mathbf{v}^2}{v_0+v_n} & \frac{-{}^t\mathbf{v}}{v_0+v_n} & -v_n \end{pmatrix}. \tag{18}$$

Thus the action of the $\mathcal{M}$ operator simply maps the n vector v_A in its opposite $-v_A$. Clearly that allows one to complete the other side of de Sitter space with $v_+ < 0$. Let us point out that the left action $\mathcal{M}h(k_\mu)$ would still map $v_0 \to -v_0$ and $v_n \to -v_N$ but would leave the other components invariant $\mathbf{v} \to \mathbf{v}$.

To summarize, an arbitrary point v on the de Sitter space-time is uniquely obtained as

$$v = (-)^\epsilon h(k_\mu).v^{(0)} = h(k_\mu)\mathcal{M}^\epsilon.v^{(0)},$$
$$\epsilon = 0 \quad \text{or } 1, \qquad h \in AN_{n-2}. \tag{19}$$

The sign $(-)^\epsilon$ corresponds to the two components of the Iwasawa decomposition. The coset space $SO(n-1,1)/SO(n-2,1)$ is isomorphic to the de Sitter space and is covered by two patches, each of these patches being isomorphic to the group AN_{n-2}.

We introduce the set $AN^c_{n-2} \equiv AN_{n-2} \cup AN_{n-2}\mathcal{M}$, such that the Iwasawa decomposition reads $SO(n-1,1) = AN^c SO(n-2,1)$ and that AN^c is isomorphic to the full de Sitter space (without any restriction on the sign of v_+). Actually, AN^c_{n-2} is itself a group. Indeed we first easily check the commutation relation between the $\mathcal{M}$ operator and AN_{n-2} group elements:

$$\mathcal{M}\, h(k_\mu) = h(k_0, -k_i)\mathcal{M},$$

where commuting $\mathcal{M}$ with h sends the 5-vector v_A to $(v_0, -\mathbf{v}, v_n)$. This implies the group multiplication on AN^c:

$$h(k_\mu)\mathcal{M}^\alpha h(q_\mu)\mathcal{M}^\beta = h(k \oplus (-)^\alpha q)\mathcal{M}^{\alpha+\beta}, \tag{20}$$

with $\alpha, \beta = 0, 1$. Finally, we point out that AN^c_{n-2} is a group but not a Lie group (because of the discrete $\mathbb{Z}_2$ component).

In the following we will focus on the $n = 5$ case looking at SO(4,1) and its subgroup AN_3 relevant for 4D deformed special relativity and quantum gravity. Consider the action of the Lorentz transformations SO(3,1) on AN_3. This is not simple when seen from the 4D perspective, i.e., from the point of view of AN_3 itself. However, it amounts to the obvious linear action of SO(3,1) on the de Sitter space-time dS_4, $\Lambda \triangleright v = \Lambda.v$, leaving the fifth component v_4 invariant. This leads to a nonlinear action of $\Lambda \in SO(3,1)$ on AN_3 (see, e.g., [25]):

$$\Lambda \triangleright h(k_\mu)\mathcal{M}^\epsilon \equiv \Lambda h(k_\mu)\mathcal{M}^\epsilon \tilde{\Lambda}^{-1} = h(k'_\mu)\mathcal{M}^{\epsilon'}, \tag{21}$$

where $\tilde{\Lambda}$, *a priori* different from Λ, is the unique Lorentz transformation ensuring that the resulting group element lives in $AN^c_3 \equiv AN_3 \cup AN_3\mathcal{M}$. An important point is that it is impossible to neglect the effect of $\mathcal{M}$. Indeed the Lorentz transformation mixes the two parts of the Iwasawa decomposition: the subgroup AN_3 is not invariant under the SO(3,1) action but the group AN^c_3 is.

It is possible to compute the "counter-boost" $\tilde{\Lambda}$ for infinitesimal Lorentz transformations [25]. This leads to the κ-Poincaré algebra presented as a nonlinear realization of the Poicaré algebra in terms of k_μ:

$$[M_i, k_j] = \epsilon^l_{ij}k_l, \qquad [M_i, k_0] = 0, \qquad [k_\mu, k_\nu] = 0$$
$$[N_i, k_j] = \delta_{ij}\left(\sinh\frac{k_0}{\kappa} - \frac{\mathbf{k}^2}{2\kappa^2}e^{k_0/\kappa}\right), \tag{22}$$
$$[N_i, k_0] = k_i e^{k_0/\kappa}.$$

Finally, we will need an integration measure on AN_3 in order to define a Fourier transform. The group AN_3 is provided with two invariant Haar measures:

$$\int dh_L = \int d^4k_\mu, \qquad \int dh_R = \int e^{+3k_0/\kappa} d^4k_\mu, \quad (23)$$

which are, respectively, invariant under the left and right action of the group AN_3. Let us point out that

$$\int d(h^{-1})_L = \int dh_R.$$

We can easily derive this measure from the 5D perspective using the parametrization (17)

$$\kappa^4 \int \delta(v_A v^A - 1)\theta(v_0 + v_4) d^5 v_A = \int d^4 k_\mu = \int dh_L, \quad (24)$$

where the $\theta(v_+)$ function imposes the $v+>0$ restriction. Indeed the SO(4,1) action on the reference vector $v^{(0)}$ generates the whole de Sitter space,

$$v = g \triangleright v^{(0)} = h\mathcal{M}^\epsilon \Lambda \triangleright v^{(0)} = h\mathcal{M}^\epsilon \triangleright v^{(0)}.$$

Therefore the natural measure on AN_3 inherited from the Haar measure on SO(4,1) is left invariant.

A crucial issue is the Lorentz invariance of the measure. Even though the measure $dh_L = d^4k_\mu$ looks Lorentz invariant, it is not, as the action of the Lorentz group on the coordinates k_μ is nontrivial and nonlinear. Actually, one can show this action does not leave the measure invariant. What causes the problem is the restriction $v_+ > 0$ (needed when inducing the measure on AN_3 from the Lorentz invariant measure on dS_4) which indeed breaks Lorentz invariance. In order to get a Lorentz invariant measure, we need to glue back the two patches $v_+ < 0$ and $v_+ > 0$ (and actually also the $v_+ = 0$ patch) and define the measure on the whole de Sitter space. In other words, we write the same measure as a measure on $AN_3^c \equiv AN_3 \cup AN_3\mathcal{M} \sim dS$:

$$\int dh_L \equiv \int_{AN_3} dh_L^+ + \int_{AN_3\mathcal{M}} dh_L^- = \int \delta(v_A v^A - 1) d^5 v. \quad (25)$$

Another way to circumvent this problem and obtain a Lorentz invariant measure is to consider a space without boundary and work on the so-called elliptic de Sitter space[1] $dS/\mathbb{Z}_2$ where we identify $v_A \leftrightarrow -v_A$, which amounts to identifying the group elements $h(k_\mu) \leftrightarrow h(k_\mu)\mathcal{M}$. This space is indeed isomorphic to AN_3 as a manifold. One way to nicely achieve this restriction at the field theory level is to consider only fields on de Sitter space (or on AN_3^c) which are invariant under the parity transformation $v_A \leftrightarrow -v_A$ [28]. In this case, we recover the measure d^4k_μ on $AN_3 \sim AN_3^c/\mathbb{Z}_2 \sim dS/\mathbb{Z}_2$.

[1]Considering deformed special relativity in three dimensions with Euclidean signature, the group field theory on SU(2) has a similar feature [26]. SU(2) being isomorphic to the sphere S^3 is indeed also covered by two patches. Note however than in this case the standard choice of coordinates is not breaking the Lorentz symmetries. To get rid of one patch, we identify the two patches and consider instead $SO(3) = SU(2)/\mathbb{Z}_2$ as in [18,27].

B. DSR field theory (in a nutshell)

We now present a DSR scalar field theory first as a group field theory. Then we recall how we can recover the scalar field theory on κ Minkowski using a generalized Fourier transform. For simplicity, we shall restrict to the case $n = 5$, so that we shall consider the noncompact and nonsemisimple groups $G = AN_3^c, AN_3$.

We consider the real scalar field ϕ: $G \to \mathbb{R}$, and define the (free) action

$$\mathcal{S}(\phi) = \int dh_L \phi(h)\mathcal{K}(h)\phi(h), \qquad \forall h \in G, \quad (26)$$

where $\mathcal{K}(h)$ is the propagator and dh_L is the left-invariant measure. Contrary to the usual group field theory philosophy, we interpret G as the momentum space.

First let us discuss the possible choices of propagators. We demand $\mathcal{K}(h)$ to be a function on G invariant under the Lorentz transformations. We have showed in the previous subsection how the Lorentz group is acting on G. It is then clear that any function $\mathcal{K}(h) = f(v_4(h))$ is a good candidate, since v_4 is by construction a Lorentz invariant quantity. Two main choices have been studied in the literature:

$$\mathcal{K}_1(h) = (\kappa^2 - \pi_4(h)) - m^2,$$
$$\mathcal{K}_2(h) = \kappa^2 - (\pi_4(h))^2 - m^2, \qquad \pi_4 = \kappa v_4. \quad (27)$$

The freedom in choosing the propagator is related to the ambiguity in choosing what we call momentum. To have a precise candidate for the notion of momentum, one needs to define first position and define momentum either as the eigenvalue of the translation operator applied to the plane wave and/or the conserved charged for the action $\mathcal{S}(\phi)$ expressed in terms of coordinates associated to the translations [28,29]. Therefore from the group field theory perspective, it is necessary to perform a Fourier transform to obtain more information.

Before introducing the Fourier transform, let us note that the action (26) is clearly Lorentz invariant if the measure is Lorentz invariant, since the propagator $\mathcal{K}(h)$ has been chosen to be a Lorentz invariant function and the transformation of the fields induced by a Lorentz transformation on the arguments h is also known, from the previous subsection. We have also seen that this measure is indeed a Lorentz invariant measure both in the case of group manifold AN_3^c and generic scalar fields, and in the case of elliptic de Sitter space or AN_3 when a restriction to symmetric fields is imposed.

The generalized Fourier transform relates functions on the group $\mathcal{C}(G)$ and elements of the enveloping algebra $\mathcal{U}(\mathfrak{an}_3)$. It is defined, respectively, for $G = AN_3^c, AN_3$ as

$$\hat{\phi}(X) = \int_{AN_3} dh_L^+ h(k_\mu)\phi^+(k) + \int_{AN_3\mathcal{M}} dh_L^- h(k_\mu)\phi^-(k), \quad X \in \mathfrak{an}_3, \quad \hat{\phi}(X) \in \mathcal{U}(\mathfrak{an}_3) \tag{28}$$

$$\hat{\phi}(X) = \int_{AN_3} dh_L h(k_\mu)\phi(k), \quad X \in \mathfrak{an}_3, \quad \hat{\phi}(X) \in \mathcal{U}(\mathfrak{an}_3) \tag{29}$$

where we used the non-Abelian plane wave $h(k_\mu)$. The inverse Fourier transform can also be introduced, if one introduces a measure d^4X. For the details we refer to [28,30]. The group field theory action on G can now be rewritten as a noncommutative field theory on κ Minkowski (to simplify the notation we restrict our attention to $G = AN_3$ and thus we implicitly consider symmetric fields)

$$\mathcal{S}(\phi) = \int dh_L \phi(h)\mathcal{K}(h)\phi(h) = \int d^4X(\partial_\mu \hat{\phi}(X)\partial^\mu \hat{\phi}(X) + m^2\hat{\phi}^2(X)). \tag{30}$$

The Poincaré symmetries are naturally deformed in order to be consistent with the nontrivial commutation relations of the κ-Minkowski coordinates. More exactly, if the Poincaré transformations act in the standard on the coordinates[2]

$$T_\mu \triangleright X_\nu = \delta_{\mu\nu}, \qquad N_i \triangleright X_j = \delta_{ij}X_0, \qquad N_i \triangleright X_0 = X_i, \qquad R_i \triangleright X_j = \epsilon_{ij}^k X_k, \qquad R_i \triangleright X_0 = 0, \tag{31}$$

its action on the product of coordinates has to be modified in order to be consistent with the nontrivial commutation relation (7), that is we demand that

$$\mathcal{T} \triangleright [X_\mu, X_\nu] = C_{\mu\nu}^\alpha \mathcal{T} \triangleright X_\alpha, \qquad \forall \mathcal{T} = T_\mu, R_i, N_i,$$

and $C_{\mu\nu}^\alpha$ is the structure constant of $\mathfrak{an}_3$. To implement this one needs to deform the coalgebra structure of the Poincaré algebra, that is one deforms the coproduct[3] Δ

$$\Delta T_\mu = T_\mu \otimes \mathbb{1} + \mathbb{1} \otimes T_\mu - \kappa^{-1}T_0 \otimes T_\mu$$
$$\Delta N_i = N_i \otimes \mathbb{1} + \mathbb{1} \otimes N_i - \kappa^{-1}T_0 \otimes N_i + \kappa^{-1}\epsilon_i^{jk}T_k \otimes R_j$$
$$\Delta R_i = R_i \otimes \mathbb{1} + \mathbb{1} \otimes R_i. \tag{33}$$

Thanks to this new coproduct, the Poincaré transformations and the commutation relations (7) are consistent, i.e., (32) is true. Moreover, using the coproduct, we can act on the plane wave and deduce the realization of the Poincaré transformations in terms of the coordinates k_μ. We recover precisely the κ algebra (22) as one could have guessed. Finally, as we mentioned earlier, the ''physical'' notion of momentum π_μ can be identified from the action of the translations on the plane wave

$$T_\mu \triangleright h(k_\nu) \equiv \pi_\mu h(k_\nu).$$

Direct calculation [28], using again the coproduct shows that

$$\pi_\mu = \kappa v_\mu.$$

We have therefore a nonlinear relation between the wave vector k_μ and the momentum π_μ. Moreover, using the 5D bicovariant differential calculus, it was also shown that the conserved charges, for the free action (30), associated to the translations are precisely π_μ [28]. With this choice of momentum the propagator $\mathcal{K}_2(h)$ becomes simply $\mathcal{K}_2(h) = \pi_\mu\pi^\mu - m^2$, thanks to the de Sitter constraint $\pi_A\pi^A = \kappa^2$.

IV. DERIVING DEFORMED SPECIAL RELATIVITY FROM GROUP FIELD THEORY

We now come to the main issue we address in this paper: to obtain a field theory on κ Minkowski (or equivalently on AN_3 momentum space) from a 4D group field theory, in particular, from one that could be related to 4D quantum gravity. We have already shown the general construction leading from a generic 4D GFT to an effective quantum field theory based on the same group manifold. Now the

[2] T_μ, N_i, R_i are, respectively, translations, boosts, and rotations.

[3] Indeed, we have, for example, for a translation

$$T_\mu \triangleright (X_\alpha X_\beta) = T_\mu \triangleright m(X_\alpha \otimes X_\beta) = m[(\Delta T_\mu) \triangleright (X_\alpha \otimes X_\beta)], \tag{32}$$

where m is the multiplication.

task is to specialize that construction to the case of physical interest.

We start from the group field theory describing topological BF theory for the noncompact gauge group SO(4,1).

There are several reasons of interest in this model. First of all, the McDowell-Mansouri formulation (as well as related ones [20]) of general relativity with cosmological constant defines 4D gravity as a BF theory for SO(4,1) plus a potential term which breaks the gauge symmetry from SO(4,1) down to the Lorentz group SO(3,1). On the one hand, this leads to the idea of understanding gravity as a phase of a fundamental topological field theory, an idea that has been put forward several times in the past. On the other hand, it suggests to try to define quantum gravity in the spin foam context as a perturbation of a topological spin foam model for SO(4,1) BF theory. These ideas could also be implemented directly at the GFT level. If one does so, the starting point would necessarily be a GFT for SO(4,1) of the type we use below. Second, as this model describes SO(4,1) BF theory in a "3rd quantized" setting, we expect any classical solution of the GFT equations to represent quantum de Sitter space on some given topology, analogous to what happens with Minkowski space in the SO(3,1) case. Such configurations would most likely be present (and physically relevant) also in a complete non-topological gravity model obtained starting from the topological one. Third, and partly as a consequence of the above, to start from the spin foam/GFT model for SO(4,1) BF theory seems to be the correct arena to build a spin foam model for 4D quantum gravity plus particles on de Sitter space [31], treating particles as arising from topological curvature defects for an SO(4,1) connection, along the lines of what has been already achieved in 3D gravity [18].

We do not describe the structure of the corresponding spin foam path integral, as the spin foam (perturbative) formulation plays no role in our construction. We start instead directly with the relevant group field theory, and work only at the level of the GFT action. As in the compact group case, we consider a gauge-invariant field on $SO(4,1)^{\times 4}$:

$$\phi(g_1, g_2, g_3, g_4) = \phi(g_1 g, g_2 g, g_3 g, g_4 g), \quad \forall g \in SO(4,1),$$

and the group field action is given by

$$S_{4d} = \frac{1}{2}\int [dg]^3 \phi(g_1, g_2, g_3, g_4)\phi(g_4, g_3, g_2, g_1) \quad (34)$$

$$- \frac{\lambda}{5!}\int [dg]^9 \phi(g_1, g_2, g_3, g_4)\phi(g_4, g_5, g_6, g_7) \times \phi(g_7, g_3, g_8, g_9)\phi(g_9, g_6, g_2, g_{10})\phi(g_{10}, g_8, g_5, g_1). \quad (35)$$

Because of the symmetry requirement, one of the field arguments in redundant, and one can effectively work with a field depending on only three group elements. This is indicated schematically above, where, we integrate only over three group elements in the kinetic term and nine in the interaction term in order to avoid redundant integrations, which would lead to divergences due to the noncompactness of the group SO(4,1). More precisely, considering the kinetic term, we can fix one of the four group elements, say g_4, to an arbitrary value (usually the identity $\mathbb{I}$) and integrate over the remaining three group elements without changing anything to the final result. Similarly, the restriction to only nine integrations in the interaction term can be understood as a partial gauge fixing, avoiding redundancies and associated divergences.

Starting with this group field theory, we want to derive the DSR field theory as a sector of the full theory. We follow the same strategy as in the 3-dimensional case and as outlined earlier for the 4-dimensional case: we search for classical solutions of the SO(4,1) group field theory and study specific 2-dimensional field variations around it. We will naturally obtain an effective field theory living on SO(4,1). On top of this, we want then to obtain, from such effective field theory, one that is restricted to the AN_3^3 (or AN_3) homogeneous space (subgroup). There are three main strategies following which this could be achieved, *a priori*:

(i) We could derive first an effective field theory on SO(4,1) and then study the possibility and mechanism for a decoupling of the AN_3^c degrees of freedom from the ones living on the Lorentz SO(3,1) sector of the initial SO(4,1) group.

(ii) We could try to identify some special classical solutions of the fundamental SO(4,1) group field theory, which are such that the effective matter field would naturally result in being localized on AN_3^c.

(iii) We could modify the initial SO(4,1) group field theory action in such a way that, after the same procedure, the resulting effective matter field is automatically localized on AN_3^c (or AN_3).

Anticipating the results of this section, we will see that the first strategy leads naturally to a DSR kinetic term, depending only on AN_3 degrees of freedom, and thus with an exact decoupling of the SO(3,1) modes. As for the second strategy, we will see that it does not work as simply as stated, and it requires necessarily a modification of the initial group field theory action, i.e., to some version of the third strategy. We will discuss some ways in which this can be implemented, but we will see that the simplest way to achieve this is to start directly with a group field theory for BF theory with gauge group AN_3^c.

A. Deformed special relativity as a phase of SO(4,1) GFT

Let us start from the action above defining the group field theory for the SO(4,1) BF theory. The first task is to write the field equations and identify classical solutions. This works as in the compact group case presented in Sec. II. We use the same ansatz:

$$\phi^{(0)}(g_i) = \sqrt[3]{\frac{4!}{\lambda}} \int_{SO(4,1)} dg \delta(g_1 g) F(g_2 g) \tilde{F}(g_3 g) \delta(g_4 g),$$

where the functions F and $\tilde{F}$ must satisfy the normalization condition $\int F\tilde{F} = 1$. Moreover, we also require that $\int F$ and $\int \tilde{F}$ be finite in order to get a meaningful effective action for the 2D field variations around the classical solutions.

The ansatz that we choose is tailored to lead us to the DSR field theory[4]:

$$F(g) = \alpha(v_4(g) + a)\vartheta(g), \qquad \tilde{F}(g) = \delta(g). \tag{36}$$

The function v_4 is defined as matrix element of g in the fundamental (nonunitary) 5-dimensional representation of SO(4,1), $v_4(g) = \langle v^{(0)}|g|v^{(0)}\rangle$, where $v^{(0)} = (0,0,0,0,1)$ is, as previously, the vector invariant under the SO(3,1) Lorentz subgroup. $\vartheta(g)$ is a cutoff function providing a regularization of F, so that it becomes an L^1 function. We first check the normalization condition $\int F\tilde{F} = \alpha(a+1)\vartheta(\mathbb{I}) = 1$, and, assuming that $\vartheta(\mathbb{I}) = 1$, we require $\alpha = (a+1)^{-1}$ in order for it to be satisfied.

Then we can derive the effective action around such classical solutions for 2D field variations just as in the compact group case given in (5):

$$S_{\text{eff}}[\psi] = \frac{1}{2} \int \psi(g)\psi(g^{-1})\left[1 - 2c^2 - \vartheta^2(g)\frac{2c(a + v_4(g))^2}{(a+1)^2}\right] - c\left(\sqrt[3]{\frac{\lambda}{4!}}\right) \int \psi(g_1)\dots\psi(g_3)\delta(g_1\dots g_3)[c + F(g_3)]$$
$$- c\left(\sqrt[3]{\frac{\lambda}{4!}}\right)^2 \int \psi(g_1)\dots\psi(g_4)\delta(g_1\dots g_4) - \frac{\lambda}{5!}\int \psi(g_1)\dots\psi(g_5)\delta(g_1\dots g_5), \tag{37}$$

where $c = \int F$. Thus the last issue to address in order to properly define this action is to compute the integral of F. The function $v_4(g)$ is invariant under the Lorentz group SO(3,1). Using the Iwasawa decomposition $g = h\Lambda$ with $h \in AN_3^c$ and $\Lambda \in SO(3,1)$, it is easy to see that the matrix element $v_4(g)$ actually only depends on h. Therefore it is natural to split the cutoff function $\vartheta(g)$ in factors independently regularizing the integrals over AN_3^c and over SO(3,1):

$$\vartheta(g) = \chi(h)\theta(\Lambda). \tag{38}$$

To keep calculations simple, we assume that we choose the function $\theta(\Lambda)$ to be a Gaussian function, or any other function peaked on $\Lambda = \mathbb{I}$, such that $\theta(\mathbb{I}) = 1$ and $\int\theta = 1$. Then using the isomorphism between AN_3^c and the de Sitter space $v_A v^A = 1$, we choose the cutoff function on AN_3^c to be L^1 and symmetric under $v_4 \leftrightarrow -v_4$: the simplest choice is to bound $|v_0| \leq V$, which automatically also bounds v_4 and $\mathbf{v}$. We get

$$c = \int F = \int dh\chi(h)\frac{a + v_4(h)}{a+1}$$
$$= \int [d^5 v_A]\delta(v_4^2 + \mathbf{v}^2 - v_0^2 - 1)\chi(v_A)\frac{a + v_4}{a+1}$$
$$= \frac{a\int_{dS}\chi}{a+1}, \tag{39}$$

since v_4 is a odd function on the de Sitter space. For our simplest choice of the χ function imposing a straightforward bound on v_0, we easily evaluate

$$\int_{dS}\chi(v) = 4\pi\int_{-V}^{V} dv_0 \int_{-\sqrt{1+v_0^2}}^{\sqrt{1+v_0^2}} dv_4\sqrt{1 + v_0^2 - v_4^2}$$
$$= \frac{4\pi^2}{3}V(V^2 + 3). \tag{40}$$

For more generic choices of cutoff functions χ, the last factor $\int_{dS}\chi$ is at most quartic[5] in the cutoff value V.

[4]We can also choose a more symmetric ansatz with $F(g) = \tilde{F}(g)$ which would correspond to a group field satisfying the reality condition. The resulting calculations would be more involved, and this is why we do not discuss in detail this choice. However, it can be easily checked that, with a similar regularization, the final result would be the same.

[5]As an example, for a cutoff function χ implementing directly a bound on v_4 and the 3-vector $\mathbf{v}$, we have

$$\int_{dS}\chi(v) = 4\pi\int_{-V}^{+V} dv_4 \int^{+V} dv \frac{v^2}{2\sqrt{v^2 + v_4^2 - 1}} \propto V^3 \ln V.$$

If we want to remove the cutoff and reabsorb all the infinities due to the noncompactness of the group, we could now send the cutoff V to ∞, and then we also send the factor a to 0, scaling it as $a \propto 1/V^3$. In this way, we keep c finite. This is the simplest method to achieve the result, but of course others can be considered. We point out that this renormalization is done at the classical level in the definition of our classical solution and not at the quantum level like in quantum field theory. In other words, this regularization is necessary in order to obtain a true and well-defined classical solution of the equations of motion, and meaningful variations around it.

After all these regularization details, in the double scaling limit[6] $a \to 0$ and $L \to \infty$ while keeping c finite, we have derived an effective theory for a field $\psi(g)$ living on SO(4,1):

$$S_{\rm eff}[\psi] = \frac{1}{2}\int \psi(g_1)\psi(g_2)[1 - 2c^2 - 2c v_4(h_1)^2\chi(h_1)^2\theta(\Lambda_1)^2]\delta(g_1 g_2) - \sqrt[3]{\frac{\lambda}{4!}}\int \psi(g_1)\dots\psi(g_3)\delta(g_1\dots g_3)[c^2 + cF(g_3)] - c\left(\sqrt[3]{\frac{\lambda}{4!}}\right)^2\int \psi(g_1)\dots\psi(g_4)\delta(g_1\dots g_4) - \frac{\lambda}{5!}\int \psi(g_1)\dots\psi(g_5)\delta(g_1\dots g_5). \tag{41}$$

We recognize the correct kinetic term for a DSR field theory. However, the effective matter field is *a priori* still defined on the full SO(4,1) momentum manifold. The only remaining issue is therefore to understand the "localization" process of the field ψ to AN_3^c. Having done this, we would truly have derived a scalar field theory in deformed special relativity from the group field theory defining topological SO(4,1) BF theory, and thus a sector of 4D quantum gravity.

Let us consider the second strategy envisaged above. A possible solution to the localization issue is, the strategy goes, to use the classical solution F itself to localize the field on the AN_3^c manifold. For example, one may require that the regularizing function $\theta(\Lambda)$ forces the SO(3,1) group element to be, say, the identity element, $\Lambda = \mathbb{I}$. The simplest choice is to use a delta function on SO(3,1). This however causes two problems. First, both $\theta(\Lambda)$ and $\theta(\Lambda)^2$ appear in the action above, and of course the square of the δ distribution is not well defined. One can devise methods to overcome this purely mathematical problem, by using suitable "smoothed" delta distributions, which achieve the same localization, but are L^2 functions. The second problem is however more fundamental. By construction, this method forces the group element g to lay in AN_3^c only in the terms containing some factors $F(g)$, i.e., depending in a nontrivial way (not as an overall constant) on the classical solution chosen. Thus the mass term and most of the interaction terms are completely transparent to this way of projecting on AN_3^c. We conclude that it is not enough to use the classical solution to achieve this reduction from the full group SO(4,1) to the submanifold AN_3^c.

We then look more carefully at the first strategy outlined above. We see immediately that the kinetic term (containing the differential operator defining the propagation of the field degrees of freedom, as well as the symplectic structure in a canonical setting), does not show any dependence on the Lorentz sector. Indeed, through our choice of classical solution, we obtained a kinetic term in $v_4(g)$ which depends only on the AN_3^c part h of the group element $g = h\Lambda$. This suggests that the SO(3,1) degrees of freedom are nondynamical and that the restriction of the domain of definition of the field ψ to AN_3^c group elements defines a dynamically stable phase of the theory. This would be trivially true if not for the fact that the interaction term does, *a priori*, depend also on the Lorentz degrees of freedom, and couples them among the different interacting fields. Therefore, we see three different simple ways to attempt to project this effective field theory down to AN_3^c:

(i) We can choose a decoupled ansatz for the perturbation field ψ and assume it has a product structure $\psi(g) = \tilde{\psi}(h)\Psi(\Lambda)$. As far as the kinetic term is concerned, the only contribution from the Lorentz sector is a constant multiplicative term $\int_{SO(3,1)} d\Lambda\Psi(\Lambda)\Psi(\Lambda)$ [and $\int \Psi\Psi\theta^2 \sim \Psi(\mathbb{I})^2$]. Thus we get the exact kinetic term for a κ-Poincaré invariant free field theory. However, the vertex term still couples the Lorentz and AN_3^c degrees of freedom and the κ-Poincaré symmetry is broken; the pure DSR-like form lost. To simplify the interaction term, we can further assume that *the dependence of the perturbation field on the Lorentz sector is trivial*, i.e., assume $\Psi(\Lambda) \equiv 1$. This naturally gives an effective field theory based only on the AN_3 subgroup and it seems that we obtain a nice DSR field theory. However, the interaction term still causes a problem. The Lorentz sector is still coupled to the DSR degrees of freedom due to the momentum conservation. Indeed, the $\delta(g_1 \dots g_n)$ constraint becomes after integration over the Lorentzian variables $\int[d\Lambda]\delta(h_1\Lambda_1\dots h_n\Lambda_n)$. This is clearly not the mo-

[6] Obviously, we do not *need* to take the limit. We could keep a, L, c all finite and define a solution parametrized by these constants. As a result, we would simply get extra constant terms in the action, e.g., terms in a^2 and $a v_4(h)$ in the propagator. The limiting procedure is implemented only in order to get a simpler form of the action.

mentum conservation of a DSR theory, which would be given by the simpler constraint $\delta(h_1 \dots h_n)$. We conclude that this ansatz does not work since it leads to a DSR interaction which violates the conservation of energy momentum.

(ii) The next possibility is *to project by hand the perturbation field on the* AN_3^c *sector*, i.e., assume $\psi(g) = \tilde{\psi}(h)\delta(\Lambda)$. This actually works and projects the whole field theory on AN_3^c:

$$S_{\rm final}[\tilde{\psi}] = \frac{1}{2}\int \tilde{\psi}(h)\tilde{\psi}(h^{-1})[1 - 2c^2 - 2cv_4(h)^2\chi(h)^2] - \sqrt[3]{\frac{\lambda}{4!}}\int \tilde{\psi}(h_1)\dots\tilde{\psi}(h_3)\delta(h_1\dots h_3)[c^2 + cv_4(h_3)\chi(h_3)] - c\left(\sqrt[3]{\frac{\lambda}{4!}}\right)^2\int \tilde{\psi}(h_1)\dots\tilde{\psi}(h_4)\delta(h_1\dots h_4) - \frac{\lambda}{5!}\int \tilde{\psi}(h_1)\dots\tilde{\psi}(h_5)\delta(h_1\dots h_5), \quad (42)$$

where all the integrations are done with the left-invariant measure dh_L on AN_3^c, which is inherited from the Haar measure on SO(4,1). The only subtlety with this approach is that the induced $\delta(h_1 \dots h_n)$ constraints are actually still δ distribution on the full group SO(4,1) and thus do not have the same density than true δ distribution on the subgroup AN_3^c. This usually leads to divergences, but we leave this technical issue aside for now.

(iii) The third alternative is to argue that *the reduction to the* AN_3 *sector happens dynamically.* This dynamical reduction could be seen in two ways. First, one can expect that transition (scattering) amplitudes involving only real particles defined on AN_3^c, i.e., with momenta in this submanifold, does not lead to creation of particles with Lorentz degrees of freedom as well, due to the form of the propagator, even if in principle they would be allowed by the enlarged momentum-conservation law coming from the interaction term, which is defined on the full SO(4,1) group. The second possibility is that a proper canonical analysis of the effective field theory would show that the SO(3,1) modes are pure gauge and can simply be fixed from the start and thus drop from the action altogether. We leave a more detailed analysis of this issue for future work. Whether or not the restriction to AN_3 is obtained automatically, in one of the above ways, or by some other procedure that will be revealed by a more detailed analysis, what is certain is that a restricted theory obtained from the above and living on AN_3^c only is dynamically stable. In fact, if we consider only excitations of the field in AN_3^c, we will never obtain excitations in SO(3,1) due to momentum conservation $\delta(g_1 \dots g_n)$ since AN_3^c is a subgroup. Therefore AN_3^c is stable under the dynamics of the field theory, and thus a restriction to fields on AN_3^c is consistent.

One can compare this situation to the case of a 2D field theory written in momentum space where the propagator depends on p_x and not on p_y:

$$S_{eg}[\psi] = \int d^2\vec{p}(-p_x^2)\psi(\vec{p})\psi(-\vec{p}) + \int [d^2p]^n\delta\left(\sum_i^n p^{(i)}\right)\prod_i^n \psi(p^{(i)}).$$

The momentum p_y does not enter the propagator and it defines a pure gauge degree of freedom, as it can be checked by straightforward canonical analysis. Therefore, we can restrict ourselves to the sector $p_y = 0$ without affecting the dynamics of the field, nor any physical content of the theory. At the end of the day, we obtain the same effective field theory (42) as above.

Finally, we have argued that this action (42) encodes the full dynamics of 2D perturbations, as emergent matter fields, of the SO(4,1) GFT, around the special classical solution we have chosen. We have thus finally derived the scalar field theory for deformed special relativity with a κ-deformed Poincaré symmetry from the SO(4,1) group field theory defining the transition amplitudes for the topological BF theory. To summarize, this was achieved in three steps:

(1) Identify the correct regularized classical solution(s) to the initial group field theory.
(2) Look at the 2-dimensional field variations around such a classical solution and write the effective action describing their dynamics.
(3) Localize the field variations on the AN_3^c group manifold relevant to deformed special relativity.

An important remark is that we have a field theory already with an in-built cutoff in momentum space due to the regularizing function $\chi(h)$, necessary to define the classical solution to the group field theory. Of course we can always send this cutoff to infinity by the double scaling limit $V \to \infty$, $a \to 0$. At the quantum level, we would anyway have to introduce such a momentum cutoff to define the perturbative expansion of the quantum field theory in term of Feynman diagrams. Here, on the other hand, the momentum cutoff is not included to regularize the Feynman diagrams, i.e., the discrete quantum histories of the theory, but it appears naturally in our derivation of the effective field theory on AN_3^c from the initial group field theory on SO(4,1). Indeed, we insist on the fact that the classical solution around which we study the group field variation cannot be defined without this momentum cutoff.

B. Starting from a restricted GFT: the AN_3 case

Having followed in detail the first strategy outlined above, and having shown the nonviability of the second, we now describe the third, and obtain a DSR-like field theory in a different way. Accordingly, instead of localizing the field variation on AN_3^c in the final step, having first derived an effective field theory on SO(4,1), we could modify our starting group field theory action in such a way that the effective field theory for perturbations is automatically localized on AN_3^c.

The first case we deal with is the simplest one in which we choose our initial fundamental theory to be itself a group field theory for 4D BF theory with AN_3^c gauge group. We can then perform the same analysis as in Sec. II, and then choose the same classical solution we have used in the previous section [now seen as a function on the AN_3^c subgroup of SO(4,1) only]. This is naturally to a field theory on AN_3^c describing a scalar field with deformed special relativity kinematics. The drawback is that the link with 4D quantum gravity is now more obscure. It is still possible that such group field theory is related to the quantization of the McDowell-Mansouri formulation of 4D gravity, but the exact relation is unclear. It still defines a topological spin foam model, thus lacking any local gravity degree of freedom; moreover, it lacks the information contained in SO(4,1) BF, e.g., the cosmological constant, and its classical solutions have no immediate space-time interpretation, contrary to that case.

Still, it represents the easiest route to a DSR field theory from GFT. The only issue that one has to be careful with in this case is the question of the measure since we have to decide whether to use the left or right invariant measure. Since the left-invariant measure is the one inherited from the Haar measure on SO(4,1), it seems to be the natural one to use. As before, we introduce the gauge-invariant group field on $(AN_3^c)^{\times 4}$:

$$\phi(h_1, \ldots, h_4) = \phi(h_1 h^{-1}, \ldots, h_4 h^{-1}), \qquad \forall h \in AN_3^c,$$

and the corresponding action

$$\begin{aligned} S_{an}[\phi] = \frac{1}{2} &\int [dh]^3 \phi(h_1, h_2, h_3, h_4)\phi(h_4, h_3, h_2, h_1) \\ &- \frac{\lambda}{5!}\int [dh]^9 \phi(h_1, h_2, h_3, h_4)\phi(h_4, h_5, h_6, h_7) \\ &\times \phi(h_7, h_3, h_8, h_9)\phi(h_9, h_6, h_2, h_{10}) \\ &\times \phi(h_{10}, h_8, h_5, h_1), \end{aligned}$$

where we have used everywhere the left-invariant measure dh_L on AN_3^c. As before we check that the "flat solution" ansatz,

$$\phi^{(0)} \equiv \sqrt[3]{\frac{4!}{\lambda}} \int dh_{(L)} \delta(h_1 h^{-1}) F(h_2 h^{-1}) \tilde{F}(h_3 h^{-1}) \delta(h_4 h^{-1}), \tag{43}$$

provides a classical solution to the group field theory as soon as $\int F\tilde{F} = 1$. Thus we should choose the same ansatz for the arbitrary functions:

$$F(h) = \frac{v_4(h) + a}{a + 1} \chi(h), \qquad \tilde{F}(h) = \delta(h), \tag{44}$$

where we choose exactly the same regularizing function $\chi(h)$ as in the previous section, e.g., the one imposing the bound $v_0(h)^2 \leq V^2$. We then look at the effective action for 2-dimensional field variation $\phi^{(0)} + \psi(h_1 h_4^{-1})$ around the classical solution. Using the left invariance of the measure, we end up of course with the same effective scalar field theory (42) living on AN_3^c.

As said, this gives the shortest path from a 4-dimensional group field theory and deformed special relativity. The natural question in this context is nevertheless the physical meaning/relevance of 4D BF theory with gauge group AN_3^c, from a 4D quantum gravity standpoint, as we discussed.

V. CONCLUSION

We have derived a scalar field theory of the deformed special relativity type, with a κ-deformed Poincaré symmetry, from the SO(4,1) group field theory defining the transition amplitudes for topological BF theory in 4 space-time dimensions. This was done directly at the GFT level, thus bypassing the corresponding spin foam formulation, in such a way that matter fields emerge from the fundamental model as perturbations around a specific phase of it, corresponding to a solution of the fundamental equations of motion, and the noncommutative field theory governs their effective dynamics. Not only is this the first example of a derivation of a DSR model for matter from a more fundamental quantum gravity model, and one further link between noncommutative geometry and quantum gravity formulated in terms of spin foam/loop quantum gravity ideas, but it is of great interest from the point of view of quantum gravity phenomenology, as we have pointed out in the Introduction. It represents, in fact, another possible way of bridging the gap between quantum gravity at Planck scale and effective (and testable) physics at low energies.

Obviously, there are many questions left unanswered in this work. Some concern purely technical details of our procedure. We have mentioned them in the bulk of this paper, so we do not repeat them. We mention here briefly a few more general ones of these open issues.

The first concern is the role of the SO(3,1) degrees of freedom in the group field theory we started from, as well as in the one we have obtained as describing the dynamics of matter. From the GFT point of view it is utterly unclear why AN_3^c should be the relevant group for the perturbations as opposed to some other subgroup of SO(4,1). One can pose this same question in terms of the classical solution we have perturbed around. What is the physical meaning of the solution we have chosen? This is unclear at present, contrary to the 3D case, where the solutions used can be related to flat geometries. As mentioned, we expect it to be related to de Sitter space, but more work is needed to understand the details of the correspondence. Related to this, it would be interesting to investigate the role of the cosmological constant in this GFT context. To start with, it seems that here the presence of a cosmological constant is encoded only in the group manifold used in the starting GFT, i.e., SO(4,1), but we have little control of how this is done. Second, we have motivated the choice of starting with this GFT model also by analogy with the McDowell-Mansouri (and related) formulation of general relativity as a SO(4,1)-gauge theory, but this works only for a strictly positive cosmological constant. It is then natural to ask what happens if we start from SO(3,2) in place of SO(4,1) in the original model and then carry out the same procedure for extracting an effective matter field theory. Further investigations are needed to establish a better link between our initial GFT model, classical solutions, and effective field theory on the one hand, and a spin foam formulation of the Freidel-Starodubstev classical gravity theory [31] and the particle observable insertions *à la* Kowalski-Glikman-Starodubtsev [20] on the other, which represent another path to deriving an effective deformed special relativity from spin foam models. Last, we have obtained a scalar field theory for matter, and thus we should now look for extensions of our procedure and a result that could give instead matter fields with nonzero spin, e.g., Dirac fermions or vector fields. Moreover, we have provided an example of the emergence of space-time (deformed) isometries from GFT, but it is natural to wonder if also gauge symmetries and thus gauge fields can be seen as emerging from some fundamental (GFT) quantum gravity model. Higher spins have already encoded in 3D GFT in [32], but never in 4D and in the usual sense of *coupling* matter degrees of freedom to quantum gravity ones, instead of having the first emerge from the second, as in the present work. Therefore, this is an area of research that is still wide open to be explored. We leave all these questions for future work.

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