# Higgs Bundles and UV Completion in F-Theory

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## Abstract

F-theory admits 7-branes with exceptional gauge symmetries, which can be compactified to give phenomenological four-dimensional GUT models. Here we study general supersymmetric compactifications of eight-dimensional Yang-Mills theory. They are mathematically described by meromorphic Higgs bundles, and therefore admit a spectral cover description. This allows us to give a rigorous and intrinsic construction of local models in F-theory. We use our results to prove a no-go theorem showing that local SU(5) models with three generations do not exist for generic moduli. However we show that three-generation models do exist on the Noether-Lefschetz locus. We explain how F-theory models can be mapped to non-perturbative orientifold models using a scaling limit proposed by Sen. Further we address the construction of global models that do not have heterotic duals. We show how one may obtain a contractible worldvolume with a two-cycle not inherited from the bulk, a necessary condition for implementing GUT breaking using fluxes. We also show that the complex structure moduli in global models can be arranged so that no dimension four or five proton decay can be generated.

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#### 1. Introduction

Recently [1, 2, 3] initiated a systematic effort to study Kaluza-Klein GUT models in F-theory. More precisely, we used an eight-dimensional gauge theory with an exceptional gauge group, coupled to ten-dimensional type IIb supergravity. The UV completion of this non-renormalizable theory is called F-theory [4]. For practical purposes however, very little is known about this non-perturbative completion. We only know the low energy gauge theory and supergravity Lagrangians, which are uniquely determined by the symmetries. To get a reliable weakly coupled description in which these Lagrangians can be trusted, the fields must be slowly varying. Thus these models have a weakly coupled description in the large volume limit, even though they are not in reach of perturbative string theory. Recent work on F-theory models includes [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20].

Despite the conceptual progress in [1, 2, 3], there were a number of unanswered questions about the actual construction of local models in F-theory. In particular, the strategy in [1] (and also [3]) relied on taking a limiting form of models with a heterotic dual. This approach yields manifestly consistent models, but it less than clear if the most general local F-theory model is recovered this way. The approach in [2] is to postulate the matter curves and the fluxes restricted to the matter curves. At first sight this looks more flexible, but in this case it is less clear if the data is mutually consistent. Given the uncertainties, it can be hard to evaluate what F-theory does or does not predict.

The first purpose of this paper is to give a rigorous and intrinsic construction of local F-theory models. The chain of logic is as follows. As mentioned above, the basic idea is that we have to construct compactifications of a supersymmetric eight-dimensional gauge theory. Such compactifications are mathematically described by meromorphic Higgs bundles. The main fact is that there is a natural isomorphism between Higgs bundles and spectral covers in an auxiliary non-compact Calabi-Yau geometry. And the last set-up is the one that allows us to make constructions, particularly of the fluxes. Moreover these spectral covers are the same as one obtains from a scaling limit of heterotic/F-theory duality. Thus, in a somewhat roundabout way, our original strategy actually recovers all possible local F-theory models. A completely parallel construction of local M-theory models will appear in [21].

Our spectral cover approach gives a precise description of the configuration space of local F-theory models, which is important for phenomenological applications. We will use this to classify the possible matter curve configurations and prove a no-go theorem, showing that the fluxes which were known to exist do not allow for a local SU(5) model with three generations. This is seen to imply that in order to find realistic models, we have to solve a Noether-Lefschetz problem, i.e. we have to tune the complex structure

<sup>&</sup>lt;sup>1</sup>This observation was made independently in [10], which appeared while this project was written up.

moduli of a local model in order to find supersymmetric solutions with three generations (which will then automatically have stabilized some of the moduli). We then write down some new classes of fluxes which are available on the Noether-Lefschetz locus, and find the first examples of three-generation models. Such more general fluxes are also available in heterotic models, where they generally get mapped to rigid bundles. In fact we will point out that heterotic constructions to date have been very special and essentially missed the landscape seen on the type II side. Along the way we discuss several other interesting issues, such as orientifold limits of F-theory models.

The second purpose of this paper is to begin the construction of global UV completions which do not have a heterotic dual. This section was originally to appear as section 5 of [5], but seemed to fit better with this paper. We will give some examples which should make the general strategy clear. We do not find any meaningful constraints on extending desired values of complex structure moduli from a local model to a global model, thereby further validating the idea of studying local models. In particular we find that it is possible to set the complex structure moduli so that no dimension four or five proton decay can be generated. But the understanding of global models is unfortunately still rather incomplete. Our discussion focuses on constructing compact models with desired 7-brane configurations, but at present we do not have any good techniques for handling global G-fluxes in general F-theory models. Constructing suitable global fluxes is again an incarnation of a Noether-Lefschetz problem, for which no really simple techniques seem to exist. One possible approach using orientifold limits is briefly mentioned in section 2.2.

## 2. Higgs bundles in F-theory

In this section we will give a detailed description of local F-theory models. Although much of this material is described implicitly or explicitly in our previous papers, writing out the chain of logic more carefully allows us to make sharper statements about the configuration space of such models.

The reader should be aware that on occasion we use two different definitions of the notion of a local model. The physical definition is that of a model in which  $M_{GUT}/M_{Pl}$  can be made parametrically small. The other definition is that of a non-compact  $CY_4$  consisting of an ALE fibration over a surface. Hopefully it is clear from the context which notion we use.

## 2.1. Local model from global model

Let us start with a global model, which is defined as a compact elliptically fibered Calabi-Yau complex four-fold with a section  $\sigma(B_3)$  (often simply written as  $B_3$ ). The elliptic fibration can be described by a Weierstrass model

$$y^2 = x^3 + fx + g (2.1)$$

where f, g are sections of  $K_{B_3}^{-4}$ ,  $K_{B_3}^{-6}$  respectively. For the purpose of detecting singularities, it is more useful to write the Weierstrass equation in generalized form as

$$y^{2} + a_{1}xy + a_{3}y = x^{3} + a_{2}x^{2} + a_{4}x + a_{6}.$$
 (2.2)

where the  $a_i$  are sections of  $K_{B_3}^{-i}$ . By completing the square and the cube, this may be written as (2.1), but the generalized form is more convenient for prescribing singular elliptic fibers along loci in  $B_3$ .

Suppose that we have a surface S of singularities in  $B_3$ . This will put certain restrictions on the sections  $a_i$  above. Let us take z to be a coordinate on the normal bundle to S in  $B_3$ , so S corresponds to z = 0. We will often denote  $c_1(NS) = -t$ . Then the order of vanishing of the  $a_i$  may increase at z = 0, so there will be conditions of the form 'z divides  $a_i$  at least  $n_i$  times,' which are characteristic of the singularity type of the elliptic fiber over z = 0. These conditions have been worked out in [22, 23] and are given in table 1 which was taken from [23]. In retrospect, the table is perhaps better understood in terms of Higgs bundles, which we will discuss later. Now to get a local model from a global model, we assign scaling dimensions to (x, y, z) and drop the irrelevant terms. Physically, this means we will be dropping certain higher order terms in the 8d gauge theory.

type	group	$a_1$	$a_2$	$a_3$	$a_4$	$a_6$	Δ
$I_0$		0	0	0	0	0	0
$I_1$	_	0	0	1	1	1	1
$I_2$	SU(2)	0	0	1	1	2	2
$I_3^{ns}$	unconven.	0	0	2	2	3	3
$I_3^s$	unconven.	0	1	1	2	3	3
$I_{2k}^{ns}$	Sp(k)	0	0	k	k	2k	2k
$I_{2k}^s$	SU(2k)	0	1	k	k	2k	2k
$I_{2k+1}^{ns}$	unconven.	0	0	k+1	k+1	2k + 1	2k + 1
$I_{2k+1}^s$	SU(2k+1)	0	1	k	k+1	2k + 1	2k + 1
II	_	1	1	1	1	1	2
III	SU(2)	1	1	1	1	2	3
$IV^{ns}$	unconven.	1	1	1	2	2	4
$IV^s$	SU(3)	1	1	1	2	3	4
$I_0^{*ns}$	$G_2$	1	1	2	2	3	6
$I_0^{*ss}$	SO(7)	1	1	2	2	4	6
$I_0^{*s}$	$SO(8)^{*}$	1	1	2	2	4	6
$I_1^{*ns}$	SO(9)	1	1	2	3	4	7
$I_1^{*s}$	SO(10)	1	1	2	3	5	7
$I_2^{*ns}$	SO(11)	1	1	3	3	5	8
$I_2^{sts}$	$SO(12)^{*}$	1	1	3	3	5	8
$I_{2k-3}^{*ns}$	SO(4k+1)	1	1	k	k+1	2k	2k + 3
$I_{2k-3}^{*s}$	SO(4k+2)	1	1	k	k+1	2k + 1	2k + 3
$I_{2k-2}^{*ns}$	SO(4k+3)	1	1	k+1	k+1	2k + 1	2k + 4
$I_{2k-2}^{*s}$	$SO(4k+4)^*$	1	1	k+1	k+1	2k + 1	2k + 4
$IV^{*ns}$	$F_4$	1	2	2	3	4	8
$IV^{*s}$	$E_6$	1	2	2	3	5	8
$III^*$	$E_7$	1	2	3	3	5	9
$II^*$	$E_8$	1	2	3	4	5	10
non-min	_	1	2	3	4	6	12

**Table 1:** Results from Tate's algorithm [22, 23]. The subscript s/ns stands for split/non-split, meaning that there is/is not a monodromy action by an outer automorphism on the vanishing cycles along the singular locus. 6

For phenomenological purposes the case of most interest is a surface S of  $I_5$  singular fibers. Then according to table 1, in order to have an SU(5) singularity along z = 0, we need the leading terms near z = 0 to be

$$a_1 = -b_5, \quad a_2 = zb_4, \quad a_3 = -z^2b_3, \quad a_4 = z^3b_2, \quad a_6 = z^5b_0$$
 (2.3)

where the  $b_i$  are generically non-vanishing, and we may have further subleading terms which vanish to higher order in z. The  $b_i$  are independent of z, so we may think of the  $b_i$  as sections of line bundles on the surface S. Now we assign scaling dimensions (1/3, 1/2, 1/5) to (x, y, z) respectively. We throw out the 'irrelevant terms' whose scaling dimension is larger than one. The resulting equation we get is

$$y^{2} = x^{3} + b_{0}z^{5} + b_{2}xz^{3} + b_{3}yz^{2} + b_{4}x^{2}z + b_{5}xy$$
(2.4)

which is exactly the equation of an  $E_8$  singularity unfolded to an SU(5) singularity. The dimension one terms give the  $E_8$  singularity and the terms with dimension smaller than one give a relevant deformation of this singularity. Thus we may extract an ALE fibration over S from any global model by taking a suitable low energy limit. Note that  $c_1(B_3)|_S = c_1(S) - t$ , and so the above equation transforms as a section of  $6c_1(S) - 6t$ . Therefore the Chern classes of the sections  $b_i$  on S are given by

$$b_i \sim (6-i)c_1(S) - t$$
 (2.5)

Note that we could have assigned different scaling dimensions to the variables, which would result in dropping additional terms in (2.4). For instance if we assign degrees (1/3, 1/2, 2/9), then the  $z^5$  term is also irrelevant and the dimension one terms give the equation of an  $E_7$  singularity. However from the results of Tate's algorithm we see that it must still be embedded in the  $E_8$  singularity (2.4), so our choice will give the most general local model. The  $E_8$  singularity is the maximal singularity that the elliptic fibration allows without destroying the Calabi-Yau property.

Part of the attraction of local F-theory models is that almost all of the observable sector is described by this one equation (2.4), plus a choice of G-fluxes. All the usual complications of global models can be hidden in the subleading corrections to this equation. This is equivalent to the statement that the local geometry is completely described by the 8d gauge theory. In the following we will analyze these local geometries in more detail.

## 2.2. Orientifold limits

In this section, we analyze IIb limits of F-theory vacua. Such limits are expected to be useful, since a number of issues (particularly global issues) are currently much better

understood in the IIb theory than in F-theory. For instance we would like to use this to analyse G-fluxes in global models. However as we will discuss the regimes of validity are not overlapping and the IIb models we get look very different from any previously considered IIb GUT-like models. Thus there is still some work to be done to understand the relation between the two pictures.

Consider again the Weierstrass equation

$$y^2 = x^3 + fx + g (2.6)$$

and its generalized form

$$y^2 + a_1 x y + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6. (2.7)$$

As in [22], we define the following quantities:

$$\begin{array}{llll} \mathbf{b}_2 & = & a_1^2 + 4a_2 & & \mathbf{b}_8 & = & \frac{1}{4}(\mathbf{b}_2\mathbf{b}_6 - \mathbf{b}_4^2) \\ \mathbf{b}_4 & = & a_1a_3 + 2a_4 & & \Delta & = & -\mathbf{b}_2^2\mathbf{b}_8 - 8\mathbf{b}_4^3 - 27\mathbf{b}_6^2 + 9\mathbf{b}_2\mathbf{b}_4\mathbf{b}_6 \\ \mathbf{b}_6 & = & a_3^2 + 4a_6 & & \end{array} \tag{2.8}$$

Then f and g may be recovered as

$$f = -\frac{1}{48}(b_2^2 - 24b_4)$$

$$g = -\frac{1}{864}(-b_2^3 + 36b_2b_4 - 216b_6)$$
(2.9)

Now supposed that we want to take a limit in the complex structure moduli space so that the axio-dilaton becomes constant almost everywhere in the IIb space-time. Since

$$j(\tau) = 4\frac{(24f)^3}{4f^3 + 27g^2} \tag{2.10}$$

this will happen when

$$\frac{f^3}{g^2} \to \text{constant}$$
 (2.11)

Inspecting (2.9), we see that the most evident way to achieve this is by scaling up  $b_2$ , or alternatively by scaling down  $b_4$  and  $b_6$ . Therefore let us consider the following scaling limit:

$$a_3 \to \epsilon \, a_3, \qquad a_4 \to \epsilon \, a_4, \qquad a_6 \to \epsilon^2 \, a_6$$
 (2.12)

Note that for our GUT models (2.4), in this limit  $b_i/b_0$  scales like  $1/\epsilon$  or  $1/\epsilon^2$ . Since  $b_i/b_0 \sim \text{Tr}(\Phi^i)$  are identified with Casimirs of the eight-dimensional Higgs field, this

means that the VEV of the Higgs field is becoming large and we can no longer trust the 8d gauge theory/F-theory description. One may still hope to get a different weakly coupled description in terms of perturbative IIb string theory. As we will discuss, this is possible, but we have to push the model through a configuration with singularities that are neither well-described by F-theory nor by perturbative type IIb.

Continuing, one finds

$$f = -\frac{1}{48} (b_2^2 - 24\epsilon b_4)$$

$$g = -\frac{1}{864} (-b_2^3 + 36\epsilon b_2 b_4 - 216\epsilon^2 b_6)$$
(2.13)

The discriminant is given by

$$\Delta = \epsilon^{2}(-b_{2}^{2}b_{8} - 8\epsilon b_{4}^{3} - 27\epsilon^{2}b_{6}^{2} + 9\epsilon b_{2}b_{4}b_{6})$$

$$\sim -\frac{1}{4}\epsilon^{2}b_{2}^{2}(b_{2}b_{6} - b_{4}^{2}) + \mathcal{O}(\epsilon^{3})$$
(2.14)

Therefore in the  $\epsilon \to 0$  limit, all the roots are located at  $b_2 = 0$  and  $b_2b_6 - b_4^2 = 0$ . The monodromies around these roots were analyzed in [24, 25], with the result that

$$O7: \mathbf{b}_2 = 0, \qquad D7: \mathbf{b}_2 \mathbf{b}_6 - \mathbf{b}_4^2 = 0$$
 (2.15)

Moreover, the j-function behaves as

$$j(\tau) \sim \frac{b_2^4}{\epsilon^2 (b_2 b_6 - b_4^2)}$$
 (2.16)

which means that the string coupling goes to zero almost everywhere. Therefore we get the following picture [24]: in the limit of complex structure moduli space that we discussed above, the Calabi-Yau four-fold becomes a constant elliptic fibration over a Calabi-Yau three-fold given by

$$\xi^2 = \mathsf{b}_2 \tag{2.17}$$

where  $b_2 \sim K_{B_3}^{-2}$ ,  $\xi \sim K_{B_3}^{-1}$ . That is, the emerging  $CY_3$  is simply the double cover over  $B_3$  with branch locus given by  $b_2 = 0$ . The orientifolding acts as

$$\xi \to -\xi, \quad y \to -y$$
 (2.18)

and the positions of the branes on this three-fold are given as above. There are two copies of the D7 locus  $b_2b_6 - b_4^2 = 0$  related by  $\xi \to -\xi$ .

Now let's apply this to our local models. The Calabi-Yau three-fold will be given by a double cover of the total space of the normal bundle  $N_S \to S$ , with branch locus given by  $\mathbf{b}_2 = 0$ . For SU(5) models we get

$$b_{2} = b_{5}^{2} + 4zb_{4}$$

$$b_{4} = z^{2}b_{3}b_{5} + 2z^{3}b_{2}$$

$$b_{6} = z^{4}b_{3}^{2} + 4z^{5}b_{0}$$

$$b_{2}b_{6} - b_{4}^{2} = z^{5}(4b_{3}^{2}b_{4} - 4b_{2}b_{3}b_{5} + 4b_{0}b_{5}^{2} + z(16b_{0}b_{4} - 4b_{2}^{2}))$$
(2.19)

where z is a local coordinate on the normal bundle  $N_S$ . Hence we find a non-compact O7-plane along the branch locus  $\mathbf{b}_2 = 0$ , five gauge D7-branes wrapped on S, as well as a non-compact flavour D7-brane. The O7-plane intersects the gauge 7-branes along the matter curve

$$\Sigma_{10} = \{b_5 = 0\} \tag{2.20}$$

which as expected carries an enhanced SO(10) singularity. The flavour D7-brane intersects the gauge D7-brane along

$$\Sigma_5 = \{ R = b_3^2 b_4 - b_2 b_3 b_5 + b_0 b_5^2 = 0 \}$$
 (2.21)

which carries an enhanced SU(6) singularity. Finally the Yukawa couplings are localized at

$$\lambda_{\text{top}} \sim \{b_5 = b_4 = 0\}, \qquad \lambda_{\text{bottom}} \sim \{b_5 = b_3 = 0\},$$
 (2.22)

which carry enhanced  $E_6$  and SO(12) singularities, respectively.

Let us now look in more detail at the points of  $E_6$  enhancement. The equation of the Calabi-Yau can be written as

$$\xi^2 = u^2 + zw (2.23)$$

where  $u = b_5$  and  $w = 4b_4$ . Thus the  $E_6$  points are conifold singularities of the Calabi-Yau three-fold. We expect that the limiting model has zero  $B_{NS}$ -field through the vanishing  $S^2$ , so that it corresponds to a non-perturbative singularity of type IIb.

Perturbative string theory breaks down at such conifold singularities, and there are extra massless states. This should be a chiral field corresponding to the zero modes of  $B_2$ ,  $C_2$  on the 'resolved' picture, or to a D3 wrapped on the vanishing  $S^3$  in the deformed picture.

In order to get a perturbative picture, we can try to resolve or deform the conifold singularity. Let us first discuss the resolutions. The two small  $\mathbf{P}^{1}$ 's are exchanged under the discrete symmetry  $\sigma: \xi \to -\xi$ , and thus the small resolution is projected out by the orientifold. The full orientifold action is given by  $\Omega(-1)^{F_L}\sigma$  where  $\Omega$  is worldsheet parity and  $(-1)^{F_L}$  maps the RR fields to minus themselves. The NS B-field is odd under  $\Omega(-1)^{F_L}$ , so it is consistent to have a non-zero value of B through the vanishing  $\mathbf{P}^1$ .

So one can 'resolve' the singularity by turning on the B-field. ( $C_2$  may also be non-zero; it is paired with  $B_2$  under SUSY). Thus there will be a description of the up-type Yukawa coupling using D1-instantons. However there is no smooth geometric picture, and  $\alpha'$  corrections would be important. The B-field may be tuned to the value 1/2 which corresponds to the quiver locus. These models are very different from the IIb SU(5) models that have been considered in the literature (see eg. [12] for a recent discussion and constructions), and more work needs to be done to connect the two pictures.

We may also ask what happens with the flux that is responsible for chiral matter in the scaling limit. Likely this yields U(1)-flux for the overall  $U(1) \subset U(5)$  in the IIb model. In F-theory, this U(1) becomes part of the larger  $E_8$  gauge symmetry, and is Higgsed by the adjoint field of the 8d gauge theory.

Instead of trying to resolve the conifold points, one can also give the  $S^3$  a finite size by deforming the branch locus to a generic section of  $K_{B_3}^{-2}$ . This is also compatible with the orientifold action and removes the conifold points. (Three-form fluxes through this  $S^3$  are not compatible with the orientifold action and can not be turned on). However this corresponds to breaking the SU(5) GUT group by giving an expectation value to a field in the 10. So although one could get a smooth geometric background this way, it comes at the cost of breaking the GUT group.

It is amusing to ask what happens for local SO(10) models when we take this limit. This corresponds to setting  $b_5 \to 0$  identically in the above equations. Then the O7-plane is reducible and consists of a component wrapping S and a component wrapped on the curve  $b_4 = 0$  in S and stretching in the normal direction. The spinors in the  $\mathbf{16}$  live on the intersection of the non-compact orientifold plane with S and are partially made of non-perturbative (p,q) strings. The local equation of the Calabi-Yau three-fold at these intersections is

$$\xi^2 = zw \tag{2.24}$$

which means that they correspond to a curve of  $A_1$  ALE singularities. Presumably again  $B_{NS}$  is zero here and they correspond to non-perturbative singularities of type IIb; indeed otherwise we would not expect massless modes of (p,q) strings here. Still this seems to be a very simple local model for producing spinor representations in the IIb language. The non-compact D7 brane intersects S along two curves, one of which is the curve above where the **16** lives, and the other is  $b_3 = 0$  which is where the **10** of SO(10) lives.

Finally we can ask what happens for  $E_6$  models. This corresponds to setting both  $b_5 \to 0$  and  $b_4 \to 0$  identically in the above equations. Then  $b_2$  vanishes identically so the limit we are trying to take does not correspond to a IIb limit (except for very special fibrations [26]).

## 2.3. Constraints from tadpole cancellation

From the local form of the singularity obtained above through the results of Tate's algorithm, we may immediately deduce the homology classes of the matter curves. Com-

puting the discriminant of (2.2), one finds

$$\Delta = z^5 b_5^4 (-b_0 b_5^2 + b_2 b_3 b_5 - b_4 b_3^2) + \mathcal{O}(z^6)$$
(2.25)

Thus the matter curves are given by

$$\Sigma_{10} = \{b_5 = 0\}, \qquad \Sigma_5 = \{R = 0\}$$
 (2.26)

which yields the following homology classes:

$$[\Sigma_{10}] = c_1 - t, \qquad [\Sigma_5] = 8c_1 - 3t$$
 (2.27)

In particular it follows that

$$[\Sigma_5] - 3[\Sigma_{10}] - 5c_1 = 0 \tag{2.28}$$

Of course we also know the precise equation of the matter curves, but even these topological constraints are already quite restrictive. Mathematically, these are necessary conditions for the local geometry to be an elliptically fibered Calabi-Yau with section.

Although it is clear from our construction that these constraints have to be satisfied, it would be more satisfactory to give them a physical interpretation. In six dimensional compactifications of F-theory such constraints can be understood more physically as a consequence of anomaly cancellation [27]. For instance the relation (2.28) is then equivalent to cancellation of the  $tr_f(F^4)$  anomaly. One expects such relations to hold also in more general F-theory settings [28]. We largely follow [27, 28] in the remainder of this subsection.

Consider the worldvolume of a 7-brane S, intersecting another 7-brane  $S_a$  over a curve  $\Sigma_a$ . Under a gauge/Lorentz transformation, in the presence of (p,q) 7-branes we get an additional contribution to the variation of the action given by

$$\delta_{\Lambda,\Theta} S \sim \int I^1_{adj,6}(\Lambda,\Theta) \wedge \delta^2(S) \wedge \delta^2(S) - \sum_{R_a} \int I^1_{R_a,6}(\Lambda,\Theta) \wedge \delta^4(\Sigma_a)$$
 (2.29)

where  $\Lambda$  is a local gauge transformation and  $\Theta$  is a local Lorentz transformation. Here  $I_{R,6}^1$  is given through the descent procedure as

$$dI^{1} = \delta I^{0}, \quad dI^{0} = I_{R,8} = \left[ \mathbf{ch}_{R}(F) \wedge \hat{\mathbf{A}}(R) \right]_{8}$$
 (2.30)

or more explicitly

$$\hat{I}_{R,8} = \frac{1}{24} \text{Tr}_R(F^4) - \frac{1}{96} \text{Tr}_R(F^2) \text{Tr}(R^2) + \frac{rk}{128} \left( \frac{1}{45} \text{Tr}(R^4) + \frac{1}{36} \text{Tr}(R^2)^2 \right)$$
(2.31)

where F is understood to be the gauge field on the gauge 7-brane wrapped on S, and  $\hat{I} = (i(2\pi)^{d/2})I$ . Further we have  $\delta^2(S) \wedge \delta^2(S_a) = m_a \delta^4(\Sigma_a)$  and  $\delta^2(S) \wedge \delta^2(S) = -c_1(S) \wedge \delta^2(S)$ . Note that intersections are frequently not transverse in F-theory and  $m_a \neq 1$ . This expression is the most straightforward generalization of the usual expression for D-branes [29, 30]. The hypermultiplet spinors are ordinary 6d spinors which do not carry R-charges, so the expression for their anomaly is the usual one. The 8d gauginos also carry R-charges which gives an extra contribution proportional to  $c_1(K_S)$ . There could be further contributions to  $\delta S$  in compact models, but here we will concentrate on the pieces that are associated to the gauge theory and have to be cancelled even in a local model.

In order to check anomaly cancellation we convert all the gauge traces to traces in the fundamental representation:

$$\operatorname{Tr}_{R}(F^{4}) = x_{R} \operatorname{Tr}_{f}(F^{4}) + y_{R} \operatorname{Tr}_{f}(F^{2})^{2}, \qquad \operatorname{Tr}_{R}(F^{2}) = n_{R} \operatorname{Tr}_{f}(F^{2})$$
 (2.32)

In F-theory, the only massless tensor field available for the Green-Schwarz mechanism is the RR field  $C_4$ . Thus one would expect that the anomaly can be cancelled by mediation of  $C_4$  if and only if the anomaly polynomial is factorizable, i.e. the matter representations occurring are such that

$$\hat{I}_{12} = \left[ \sum_{0,a} n_a \delta^2(S_a) \wedge (2 \operatorname{Tr}_f(F^2) - \frac{1}{2} \operatorname{Tr}(R^2)) \right]^2, \tag{2.33}$$

The corresponding tadpole cancellation condition is the well-known constraint:

$$N_{D3} = \frac{\chi(Y_4)}{24} - \frac{1}{8\pi^2} \int_{Y_4} \mathsf{G} \wedge \mathsf{G}$$
 (2.34)

Since all three terms receive unknown contributions from infinity, we do not have to worry about this condition in a local model.

However this leaves a puzzle. The  $\operatorname{Tr}_f(F^4)$  anomalies are non-zero and localized at different places in the internal space. So how do these pieces get cancelled exactly? There must be something mediating them. In perturbative type IIb, the  $\operatorname{Tr}_f(F^4)$  and  $\operatorname{Tr}(R^4)$  anomalies on branes are cancelled by mediation of the RR fields  $C_0/C_8$ . However in F-theory these fields are massive and do not appear as propagating fields in the effective action. Nevertheless it seems clear what must happen: in general F-theory compactifications integrating out the massive modes of the RR fields  $C_0$  and  $C_8$  leaves an effective interaction whose variation cancels the  $\operatorname{Tr}_f(F^4)$  anomalies.

A similar issue in fact also arises in M-theory on  $G_2$  manifolds and has been analyzed there [31] (see also [21] for a discussion). In the M-theory setting, chiral fermions are localized at points on the worldvolume of the gauge brane. In type IIa the corresponding

anomalies would be mediated by the RR gauge field, but in M-theory this field is massive. Nevertheless there is a residual interaction  $\int K \wedge \omega^{(5)}$  which transforms under gauge transformations, and the Gauss law for  $K \sim dA_{RR}^{(1)}$  is satisfied precisely when the  $\text{Tr}_f(F^3)$  anomalies are cancelled.

We have not precisely worked out the analogous statements in F-theory. The problem is that if we apply the analogous trick, rewriting  $\int C_0 \wedge F^4 \sim -\int dC_0 \wedge \omega_7$ , it does not yield an interaction that is invariant under Sl(2,Z) transformations, so it is incomplete. However for our purposes we don't really need to work this out in detail, because we can use the IIb orientifold limit identified in section 2.2 to show that the expected constraints have to be satisfied. In the IIb limit the anomaly is cancelled by  $C_0/C_8$  exchange as usual, and we get the following modified Bianchi identity:

$$dF_1/2\pi = \sum_{D7} n_a \delta^2(S_a) - 8 \sum_{O7} \delta^2(O7)$$
 (2.35)

Here we use the 'upstairs' picture, that is we write the relation on the covering space before taking the orientifold quotient. (F-theory corresponds more naturally to the 'downstairs' picture).

Now the integral of  $dF_1$  over any closed two-cycle is zero. Let us integrate over any curve  $\Sigma_b$  in S, and let us write (2.35) more suggestively as

$$dF_1/2\pi = 5\delta^2(S) + \delta^2(S_a) + 5\delta^2(S') - 8\delta^2(O7) + \text{other}$$
(2.36)

where S' is the mirror of S under the orientifold action, the O7-plane is the one intersecting S over  $\Sigma_{10}$  (where it also intersects S'), and  $S_a$  is the part of the  $I_1$  locus intersecting S over  $\Sigma_5$ . Then we find

$$0 = -5c_1(S) \cdot \Sigma_b + \Sigma_5 \cdot \Sigma_b + (5-8)\Sigma_{10} \cdot \Sigma_b$$
(2.37)

or equivalently

$$[\Sigma_5] - 3[\Sigma_{10}] - 5c_1 = 0 \tag{2.38}$$

in  $H_2(S, \mathbf{Z})$ , which is what we wanted to show. More generally we expect the relation

$$\sum_{R_a} x_{R_a} [\Sigma_a] - \frac{1}{2} x_{\text{adj}} c_1(S) = 0$$
 (2.39)

to be equivalent to cancelling the  $\operatorname{Tr}_f(F^4)$  anomalies, but we have not been able to show this in full generality. As a special case, in six-dimensional compactifications of F-theory the above homology classes are all proportional to the class of a point, and this relation was verified in [27].

Following [28], we may get a second constraint by using a further relation in F-theory models:

$$\Delta = -12K_{B_3} \tag{2.40}$$

This is also a kind of 7-brane tadpole cancellation (eg. on K3 it restricts the total number of 7-branes to be 24), but it differs from (2.35). Since we have an SU(5) singularity along S, we may write

$$\Delta = 5[S] + \Delta' \tag{2.41}$$

If we assume there are only matter curves for hypermultiplets in the  $\mathbf{5}$  or  $\mathbf{10}$ , as is generically the case, then by intersecting with S we obtain

$$-5t + 4\Sigma_{10} + \Sigma_5 = -12K_{B_3}|_S \tag{2.42}$$

Here we used  $[S] \cdot [S] = c_1(NS)|_S = -t$ . The intersection multiplicities can be read from the explicit form of the discriminant (2.25) (the coefficient of  $[\Sigma_{10}]$  can presumably be understood from the fact that the charge of an orientifold plane is -4 in the 'downstairs' picture). Further applying the adjunction formula  $K_{B_3}|_S = K_S + t$ , we find that

$$7t + 4\Sigma_{10} + \Sigma_5 = -12K_S \tag{2.43}$$

Together with the earlier constraint (2.38), it then follows that the homology classes of the matter curves are given by

$$[\Sigma_{10}] = c_1(S) - t, \qquad [\Sigma_5] = 8c_1 - 3t$$
 (2.44)

exactly as promised.

#### 2.4. Higgs bundles, spectral covers and ALE-fibrations

There are several equivalent descriptions of the supersymmetric configurations of an 8d gauge theory. We may describe such a configuration as an ALE fibration, which is how it arises in F-theory in 'closed string' variables. However we may also think of it more intrinsically in terms of field configurations of the adjoint scalars and gauge field. This gives us the Higgs bundle picture. Finally we may replace the Higgs and gauge fields by their eigenvalues. This gives us the spectral cover picture, or a fibered weight diagram. The latter yields conventional B-branes in an auxiliary non-compact Calabi-Yau three-fold X. The description of B-branes in a Calabi-Yau is already a well-developed subject and so this picture is the most convenient for doing actual constructions and calculations. In this section, we spell out the spectral cover description and its relation to the other pictures in a bit more detail.

Much of the structure discussed here has been discussed in the heterotic setting, but the main point is that it is in fact *intrinsic* to the 8d supersymmetric Yang-Mills theory and therefore applies to an arbitrary local F-theory geometry, or any other UV completion of 8d Yang-Mills theory. Moreover the spectral cover description allows us to tie up some technical loose ends from our previous papers. A completely analogous construction can be made in 7d supersymmetric Yang-Mills theory [21] and leads to the construction of local models in M-theory, in the large volume limit where the Yang-Mills theory gives an accurate description. One can also apply the dictionary for ALE fibrations over a Riemann surface. This is essentially classic geometric engineering.

## 2.4.1. The dictionary

Given an ALE fiber over a point  $p \in S$ , we may choose a basis  $\alpha_i$  of  $H_2(ALE_p, \mathbf{Z})$  corresponding to the fundamental roots of the corresponding ADE Lie algebra (obviously this depends on a choice of Weyl chamber). We may choose a dual basis  $\omega^j$  of  $H^2(ALE_p, \mathbf{Z})$  satisfying

$$\int_{\alpha_i} \omega^j = \delta^{ij} \tag{2.45}$$

The Cartan generators for the adjoint fields arise from deformations of the complex structure

$$\delta\Omega^{4,0} = \Phi_j^{2,0} \wedge \omega^j \tag{2.46}$$

and the gauge fields arise from deformations of the three-form field

$$\delta C_3 = A_j \wedge \omega^j \tag{2.47}$$

Further, the non-abelian generators arise from membranes wrapped on the vanishing cycles of the ALE. Thus in F-theory, an ALE fibration fibered over a surface S yields precisely the data of a supersymmetric 8d gauge theory compactified on S: a gauge field A on a bundle E on S, and a 'Higgs field'  $\Phi$  which is a section of

$$K_S \otimes Ad(G)$$
 (2.48)

where G is the structure group of the bundle E.

The conditions for supersymmetry in the 8d gauge theory are obtained by dimensional reduction. Namely we start with the Hermitian-Yang-Mills equations in 10d, and assume fields are invariant under translation along a complex line. Then we can write the gauge field as

$$\mathsf{A}^{0,1} = A_{\bar{1}}(z^1, z^2)d\bar{z}^1 + A_{\bar{2}}(z^1, z^2)d\bar{z}^2 + \Phi_{\bar{3}}(z^1, z^2)d\bar{z}^3 \tag{2.49}$$

The F-terms are

$$\mathsf{F}^{0,2} = 0 \qquad \Rightarrow \qquad F^{0,2} = 0, \qquad \bar{D}_A \Phi = 0$$
 (2.50)

and the D-terms are

$$g^{i\bar{j}}\mathsf{F}_{i\bar{j}} = 0 \qquad \Rightarrow \qquad g^{i\bar{j}}F_{i\bar{j}} + g^{i_1\bar{j}_1}g^{i_2\bar{j}_2}[\Phi^{\dagger}_{\bar{i}_1\bar{j}_2}, \Phi_{i_1i_2}] = 0$$
 (2.51)

where  $\Phi_{i_1i_2} = \Phi_{\bar{3}}\Omega_{i_1i_23}g^{3\bar{3}}$  is a (2,0) form. The *D*-term is the moment map for gauge transformations acting on the pair  $(A,\Phi)$  with respect to the Kähler form associated to the metric

$$g(\mathsf{A},\mathsf{A}) = \int |\mathsf{A}^{0,1}|^2 = \int |A^{0,1}|^2 + |\Phi^{2,0}|^2 \tag{2.52}$$

The F- and D-term equations are called Hitchin's equations or the Yang-Mills-Higgs equations. They are the critical points of two functionals, the holomorphic Chern-Simons functional and the D-term potential:

$$W = \frac{1}{4\pi} \int_{S} \text{Tr} (A + \Phi) \bar{\partial} (A + \Phi) + \frac{2}{3} (A + \Phi)^{3}$$

$$V_{D} \sim \frac{1}{2} \int |J \wedge F + [\Phi, \Phi^{\dagger}]|^{2}$$
(2.53)

In ALE fibrations, only the Cartan generators of  $\Phi$  have a non-vanishing VEV, and so we have  $[\Phi, \Phi^{\dagger}] = 0$ . Such solutions are said to be regular. Field configurations with  $[\Phi, \Phi^{\dagger}] \neq 0$  should have an alternative description as space-filling 9-branes satisfying the Hermitian Yang-Mills equations. In the following we will assume that  $[\Phi, \Phi^{\dagger}] = 0$ . Then this data defines a Higgs bundle [32, 33].

Since  $[\Phi, \Phi^{\dagger}] = 0$ , the real and imaginary parts of  $\Phi$  can be simultaneously diagonalized and we may try to replace  $\Phi$  by its spectral data, i.e. its eigenvalues and eigenvectors. For convenience we temporarily focus on SU(n) gauge groups, though analogous constructions exist for any gauge group. We let s denote a coordinate on the canonical bundle  $K_s$ . The Hitchin map is the map that sends the Higgs field  $\Phi$  to its Casimirs. In the SU(n) case, the Casimirs are the coefficients of the polynomial

$$\det(sI - \Phi) = 0 \tag{2.54}$$

This polynomial equation makes sense globally on a non-compact CY three-fold X, consisting of the total space of the canonical bundle  $K_S \to S$ . For a generic point on S the roots  $\lambda_i$  of this polynomial give us n points on the fiber of  $K_S$ . Thus the n roots trace out a complex surface C which covers the zero section n times. This is the spectral cover for the fundamental representation of SU(n). Since we will be interested in non-compact covers, we should allow simple poles for the Higgs fields. We can get rid of the poles in (2.54) by multiplying with a suitable section. Thus instead of (2.54) we will write the degree n equation

$$0 = b_0 s^n + b_1 s^{n-1} + b_2 s^{n-2} + \ldots + b_n$$
(2.55)

Since s = 0 is marked, the only coordinate transformations allowed are rescaling. For SU(n) gauge groups we further want to impose that all the roots add up to zero. Since we have

$$\lambda_1 + \ldots + \lambda_n = b_1, \tag{2.56}$$

therefore we set  $b_1 = 0$ . The surface C is non-compact. Along the locus  $b_0 = 0$ , two of the roots go off to infinity. Let us denote the divisor  $b_0 = 0$  on S by  $\eta$ . Since s is a coordinate on  $K_S$ , the  $b_i$  are then seen to be sections of

$$b_i \sim \eta - i c_1(S) \tag{2.57}$$

We further have to describe the gauge field A in this picture, or equivalently the bundle E. To do this, it is useful to think of  $\Phi$  as a map

$$\Phi: E \to E \otimes K_S \tag{2.58}$$

Then under the action of  $\Phi$ , each fiber of E can be decomposed into its eigenspaces  $\bigoplus_i \mathbf{C} |i\rangle$ . Let us denote coordinates on the total space  $K_S \to S$  by pairs (p, s) where  $p \in S$  and s is the coordinate on the fiber. The assignment

$$(p, \lambda_i) \to \mathbf{C} |i\rangle$$
 (2.59)

yields a line bundle L on C called the spectral line bundle.<sup>2</sup> Furthermore since  $\bar{D}_A \Phi = 0$ ,  $\bar{D}_A$  commutes with the action of  $\Phi$  on E, and we can we can simultaneously diagonalize  $\bar{D}_A$ . Thus we get a holomorphic connection on L, and we can pick a section  $|i\rangle \in \mathbb{C} |i\rangle$  by parallel transport. Note that the spectral cover and line bundle in  $K_S$  satisfy the usual requirements of a B-brane in the large volume limit: a holomorphic cycle with a holomorphic bundle on it.

Conversely, given a spectral cover and a spectral line bundle, we may recover the Higgs field  $\Phi$  and the bundle E. We may represent  $\Phi$  as

$$\Phi = \sum_{i} \lambda_{i} \Pi_{i} \tag{2.60}$$

where  $\Pi_i$  is the projection on  $\mathbb{C}|i\rangle$ . More formally we can pull-back  $\Phi$  to the total space of the canonical bundle. Then we may write it as the canonical section

$$\pi^* \Phi(p, s) = sI \tag{2.61}$$

where I is the identity operator. Therefore given the spectral data, we may recover the Higgs bundle as:

$$E = p_{C*}L, \qquad \Phi = p_{C*}s$$
 (2.62)

The amount of the following precisely, let us denote  $R = K_{C/S}$  the ramification divisor, and  $s \in H^0(C, p_C^*K_S)$  the tautological eigenvalue section, whose value at a point (p, s) is given by s. Then  $L \otimes \mathcal{O}(-R)$  is the kernel of  $p_C^*\Phi - sI : p_C^*E \to p_C^*E \otimes p_C^*K_S$ .

The gauge field A is obtained as the push-forward of a connection on L. Furthermore if we want an SU(n) bundle rather than a U(n) bundle, then we also need to require

$$\det(p_{C*}L) = \mathcal{O} \tag{2.63}$$

where  $\mathcal{O}$  is the trivial line bundle. This gives a topological constraint on the allowed spectral line bundles.

We may go back and forth between this description and the ALE fibration. For SU(n) gauge groups the  $A_{n-1}$ -ALE fibration is defined by the following equation

$$y^{2} = x^{2} + b_{0}s^{n} + b_{2}s^{n-2} + \ldots + b_{n}$$
(2.64)

As far as the variation of Hodge structure is concerned, the quadratic terms  $x^2$  and  $y^2$  are irrelevant and may be dropped, recovering our previous equation. This argument is well known from Landau-Ginzburg models, where we 'integrate out' the fields with a quadratic potential.

Furthermore in terms of the ALE fibration  $Y_4$ , the spectral line bundle is encoded as G-flux. Let us think of (2.64) as a conic bundle fibered over the complex plane parametrized by s. We have a map

$$p_R: R \to C \tag{2.65}$$

where R is obtained from C by attaching a line (with equation y = x) to each point in the fiber of the covering  $C \to S$ . Furthermore we have a map

$$i: R \to Y_4 \tag{2.66}$$

which embeds these lines in the ALE (2.64), each line sitting at the corresponding point  $s = \lambda_i$  in the s-plane. Let us decompose the flux of the spectral line bundle as

$$c_1(L) = \frac{1}{2}c_1(K_{C/S}) + \gamma \tag{2.67}$$

where  $K_{C/S} = K_C - p_C^* K_S$  is the ramification divisor, and  $p_{C*} \gamma = 0$ . Then the spectral line bundle and the G-flux are related by

$$G = i_* p_R^* \gamma - q P_{Y_4}(S) \in H^{2,2}(Y_4)$$
(2.68)

Here  $P_{Y_4}(S)$  is the Poincaré dual to the zero section S in  $Y_4$  (which can be dually represented by an ALE fiber), and q is determined by requiring that  $\int_S G = 0$ , which gives  $q = \gamma \cdot_C \Sigma_E$ . Given this explicit expression it is not too hard to check that such fluxes are

always primitive, i.e. satisfy  $J \wedge G = 0$  on  $Y_4$ , if  $p_{C*}\gamma = 0$ . For U(n) bundles, we need to make sure that if J contains a piece  $\pi^*J_S$  pulled-back from S, then  $p_{C*}\gamma \cdot J_S = 0$ .

We may state this more naturally as follows. We have a single charge lattice  $\Lambda = \bigoplus_i \mathbf{Z} |i\rangle$  varying over S, which can be given two equivalent interpretations. In the ALE picture  $\Lambda$  is identified with  $H_2(ALE, \mathbf{Z})$ . In the spectral cover picture it is identified with  $\bigoplus_i \mathbf{Z}e_i$ , where  $e_i$  are the nodes of the corresponding ADE Dynkin diagram. Similarly the dual lattice  $\Lambda^*$ , which is actually isomorphic to  $\Lambda$  because ADE lattices are self-dual, is given by  $H^2(ALE, \mathbf{Z})$  in the ALE picture, or alternatively by  $\bigoplus_i \mathbf{Z}e_i^*$ . These two local systems are naturally isomorphic. The  $b_i/b_0$ 's correspond to the invariant polynomials of a meromorphic section of  $\Lambda^* \otimes K_S$ . The flux of the spectral line bundle or equivalently the G-flux corresponds to a generator of

$$H^2(S, \Lambda^*) \cap H^{1,1}(S, \Lambda^* \otimes \mathbf{C})$$
 (2.69)

## 2.4.2. Other associated spectral covers

The spectral cover we have considered so far should really be called  $C_E$ , to indicate that it corresponds to the fundamental representation. We can also construct spectral covers for other representations, which typically describe equivalent data. One important cover that we will need is the spectral cover  $C_{\Lambda^2 E}$  for the anti-symmetric representation of SU(n). This has  $\frac{1}{2}n(n-1)$  sheets. Each sheet intersects a fiber of  $K_S$  in the points

$$\Lambda^2 E: \qquad \lambda_i + \lambda_j, \quad i < j \tag{2.70}$$

where addition is defined in the obvious way in each fiber. In fact it is not hard to write down an explicit equation using mathematica. For the case n = 5, the cover is defined by the degree 10 equation

$$0 = s^{10} + 3s^8c_2 - s^7c_3 + s^6(3c_2^2 - 3c_4) + s^5(-2c_2c_3 + 11c_5) + s^4(c_2^3 - c_3^2 - 2c_2c_4) + s^3(-c_2^2c_3 + 4c_3c_4 + 4c_2c_5) + s^2(-c_2c_3^2 + c_2^2c_4 - 4c_4^2 + 7c_3c_5) + s(c_3^3 + c_2^2c_5 - 4c_4c_5) - c_3^2c_4 + c_2c_3c_5 - c_5^2$$
(2.71)

where  $c_i = b_i/b_0$  and the whole equation should be multiplied with  $b_0^3$  in order to remove the denominators. We denote the intersection of  $C_{\Lambda^2 E}$  with the zero section s = 0 by  $\Sigma_{\Lambda^2 E}$ . The surface  $C_{\Lambda^2 E}$  is singular when two of the eigenvalues coincide, i.e.  $\lambda_i + \lambda_j = \lambda_k + \lambda_l$  for some i, j, k, l. This happens in codimension one, so the matter curve  $\Sigma_{\Lambda^2 E}$  is

<sup>&</sup>lt;sup>3</sup>Note that the subscript here indicates the representation of the holonomy group, not the unbroken gauge group. In our discussion later however we will instead use the subscript to denote the representation under the GUT group, as in our previous papers. Thus in our SU(5) examples later we will have  $\Sigma_{\Lambda^2 E} = \Sigma_5$  and  $\Sigma_E = \Sigma_{10}$ .

also singular at isolated points. The spectral line bundle on this cover is given fiberwise by

$$L_{\Lambda^2 E}: \qquad (p, \lambda_i + \lambda_j) \to \mathbf{C} |i\rangle \wedge |j\rangle$$
 (2.72)

It is not really a line bundle but a (torsion-free) sheaf, its rank jumping up at the singular locus, and one has to desingularize in order to define things unambiguously. Still this data is determined uniquely by the spectral line bundle for the cover of the fundamental representation, as follows.

In order to write an unambiguous formula it is more natural to think about unembedded covers [34]. We take pairs of points  $(q_1, q_2) \in C_E \times_S C_E$ , and remove the diagonal where  $q_1 = q_2$ . Then we define the quotient<sup>4</sup>

$$\tilde{C}_{\Lambda^2 E} = \{ (q_1, q_2) \in C_E \times_S C_E \mid q_1 \neq q_2 \} / \mathbf{Z}_2$$
 (2.73)

where the  $\mathbb{Z}_2$  action interchanges  $(q_1, q_2) \to (q_2, q_1)$ . This cover is embedded in  $X \times_S X/\mathbb{Z}_2$ , but not in X, and provides a resolution of  $C_{\Lambda^2 E}$ . There is a natural map

$$C_E \times_S C_E - \operatorname{diag}(C_E) \rightarrow \tilde{C}_{\Lambda^2 E} \rightarrow C_{\Lambda^2 E}$$
 (2.74)

The last map is given fiberwise by sending  $(\lambda_i, \lambda_j) \to \lambda_i + \lambda_j$ . The pairs  $(\lambda_i, \lambda_j)$  and  $(\lambda_k, \lambda_l)$  are distinct in  $\tilde{C}_{\Lambda^2 E}$  even when  $\lambda_i + \lambda_j = \lambda_k + \lambda_l$  in  $C_{\Lambda^2 E}$ . The inverse image of  $\Sigma_{\Lambda^2 E}$  in  $\tilde{C}_{\Lambda^2 E}$  is its normalization  $\tilde{\Sigma}_{\Lambda^2 E}$ . The spectral line bundle  $L_E$  on  $C_E$  gets mapped to a smooth line bundle on  $\tilde{C}_{\Lambda^2 E}$ :

$$L_E \times L_E \quad \to \quad \tilde{L}_{\Lambda^2 E} \quad \to \quad L_{\Lambda^2 E}$$
 (2.75)

It only gets mapped to a sheaf  $L_{\Lambda^2 E}$  on  $C_{\Lambda^2 E}$  because the map  $\tilde{C}_{\Lambda^2 E} \to C_{\Lambda^2 E}$  is two-to-one at the singular locus, but this is irrelevant since we should work with the non-singular surface  $\tilde{C}_{\Lambda^2 E}$ . This construction should be interpreted as follows. The spectral line bundle on  $C_{\Lambda^2 E}$  is the set of eigenlines  $|i\rangle \wedge |j\rangle$  of  $\Lambda^2 E$  under the action of the Higgs field. When  $\lambda_i + \lambda_j = \lambda_k + \lambda_l$  the cover  $C_{\Lambda^2 E}$  is singular, so there is an ambiguity in assigning eigenlines of  $\Lambda^2 E$  to eigenvalues of  $\Phi_{\Lambda^2 E}$  in a neighbourhood of the singular locus. This ambiguity is naturally resolved by recalling that the assignment of eigenlines to eigenvalues was unambiguous for E (assuming  $C_E$  is smooth), in other words it is naturally resolved by requiring that  $L_{\Lambda^2 E}$  descends from a smooth line bundle on  $\tilde{C}_{\Lambda^2 E}$ . As emphasized in [3], this means that keeping track of the gauge indices implies that the hypermultiplet at the intersection really couples to  $\tilde{L}_{\Lambda^2 E}$ . Thus the hypermultiplet propagates on the normalized matter curve  $\tilde{\Sigma}_{\Lambda^2 E}$  rather than on  $\Sigma_{\Lambda^2 E}$  itself.

<sup>&</sup>lt;sup>4</sup>Strictly we have to take the closure and then take the quotient. We oversimplified this issue here and in the remainder in order to avoid too much notation.

Similarly we may construct spectral covers for other representations. For instance the spectral cover for the symmetric representation  $C_{S^2E}$  is given fiberwise by

$$S^2E: \qquad \lambda_i + \lambda_j, \quad i \le j \tag{2.76}$$

We will not have any need for these other coverings in this paper.

## 2.4.3. Fermion zero modes

Now that we have a description of configurations in the 8d gauge theory in terms of holomorphic cycles and bundles on them, we would like to describe the zero modes of the Dirac operator. In holomorphic geometry the Dirac operator splits into a Dolbeault operator

$$\bar{D} = \bar{\partial} + A^{0,1} + \Phi_{\Omega}^{2,0} \tag{2.77}$$

and its adjoint  $\bar{D}^{\dagger}$ . Here  $\Phi_{\Omega}^{2,0}$  is our  $\Phi^{2,0}$  Higgs field contracted with the anti-holomorphic (0,3) form on the non-compact Calabi-Yau X, yielding a (0,1) form whose index lies in the normal direction to S in X. Since supersymmetry is preserved  $\Omega$  is covariantly constant and hence the normal bundle is identified with the canonical bundle. The spinor configuration space together with the  $\bar{D}$  operator yield a complex whose cohomology is computed by Ext-groups. On the other hand, the zero modes of  $D = \bar{D} + \bar{D}^{\dagger}$  are also in one-one correspondence with the cohomology of  $\bar{D}$ . Let us denote by i,j the embedding of divisors into X, and assume that R, R' are sheaves on these divisors. Then the wave functions of fermion zero modes are in one-one correspondence with generators of Ext groups:

$$\operatorname{Ext}_{X}^{p}(i_{*}R, j_{*}R') \tag{2.78}$$

As usual, the index p correlates with the 4d chirality as  $(-1)^p$ . For p = 1, 2 the 4d part of the wave function belongs to a chiral (anti-chiral) superfield, and for p = 0, 3 we get four-dimensional gauginos (or possibly ghosts if suitable stability conditions are not satisfied). These cohomology groups are of course localized on the intersection of the supports.

The Ext-groups naturally give a unified description of all the possibilities. Let us assume that the full spectral cover splits up into a multiple of the zero section (the 'gauge brane') and some additional non-compact pieces (the 'flavour branes'). If we assume that i embeds the zero section in X and j embeds the remainder of the spectral cover in X, then this reduces to

$$\operatorname{Ext}_{\mathbf{Y}}^{p}(i_{*}R, j_{*}R') \sim H^{p-1}(\Sigma, R^{\dagger} \otimes R' \otimes K_{S}|_{\Sigma})$$
(2.79)

for the case of intersecting branes, and

$$\operatorname{Ext}_{X}^{p}(i_{*}R, i_{*}R') \sim H^{p-1}(S, R^{\dagger} \otimes R' \otimes K_{S}) \oplus H^{p}(S, R^{\dagger} \otimes R')$$
 (2.80)

for the case of coincident branes, as was deduced in [1, 2]. To be more precise, there is a 'spectral sequence' which starts with the right-hand-side and may possibly lift some of the zero modes to arrive at the left-hand-side, as was actually noted in [1]. In typical examples this lifting does not happen, and so we can take this relation to be an equality. The number of moduli of the configuration is given by

$$N_{mod} = \operatorname{Ext}_{X}^{1}(j_{*}L_{E}, j_{*}L_{E}) \sim h^{2,0}(C) \oplus h^{0,1}(C).$$
(2.81)

Here we assumed that the spectral cover is smooth. Then the number of moduli is independent of the spectral line bundle on C, which just cancels in the formula. Similarly the number of adjoints is given by

$$N_{adj} = \text{Ext}_X^1(i_*\mathcal{O}_S, i_*\mathcal{O}_S) \sim h^{2,0}(S) \oplus h^{0,1}(S)$$
 (2.82)

Further, the unambiguous formula for the amount of chiral matter on  $\tilde{\Sigma}_{\Lambda^2 E}$  is given by

$$\operatorname{Ext}^{1}(i_{*}\mathcal{O}_{S}, j_{*}\nu_{*}\tilde{L}_{\Lambda^{2}E}) \sim H^{0}(\tilde{\Sigma}_{\Lambda^{2}E}, \tilde{L}_{\Lambda^{2}E} \otimes \nu^{*}K_{S}|_{\tilde{\Sigma}_{\Lambda^{2}E}})$$
 (2.83)

where  $\nu$  is the normalization  $\tilde{\Sigma}_{\Lambda^2 E} \to \Sigma_{\Lambda^2 E}$ . This recovers the answer found in [3].

The spectral cover description also allows us to give a precise mathematical definition of the classical Yukawa couplings (and higher dimension couplings as well), at least up to field redefinitions. It is simply given by the Yoneda pairing:

$$\operatorname{Ext}^{p_1}(i_{1*}R_1, i_{2*}R_2) \times \operatorname{Ext}^{p_2}(i_{2*}R_2, i_{3*}R_3) \times \operatorname{Ext}^{3-p_1-p_2}(i_{3*}R_3, i_{1*}R_1) \to \mathbf{C}$$
 (2.84)

Again this expression summarizes all the possibilities, with wave functions either localized in the bulk or on 7-brane intersections. One should be careful about drawing conclusions from such computations however. The usual warnings about the relation with the physical Yukawa couplings (which depend on the Kähler potential and may receive loop corrections) apply.

## 2.4.4. $E_8$ Higgs bundles

Now we return to the case of primary interest. In local F-theory models we are dealing with fibrations by  $E_8$  ALE spaces, or equivalently with  $E_8$  Higgs bundles. The relevant spectral cover is the one for the adjoint representation, which we will simply call 'the' spectral cover. The adjoint representation is 248 dimensional, of which eight are Cartan generators. Thus the full spectral cover will have 248 sheets. In order to break to an SU(5) GUT group, we turn on an  $Sl(5, \mathbb{C})$  Higgs bundle. The adjoint representation of  $E_8$  decomposes as

$$248 = (24,1) + (1,24) + (5,10) + (\overline{5},\overline{10}) + (10,\overline{5}) + (\overline{10},5)$$
 (2.85)

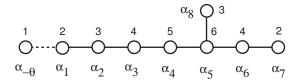


Figure 1: The extended  $E_8$  Dynkin diagram and Dynkin indices.

Thus the  $E_8$  spectral cover breaks up into several pieces, which can be labelled by representations of the holonomy group of the Higgs bundle. Clearly the relevant spectral covers are those for the fundamental representation and for the anti-symmetric representation of SU(5).

Referring back to the general form of a local SU(5) model as derived using Tate's algorithm:

$$y^{2} = x^{3} + b_{0}z^{5} + b_{2}xz^{3} + b_{3}yz^{2} + b_{4}x^{2}z + b_{5}xy$$
(2.86)

The parameters can be identified with the following Casimirs of a meromorphic  $Sl(5, \mathbf{C})$  Higgs bundle:

$$C_i(\Phi) \sim \text{Tr}(\Phi^i) \sim b_i/b_0$$
 (2.87)

To see this, the singularity (2.86) is generically of type  $A_4$ , but by sequentially tuning the  $b_i$  to zero we get successively SO(10),  $E_6$ ,  $E_7$  and an  $E_8$  singularity. Since the holonomy group of the Higgs bundle is the commutant of the gauge group in  $E_8$ , then the parameters must correspond to the indicated Casimirs. (A more precise way to see this [1] is by using the F-theory/heterotic duality map). We see that there exists a canonical map between the parameters in the ALE fibration, and an SU(5) spectral cover in  $K_S \to S$  defined by

$$b_0 s^5 + b_2 s^2 + \ldots + b_5 = 0 (2.88)$$

Note that  $\eta$  is related to our earlier t by  $\eta = 6c_1 - t$ . The five roots  $\{\lambda_1, \ldots, \lambda_5\}$  of this polynomial determine the sizes of all the cycles of the  $E_8$  ALE space. Recall that

$$\Phi |i\rangle = \lambda_i |i\rangle \tag{2.89}$$

As discussed in section 5.1 of [5], we can take the five roots of the polynomial to correspond to the periods of the following cycles (up to Weyl permutations)

$$|1\rangle = \alpha_4 \qquad |4\rangle = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 
|2\rangle = \alpha_3 + \alpha_4 \qquad |5\rangle = \alpha_{-\theta} + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 
|3\rangle = \alpha_2 + \alpha_3 + \alpha_4 \qquad (2.90)$$

The sizes of the cycles  $\{\alpha_5, \ldots, \alpha_8\}$  are taken to be zero, generating an SU(5) GUT group, and all other cycles are obtained as linear combinations. The matter curve  $\Sigma_{10}$ 

corresponds to  $\lambda_i = 0$  for some i, the matter curve  $\Sigma_{\mathbf{5}}$  corresponds to  $\lambda_i + \lambda_j = 0$  for some i, j, etc. The top Yukawa is localized at  $\lambda_i = \lambda_j = \lambda_i + \lambda_j = 0$ , the bottom at  $\lambda_i + \lambda_j = \lambda_k + \lambda_l = \epsilon_{ijklm}\lambda_m = 0$ , and the  $\mathbf{5} \cdot \overline{\mathbf{5}} \cdot \mathbf{1}$  at  $\lambda_i + \lambda_j = \lambda_j + \lambda_k = \lambda_i - \lambda_k = 0$ .

## 2.5. Construction of fluxes

Let us briefly recap what we saw above. Local models in F-theory correspond to ALE fibrations over a surface S with G-flux. Physically we expect that this data, the ALE fibration and the G-flux, can be described as configurations in an 8d supersymmetric Yang-Mills theory compactified on S, i.e. a Higgs bundle. This is indeed the case, and moreover this data is also equivalent to a covering  $C_E$  of the zero section in an auxiliary non-compact Calabi-Yau three-fold  $X = (K_S \to S)$ , together with a holomorphic line bundle. In a IIb-like language, we can call this covering a non-compact flavour brane, whose intersection with the gauge brane (which is wrapped on the zero section of X) yields the matter curve  $\Sigma_{10}$ . The group theory of  $E_8$  implies that there is a second flavour-brane  $C_{\Lambda^2 E}$ , completely determined by the first covering, whose intersection with the gauge brane yields the matter curve  $\Sigma_5$ .

Constructing the branes, or equivalently the ALE fibration, is easy: we only need to specify the  $b_i$ , which are sections of line bundles on S with Chern classes given by  $\eta - ic_1$ . In order to get chiral matter however we must actually turn on a flux on  $C_E$ , which will determine a unique flux on  $C_{\Lambda^2 E}$  by group theory. In this subsection we discuss the issue of constructing such fluxes.

In order to facilitate the analysis we will compactify the local Calabi-Yau to

$$\bar{X} = \mathbf{P}(\mathcal{O} \oplus K_S) \tag{2.91}$$

 $\bar{X}$  is certainly not a Calabi-Yau; the Calabi-Yau metric diverges at infinity. We denote by  $\mathcal{O}(1)$  the line bundle on  $\bar{X}$  which restricts to the eponymous line bundle on each  $\mathbf{P}^1$ -fiber. We may choose homogeneous coordinates  $(u_1, u_2)$  on the  $\mathbf{P}^1$ -bundle which are sections of  $\mathcal{O}(1)$  and  $\mathcal{O}(1) \otimes K_S$  respectively. The coordinate s used previously is identified with  $u_2/u_1$ .

The spectral cover  $C \subset X$  is compactified to a compact surface  $\overline{C} \subset X$  by adding a divisor  $\eta_{\infty}$  at infinity. The equation

$$b_0 s^5 + b_2 s^3 + b_3 s^2 + b_4 s + b_5 = 0 (2.92)$$

has a double zero at  $u_1 = 0$ . Therefore  $\eta_{\infty}$  covers  $\eta$  exactly once and is isomorphic to it. We denote the cohomology class dual to the zero section (the Poincaré dual of S in  $\bar{X}$ ) by  $s_0$  and the class of the section at infinity by  $s_{\infty} = c_1(\mathcal{O}(1))$ . Then we have  $s_{\infty} = s_0 + c_1(TS)$  and  $0 = s_0 \cdot s_{\infty} = s_0 \cdot (s_0 + c_1(TS))$ .

We would like to lift line bundles on C to line bundles on the compact surface  $\bar{C}$  which are easier to study. If the genus of  $\eta_{\infty}$  is non-zero, then there may be line bundles on C

that cannot be lifted to  $\bar{C}$ . However any algebraic line bundle on C can be lifted. To see this, any algebraic line bundle on C is of the form  $\mathcal{O}(D)_C$  for some divisor D. Let  $\bar{D}$  be the closure of D in  $\bar{C}$ . Then  $\mathcal{O}(\bar{D})_{\bar{C}}$  gives a lift of  $\mathcal{O}(D)_C$  as desired. Moreover global G-fluxes are algebraic and yield an algebraic class in the local model. Therefore from here on we may restrict our attention to extendable line bundles.

Now consider a spectral line bundle L on  $\bar{C}$ . The corresponding Higgs bundle is given by  $E = p_{C*}L$ ,  $\Phi = p_{C*}s$ . We have

$$c_1(p_{C*}L) = p_{C*}c_1(L) - \frac{1}{2}p_{C*}r$$
(2.93)

where r is the ramification divisor,  $r = -c_1(\bar{C}) + p_C^*c_1(S)$ . Explicitly we find that

$$r = (n-2)s_0 + p_C^*(\eta - c_1(S))$$
(2.94)

If  $c_1(p_{C*}L)$  is not zero, then we have a  $Gl(n, \mathbf{C})$  Higgs bundle rather than an  $Sl(n, \mathbf{C})$  Higgs bundle. For phenomenological applications we want the latter, so we need to impose the restriction  $c_1(p_{C*}L) = 0$  on the allowed spectral line bundles. Then it is convenient to decompose

$$c_1(L) = \frac{1}{2}r + \lambda\gamma \tag{2.95}$$

where  $\lambda$  is a parameter. The condition  $c_1(p_{C*}L) = 0$  is then equivalent to  $p_{C*}\gamma = 0$ . The class r/2 is generally not integer quantized. Since  $c_1(L)$  must be integer quantized,  $\gamma$  must compensate and can generally not be an integer class either, but it will always be a rational linear combination of integer classes.

So our task is to find integer classes  $\gamma$  with  $p_{C*}\gamma = 0$  in  $H^2(\bar{C}, \mathbf{Z})$ . To get a supersymmetric configuration,  $\gamma$  must further be of Hodge type (1,1). As we will now argue, for generic complex structure moduli there exists only one such class (up to multiplication by an integer), and we can write it down explicitly.

As for line bundles on  $\bar{C}$ , we can use the Lefschetz-Noether theorem.  $\bar{C}$  is the zero locus of a section of  $\mathcal{O}(n) \otimes L_{\eta-nc_1}$ , which is usually an ample line bundle since  $\mathcal{O}(n)$  is ample and  $L_{\eta-nc_1}$  is effective and non-zero. Therefore there is an injective map  $i^*$ :  $H^{1,1}(\bar{X}) \to H^{1,1}(\bar{C})$ . As a result,  $H^{1,1}(\bar{C})$  splits into two pieces, the classes inherited from  $\bar{X}$  and the primitive classes. The Noether-Lefschetz theorem says that when  $\bar{C}$  is ample, then for 'generic' complex structure moduli there are no primitive classes in  $H^{1,1}(\bar{C})$ .

Let us write down the inherited class explicitly. The cohomology group  $H^2(\bar{X}, \mathbf{Z})$  is spanned by  $s_0$  and  $\pi^*H^2(S, \mathbf{Z})$ , i.e. the pull-back of classes on S to  $\bar{X}$ . Therefore the inherited classes in  $H^{1,1}(\bar{C})$  are spanned by the class of the matter curve  $\Sigma_{10}$ , as well as any class on S pulled back to  $\bar{C}$ . (In particular,  $c_1(\bar{C})$  is in this span, by the adjunction formula, and so is  $\eta_{\infty}$ ). Our class  $\gamma$  will be a linear combination of those, but it also needs to satisfy  $p_{C*}\gamma = 0$ , which is clearly not satisfied by any class pulled back from S. Thus

we can single out the class  $[\Sigma_E]$  and subtract the 'trace', i.e. we single out the following unique linear combination:

$$\gamma_u = n[\Sigma_E] - p_C^* p_{C*}[\Sigma_E] = n[\Sigma_E] - p_C^* (\eta - nc_1)$$
(2.96)

We used the subscript u on  $\gamma_u$  to indicate that this class is universal, i.e. it always exist in a local model. In the last equality we just the fact that  $p_{C*}[\Sigma_E]$  is just the class  $[\Sigma_E]$  sitting inside  $H^2(S)$ , and since it is given by  $b_n = 0$  it follows that it can also be written as  $\eta - nc_1$ .

Let us define a line bundle using this class. Its first Chern class will be given by

$$c_1(L) = \frac{1}{2}r + \lambda \gamma_u \tag{2.97}$$

with  $\gamma_u$  as defined above and  $\lambda$  a parameter. From our explicit expressions for r and  $\gamma_u$ , we see that  $c_1(L)$  is an integer class when  $\lambda$  is an integer and n is even, or when  $\lambda = \frac{1}{2} + \frac{1}{2}$  integer and n is odd. For this corresponding choice of spectral line bundle, we can deduce the net amount of chiral matter. It is given by

$$N_{\text{chiral}} = -\chi(i_* \mathcal{O}_S, j_* L) = +\chi(L \otimes K_S|_{\Sigma_{10}}) = \lambda \int_{\Sigma} \gamma_u$$
 (2.98)

where in the last equality we used the Riemann-Roch formula and the fact that  $(r/2 + c_1(K_S))|_{\Sigma} = -c_1(\Sigma)/2$ . We have

$$\Sigma \cdot_{\bar{C}} \Sigma = S_0 \cdot_{\bar{X}} S_0 \cdot_{\bar{X}} \bar{C} = -c_1(S_0) \cdot_{S_0} \Sigma \tag{2.99}$$

Further we have  $\Sigma \cdot_{\bar{C}} p^* \alpha = \alpha \cdot_{S_0} \Sigma$  for any  $\alpha \in H^2(S, \mathbf{Z})$ . Applying this with  $\alpha = \eta - nc_1$ , we see that

$$\gamma_u \cdot_{\bar{C}} \Sigma = -\eta \cdot_{S_0} \Sigma \tag{2.100}$$

Therefore we find

$$N_{\text{chiral}} = \lambda \int_{\Sigma} \gamma_u = -\lambda \eta (\eta - nc_1)$$
 (2.101)

This is of course the same formula as encountered in spectral cover constructions in the heterotic string [37].

Therefore the only fluxes available for general complex structure moduli will give the conventional chirality formula known from the heterotic string. We do not see more general options in the local F-theory set-up. We will call such fluxes *inherited* or *universal*. However, there do exist more general fluxes, both in the heterotic setting and in the F-theory setting. The point is that general fluxes are not supersymmetric for generic Higgs

bundle moduli, and thus are not among the fluxes that we found above. For special values of the moduli (which is called the Noether-Lefschetz locus) there are additional supersymmetric fluxes available, and turning on such fluxes would therefore automatically stabilize some of the moduli. We will call such fluxes primitive or *non-inherited*. Generic fluxes are non-inherited. They exist for both F-theory spectral covers and heterotic spectral covers, where they give rise to rigid bundles on the  $CY_3$  after Fourier-Mukai transform. But they are harder to write down and have not really been analyzed in either context.

#### 2.6. Further constraints

In the previous sections we encountered a number of constraints that must be satisfied for consistency of the local model. Now we would like to consider imposing a few further constraints, that are not needed for consistency but are likely needed to get a realistic and calculable four-dimensional model. We will concentrate on SU(5) models, so there is a matter curve

$$[\Sigma_{10}] = c_1 - t \tag{2.102}$$

which must be effective and non-zero.

The Kähler class J is a generator of  $H^2(B_3, \mathbf{R})$  that must be positive on all the effective cycles of the geometry. Modulo these positivity constraints, there is an independent scale in the geometry for every generator of  $H^2(B_3, \mathbf{R})$ . In order to get a model that is calculable and predictive, we need some small parameters that we can expand in. The main requirement that we want to make is that  $M_{GUT}/M_{Pl}$  is unbounded from below, where  $M_{GUT} \sim V_S^{1/4}$  and  $M_{Pl} \sim V_{B_3}^{1/6}$ . Now it is a priori possible that in a given model we can take  $V_{B_3} \to \infty$  while keeping  $V_S$  finite, therefore decoupling the GUT and Planck scales, but we cannot take  $V_S \to 0$  while keeping  $V_{B_3}$  finite. However this would depend on the geometry of  $B_3$ , and moreover would normally leave additional scales in the model with physics that cannot be decoupled from the visible sector. Needless to say that would not be beneficial for the predictiveness of the model.<sup>5</sup> To get a predictive model in which the visible sector is largely independent of the rest of  $B_3$ , we will require that one can take  $V_S \to 0$  while keeping  $V_{B_3}$  (or any other cycles not inside S) finite. Moreover this yields a local constraint that can be checked without knowing the compactification manifold  $B_3$ .

There are two ways in which we could take  $V_S \to 0$  while keeping other cycles fixed. The first is that we could require S to contract to a point. This will turn out to be a very strong condition which will essentially single out a unique model. We could also require S to contract to a curve of singularities. This is a less stringent condition, but together with some other physical constraints will still rule out a good deal of models.

Thus our first assumption is as follows:

<sup>&</sup>lt;sup>5</sup>Perhaps an exception would be if  $B_3$  is fibered over S, but this can be excluded by condition (3) below.

1. Contractibility. S can be contracted to a point. By Grauert's criterion [38], this means that the class t must be ample, in particular  $t \cdot C > 0$  for any curve C in S.<sup>6</sup>

We can draw some immediate conclusions from this assumption. Since  $c_1 - t$  is effective and non-zero, and t is ample,  $c_1$  must be effective and non-zero. Therefore  $K_S^{-n}$  cannot have sections for any positive n and the Kodaira dimension is  $-\infty$ . From the classification of surfaces, we then know that S is related to  $\mathbf{P}^2$  or a ruled surface (i.e. a  $\mathbf{P}^1$ -fibration over a Riemann surface of genus g) by a sequence of blow-ups and blow-downs.

The ruled surfaces have  $h^{1,0}(S) = g$ , which would lead to massless adjoint fields in the effective four-dimensional theory if g > 0. This looks phenomenologically undesirable, so we will exclude this possibility with our second assumption:

2. No adjoint scalars. The Hodge numbers of S must satisfy  $h^{0,1}(S) = 0$  and  $h^{2,0}(S) = 0$ .

Then S is either  $\mathbf{P}^2$  or can be obtained by a sequence of blow-ups from a Hirzebruch surface  $\mathbf{F}_r$ . Note this includes all the del Pezzo surfaces. The divisors on  $\mathbf{F}_r$  are generated by b, f and  $E_i$ , with the intersection numbers

$$b \cdot b = -r$$
,  $b \cdot f = 1$ ,  $f \cdot f = 0$ ,  $b \cdot E_i = f \cdot E_i = 0$ ,  $E_i \cdot E_j = -\delta_{ij}$  (2.103)

By exchanging b and b + f, we may take  $r \ge 0$ . Further we have

$$c_1(\mathbf{F}_r) = 2b + (r+2)f - \sum_{i=1}^k E_i$$
 (2.104)

Similarly we may write

$$t = n_b b + n_f f - \sum_{i=1}^k n_i E_i$$
 (2.105)

Let us first assume we are on  $\mathbf{F}_r$ , with no blow-ups. From ampleness of t we get  $n_b > 0$ ,  $-n_b r + n_f > 0$ . Since  $c_1 - t$  is effective and non-zero, we also get  $n_b \leq 2$  and  $n_f \leq r + 2$ , with strict inequality for  $n_f$  if  $n_b = 2$  or vice versa. Then we either have  $n_b = 1$  and  $r < n_f \leq r + 2$ , or we have  $n_b = 2$ ,  $n_f = r + 1$  and r = 0 or 1.

We may add a further reasonable assumption which eliminates most of these models. Currently, there is only one known mechanism for breaking the GUT group while preserving the standard GUT relations at leading order [5, 6]. This mechanism requires a

<sup>&</sup>lt;sup>6</sup>Even though we have seen that for many purposes S can be regarded as living inside the total space of the canonical bundle, this criterion has nothing to do with contractibility in the auxiliary local Calabi-Yau. We will see this more explicitly in the global examples later.

-2-class on S (i.e. a class with  $x \cdot x = -2$ ) in order to avoid massless lepto-quarks. This class must further be orthogonal to any classes that are inherited from  $B_3$  in order to avoid Higgsing hypercharge or loosing the standard SU(5) relations between the gauge couplings at leading order. Let us take this as our third assumption:

3. GUT breaking using fluxes. There must exist a -2-class  $x \in H^2(S, \mathbf{Z})$  which is orthogonal to any class inherited from  $H^2(B_3, \mathbf{Z})$ . In particular,  $x \cdot c_1 = x \cdot t = 0$ 

Let us again consider the Hirzebruch surfaces. Then  $h^2(\mathbf{F}_r) = 2$  so by condition (3) it follows that t must be a rational multiple  $ac_1$  of  $c_1$ . Since  $t \cdot f > 0$ , the coefficient a must be positive, and since  $c_1 - t$  must be effective, the coefficient a must be  $\leq 1$ . But this happens only for r even in which case  $c_1$  is divisible by 2, so this leaves

$$S = \mathbf{F}_r \quad \text{with } r \text{ even}, \qquad \Sigma_{\mathbf{10}} = \frac{1}{2}c_1$$
 (2.106)

But now by condition (1) we get  $t \cdot b = -r + 2 > 0$ , so this leaves only  $S = \mathbf{F}_0$  and  $t = \frac{1}{2}c_1$ . Note that  $\mathbf{P}^2$  is also ruled out by condition (3).

Now we consider the case of Hirzebruch surfaces with at least one blow-up. Again we have the constraints above from  $t \cdot b > 0$ ,  $t \cdot f > 0$  and  $c_1 - t$  effective. However we also get  $t \cdot E_i = n_i > 0$  and  $t \cdot (f - E_i) = n_b - n_i > 0$ . Hence we must have

$$t = 2b + (r+1)f - \sum_{i=1}^{k} E_i$$
 (2.107)

From  $t \cdot b = -r + 1 > 0$  we find that r = 0. Moreover,  $\mathbf{F}_r$  with one blow-up is actually the same surface as  $\mathbf{F}_1$  with one blow-up, so r = 0 is ruled out as well, and therefore all cases with blow-ups are ruled out. So we conclude that assumptions (1)-(3) leave a unique possibility for S and t:

$$(1) + (2) + (3) \Rightarrow S = \mathbf{F}_0, \qquad t = \frac{1}{2}c_1, \qquad \Sigma_{\mathbf{10}} = \frac{1}{2}c_1$$
 (2.108)

We will study this case in more detail later in the paper. In particular we will show how to engineer three-generation models and how to embed it in a global model.

It is evident by now that condition (1) in particular is quite strong. In order to have  $V_S \to 0$  while keeping other cycles fixed, we can also replace assumption (1) by:

1'. Contractibility. S can be contracted to a curve, i.e. S admits a fibration  $F \to S \to B$  where the fibers F can be contracted to a curve B of singularities.

In this case t is not necessarily ample, but t must be ample when restricted to the components that are being contracted [39, 40]. Therefore,  $t \cdot C > 0$  when C is the general fiber F or an irreducible component of the singular fibers.

A priori the base B of the fibration can have any genus g, in which case we would have  $h^{1,0}(S) = g$ . Again by assumption (2) the base B is restricted to be  $\mathbf{P}^1$ . Likewise, the fibers must be rational: the curve  $c_1 - t = \Sigma_{10}$  is effective and the fiber F moves, so there must be some copy of F that is not contained in  $\Sigma_{10}$ . Therefore we must have  $(c_1 - t) \cdot F \geq 0$ . Since F gets contracted, we have  $t \cdot F > 0$ , and it follows that  $c_1 \cdot F > 0$ . But by the adjunction formula we have  $c_1 \cdot F = 2 - 2g(F)$ , hence  $c_1 \cdot F = 2$  and F is a  $\mathbf{P}^1$  as promised.

So our S is a ruled surface with rational base and fibers. From the classification of surfaces, we know that S is rational and can be obtained by blowing up some points on a Hirzebruch surface  $\mathbf{F}_r$ . The most general possibility is obtained by blowing up some points on a conic bundle, which allows for the possibility of a multiple fiber. Our argument below is actually even more constraining when there are multiple fibers, so in what follows we will focus on the case that none of the fibers is multiple.

If S is a Hirzebruch surface then we can run the previous argument. Using condition (3) it follows that  $t = \frac{1}{2}c_1$  and r is even. Apart from these we must consider possible blow-ups of  $\mathbf{F}_r$ . Again we write

$$t = n_b b + n_f f - \sum_{i=1}^k n_i E_i (2.109)$$

Under assumption (1') we can no longer conclude that  $t \cdot b$  must be positive, but we still know that t must be positive on  $f, E_i$  and  $f - E_i$ . From  $t \cdot f > 0$ ,  $t \cdot E_i > 0$  and  $t \cdot (f - E_i) > 0$  we get  $n_b > 0$ ,  $n_i > 0$ , and  $n_b - n_i > 0$ . From  $c_1 - t$  effective and non-zero we get  $n_b \leq 2$  and  $n_f \leq r + 2$ , with strict inequality for  $n_f$  if  $n_b = 2$  and  $n_i = 1$ . Therefore the only possibility is

$$S = B_k(\mathbf{F}_r), \qquad t = 2b + n_f f - \sum_{i=1}^k E_i, \qquad \Sigma_{10} = (r + 2 - n_f)f$$
 (2.110)

with  $n_f < r + 2$ . Here we used  $B_k$  to denote blowing-up k times. These possibilities also satisfy condition (3), since there are classes of the form  $f - E_i - E_j$  and  $E_i - E_j$  which are orthogonal to  $c_1$  and t. Moreover we can't do too many blow-ups. Recall that the sections  $b_i$  specifying an SU(5) model live in  $c_1 - t$ , ...,  $6c_1 - t$ , so these line bundles need to admit sections.

So we conclude that under assumptions (1'), (2) and (3), we get the following possibilities for S and t:

$$S = \mathbf{F}_r$$
 with  $r$  even,  $t = \frac{1}{2}c_1$ ,  $\Sigma_{10} = \frac{1}{2}c_1$  (2.111)

$$S = B_k(\mathbf{F}_r), \qquad t = 2b + n_f f - \sum_{i=1}^k E_i, \qquad n_f < r + 2.$$
 (2.112)

The remaining possibility  $S = \mathbf{P}^2$  is still ruled out by assumption (3).

We may consider adding one final assumption. We will soon see though that this assumption has an important loophole, so it will be weakened significantly.

4. Three generations. The net number of generations is given by

$$-\lambda(6c_1 - t) \cdot_{dP} (c_1 - t) \tag{2.113}$$

where  $\lambda \in \mathbf{Z} + \frac{1}{2}$ . As we argued, this is the only universal formula one can write down. However this does not represent the most general configuration of local F-theory models and will be revisited in section 2.8.

Let's apply this to all the possibilities we found. For the Hirzebruch surfaces with  $t = \frac{1}{2}c_1$  we find that the minimal number of generations is eleven. For the blow-ups of Hirzebruch surfaces with t as in (2.112), we find that the minimal number of generations is  $5 \times (r + 2 - n_f)$ , and in general it is always divisible by 5. We conclude it is not possible to make a local three generation SU(5) model under these assumptions.

If we drop condition (1) or (1') it is not hard to find three-generation models. For instance, the  $dP_8$  example in [1] with  $\eta = 6c_1(S)$  is consistent and satisfies assumptions (2) and (3), but it does not satisfy assumption (1) or (1') since it has t = 0, and is therefore not a truly local model.

It may be interesting to point out that the three generation SO(10) models in [1] (which have  $\eta = 4c_1 + E$  where E is any -1-curve, and  $\Sigma_{16} = [E]$ ) do satisfy the conditions (1),(2) and (3) for  $dP_k$  with  $2 \le k \le 7$ . However a fully satisfactory way of breaking the SO(10) GUT group in these models while preserving gauge coupling unification has not yet been identified.

To summarize this subsection, under conditions (1) - (3) we only found one possibility for S and t, listed in (2.108). Under assumptions (1') - (3) we only found the possibilities listed in (2.111) and (2.112). Using the inherited fluxes (assumption (4)), none of these models could account for three generations. In the following subsections, we will examine some possible loopholes in our assumptions.

## 2.7. Another way to break the GUT group

In [5, 6] the GUT group was broken to the Standard Model gauge group by turning on an abelian flux. A priori there exists a second possibility: one may also break the GUT group to the Standard Model by turning on an abelian Higgs field. To do this, we take a degree six spectral cover

$$b_0 s^6 + b_2 s^4 + \ldots + b_6 = 0 (2.114)$$

which generically breaks  $E_8$  to  $SU(3) \times SU(2)$ . Note that  $b_1$  must vanish if the structure group is to be in SU(6) rather than U(6). Now if the  $b_i$  are such that this equation factorizes

$$(c_0s + c_1)(d_0s^5 + d_1s^4 + \dots + d_5) = 0 (2.115)$$

where  $c_0d_1 + c_1d_0 = 0$ , then the structure group of the Higgs field commutes with  $SU(3) \times SU(2) \times U(1)$ .

The matter curves are easy to find. Consider first an irreducible degree 6 spectral cover (this was worked out in [3]). One uses the following decomposition of the adjoint of  $E_8$  under  $SU(3) \times SU(2) \times SU(6)$ :

248 = 
$$(8,1,1) + (1,3,1) + (1,1,35)$$
  
  $+(3,2,6) + (\bar{3},2,\bar{6}) + (3,1,15) + (\bar{3},1,\bar{1}5) + (1,2,20).$  (2.116)

We have the following matter curves:

$$\begin{aligned}
(\mathbf{3}, \mathbf{2}) &\to \{b_6 = 0\} \\
(\mathbf{3}, \mathbf{1}) &\to \{b_0 b_5^3 - b_2 b_3 b_5^2 + b_4 b_5 b_3^2 - b_3^3 b_6 = 0\} \\
(\mathbf{1}, \mathbf{2}) &\to \{b_6 (b_2^2 - 4b_4 b_0) + b_0 b_5^2 - b_2 b_5 b_3 + b_4 b_3^2 = 0\}
\end{aligned} (2.117)$$

In the reducible case we simply substitute the  $b_i$  for the appropriate bilinears in  $c_i$  and  $d_j$ .

In addition there can be hypercharged scalars. To see this, recall that the moduli of the spectral cover are counted by

$$\operatorname{Ext}_{X}^{1}(i_{*}L, i_{*}L) \tag{2.118}$$

When the spectral cover is reducible, this decomposes as

$$\sum_{m,n} \operatorname{Ext}_{X}^{1}(i_{m*}L_{m}, i_{n*}L_{n}) \tag{2.119}$$

where m, n run over the irreducible components. The off-diagonal zero modes are clearly charged under the extra U(1)'s since their VEVs smooth the spectral cover and break these U(1)'s. These modes are localized at the intersection  $c_0s + c_1 = d_0s^5 + d_1s^4 + \ldots + d_5$ . In order to get a realistic model, such hypercharged scalars must obtain a mass, i.e. we must obstruct the deformation of two irreducible components of the cover into a single smooth piece. This could be done by turning on spectral line bundles on the irreducible

components that do not arise as the limit of a line bundle on the smooth deformation. Possibly some other mechanism like an instanton effect can also be used.

There is unfortunately one issue with this scenario. The polynomial  $b_6$  is a section of  $N_S$ . Thus not only  $c_1 - t$  should have a section, but also -t. By our classification of the possible pairs (S, t), even with condition (3) dropped, such an S can not be contractible, which means it is not a true local model. Thus this mechanism can only be used if we drop the requirement that  $M_{GUT}/M_{Pl}$  can be made parametrically small.

A second concern is that there is a potential D-term instability. The U(1) will be non-anomalous if we succeeded in lifting the charged moduli mentioned above, but there may be a bare FI term. This FI term is given by the moment map

$$\zeta \sim G \wedge J \sim F \wedge J \tag{2.120}$$

where F is the flux of the spectral line bundle/sheaf on the degenerate spectral cover. We've argued that spectral line bundles on smooth spectral covers with  $c_1(p_{C*}L) = 0$  are primitive. This will also hold for line bundles on degenerate spectral covers if they are obtained by taking a limit of a line bundle on a smooth spectral cover, because primitiveness is a closed condition. However these are precisely not the line bundles we want, because they would not lift the hypercharged scalars. Hence on degenerate spectral covers one would need to look at specific models in more detail.

## 2.8. Non-inherited fluxes

The results of section 2.6 clarify our options. Dropping the first assumption is a priori possible and leads to consistent models, but it would mean we can not make an expansion in  $M_{GUT}/M_{Pl}$  and therefore would greatly diminish the predictiveness of F-theory GUTs. Dropping assumption (3) means that we need an alternative mechanism to break the GUT group while preserving the standard SU(5) relations at leading order. We made such an alternative proposal in section 2.7 but it did not seem to be compatible with contractibility of S. Therefore we are led to drop assumption (4) and investigate the possibilities of non-inherited G-fluxes.

The argument in section 2.6 that the inherited fluxes are not sufficient by no means rules out three generation models. Rather it means that we need to look at more general fluxes that are not critical points of the holomorphic Chern-Simons superpotential for generic moduli, and we have to do more work to show that there exists a stable supersymmetric minimum. In mathematics circles this would be called a Noether-Lefschetz problem. On the other hand, the first three assumptions already ruled out all but a handful of 7-brane configurations, listed in (2.108), (2.111) and (2.112) in section 2.6. Thus in contrast to eg. heterotic model building, we have a very restricted set of possibilities to start with and we know all the continuous parameters.

Let us first ignore the requirement of supersymmetry, and simply ask if there are any

fluxes available, not necessarily of type (1,1), which might give three generations. This will be the case if we can show there exists a class  $\gamma_2 \in H^2(\bar{C}, \mathbf{Z})$  which is orthogonal to  $p_C^*H^2(S, \mathbf{Z})$  and satisfies  $\gamma_2 \cdot \Sigma_{10} = 1$ , since then we can add some integer multiple of it to  $\frac{1}{2}r + \frac{1}{2}\gamma_u$  and get any number of generations we want. Equivalently,  $\gamma_2$  must satisfy

$$\gamma_2 \cdot \gamma_u = \gamma_2 \cdot (5\Sigma_{10} - p_C^* p_{C*} \Sigma_{10}) = 5$$

$$(2.121)$$

Now the lattice  $H^2(\bar{C}, \mathbf{Z})$  modulo torsion is unimodular by Poincaré duality. Thus if  $\gamma_u$  is primitive in the sense that it is not an integer multiple of a smaller integer class, then there exists an  $\alpha \in H^2(\bar{C}, \mathbf{Z})$  such that

$$\alpha \cdot \gamma_u = 1 \tag{2.122}$$

Then defining  $\gamma_2 = 5\alpha - p_C^* p_{C*} \alpha$ , we have  $\gamma_2 \cdot \gamma_u = 5$  and  $p_{C*} \gamma_2 = 0$  as required. Therefore we are guaranteed that the required fluxes exist if  $\gamma_u$  is primitive in the sense above. In a unimodular lattice, a sufficient condition is that  $\gamma_u \cdot \gamma_u$  is not divisible by any square. It is easy to see that  $\gamma_u \cdot \gamma_u = 5\gamma_u \cdot \Sigma_{10}$  which we have already computed in section 2.6. For  $S = \mathbf{P}^1 \times \mathbf{P}^1$  with  $t = \frac{1}{2}c_1$ , it is equal to  $5 \times 22$  which does not have any squares. So we conclude from a purely topological argument that it is possible to obtain three generations, although this argument cannot establish that there is a supersymmetric minimum for finite values of the Higgs bundle moduli.

In the following we would like to give a fairly general construction of algebraic classes that satisfy  $\alpha \cdot_{\bar{C}} \gamma = 1$ . We will apply it to  $S = \mathbf{P}^1 \times \mathbf{P}^1$ , but it should be clear that with some simple substitutions it can also be applied to the other cases. Thus we will finally establish some examples of supersymmetric SU(5) models with three generations and S contractible.

The strategy is as follows: we first take a curve  $\alpha_0 \in H^2(S, \mathbf{Z})$  such that  $\alpha_0 \cdot_S \Sigma_{10} = 1$ . Then we will construct a curve  $\alpha \in \bar{X}$  which does not intersect  $\Sigma_{10}$ , and which covers  $\alpha_0$  exactly once. Finally we will require  $\bar{C}$  to contain  $\alpha$  by tuning the complex structure moduli. The result is an algebraic class in  $H^2(\bar{C}, \mathbf{Z})$  with  $\alpha \cdot_{\bar{C}} \gamma = 1$ . We can then construct an additional flux  $\gamma_2$  as above by subtracting the trace, and and define a spectral line bundle with

$$c_1(L) = \frac{1}{2}r + \frac{1}{2}\gamma_u + n\gamma_2 \tag{2.123}$$

By adjusting n, we then get any number of generations we want.

In the case of interest, we have  $S = \mathbf{P}^1 \times \mathbf{P}^1$ . We denote the coordinates on S by  $(z_1, z_2; w_1, w_2)$ , and the two rulings by  $H_1$  and  $H_2$ , with intersection numbers  $H_1^2 = H_2^2 = 0$ ,  $H_1 \cdot H_2 = 1$ . As we deduced above, the matter curve should be given by  $\Sigma_{\mathbf{10}} = [H_1 + H_2]$ , and we need a class with  $\alpha_0 \cdot \Sigma_{\mathbf{10}} = 1$ . Thus a simple choice is to pick  $\alpha_0 = H_1$ , though clearly there are additional options. In equations it is given by (say)  $w_1 = 0$ .

Now we need to construct  $\alpha$ . As before we use coordinates  $(u_1, u_2)$  on the  $\mathbf{P}^1$ -fibers of  $\bar{X} = \mathbf{P}(\mathcal{O} \oplus K_S)$ . Then we define  $\alpha$  by the following two equations:

$$\alpha: w_1 = 0, \qquad u_1 = P_2(z, w)u_2$$
 (2.124)

where  $P_2(z, w)$  is a section of  $\mathcal{O}(2, 2)$ . Note that  $u_2 = 0 \Rightarrow u_1 \neq 0$ , so  $\alpha$  does not intersect the zero section  $u_2 = 0$ . This is needed if we want  $\bar{C}$  to contain  $\alpha$ , because the intersection of  $\bar{C}$  with  $u_2 = 0$  is by definition  $\Sigma_{10}$ , and we promised to construct a class which does not intersect  $\Sigma_{10}$ . Also,  $\alpha$  covers  $\alpha_0$  precisely once.

Therefore we now need to show that we can tune the complex structure moduli so that  $\bar{C}$  contains  $\alpha$ . The equation of  $\bar{C}$  is given by

$$b_0(z, w)u_2^5 + b_2(z, w)u_1^2u_2^3 + \ldots + b_5(z, w)u_1^5 = 0.$$
(2.125)

We simply substitute the equations for  $\alpha$  in order to get a restriction on the coefficients of  $\bar{C}$ . Clearly we find that

$$w_1$$
 divides  $b_0(z, w) + b_2(z, w)P_2(z, w)^2 + \dots + b_5(z, w)P_2(z, w)^5$  (2.126)

This can be satisfied by leaving  $b_2, \ldots, b_5$  arbitrary, and putting

$$b_0(z,w) = -\left[b_2(z,w)P_2(z,w)^2 + \ldots + b_5(z,w)P_2(z,w)^5\right]_{w_1 \to 0} + \mathcal{O}(w_1)$$
 (2.127)

The only thing left to check is that  $\bar{C}$  is generically smooth, so that our calculations of the chiral spectrum apply. But this is fairly obvious because generically the derivatives of equation (2.125), even with (2.127), give independent equations.

Therefore we have constructed a (1,1) class  $\gamma_2 = 5\alpha - p_C^* p_{C*} \alpha$  with the desired properties. Defining a spectral line bundle as in (2.123) with n=8, we find precisely three chiral generations on  $\Sigma_{10}$ .

#### 2.9. 'Flux' vacua in the heterotic string and F-theory

As we already remarked, much of the structure of F-theory vacua is identical with that of the heterotic string. BPS instantons effects, branes and flux superpotentials, which are some of the main ingredients inducing potentials for the moduli, can be related under the duality. In particular, semi-realistic heterotic models appear to have an enormous number of 'flux' vacua as well. We put 'flux' in quotation marks here because after Fourier-Mukai transform, we get a smooth non-abelian bundle on the Calabi-Yau three-fold without any U(1) fluxes. These extra 'flux' vacua are obtained by using spectral line bundles that are not inherited, and generically should stabilize all vector bundle moduli. There

is a landscape of such vacua and a priori it is not clear why we should exclude these possibilities. The method we used for constructing such more general fluxes in section 2.8 can also be used in the heterotic string.

Thus landscapes seem to be quite generic properties of superpotentials in string theory. It would be interesting to study these vacua microscopically. In the heterotic setting there are no RR fields so one could try to use conventional CFT techniques. Perhaps one may then find a reason to exclude most of them, although it is currently not clear why that would be the case.

As in type II settings, this leads to philosophical problems: we don't really understand moduli stabilization and the cosmological constant problem very well, it is practically impossible to enumerate all the vacua that seem to exist at the effective field theory level, and one of the solutions that have been proposed to solve the cosmological constant problem is NP hard [41]. A possible way out was promoted in [42]: if  $M/M_{Pl}$  can be parametrically small, where M is some scale relevant for particle physics like the GUT scale, then we can prevent the unknown physics responsible for solving gravity-related problems from feeding back into physics at the scale M. This may allow us to discuss phenomenology without having to solve the cosmological constant problem and other problems related to gravity. But combining this principle with GUTs leads us to F-theory; this idea cannot be implemented in the heterotic string.

### 2.10. Conclusions

We have clarified the rules for constructing local models in F-theory. Such models can be defined by specifying suitable spectral data (a type of B-brane) in an auxiliary Calabi-Yau geometry. We classified the possible matter curves for local SU(5) models. We have constructed the first truly local SU(5) models with three generations. It is still an open problem to construct local SU(5) models with exactly the MSSM spectrum, or some acceptable extension. We explained how to connect F-theory models to a IIb picture by taking an orientifold limit.

We also found that it seems to be impossible for a local SU(5) model with completely unstabilized Higgs field moduli to have three generations. From a physical perspective, this is good news since there are many indications that we do not want a generic model, such as problems associated to dimension four and five proton decay. Thus requiring a three-generation model automatically stabilizes some of the moduli. Requiring the precise MSSM spectrum will likely stabilize even more moduli. On the other hand, this also makes the problem of constructing realistic local models much more challenging.

Along the way we have encountered a number of constraints that the matter curves and fluxes on the matter curves must satisfy in a consistent local model, from topological and integral (such as anomaly cancellation (2.28) and the fact that the flux must lift to the integral class of a line bundle L with  $c_1(p_{C*}L) = 0$ ), to analytic (such as the forced singularities on  $\Sigma_{\Lambda^2 E}$  and constraints on the moduli entering the matter curves for non-

inherited fluxes, so that L is a holomorphic line bundle). Probably we have not found them all, since some of these constraints look very non-trivial from the point of view of the brane carrying the unbroken gauge group. However we have shown that there is an isomorphism relating a configuration  $(E, \Phi)$  in the 8d gauge theory to its spectral data  $(C_E, L_E)$ , and the constraints on the spectral data are few and simple to understand.

In the next section we will make the first strides towards embedding our local models in a global model.

#### 3. Compactification

In order to get a finite four-dimensional Planck scale we should embed our local models into a compact elliptically fibered CY four-fold. In the philosophy of local model building, the goal of this pursuit is not to find 'the' UV completion of the local model. Indeed as we reviewed earlier, it is not even clear that this is an answerable question. Rather it is to ascertain that all the ingredients used can in principle be combined in a UV complete model, and there are no obvious constraints from UV completion that would rule them out. In doing so there are many issues to be addressed. Our aim here is rather modest; we would like to discuss a simple set of compactifications which implement a few of the requirements for viable local GUTs, and which make clear how such constructions work in general. In particular we would like to construct compactifications in which GUT breaking by fluxes can be implemented, and in which dangerous proton decay channels can be avoided.

Our discussion will have one important caveat. We will freely assume that appropriate fluxes may be found which give precisely the Standard Model spectrum on the matter curves we engineer. As we emphasized in section 2, it has not yet been shown that this can actually be done in a local model, let alone in a global model. The point of our discussion is not to understand the fluxes, but rather some of the constraints on the geometry of the four-fold arising from phenomenological requirements.

# 3.1. First example: cubic surface in $\mathbf{P}^3$ .

Let us discuss simple compactifications of local toy models with SU(5) GUT group. Our GUT brane should be wrapped on a Del Pezzo surface  $S_2 \subset B_3$ , such that some homology classes in the Del Pezzo become boundaries when embedded in  $B_3$ . For simplicity we will take  $B_3$  to be  $\mathbf{P}^3$  in our first example, although much of what we will say can clearly be adapted to more general Fano three-folds. Then we can take  $S_2$  to be a quadric surface  $Q_2(z_1, z_2, z_3, z_4) = 0$  (i.e.  $\mathbf{P}^1 \times \mathbf{P}^1$ ) or a cubic surface  $Q_3(z_1, z_2, z_3, z_4) = 0$  (i.e. a Del Pezzo 6). For definiteness we take the cubic.

Recall again the Tate form of the Weierstrass equation

$$y^2 + a_1 x y + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6 (3.1)$$

where the  $a_i$  are sections of  $K_{B_3}^{-i}$ . In the case of  $B_3 = \mathbf{P}^3$ , the  $a_i$  are polynomials of degree 4i. As we discussed in section, in order to get an  $I_5$ -locus along  $Q_3 = 0$ , as well as matter curves and Yukawa coupling localized along certain prescribed submanifolds, we must impose certain restrictions on the  $a_i$  which can be read from table 1.

Now let us try to impose various constraints.

1. SU(5) gauge group on  $Q_3 = 0$ . According to table 1 this implies the following leading form for the  $a_i$ :

$$a_1 = P_4, a_2 = sP_5, a_3 = s^2P_6, a_4 = s^3P_7, a_6 = s^5P_9 (3.2)$$

where  $s = Q_3$  and the  $P_n$  are generic polynomials of degree n on  $\mathbf{P}^3$ . They are identified with the sections  $b_{9-n}$  that appeared in the general discussion. In fact we are clearly allowed to add further subleading terms, eg.  $a_1 = P_4 + sT_1$ ,  $a_2 = sP_5 + s^2T_2$ , et cetera. Such additional subleading terms do not affect behaviour near the  $I_5$  locus, but they do provide additional flexibility in building a global model. To keep things simple, we will set them to zero. Then the discriminant is computed to be

$$\Delta = s^5 P_4^4 (-P_9 P_4^2 + P_6 P_7 P_4 - P_5 P_6^2) + \mathcal{O}(s^6)$$
(3.3)

which vanishes to 5th order along  $Q_3 = 0$ , as required.

2. Matter curves. The discriminant vanishes to higher order along  $\Sigma_{10} = \{Q_3 = P_4 = 0\}$  and  $\Sigma_5 = \{Q_3 = R_{17} = 0\}$ , where

$$R_{17} \equiv P_9 P_4^2 - P_6 P_7 P_4 + P_5 P_6^2. \tag{3.4}$$

Note that

$$\Lambda^2 T_{Q_3} = \mathcal{O}(1)|_{Q_3}, \qquad N_{Q_3} = \mathcal{O}(3)|_{Q_3}$$
 (3.5)

and hence the cohomology classes dual to  $\Sigma_{10}$  and  $\Sigma_{5}$  on  $Q_{3}$  are given by  $c_{1}-t$  and  $8c_{1}-3t$ . Of course this all fits in the general discussion in sections 2.1 and 2.3.

3. Yukawa couplings and dimension four proton decay. The up-type Yukawa couplings are localized at  $\{Q_3 = P_4 = P_5 = 0\}$  and the down type Yukawas are localized at  $\{Q_3 = P_4 = P_6 = 0\}$ . Methods for suppressing proton decay in F-theory were discussed in [28, 5, 6]. Here we will see how they can be implemented in global models.

In order to prevent dimension four proton decay, we want to make sure that

$$\begin{array}{ccc}
\mathbf{10}_{m} \cdot \overline{\mathbf{5}}_{m} \cdot \overline{\mathbf{5}}_{m} \\
\mathbf{10}_{m} \cdot \overline{\mathbf{5}}_{h} \cdot \overline{\mathbf{5}}_{h}
\end{array} \Rightarrow \text{absent}, \qquad \mathbf{10}_{m} \cdot \overline{\mathbf{5}}_{m} \cdot \overline{\mathbf{5}}_{h_{d}} \Rightarrow \text{present}$$
(3.6)

This can be done by splitting  $\Sigma_5$  into two pieces, one supporting the matter fields and another supporting the Higgses. This means we have to tune the  $P_n$  so that the polynomial R factorizes modulo  $Q_3$ . In terms of ideals, we require a decomposition

$$\langle Q_3, R \rangle = I_{\overline{5}_m} \cap I_h \tag{3.7}$$

To secure the absence of R-parity violating down-type Yukawa couplings we must make sure that whenever we have an intersection of  $\Sigma_{\mathbf{10}_m}$  with  $\Sigma_{\mathbf{\overline{5}}_m}$ , there is also a branch of  $\Sigma_{\mathbf{\overline{5}}_h}$  intersecting at that point. Since R=0 has a double point at such intersections, we can also say that whenever  $\Sigma_{\mathbf{10}_m}$  intersects with  $\Sigma_{\mathbf{\overline{5}}_m}$ , then  $\Sigma_{\mathbf{\overline{5}}_m}$  is not allowed to have a double point (the second order vanishing of R instead being due to a branch of  $\Sigma_{\mathbf{\overline{5}}_h}$  coming in and intersecting there). Similarly in order to avoid the couplings  $\mathbf{10}_m \cdot \mathbf{\overline{5}}_h \cdot \mathbf{\overline{5}}_h$ , we want to avoid double points on  $\Sigma_{\mathbf{\overline{5}}_{h_d}}$  which also meet  $\Sigma_{\mathbf{10}_m}$ .

We don't know the general solution to this algebraic problem. But to see that it can be achieved, we will exhibit one simple solution that exists for more general SU(5) models as well. We take

$$P_{6} = H_{6}^{d} \mod Q_{3} \qquad P_{9} = H_{1}^{u} T_{2} H_{6}^{d} \mod Q_{3} P_{7} = H_{1}^{u} T_{6} \mod Q_{3} \qquad P_{5} = H_{1}^{u} T_{4} \mod Q_{3}$$

$$(3.8)$$

for some  $T_i$  and  $H_i$  of the appropriate degree, but otherwise arbitrary. Then we take  $\Sigma_{\mathbf{5}_h} = \{Q_3 = H_1^u H_6^d = 0\}$ , i.e. the Higgs curve is actually reducible, with only up type Yukawa couplings on  $H_1^u = 0$  (since  $H_1^u = 0$  implies  $P_5 = 0$ ) and only down type Yukawa couplings on  $H_6^d = 0$ . When we discuss dimension five proton decay we will see why that is a good thing to have. Now we can factorise  $\Sigma_{\mathbf{5}}$  as

$$R_{17} = (H_1^u H_6^d) \cdot M_{10} \mod Q_3, \qquad M_{10} = T_2 P_4^2 - T_6 P_4 + T_4 P_6$$
 (3.9)

and  $M_{10}$  has no double points at  $Q_3 = P_4 = P_6 = 0$ . Moreover the up and down-type Yukawa's are still present. For instance the up-type Yukawa's come from  $Q_3 = P_4 = H_1^u = 0$ , which consists of  $3 \cdot 4 \cdot 1 = 12$  points.

There are additional cubic couplings of the form

$$\overline{\mathbf{5}}_m \cdot \mathbf{5}_{h_u} \cdot \mathbf{1}, \qquad \overline{\mathbf{5}}_{h_d} \cdot \mathbf{5}_{h_u} \cdot \mathbf{1}$$
 (3.10)

The singlets correspond to Higgs field moduli (which are complex structure moduli of the Calabi-Yau four-fold). At least three of them should give rise to right-handed neutrinos, with the first coupling in (3.10) corresponding to the usual Yukawa couplings for neutrinos. The number of moduli appearing in such couplings is the difference between the number of moduli describing  $\Sigma_{\bar{\mathbf{5}}_m}$  and  $\Sigma_{\mathbf{5}_{h_u}}$  separately or as a single smooth curve, which yields 65 singlets in our example. The problem of getting Majorana masses of the right order of magnitude is a problem of moduli stabilization for the Higgs fields. The couplings on the right give rise to the minimal extension of the MSSM with a dynamical  $\mu$ -parameter. There are additional constraints from dimension five proton decay however, as we discuss next.

4. Dimension five proton decay. We further want to eliminate dimension five proton decay. This proceeds through mediation of massive KK triplets  $T_u, T_d$  propagating on curves supporting a hypermultiplet in the 5. The possible channels are given by

$$QQ \xrightarrow{\lambda_u} T_u \xrightarrow{m^{ab}} T_d \xrightarrow{\lambda_d} QL$$

$$\Sigma_{\mathbf{10}_m} \times \Sigma_{\mathbf{10}_m} \qquad \Sigma_{\mathbf{5}}^a \qquad \Sigma_{\mathbf{5}}^b \qquad \Sigma_{\mathbf{10}_m} \times \Sigma_{\mathbf{\overline{5}}_m}$$

$$(3.11)$$

In order to prevent such processes, we have to shut off at least one of the interactions in this chain. If  $H_u$  and  $H_d$  propagate on the same matter curve, and if we assume the existence of classical up-type and down-type Yukawa couplings for the Standard Model, then such decays are unavoidable. Since we want to keep the classical Yukawa couplings, we require the existence of a decomposition

$$I_h = I_u \cap I_d \tag{3.12}$$

so that we can shut off the coupling  $m^{ab}$ . If we allow  $\Sigma_u$  and  $\Sigma_d$  to intersect, then there could either be a branch of  $\Sigma_{\mathbf{10}_m}$  also intersecting there; or it can correspond to a  $\mathbf{5}_u \cdot \mathbf{\bar{5}}_d \cdot \mathbf{1}$  coupling. As long as the VEV of the singlet vanishes we do not have the troublesome mass terms linking triplets localized on  $\Sigma_u$  and  $\Sigma_d$ . In either case the existence of a classical  $\mu$ -term is excluded. Our example corresponds to the latter case: there are  $3 \cdot 1 \cdot 6 = 18$  intersection points on  $H_1^u = H_6^d = 0$  corresponding to the couplings  $\mathbf{5}_u \cdot \mathbf{\bar{5}}_d \cdot \mathbf{1}$ .

There are several possible alternate channels for dimension five proton decay (3.11). The most dangerous are cases where  $\Sigma_{\bf 5}^a = \Sigma_{\bf 5}^b$ , because then mass terms  $m^{ab}$  between  $T_u$  and  $T_d$  cannot be avoided. The case  $\Sigma_{\bf 5}^a = \Sigma_{\bf 5}^b = \Sigma_{\bf 5}_m$  is harmless by the solution to dimension four proton decay, which shuts off the interactions  $\lambda_d$ . The case  $\Sigma_{\bf 5}^a = \Sigma_{\bf 5}^b = \Sigma_{\bf 5}_{h_d}$  requires shutting off the interactions  $\lambda_u$ . The curve  $\Sigma_{\bf 10}_m$  is positive in our example and therefore certainly intersects  $\Sigma_{\bf 5}_{h_d}$ . However in our solution to the dimension four problem, by design any such intersection has  $P_4 = P_6 = 0$  and therefore corresponds to a  $\lambda_d$  coupling, not a  $\lambda_u$  coupling, so this channel is not available. Finally there is the case  $\Sigma_{\bf 5}^a = \Sigma_{\bf 5}^b = \Sigma_{\bf 5}_{h_u}$ , which requires shutting off the interaction  $\lambda_d$ . In our example, by design any intersection point between  $\Sigma_{\bf 5}_{h_u}$  and  $\Sigma_{\bf 10}$  yields an up-type Yukawa, so the potentially troublesome interactions are again absent.

The remaining possible channels have  $\Sigma_{\bf 5}^a \neq \Sigma_{\bf 5}^b$ . Assuming both the  $\lambda_u$  and  $\lambda_d$  couplings are present (which they need not necessarily be), the problem is to shut off the interactions  $m^{ab}$ . This depends on the existence of intersections of  $\Sigma_{\bf 5}^a$  and  $\Sigma_{\bf 5}^b$  which give rise to a coupling  ${\bf 5} \cdot {\bf \bar 5} \cdot {\bf 1}$ . If such intersections are present,  $m^{ab}$  is proportional to the VEV of the singlet, which is a complex structure modulus. As long as the dynamics of moduli stabilization is such that the VEV of this field remains zero, there will be no proton decay through this channel.

Consider for instance  $\Sigma_{\mathbf{5}}^a = \Sigma_{\mathbf{5}_m}$  and  $\Sigma_{\mathbf{5}}^b = \Sigma_d$  (the other cases being similar). The curves  $\Sigma_{\mathbf{5}_m}$  and  $\Sigma_d$  can intersect in two ways. Either there is also a branch of  $\Sigma_{\mathbf{10}}$  intersecting there, which corresponds to the down type Yukawa's that we want to have; or it corresponds to a  $\mathbf{5} \cdot \mathbf{\bar{5}} \cdot \mathbf{1}$  coupling. In our example with  $I_{h_d} = \langle Q_3, H_6^d \rangle$ , intersection points where  $H_6^d = 0$  and  $M_{10} = 0$  have either  $P_4 = 0$  or  $T_2 P_4 - T_6 = 0$ . In the former case it meets with  $\Sigma_{\mathbf{10}}$ , and there are  $3 \cdot 6 \cdot 4 = 72$  such intersection points; in the latter case it corresponds to the coupling to singlets whose VEV must remain zero, and this accounts for  $3 \cdot 6 \cdot 6 = 108$  intersection points.

Hence we see in this simple example that there is enough room in complex structure moduli space to implement our geometric requirements for absence of dimension four and five proton decay. In fact although we did not write the most general solution above, it seems a solution along these lines is required. In order to eliminate the double points from  $\Sigma_{\overline{\bf 5}_m}$  and  $\Sigma_{\overline{\bf 5}_{h_d}}$ , we should factor out  $P_6$  from R, and in order to avoid dimension five proton decay, we should make sure that  $\Sigma_{\overline{\bf 5}_{h_d}}$  is contained in  $P_6 = 0$  and  $\Sigma_{{\bf 5}_{h_u}}$  is contained in  $P_5 = 0$ .

The solution we provided though required the VEVs of certain complex structure moduli to remain vanishing. This is a requirement we must impose on the moduli stabilization mechanism, which we have not considered here, and on the face of it does not seem particularly natural (although it is technically natural). One might speculate there are extra supersymmetric fluxes available for these values of the moduli which we should turn on in order to recover the precise Standard Model spectrum. That would be a nice way to really explain moduli stabilization and lack of proton decay in our models, but it seems currently unclear why that should be the case. An alternative approach would be to ensure that the potentially troublesome intersection points are all absent, which seems much harder to arrange, or to implement the approach of [28], which requires decomposing the SU(5) Casimirs into those of a smaller holonomy group and then making a small deformation, so that one has additional U(1)'s available.

# 3.2. Second example: a contractible $\mathbf{P}^1 \times \mathbf{P}^1$ .

In our previous example the del Pezzo was not contractible in  $B_3$ . The main purpose of this subsection is to give a simple example of a del Pezzo S which has two-cycles not inherited from  $B_3$  (necessary for allowing GUT breaking fluxes), and which is also contractible in  $B_3$ . This is an explicit realization of case (2.108) discussed in section 2.6.

The example is as follows. We will take  $B_3$  to be the blow-up of  ${\bf P}^3$  along a curve C defined by

$$C = \{Q_2 = 0\} \cap \{Q_3 = 0\} \tag{3.13}$$

The corresponding ideal is denoted as  $I_C$  and the blow-up along this ideal as  $B_3 = \tilde{\mathbf{P}}^3$ . We have

$$K_{\tilde{\mathbf{P}}^3} = i^* K_{\mathbf{P}^3} + \tilde{C} \tag{3.14}$$

where  $\tilde{C}$  is the exceptional divisor (a  $\mathbf{P}^1$ -fibration over C, given by projectivising the normal bundle). Sections of the anti-canonical bundle  $K_{\tilde{\mathbf{p}}_3}^{-1}$  are sections of  $K_{\mathbf{p}_3}^{-1}$  which are also in the ideal  $I_C$ . In particular there are non-trivial sections in  $K_{\tilde{\mathbf{p}}_3}^{-4}$  and  $K_{\tilde{\mathbf{p}}_3}^{-6}$ , and so we can write a Weierstrass equation and construct elliptic fibrations over  $\tilde{\mathbf{P}}^3$  which are Calabi-Yau.

In this example, the del Pezzo on which the gauge branes are wrapped will be the Hirzebruch surface  $\mathbf{F}_0 = \mathbf{P}^1 \times \mathbf{P}^1$ , here defined by  $Q_2 = 0$ . As mentioned, the reason for picking this model as our next example is that the proper transform of  $S = \{Q_2 = 0\}$  is contractible. To see this, let us first check that the normal bundle is indeed negative. The normal bundle of S in  $\mathbf{P}^3$  is  $\mathcal{O}(2)|_S$ . After blowing up C, the new normal bundle is  $\mathcal{O}(2)|_S \otimes \mathcal{O}(-\tilde{C})|_S$ , where  $\tilde{C}$  is the exceptional divisor. But the intersection of  $\tilde{C}$  with S is in  $\mathcal{O}(3)|_S$ , so

$$\mathcal{O}(2) \otimes \mathcal{O}(-\tilde{C})|_{S} \sim \mathcal{O}(-1)|_{S}.$$
 (3.15)

Hence the normal bundle is negative, a necessary condition for being contractible.

By looking at this example slightly differently, one may establish that S is indeed contractible. Consider a cubic hypersurface Q in  $\mathbf{P}^4$  vanishing to second order at a point p. Let T be the tangent space to  $\mathbf{P}^4$  at p. We identify T with an open subset of  $\mathbf{P}^4$ , and write the Taylor expansion of Q at p as:

$$Q = Q_2 + Q_3, (3.16)$$

with  $Q_2$ ,  $Q_3$  as in (3.13). If we also identify the  $\mathbf{P}^3$  of (3.13) with the projectivization of this T, we see that the set of lines in Q through its singular point p can be identified with the curve C.

Consider the projection  $\widetilde{\mathbf{P}}^4 \to \mathbf{P}^3$  with center p, where  $\widetilde{\mathbf{P}}^4$  is the blowup of  $\mathbf{P}^4$  at p. It restricts to a surjective morphism  $\pi : \widetilde{Q} \to \mathbf{P}^3$ , where  $\widetilde{Q}$  is the blowup of Q at p. The exceptional divisor  $S_0$  in  $\widetilde{Q}$  is mapped by  $\pi$  isomorphically to a quadric surface in  $\mathbf{P}^3$  that can be identified with our surface S. On the other hand, the inverse image of each point of  $C \subset \mathbf{P}^3$  is the corresponding line in Q. We thus have an identification of  $\widetilde{Q}$  with  $B_3 = \widetilde{\mathbf{P}}^3$ , showing that S can indeed be blown down to the singular point p of the cubic threefold Q.

In order to break the GUT group without generating a mass for hypercharge, we need a class in S which is topologically trivial in  $B_3$ . For  $S = \mathbf{P}^1 \times \mathbf{P}^1$  there is a unique candidate, the difference between the two rulings. It's not hard to see that the two  $\mathbf{P}^1$ 's yield equivalent classes in  $B_3$ : they have the same intersection number with the transform of the hyperplane class in  $\mathbf{P}^3$ , as well as with the exceptional divisor.

Now let us write explicitly the elliptic fibration. Once again we recall the Tate form of the Weierstrass equation

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6 (3.17)$$

and repeat the exercise of the previous subsection. The  $a_i$  are sections of  $K_{\tilde{\mathbf{p}}_3}^{-i}$ , which means they are polynomials of degree 4i on  $\mathbf{P}^3$  which vanish to ith order along C. Let us define  $s = Q_2$  and  $u = Q_3$ . A polynomial which vanishes to ith order along C is a sum of terms each of which is of degree at least i in s and u.

1. SU(5) gauge group on  $Q_2 = 0$ . According to table 1 this implies the following form for the  $a_i$ :

$$a_1 = P_1 u + P_2 s$$
  $a_4 = s^3 (P_7 u + P_8 s)$   
 $a_2 = s(P_3 u + P_4 s)$   $a_6 = s^5 (P_{11} u + P_{12} s)$  (3.18)  
 $a_3 = s^2 (P_5 u + P_6 s)$ 

Here we included various subleading terms  $(P_2, P_4, P_6, P_8, P_{12})$  which are not directly needed, and which will not show up in the analysis of the matter curves and Yukawa couplings.

2. Matter curves. The discriminant is given by

$$\Delta = s^5 u^7 P_1^4 R_{13} + \mathcal{O}(s^6) \tag{3.19}$$

where

$$R_{13} = -P_1^2 P_{11} + P_1 P_5 P_7 - P_3 P_5^2 (3.20)$$

If we recall that  $\tilde{C}$  is a  $\mathbf{P}^1$ -fibration over C, then the intersection of  $\tilde{C}$  with s=0 gives the section at 'zero' and with u=0 gives the section at 'infinity.' Hence after blowing up along C, the surfaces s=0 and u=0 no longer intersect; instead they intersect  $\tilde{C}$  along two disjoint curves. Thus from the discriminant we read off that the matter curves are given by

$$\Sigma_{10} = \{ Q_2 = P_1 = 0 \} \tag{3.21}$$

which is generically a rational curve, and

$$\Sigma_5 = \{ Q_2 = R_{13} = 0 \}. \tag{3.22}$$

Recall we showed above that  $N_S = \mathcal{O}(-1)|_S$ , and it is not hard to see that  $c_1 \sim \mathcal{O}(2)|_S$ . Therefore the homology classes of the matter curves are given by  $\mathcal{O}(1)|_S \sim c_1 - t$  and  $\mathcal{O}(13)|_S \sim 8c_1 - 3t$ , in full agreement with the general discussion.

3. Yukawa couplings and proton decay. As is familiar by now, the Yukawa couplings are localized at  $\lambda_{up} \sim \{Q_2 = P_1 = P_3 = 0\}$  and  $\lambda_{down} \sim \{Q_2 = P_1 = P_5 = 0\}$ . The discussion of the first example goes through if we choose the analogous factorization:

$$P_{11} = P_5 T_5 H_1^u \mod Q_2, \qquad P_7 = T_6 H_1^u \mod Q_2, \qquad P_3 = T_2 H_1^u \mod Q_2.$$
 (3.23)

With this factorization we have

$$R_{13} = (P_5 H_1^u) M_7, \qquad M_7 = -P_1^2 T_5 + P_1 T_6 - T_2 P_5$$
 (3.24)

and we identify  $\Sigma_{\mathbf{\bar{5}}_{h_d}} = \{Q_2 = P_5 = 0\}$ ,  $\Sigma_{\mathbf{\bar{5}}_{h_u}} = \{Q_2 = H_1^u = 0\}$ , and  $\Sigma_{\mathbf{\bar{5}}_m} = \{Q_2 = M_7 = 0\}$ . We refer to the discussion in the first example for why this eliminates the classical dimension four and five proton decay.

As we saw, in local models of this type it is possible to engineer three generations by putting a mild restriction on  $P_{11}$ . Whether there exist fluxes which yield the Standard Model spectrum, and whether these fluxes can be extended globally is an open question.

This example is really a special case of a more general construction. Consider blowing up a Fano three-fold along a curve C, and assume that the blow-up still admits a CY  $T^2$ -fibration. The proper transform of the surface S of minimal degree containing C is usually contractible, by the reasoning around equation (3.15). Moreover such a surface will typically have homology classes which are not inherited from the ambient space, as in the example above, if the surface had such classes before blowing up. If the degree of the surface is not too large, we can prescribe an  $I_5$  fibration along it.

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## References

- [1] R. Donagi and M. Wijnholt, "Model Building with F-Theory," arXiv:0802.2969 [hep-th].
- [2] C. Beasley, J. J. Heckman and C. Vafa, "GUTs and Exceptional Branes in F-theory I," JHEP **0901**, 058 (2009) [arXiv:0802.3391 [hep-th]].
- [3] H. Hayashi, R. Tatar, Y. Toda, T. Watari and M. Yamazaki, "New Aspects of Heterotic–F Theory Duality," Nucl. Phys. B **806**, 224 (2009) [arXiv:0805.1057 [hep-th]].
- [4] C. Vafa, "Evidence for F-Theory," Nucl. Phys. B **469**, 403 (1996) [arXiv:hep-th/9602022].
- [5] R. Donagi and M. Wijnholt, "Breaking GUT Groups in F-Theory," arXiv:0808.2223 [hep-th].
- [6] C. Beasley, J. J. Heckman and C. Vafa, "GUTs and Exceptional Branes in F-theory II: Experimental Predictions," JHEP **0901**, 059 (2009) [arXiv:0806.0102 [hep-th]].
- [7] R. Tatar and T. Watari, "GUT Relations from String Theory Compactifications," Nucl. Phys. B 810, 316 (2009) [arXiv:0806.0634 [hep-th]].
- [8] J. J. Heckman and C. Vafa, "Flavor Hierarchy From F-theory," arXiv:0811.2417 [hep-th].
- [9] J. J. Heckman, G. L. Kane, J. Shao and C. Vafa, "The Footprint of F-theory at the LHC," arXiv:0903.3609 [hep-ph].
- [10] H. Hayashi, T. Kawano, R. Tatar and T. Watari, "Codimension-3 Singularities and Yukawa Couplings in F-theory," arXiv:0901.4941 [hep-th].
- [11] B. Andreas and G. Curio, "From Local to Global in F-Theory Model Building," arXiv:0902.4143 [hep-th].
- [12] R. Blumenhagen, V. Braun, T. W. Grimm and T. Weigand, "GUTs in Type IIB Orientifold Compactifications," arXiv:0811.2936 [hep-th].
- [13] A. Font and L. E. Ibanez, "Yukawa Structure from U(1) Fluxes in F-theory Grand Unification," JHEP **0902**, 016 (2009) [arXiv:0811.2157 [hep-th]].
- [14] R. Blumenhagen, "Gauge Coupling Unification In F-Theory Grand Unified Theories," Phys. Rev. Lett. **102**, 071601 (2009) [arXiv:0812.0248 [hep-th]].
- [15] J. L. Bourjaily, "Local Models in F-Theory and M-Theory with Three Generations," arXiv:0901.3785 [hep-th].

- [16] J. Jiang, T. Li, D. V. Nanopoulos and D. Xie, "F-SU(5)," arXiv:0811.2807 [hep-th].
- [17] A. Collinucci, "New F-theory lifts," arXiv:0812.0175 [hep-th].
- [18] A. Collinucci, F. Denef and M. Esole, "D-brane Deconstructions in IIB Orientifolds," JHEP **0902**, 005 (2009) [arXiv:0805.1573 [hep-th]].
- [19] J. J. Heckman, J. Marsano, N. Saulina, S. Schafer-Nameki and C. Vafa, "Instantons and SUSY breaking in F-theory," arXiv:0808.1286 [hep-th].
- [20] J. Marsano, N. Saulina and S. Schafer-Nameki, "An Instanton Toolbox for F-Theory Model Building," arXiv:0808.2450 [hep-th].
- [21] T. Pantev and M. Wijnholt, "Hitchin's Equations and M-Theory Phenomenology," to appear.
- [22] J. Tate, "Algorithm for determining the type of a singular fiber in an elliptic pencil," Modular Functions of One Variable IV, Lecture Notes in Mathematics, vol. 476 (1975), Berlin / Heidelberg: Springer, pp. 33-52.
- [23] M. Bershadsky, K. A. Intriligator, S. Kachru, D. R. Morrison, V. Sadov and C. Vafa, "Geometric singularities and enhanced gauge symmetries," Nucl. Phys. B 481, 215 (1996) [arXiv:hep-th/9605200].
- [24] A. Sen, "Orientifold limit of F-theory vacua," Phys. Rev. D **55**, 7345 (1997) [arXiv:hep-th/9702165].
- [25] A. Sen, "F-theory and Orientifolds," Nucl. Phys. B 475, 562 (1996) [arXiv:hep-th/9605150].
- [26] K. Dasgupta and S. Mukhi, "F-theory at constant coupling," Phys. Lett. B 385, 125 (1996) [arXiv:hep-th/9606044].
- [27] V. Sadov, "Generalized Green-Schwarz mechanism in F theory," Phys. Lett. B 388, 45 (1996) [arXiv:hep-th/9606008].
- [28] R. Tatar and T. Watari, "Proton decay, Yukawa couplings and underlying gauge symmetry in string theory," Nucl. Phys. B **747**, 212 (2006) [arXiv:hep-th/0602238].
- [29] Y. K. Cheung and Z. Yin, "Anomalies, branes, and currents," Nucl. Phys. B 517, 69 (1998) [arXiv:hep-th/9710206].
- [30] R. Minasian and G. W. Moore, "K-theory and Ramond-Ramond charge," JHEP 9711, 002 (1997) [arXiv:hep-th/9710230].
- [31] E. Witten, "Anomaly cancellation on G(2) manifolds," arXiv:hep-th/0108165.
- [32] N.J. Hitchin, "The Self-Duality Equations on a Riemann Surface," Proc. London Math. Soc. (3) 55 (1987) 59-126.

- [33] C.T. Simpson, "Higgs Bundles and Local Systems," Publications Mathématiques de l'IHES, 75 (1992), p.5-95.
- [34] R. Donagi, "Spectral covers," arXiv:alg-geom/9505009.
- [35] R. Donagi, Y. H. He, B. A. Ovrut and R. Reinbacher, "The particle spectrum of heterotic compactifications," JHEP **0412**, 054 (2004) [arXiv:hep-th/0405014].
- [36] R. Blumenhagen, S. Moster, R. Reinbacher and T. Weigand, "Massless spectra of three generation U(N) heterotic string vacua," JHEP **0705**, 041 (2007) [arXiv:hep-th/0612039].
- [37] G. Curio, "Chiral matter and transitions in heterotic string models," Phys. Lett. B 435, 39 (1998) [arXiv:hep-th/9803224].
- [38] H. Grauert, "Ueber Modifikationen und exzeptionelle analytische Mengen," Math. Ann., 146 (1962), 331-368.
- [39] V. Ancona, "Faisceaux amples sur les espaces analytiques." (French. English summary) [Ample sheaves on analytic spaces] Trans. Amer. Math. Soc. 274 (1982), no. 1, 89100.
- [40] T. Peternell [Manuscripta Math. 37 (1982), 1926; MR0649561 (84d:32042); erratum, ibid. 42 (1983), 259263].
- [41] F. Denef and M. R. Douglas, "Computational complexity of the landscape. I," Annals Phys. **322**, 1096 (2007) [arXiv:hep-th/0602072].
- [42] H. Verlinde and M. Wijnholt, "Building the Standard Model on a D3-brane," JHEP **0701**, 106 (2007) [arXiv:hep-th/0508089].