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LTB spacetimes in terms of Dirac observables

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Abstract
The construction of Dirac observables, that is, gauge-invariant objects, in general relativity is technically more complicated than in other gauge theories such as the standard model due to its more complicated gauge group which is closely related to the group of spacetime diffeomorphisms. However, the explicit and usually cumbersome expression of Dirac observables in terms of gauge noninvariant quantities is irrelevant if their Poisson algebra is sufficiently simple. Precisely that can be achieved by employing the relational formalism and a specific type of matter proposed originally by Brown and Kuchař, namely pressureless dust fields. Moreover one is able to derive a compact expression for a physical Hamiltonian that drives their physical time evolution. The resulting gauge-invariant Hamiltonian system is obtained by Higgs-ing the dust scalar fields and has an infinite number of conserved charges which force the Goldstone bosons to decouple from the evolution. In previous publications we have shown that explicitly for cosmological perturbations. In this paper we analyse the spherically symmetric sector of the theory and it turns out that the solutions are in one-to-one correspondence with the class of Lemaître–Tolman–Bondi metrics. Therefore, the theory is capable of properly describing the whole class of gravitational experiments that rely on the assumption of spherical symmetry.

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1. Introduction

Until now general relativity (GR) is the most successful theory in describing observations involving gravitational interactions on macroscopic scales. It has been tested in various experimental settings and so far many predictions of GR have been confirmed to great accuracy. But next to the complicated highly nonlinear structure of Einstein's equations, there is one aspect in GR which has to be treated with special care as compared to other field theories such as Maxwell's theory: the issue of observables. Observables are those quantities that respect the gauge symmetry of the theory; for electrodynamics this would simply be all $U(1)$-invariant quantities. In contrast for GR, the gauge group is closely related to the group of spacetime diffeomorphisms, denoted by Diff(M), reflecting the independence of physics on spacetime coordinates. We are dealing with a background-independent theory where 'space' and 'time' do not have any *a priori* physical meaning.

In the canonical picture of GR [1], where one performs a (3+1)-split of the four-dimensional spacetime (M, g), gauge invariance carries over to the conditions that phase space functions have to commute with the Hamiltonian and spatial diffeomorphism constraints of GR[4]. The reason why it is so complicated to extract the gauge-invariant content of GR is that the Poisson algebra formed by the constraints of GR, the so-called Dirac algebra, is an extremely difficult one. It is not only non-Abelian but is not even an honest Lie-algebra: instead of structure constants it involves structure functions, i.e. phase space-dependent quantities. However, even if observables were known there would still be a conceptual task to solve which is often referred to as the problem of time in GR [4, 5] and concerns the evolution of observables. Speaking about dynamics of GR we usually refer to Einstein's equations. In the canonical picture these can be obtained from a canonical Hamiltonian and the Hamiltonian and diffeomorphism constraint equations. However, as a consequence of diffeomorphism invariance the canonical Hamiltonian is a linear combination of constraints only[5]. Hence, the Hamiltonian equations generated by this canonical Hamiltonian will be identical to zero in the case of observables. Consequently, Einstein's equations do not describe what we usually would call physical evolution but rather describe how the metric changes under gauge transformations. What is needed for observables is a gauge-invariant version of Einstein's equation, generated by a so-called physical Hamiltonian and describing non-trivial evolution of observables.

A framework that allows to construct observables and analyse their evolution for constrained systems such as GR is the so-called relational formalism. The basic idea is to take the background-independent nature of GR seriously and define observables not with respect to unphysical spacetime points in M but to use relations between dynamical fields instead. These ideas date back to the seminal work of Bergmann and Komar [6–8] from the 1960s. Its conceptual foundations were very much improved in the 1990s (see [9, 10] and references therein). The corresponding mathematical framework was also developed in the 1990s (see e.g. [11] and references therein and rediscovered more recently in [12, 13]). Once appropriate dynamical fields are chosen as clocks[6] in order to give space and time a physical meaning, one can at least formally write down an expression for observables associated with any phase space function f. In [14, 15] a perturbative scheme to compute these observables was developed and its application to perturbation theory around Minkowski space and cosmological perturbation theory for different choices of clocks was discussed. However, the explicit form is a power series in these clock variables with coefficients involving multiple Poisson brackets of the constraints and f generally leading to a rather complicated dependence on the physical time parameter that in most cases cannot be written down in a closed analytic form.

However, as pointed out, for instance in [16], the precise expression of the Dirac observables in terms of nongauge-invariant objects, which generically involves a hopelessly

[4] See [2] or [3] for a detailed description of how to deal with gauge symmetries in physical theories.

[5] In the absence of boundaries such as in asymptotically flat situations.

[6] Whenever we talk of clocks we mean devices (fields) to measure temporal *and* spatial distances.

complicated, infinite series of multiple Poisson brackets, is in fact irrelevant from the point of view of the physical (or reduced) phase space that we are actually interested in. What is relevant is the Poisson algebra of those Dirac observables and the physical Hamiltonian which drives their physical time evolution. General expressions for an arbitrary first-class system and for general choices of clocks were derived in [16]. From a practical point of view and especially with regard to quantization, it is of course desirable to have a sufficiently simple gauge-invariant Poisson algebra $\mathfrak{A}_{\rm phys}$ and a compact expression for the resulting physical Hamiltonian $\rm H_{phys}$. The structure of both $\mathfrak{A}_{\rm phys}$ and $\rm H_{phys}$ depends sensitively on the choice of clocks and for generic choices both are beyond mathematical control. One therefore should analyse first which type of matter simplifies their structure.

First steps in this direction have been performed in [17] where a phantom scalar field, i.e. a scalar field with a negative kinetic energy term, has been used as a clock and the corresponding physical Hamiltonian has been derived. The choice of a scalar field with vanishing potential as a clock field was motivated entirely by mathematical considerations based on the Brown–Kuchař mechanism [18] which allows us to deparametrize the Hamiltonian constraint. From a physical point of view, introducing some *ad hoc* matter component that adopts the role of clocks for GR might look artificial and even dangerous, especially if that matter is in conflict with the usual energy conditions that stabilize the system. However, the way that the relational formalism works is in fact very similar to the Higgs mechanism. In the gauge-invariant formulation, the scalar field completely disappears. What remains is the corresponding physical Goldstone boson and what one has to worry about is that this additional degree of freedom is not in conflict with observation, in particular, that the energy conditions hold in the gauge-invariant description.

A more physically motivated choice of clocks which have the additional advantage of implementing the Brown–Kuchař mechanism is defined by the original Lagrangian for pressure-free dust due to Brown and Kuchař [18]. This choice of matter is a sense physically distinguished because it can be considered in a precise sense as a congruence of mutually noninteracting, freely falling observers which only interact gravitationally and which define the dynamical reference frame for GR. From the mathematical point of view, the essential point in the construction is that the constraints of the coupled system, including gravity, matter and dust, can be written in a deparametrized form[7]. As was shown explicitly in [19], this achieves the goal of drastically simplifying the Poisson algebra of physical observables and the physical Hamiltonian and furthermore ensures that the associated physical Hamiltonian is time independent. In contrast to [18], the requirement that the physical Hamiltonian should be positive definite, or equivalently physical time is required to run forward rather than backwards, demands the dust needs to be phantom dust. This is exactly the same reason why the phantom occurred in the previous work [17] and ensures that the energy–momentum tensor of observable matter obeys the usual energy conditions. It was also shown that the resulting theory is in good agreement with current cosmological observations in [20] as far as FRW spacetimes as well as perturbations around FRW are considered. The reason for why that happens is the aforementioned Higgs mechanism by which the phantom dust completely disappears from the physical particle spectrum together with another nice feature of this particular choice of clocks which is absent for generic choices. The physical Hamiltonian system possesses an infinite number of conserved charges which can be considered as physical

[7] A system of constraints C_I is said to be deparametrizable if one can find a local coordinate chart on phase space with two mutually commuting sets of canonical pairs denoted by (q^a, p_a) and (T^I, P_I) such that in this chart the constraints can be written in the locally equivalent form $C_I = P_I + h_I$, where h_I depends only on (q^a, p_a). Equivalence has to be understood in the sense that the constraint hypersurface, spanned by the deparametrized constraints, is exactly the same as the one spanned by the original set of constraints.

energy and momentum densities of the dust respectively and which enforces the corresponding gravitational Goldstone boson modes to decouple in a mathematically precise sense from the physical time evolution. In particular, it is consistent to tune those charges to be arbitrarily small so that their corrections to Einstein's equations for the non-Goldstone modes are also arbitrarily small. In this way the dust comes as close as possible to the mathematical idealization of a test observer while taking its gravitational interaction into account. Of course, one could also use gravitational clocks, that is, certain components of the metric tensor and thus avoid the Goldstone modes altogether. However, while that works well for the linearized theory, taking into account the full nonlinearities of Einstein's theory leads to equations of motion for the physical gravitational modes that have no resemblance with the classical Einstein equations whatsoever. Similar to the vast number of proposals from cosmologists, particle physicists and string theorists for possible extensions of the field content of the standard model such as inflatons, axions, dark matter, supersymmetric extensions, dilatons, Kaluza Klein modes, etc, we consider the existence of a fundamental dust field in addition to standard matter as an at least theoretical possibility which of course has to be tested in experiments. The fact that the dust only interacts gravitationally makes it in principle a perfect dark matter candidate[8].

The topic addressed in this paper is the spherically symmetric sector of the theory presented in [19]. We will show in detail that a gauge-invariant version of Einstein's equations specialized to the spherically symmetric case can be mapped to a family of Lemaître–Tolman–Bondi (LTB) solutions. The only modification is that the equations contain a phantom dust energy density instead of the usual dust energy density and we will discuss explicitly which consequences this has.

As pointed out in [16], the existence of gauge-invariant version of Einstein's equations and thus a reduced phase space formulation of GR involving an algebra of observables and a physical Hamiltonian is of advantage when a quantization of gravity is concerned because one sidesteps the difficulties involved in the anomaly-free quantization of constraints and the construction of the physical Hilbert spaces. Recently, this strategy has been used in order to present a reduced phase space quantization [21] for loop quantum gravity [22, 23]. This framework developed for the full theory can of course be applied also to the spherically symmetric sector.

The paper is structured as follows.

In section 2 we review the results of [19] for the full theory. In section 3 we specialize the observables and the physical Hamiltonian to spherically symmetric spacetimes and in section 4 derive the corresponding equations of motion. These equations are a gauge-invariant version of Einstein's equations in the case of spherical symmetry. Section 5 discusses the solution of these equations which belong to a family of LTB-solutions. In sections 6 and 7 we discuss further properties of these solutions such as their semistatical behaviour as well as the occurrence of singularities. For the latter we analyse in particular the effect of the phantom dust energy–momentum that occurs as a source in these equations. Finally in

[8] One could call it a NIMP (non-interacting massless particle) as compared to a WIMP (weakly interacting massive particle) which is considered as the most favourite (cold) dark matter candidate. Massless here refers to the fact that the dust Lagrangian does not contain a usual mass (or potential) term. However, the foliation defined by the dust is always space-like and the corresponding foliation vector field has a unit time-like norm with respect to the physical metric (but it is not necessarily normal to the foliation). That is, the dust moves along unit time-like geodesics which also follows from the fact that the dust Lagrangian can be interpreted as the sum (or integral) of relativistic point particle Lagrangians (one for each flow line of the congruence) with variable mass depending on the flow line. This mass distribution is just the energy density of the dust. See [18] for details. Hence, the dust moves at nonrelativistic speeds as required by realistic dark matter models. Of course the dust energy density has the wrong sign to explain the anomalous galactic rotation curves but we should recall that the dust is anyway like a Higgs boson which disappears from the observable particle spectrum. There is no obstacle in adding one's favourite *observable* dark matter model.

section 8 we conclude, discuss the implications of the work done in this paper and give an outlook. Several appendices on the boundary conditions imposed on gauge-invariant variables, various spherically symmetric coordinate systems employed in the literature and a comparison with the covariant derivation of the LTB solutions complete the paper.

2. Brown–Kuchař dust reduction of general relativity

In this section we summarize the analysis performed in [19] which builds on the seminal work [18].

The Brown–Kuchař dust Lagrangian is given by

$$S_{\mathrm{dust}} = -\frac{1}{2}\int_M \mathrm{d}^4X\sqrt{|\det(g)|}\rho[g^{\mu\nu}U_\mu U_\nu + 1], \tag{2.1}$$

which is coupled to the standard Einstein Hilbert action S_{EH}:

$$S_{\mathrm{EH}} = \frac{1}{\kappa}\int_M \mathrm{d}^4X\sqrt{|\det(g)|}\,{}^{(4)}R. \tag{2.2}$$

Standard matter Lagrangians can be added in the usual way, but they do not couple to the dust. Here M is a four-dimensional manifold which can topologically be identified with $\mathbb{R}\times\mathcal{X}$ for some three-dimensional manifold $\mathcal{X}$ of arbitrary topology and X are local coordinates on M. Next, $g_{\mu\nu}(X)$ with $\mu,\nu = 0, 1, 2, 3$ denotes a (pseudo-)Riemannian metric on M. $U \in T^*M$ is a one-form defined as the differential $U = -\mathrm{d}T + W_j\,\mathrm{d}S^j$, $j = 1, 2, 3$, for some scalar fields $T, W_j, S^j \in C^\infty(M)$. So finally, action (2.1) is a functional of $g_{\mu\nu}$ and eight scalar fields ρ, T, W_i, S^i. ${}^{(4)}R$ is the Ricci scalar corresponding to the metric g.

We need the Hamiltonian formulation of that system which can be derived by performing a usual ADM-(3+1)-split with respect to a foliation of $M \cong \mathbb{R}\times\mathcal{X}$. We denote the momentum conjugate to the ADM 3-metric q_{ab} by p^{ab}. The momenta conjugate to the configuration variables ρ, W_j, T, S^j are denoted by Z, Z^j, P, P_j, respectively. Latin indices range from 1 to 3. A detailed Dirac analysis of the occurring constraints shows that the coupled system is second class. Hence, in order to proceed, one passes on to the corresponding Dirac bracket and solves the second-class constraints explicitly. As a consequence it turns out that the momenta conjugate to ρ and W_j vanish and that these configuration variables can be expressed in terms of the remaining phase space variables. Explicitly, we have

$$Z := 0, \qquad Z^j := 0, \qquad W_j := -\frac{P_j}{P}, \qquad \rho^2 := \frac{P^2}{\sqrt{\det q}}[q^{ab}U_aU_b + 1]. \tag{2.3}$$

In general, analysing second-class constraint systems is a very hard task due to the complicated structure of the corresponding Dirac bracket. However, it turns out that for the system $S_{\mathrm{EH}} + S_{\mathrm{dust}}$, when restricting attention to the geometry variables (q_{ab}, p^{ab}) as well as the remaining dust variables (S^j, P_j) and (T, P), the Dirac bracket in this sector reduces to the standard Poisson bracket again, which makes a further analysis of this system tractable.

We end up with a first-class system possessing the following constraints:

$$c^{\mathrm{tot}} = c + c^{\mathrm{dust}} \tag{2.4}$$

$$c_a^{\mathrm{tot}} = c_a + c_a^{\mathrm{dust}}, \tag{2.5}$$

where the geometry and dust contributions can be written as

$$c_a := -\frac{2}{\kappa}q_{ac}D_bp^{bc} \tag{2.6}$$

$$c := \frac{1}{\kappa}\frac{1}{\sqrt{\det q}}\left[q_{ac}q_{bd} - \frac{1}{2}q_{ab}q_{cd}\right]p^{ab}p^{cd} - \sqrt{\det(q)}R(q) \tag{2.7}$$

$$c^{\text{dust}} := P\sqrt{1 + q_{ab}U_aU_b} \tag{2.8}$$

$$c_a^{\text{dust}} := PT_{,a} + P_jS^j_{,a}. \tag{2.9}$$

Here D_a is the covariant differential compatible with q_{ab} and $R(q)$ denotes the Ricci scalar of q_{ab}. Here P takes only nonpositive values and thus the energy density of the dust is negative or zero which is why we call it phantom dust in contrast to [18]. The reasons for that can be summarized as follows. If P would take positive values then on the constraint surface we would have $c < 0$. As we will see, the derived physical Hamiltonian is approximated by $|c| = -c$ which in the limit of flat space would be the negative of the standard model Hamiltonian. One could cure this by letting time run backwards and defining the physical Hamiltonian by $-|c| = c$ but then energy would be unbounded from below. Furthermore, the dynamical foliation generated by the dust would be past oriented if we choose the other sign (see [19] for a detailed discussion).

Concerning the interpretation of (2.1) we refer the reader to [18], where the authors describe that the dust action can actually be derived as a field theoretic generalization of the concept of free massive relativistic particles moving on geodesics of the gravitational field created by the entire collection of particles. To get an idea why this interpretation is possible, one can check that the integral curves of $U^\mu = g^{\mu\nu}U_\nu$ describe geodesics of $g_{\mu\nu}$, S^j is constant along each of these curves and T describes proper time. So $S^j =$ const. labels a geodesic and $T =$ const. is an affine parameter along the geodesic. This is exactly the reason why it so convenient to use the fields T, S^j as a physical reference frame.

The canonical Hamiltonian that generates Hamiltonian equations for $(q_{ab}, p^{ab}), (P, T)$ which are, together with the constraints in (2.4) and (2.5), equivalent to the ten Einstein equations obtained in the Lagrangian framework is given by

$$\mathbf{H}_{\text{can}} := \int_{\mathcal{X}} \mathrm{d}^3x\,(n(x)c(x) + n^a(x)c_a(x)), \tag{2.10}$$

where n is the so-called lapse function and n^a the so-called shift vector which play the role of Lagrange multipliers.

Now we come to the crucial observation made by Brown and Kuchař in [18] that the system $S_{\text{EH}} + S_D$ is indeed (partially) deparametrizable. To see this, note first that the total Hamiltonian constraint (2.4) can be written in an equivalent form:

$$c^{\text{tot}} = c + P\sqrt{1 + \frac{q^{ab}c_a^{\text{dust}}c_b^{\text{dust}}}{P^2}} \simeq c + P\sqrt{1 + \frac{q^{ab}c_ac_b}{P^2}}. \tag{2.11}$$

On the constraint hypersurface, the constraints c^{tot} and c_a^{tot} can be solved for the momenta P and P_j, respectively, and thus be written down in completely equivalent[9] form as

$$\tilde{c}^{\text{tot}} = P + h \qquad h = \sqrt{c^2 - q^{ab}c_ac_b} \tag{2.12}$$

$$\tilde{c}_j^{\text{tot}} = P_j + h_j \qquad h_j = S_j^a[c_a - hT_{,a}], \tag{2.13}$$

where we assumed that $S^j_{,a}$ is nondegenerate and defined its inverse S_j^a. As shown in [19], this condition is gauge invariant under the gauge transformations generated by (2.12) and (2.13).

[9] By equivalence we mean that the constraint hypersurface generated by both constraints are the same.

At least the total Hamiltonian constraint is in deparametrized form now because h depends only on the gravitational variables q_{ab}, p^{ab}. Moreover, since the constraints are linear in the dust momenta, (2.12) and (2.13) form a strongly Abelian first-class constraint algebra. As a consequence also $\{h(x), h(y)\} = 0$ because h commutes with P. However, only the total Hamiltonian constraint c^{tot} is of deparametrized form but not the total diffeomorphism constraint, so Poisson brackets between either $h(x)$ and $h_j(y)$ or $h_j(x)$ and $h_j(y)$ will not vanish in general. So we achieved a partially deparametrized form for the coupled system of gravity and dust. Note that this can also be obtained when additional, for instance standard model, matter would be coupled to gravity and dust. The only difference in this case will be that the constraints c and c_a above will then not only include gravitational contributions but also consist of a sum of gravity plus the additional matter contributions. An example where additional to gravity and dust a K.G.–scalar field was considered can be found in [19, 20].

We are now in the position to define distinguished coordinates on the reduced phase space defined by the first-class system defined by (2.12) and (2.13). It is clear that T, S^j are pure gauge and that P, P_j can be solved for the gravitational (and standard matter) field variables. Therefore, it is natural to introduce a four-parameter family of gauge-fixing conditions defined by $T(x, t) = \tau$, $S^j(x, t) = \sigma^j$ where $\tau \in \mathbb{R}$ and σ takes values in the dust space $\mathcal{S} = S(\mathcal{X})$ which by assumption is diffeomorphic to $\mathcal{X}$. The dust space labels the geodesics and τ is an affine parameter along the geodesics.

The relational framework can now be applied and the results can be described as follows (see [19] for all the details and the derivations).

Let $S^a_j(x)$ be the inverse of $S^j_{,a}(x)$ and $J(x) := \det(\partial S/\partial x)$. Consider

$$\begin{aligned} \tilde{q}_{ij}(\sigma) &= \int_{\mathcal{X}} \mathrm{d}^3x |J(x)| \delta(\sigma^j - S^j(x)) S^a_i S^b_j q_{ab} \\ \tilde{p}^{ij}(\sigma) &= \int_{\mathcal{X}} \mathrm{d}^3x |J(x)| \delta(\sigma^j - S^j(x)) \frac{S^i_{,a} S^j_{,b} p^{ab}}{J}(x). \end{aligned} \tag{2.14}$$

It maybe checked explicitly that

$$\{\tilde{p}^{ij}(\sigma), \tilde{q}_{kl}(\sigma')\} = \kappa \delta^i_{(k} \delta^j_{l)} \delta(\sigma, \sigma'). \tag{2.15}$$

The interpretation of (2.14) is obvious. These are the coordinate transformations of (q_{ab}, P^{ab}) respectively into the dynamical coordinate system defined by $x_\sigma = S^{-1}(\sigma)$. One can arrive at these expressions independently by symplectic reduction which explains why $(\tilde{q}_{ij}, \tilde{p}^{ij})$ continue to be a conjugate pair.

Next let

$$\begin{aligned} Q_{jk}(\tau, \sigma) &:= \sum_{n=0}^{\infty} \frac{1}{n!} \{\tilde{h}_\tau, \tilde{q}_{jk}\}_{(n)} \\ P^{jk}(\tau, \sigma) &:= \sum_{n=0}^{\infty} \frac{1}{n!} \{\tilde{h}_\tau, \tilde{p}^{jk}\}_{(n)}, \end{aligned} \tag{2.16}$$

where we introduced

$$\tilde{h}_\tau := \int_{\mathcal{S}} \mathrm{d}^3\sigma (\tau - \tilde{T}(\sigma)) \tilde{h}(\sigma) \qquad \text{with} \quad \tilde{h} = h(\tilde{q}_{jk}, \tilde{p}^{jk}). \tag{2.17}$$

It may be checked explicitly that (2.16) has vanishing Poisson brackets with all constraints.

These expressions can no longer be described in a compact form; they are hopelessly complicated to evaluate as functions of $\tilde{q}$, $\tilde{p}$. However, as stressed in the introduction, all we need is their Poisson algebra and their time evolution. To that end, set $Q_{ij}(\sigma) := Q_{ij}(\tau = 0, \sigma)$, $P^{ij}(\sigma) := P^{ij}(\tau = 0, \sigma)$. Then it may be checked explicitly that

$$\{P^{ij}(\sigma), Q_{kl}(\sigma')\} = \kappa\delta^i_{(k}\delta^j_{l)}\delta(\sigma, \sigma') \tag{2.18}$$

still forms a canonical pair, all other Poisson brackets vanishing. Furthermore, let us define

$$H(\sigma) = \sqrt{C^2 - Q^{ij}C_iC_j}(\sigma), \tag{2.19}$$

where

$$C := \tilde{c}(\tilde{q}_{jk} = Q_{jk}, \tilde{p}^{jk} = P^{jk}), \qquad C_i := \tilde{c}_i(\tilde{q}_{jk} = Q_{jk}, \tilde{p}^{jk} = P^{jk}), \tag{2.20}$$

and $\tilde{c}, \tilde{c}_i$ are just c, c_a expressed in the dust coordinate system. Then it may be checked that the *physical Hamiltonian* generating time evolution for all observables $f(Q_{ij}, P^{ij})$ is given by

$$\mathbf{H}_{\text{phys}} := \int_{\mathcal{S}} \mathrm{d}^3\sigma\, H(\sigma). \tag{2.21}$$

When looking at its variation

$$\delta\mathbf{H}_{\text{phys}} = \int_{\mathcal{S}} \mathrm{d}^3\sigma \left[\left(\frac{C}{H}\right)\delta C - \left(\frac{Q^{ij}C_j}{H}\right)\delta C_i + \frac{1}{2H}C_iC_jQ^{ik}Q^{jl}\delta Q_{ij}\right] \tag{2.22}$$

$$=: \int_{\mathcal{S}} \mathrm{d}^3\sigma \left(N\delta C + N^i\delta C_i + \frac{1}{2}HN^iN^j\delta Q_{ij}\right), \tag{2.23}$$

one sees that the physical equations of motion generated by $\mathbf{H}_{\text{phys}}$ are almost equivalent to the ones generated by the canonical Hamiltonian $\mathbf{H}_{\text{can}}$ with the identification $q_{ab}(x) \to Q_{ij}(\sigma)$, $p^{ab}(x) \to P^{ij}(\sigma)$ modulo the following important differences. First, lapse N and shift N^i are not phase space-independent functions as for $\mathbf{H}_{\text{can}}$ where they only encode the arbitrariness of the foliation. Rather they are *observable* phase space functions composed out of the elementary fields Q_{ij}, P^{ij} as

$$N := \frac{C}{H}, \qquad N^i = -\frac{Q^{ij}C_j}{H}. \tag{2.24}$$

Second, there is one additional contribution proportional to the Hamiltonian density $H(\sigma)$. But $H(\sigma)$ is a conserved quantity in the theory and can be freely chosen on the initial value hypersurface, so we may tune this term to alter the equations of motion as little as we like.

The physical Hamiltonian has an infinite number of conserved charges, namely energy and momentum density $H(\sigma), C_j(\sigma)$. The latter ones generate *active* diffeomorphisms of the dust space $\mathcal{S}$; they are to be considered as symmetries of the system rather than gauge transformations generated by the *passive* diffeomorphisms of $\mathcal{X}$. Likewise, the former are related in an intricate way to time reparametrization invariance in general relativity.

3. Spherical symmetry

Now we want to specialize the general theory to spherically symmetric spacetimes. We will work directly at the gauge-invariant level and assume that what one usually measures in physical experiments are not the gauge variant 3-metrics q_{ab} and their canonically conjugate momenta p^{ab} in some unphysical coordinate system x^a but rather the *physical* metrics Q_{ij} and their *physical* canonically conjugate momenta P^{ij} measured with respect to the *physical reference frame* $\mathcal{S}$ given by the dust fields. There is nothing to debate about the fact that whenever we perform experiments we measure physical gauge-invariant quantities and not the kinematical quantities q_{ab}, p^{ab}. Hence we will require spherical symmetry with respect to the *physical* coordinate system σ.

Thus, spherically symmetric spacetimes $M = \mathbb{R} \times \mathcal{S}$ will be characterized by a triplet of Killing vector fields $\{\vec{\xi}_1, \vec{\xi}_2, \vec{\xi}_3\}$ on $\mathcal{S}$ whose commutator algebra is isomorphic to the Lie algebra $so(3)$. As usual, for such spacetimes, it is always possible to find a coordinate chart in which the physical 4-metric $G_{\mu\nu}$ takes the special form

$$G_{\mu\nu} = \begin{pmatrix} -N^2 + Q_{ij} N^i N^j & N^i \\ N^j & Q_{ij} \end{pmatrix}, \tag{3.1}$$

where $i, j = 1, 2, 3$, and

$$Q_{ij}(\sigma_r) = \mathrm{diag}\big[\Lambda^2(\sigma_r), \quad R^2(\sigma_r), \quad R^2(\sigma_r)\sin^2\sigma_\theta\big] \tag{3.2}$$

$$P^{ij}(\sigma_r) = \sin\sigma_\theta \,\mathrm{diag}\left[\frac{P_\Lambda(\sigma_r)}{2\Lambda(\sigma_r)}, \quad \frac{P_R(\sigma_r)}{4R(\sigma_r)}, \quad \frac{P_R(\sigma_r)}{4R(\sigma_r)}\sin^{-2}\sigma_\theta,\right], \tag{3.3}$$

where $(\sigma_r, \sigma_\theta, \sigma_\phi)$ are spherical coordinates of a *physical* coordinate system on the spatial slices and (Λ, P_Λ) and (R, P_R) respectively are conjugate pairs. Hence, the only nonvanishing Poisson brackets are given by

$$\{P_\Lambda(\sigma_r), \Lambda(\sigma_r{}')\} = \frac{\kappa}{4\pi}\delta(\sigma_r, \sigma_r{}'), \qquad \{P_R(\sigma_r), R(\sigma_r{}')\} = \frac{\kappa}{4\pi}\delta(\sigma_r, \sigma_r{}'). \tag{3.4}$$

In contrast to the gauge-variant formalism lapse N and shift N^i are not arbitrary phase space-independent functions but are given by (2.24).

Using these fields the geometry parts of the Hamiltonian and spatial diffeomorphism constraints reduce to

$$C = \frac{1}{\kappa}\sin\sigma_\theta \Lambda R^2\left[\frac{P_\Lambda{}^2}{8R^4} - \frac{P_\Lambda P_R}{4\Lambda R^3} + \frac{2}{\Lambda^2}\left(\frac{R'}{R}\right)^2 + \frac{4}{\Lambda^2}\frac{R''}{R} - \frac{4R'\Lambda'}{\Lambda^3 R} - \frac{2}{R^2}\right] \tag{3.5}$$

$$C_{\sigma_r} = \frac{1}{\kappa}\sin\sigma_\theta[-\Lambda P_\Lambda{}' + R' P_R] \tag{3.6}$$

$$C_{\sigma_\theta} = C^{\mathrm{geo}}_{\sigma_\phi} = 0. \tag{3.7}$$

Hence, the physical Hamiltonian $\mathbf{H}_{\mathrm{phys}}$, which generates the evolution of observables, specialised to the case of spherically symmetric spacetimes, reads

$$\mathbf{H}_{\mathrm{phys}} = \int_{\mathcal{S}} \mathrm{d}^3\sigma \sqrt{C^2 - \frac{1}{\Lambda^2}C_{\sigma_r}^2}. \tag{3.8}$$

4. Equations of motion

Now we want to discuss the physical equations of motion for the spherically symmetric case. Physical time evolution is generated by the physical Hamiltonian (2.21), so as a first step we need to compute its variation. This was already demonstrated for the general case at the end of section 2 and now we want to specialize this result to the spherically symmetric case. For this purpose we must use the physical Hamiltonian $\mathbf{H}_{\mathrm{phys}}$ that has been specialized to the spherically symmetric case in (3.8) and compute its variation while considering Λ, R, P_Λ and P_R as the basic variables. This results in the following first-order equations of motion:

$$\begin{aligned} \dot{\Lambda} &= \{\mathbf{H}_{\rm phys}, \Lambda\} \qquad & \dot{R} &= \{\mathbf{H}_{\rm phys}, R\} \\ \dot{P}_\Lambda &= \{\mathbf{H}_{\rm phys}, P_\Lambda\} \qquad & \dot{P}_R &= \{\mathbf{H}_{\rm phys}, P_R\}. \end{aligned} \tag{4.1}$$

The explicit derivation of physical equations of motion from the variation $\delta\mathbf{H}_{\rm phys}$ of the physical Hamiltonian has been done for full GR in [19] and it was shown that suitable boundary conditions can be chosen so that boundary terms can be neglected in that calculation. This carries over to the spherically symmetric case. A discussion of boundary conditions in the latter case is given in appendix A. Denoting derivatives with respect to τ and σ_r with dot and slash, respectively, the first-order equations of motion explicitly read as

$$\dot{\Lambda} = \frac{N\Lambda P_\Lambda}{4R^2} - \frac{NP_R}{4R} + (N^{\sigma_r})'\Lambda + N^{\sigma_r}\Lambda' \tag{4.2}$$

$$\dot{R} = -\frac{NP_\Lambda}{4R} + N^{\sigma_r}R' \tag{4.3}$$

$$\dot{P}_\Lambda = -\frac{NC(\sigma_r)}{4\pi\Lambda} - \frac{NP_\Lambda P_R}{4\Lambda R} - \frac{4N'RR'}{\Lambda^2} + \frac{4NR''R}{\Lambda^2} - \frac{4NRR'\Lambda'}{\Lambda^3} + N^{\sigma_r}P_\Lambda{}' - (N^{\sigma_r})^2\Lambda H \tag{4.4}$$

$$\begin{aligned} \dot{P}_R = &-\frac{2NC(\sigma_r)}{4\pi R} + \frac{N\Lambda P_\Lambda{}^2}{2R^3} - \frac{3NP_\Lambda P_R}{4R^2} + \frac{4NR'^2}{\Lambda R} + \frac{4NR''}{\Lambda} - \frac{4NR'\Lambda'}{\Lambda^2} \\ &- \frac{4N\Lambda}{R} - \frac{4N''R}{\Lambda} - \frac{4N'R'}{\Lambda} + \frac{4N'R\Lambda'}{\Lambda^2} + (N^{\sigma_r})'P_R + N^{\sigma_r}P_R. \end{aligned} \tag{4.5}$$

We obtain the same equations when we specialize the first-order Hamiltonian equations for full GR given in [19] to spherical symmetry and additionally set the K.G.–scalar field contributions to zero. Formally these equations of motion coincide with the ones generated by $\mathbf{H}_{\rm can}$ for the gauge-variant 3-metrics and momenta up to the term proportional to the Hamiltonian density H in (4.4), but there is one important difference: N and N^{σ_r} are not free functions anymore. Rather they are phase space-dependent functions given by (2.24) and choosing different N, N^i is equivalent to working in a different *physically distinguishable* coordinate system. This again is due to the fact that Diff($\mathcal{S}$) is the group of *active diffeomorphisms on* $\mathcal{S}$ as opposed to the passive diffeomorphism on $\mathcal{X}$.

The Hamiltonian density $H(\sigma)$ as well as C^{σ_r} are constants of motion [19]. The latter corresponds to the momentum density of the dust denoted by $\epsilon^{\sigma_r}(\sigma)$, that is, $C^{\sigma_r}(\sigma) = -\epsilon^{\sigma_r}(\sigma)$. In the following, we will restrict our discussion to the case of vanishing dust momentum density. Then the shift vector N^{σ_r} being proportional to C^{σ_r} is vanishing. Furthermore the Hamiltonian density is given by $H = \sqrt{C^2 - Q^{ij}C_iC_j} = C$ and we automatically get unit lapse $N = 1$. This means that equations (4.2)–(4.5) simplify significantly; all terms proportional to N^{σ_r}, N' and N'' vanish. Using that $H(\sigma)$ is a constant of motion, meaning that it does not evolve in physical time τ, we can write

$$H(\sigma) = \sin\sigma_\theta\epsilon(\sigma_r) \qquad \text{for} \quad \epsilon(\sigma_r) > 0. \tag{4.6}$$

From $H(\sigma) = -P(\sigma) = \rho_{\rm dust}\sqrt{\det(Q)}$, we obtain

$$\rho_{\rm dust}(\tau, \sigma_r) = -\frac{\sin\sigma_\theta\epsilon(\sigma_r)}{\sqrt{\det Q}} = -\frac{\epsilon(\sigma_r)}{\Lambda(\tau, \sigma_r)R^2(\tau, \sigma_r)}. \tag{4.7}$$

Hence, we can write

$$C(\sigma_r) := \int d\sigma_\theta\, d\sigma_\phi C(\sigma) = -\int d\sigma_\theta\, d\sigma_\phi \sin\sigma_\theta \Lambda(\sigma_r) R^2(\sigma_r)\rho_D(\tau,\sigma_r)$$
$$= -4\pi\rho_D(\sigma_r)\Lambda(\sigma_r)R^2(\sigma_r). \tag{4.8}$$

For further analysis we want to transform the first-order system (4.2)–(4.5) into a second-order system. To this end one can solve (4.2) and (4.3) for the momenta and then use these expressions and their τ-derivatives in (4.4) and (4.5). Setting $N = 1$ and $N^{\sigma_r} = 0$ and using (4.8), this results in

$$\frac{\ddot\Lambda}{\Lambda} + \frac{\ddot R}{R} - \frac{\dot\Lambda\dot R}{\Lambda R} - \frac{\Lambda' R'}{\Lambda^3 R} - \left(\frac{\dot R}{R}\right)^2 + \frac{1}{\Lambda^2}\left(\frac{R'}{R}\right)^2 + \frac{2R''}{\Lambda^2 R} - \frac{1}{R^2} = -\frac{\rho_D}{2}, \tag{4.9}$$

$$\frac{2\ddot R}{R} + \frac{2R''}{\Lambda^2 R} - \frac{2\dot\Lambda\dot R}{\Lambda R} - \frac{2\Lambda' R'}{\Lambda^3 R} = -\frac{\rho_D}{2}. \tag{4.10}$$

Furthermore we can use that $H(\sigma) = -\rho_{\rm dust}\sqrt{\det(Q)}$ yielding

$$-\left(\frac{\dot R}{R}\right)^2 + \frac{1}{\Lambda^2}\left(\frac{R'}{R}\right)^2 - \frac{2\dot R\dot\Lambda}{\Lambda R} - \frac{2R'\Lambda'}{\Lambda^3 R} + \frac{2R''}{\Lambda^2 R} - \frac{1}{R^2} = -\frac{\rho_D}{2}, \tag{4.11}$$

and the vanishing of the momentum density

$$\dot R' - \frac{\dot\Lambda}{\Lambda}R' = 0. \tag{4.12}$$

Equations (4.9)–(4.11) are actually not entirely independent of each other; in fact it is sufficient to keep (4.10) and (4.11). One can check that solutions to these two equations also satisfy (4.9).

5. Solving the equations of motion

In this section we will discuss the solution of the equations of motion. We will mainly follow [24] for the derivation of this solution using a slightly different notation here. In order to solve (4.9)–(4.12), it is first important to note that the (physical τ-) time dependences of $\Lambda(\tau,\sigma_r)$ and $R(\tau,\sigma_r)$ are not independent from each other. We can solve (4.12) and get

$$\Lambda(\tau,\sigma_r) = R'(\tau,\sigma_r)\frac{1}{\sqrt{1+E(\sigma_r)}}, \tag{5.1}$$

where $E(\sigma_r)$ is a so far arbitrary, time-independent function[10] of σ_r. Exploiting this, we can rewrite (4.10) and (4.11) as

$$\frac{2\ddot R}{R} + \left(\frac{\dot R}{R}\right)^2 - \frac{E}{R^2} = 0 \tag{5.2}$$

$$\frac{\ddot R}{R} - \frac{\dot R'\dot R}{R'R} + \frac{E'}{2R'R} = -\frac{\rho_D}{4}. \tag{5.3}$$

First, by multiplying (5.2) by $R^2\dot R$ we get

$$2R\dot R\ddot R + \dot R^3 - E\dot R = R(2\dot R\ddot R) + \dot R(\dot R^2 - E) = \frac{d}{d\tau}[R(\dot R^2 - E)] = 0$$
$$\Rightarrow \dot U = 0, \tag{5.4}$$

[10] It will become clear later on why we have chosen this rather complicated form for the integration constant.

where we defined $U = R(\dot{R}^2 - E)$. U does not depend on τ so we can write $U(\tau, \sigma_r) = F(\sigma_r)$ for a so far arbitrary σ_r-dependent function $F(\sigma_r)$. By differentiating U with respect to σ_r, we get

$$U' = F' = R'(\dot{R}^2 - E) + 2R\dot{R}\dot{R}' - E'R. \tag{5.5}$$

Dividing this by R^2R' we see that F' is equal to the left-hand side of (5.2) minus two times the left-hand side of (5.3). Thus, we can rewrite F' as

$$F' = \tfrac{1}{2}R'R^2\rho_D = \tfrac{1}{2}\sqrt{1+E}\Lambda R^2\rho_D = -\tfrac{1}{2}\sqrt{1+E}\epsilon(\sigma_r), \tag{5.6}$$

where we used (5.1) in the first equality and (4.7) in the second one. Integrating the equation above once, we end up with

$$F(\sigma_r) = -\frac{1}{2}\int_0^{\sigma_r} \mathrm{d}\lambda\, \epsilon(\lambda)\sqrt{1+E(\lambda)} + \alpha =: -\frac{1}{2}M(\sigma_r) + \frac{1}{2}M(0), \tag{5.7}$$

where $M(0)$ is a so far arbitrary (constant) mass. $M(\sigma_r)$ can be interpreted as the effective gravitating mass (multiplied by 1/4 since the relation between the Schwarzschild radius and the central mass involves a factor of 2) of the dust inside a sphere with the radial label σ_r. So finally the equation $U = F$ yields

$$\dot{R} = \pm\sqrt{\frac{F}{R} + E}. \tag{5.8}$$

We arrive at the following general form of the metric

$$\begin{aligned}
\mathrm{d}s^2 &= -\mathrm{d}\tau^2 + \frac{R'^2(\tau,\sigma_r)}{1+E(\sigma_r)}\mathrm{d}\sigma_r{}^2 + R^2(\tau,\sigma_r)\,\mathrm{d}\Omega^2 \\
\dot{R}(\tau,\sigma_r) &= \pm\sqrt{E(\sigma_r) + \frac{F(\sigma_r)}{R(\tau,\sigma_r)}} \\
\rho_D(\tau,\sigma_r) &= \frac{2F'(\sigma_r)}{R'(\tau,\sigma_r)R^2(\tau,\sigma_r)},
\end{aligned} \tag{5.9}$$

with $E(\sigma_r) > -1$ and $F(\sigma_r)$ must be chosen such that the expression under the square root in the second line is non-negative for all $R(\tau, \sigma_r)$ (see section 7 for further elaboration on this topic).

The solution for $R(\tau, \sigma_r)$ for general $E(\sigma_r)$ can only be given in parametric form; distinguishing the cases $E > 0$, $E = 0$, $E < 0$ and using the conventions of [25] one finds

- $E(\sigma_r) > 0$:

$$\begin{aligned}
R(\tau,\sigma_r) &= \frac{F(\sigma_r)}{2E(\sigma_r)}(\cosh(\eta) - 1) \\
(\sinh(\eta) - \eta) &= \frac{2[E(\sigma_r)]^{\frac{3}{2}}(\beta(\sigma_r) - \tau)}{F(\sigma_r)}
\end{aligned} \tag{5.10}$$

- $E(\sigma_r) < 0$:

$$\begin{aligned}
R(\tau,\sigma_r) &= \frac{F(\sigma_r)}{2(-E(\sigma_r))}(1 - \cos(\eta)) \\
(\eta - \sin(\eta)) &= \frac{2[-E(\sigma_r)]^{\frac{3}{2}}(\beta(\sigma_r) - \tau)}{F(\sigma_r)}
\end{aligned} \tag{5.11}$$

- $E(\sigma_r) = 0$:

$$R(\tau,\sigma_r) = \left[\tfrac{3}{2}\sqrt{F}(\beta(\sigma_r) - \tau)\right]^{2/3}. \tag{5.12}$$

Here $\beta(\sigma_r)$, $E(\sigma_r)$ and $F(\sigma_r)$ are so far arbitrary functions of σ_r which allow a coordinate choice by a particular form of $\beta(\sigma_r)$ and two further physical quantities. As mentioned above F in (5.7) can be understood as the effective gravitating mass within the radius σ_r and $E(\sigma_r)$ determines the time evolution of R as well as the local geometry. For a more detailed discussion, see for instance [26]. The time τ at which $R(\tau, \sigma_r)$ is equal to zero is $\beta(\sigma_r)$ and one calls $\tau \geqslant \beta(\sigma_r)$ the big bang time whereas $\tau \leqslant \beta(\sigma_r)$ is referred to as the recollapse time. Often one refers to these three cases as elliptic, parabolic and hyperbolic for $E(\sigma_r) < 0$, $E(\sigma_r) = 0$ and $E(\sigma_r) > 0$, respectively. The solution in (5.9) can easily be identified as the whole class of Lemaître–Tolman–Bondi (LTB) metrics (see appendix B.3), so we managed to map the problem of solving the physical τ-evolution of gauge-invariant 3-metrics Q_{ij} to the problem of analysing the LTB class of solutions in standard GR. However, it is important to note that opposed to the standard LTB models, the mass term $M(\sigma_r)$ enters with the opposite sign in $F(\sigma_r) = \frac{1}{2}(M(0) - M(\sigma_r))$. That means the effective gravitating mass will decrease when going further away from the central mass. This happens because the dust clocks, which have been chosen as a physical reference system, have negative energy.

We can fix the coordinate choice included in $\beta(\sigma_r)$ and the mass $M(0)$ in (5.7) by requiring that in the vacuum case the general solution above should reduce to the Lemaître solution [27] given by

$$\mathrm{d}s^2 = -\,\mathrm{d}\tau^2 + \frac{R_s}{R_{\mathrm{Lem}}(\tau, \sigma_r))}\,\mathrm{d}{\sigma_r}^2 + R^2_{\mathrm{Lem}}(\tau, \sigma_r)\,\mathrm{d}\Omega^2 \qquad \text{with}$$

$$R_{\mathrm{Lem}}(\tau, \sigma_r) = \left[\frac{3}{2}\sqrt{R_s}(\sigma_r - \tau)\right]^{2/3}, \tag{5.13}$$

where $R_s = 2\,MG/c^2$ denotes the Schwarzschild radius which simplifies in units $h = c = G = 1$ to $R_s = 2\,M$. The Lemaître solution is diffeomorphic to the Schwarzschild solution and can be obtained from the latter through a coordinate transformation into a comoving coordinate system. So this metric describes the local coordinate system of a free-falling observer in a spherically symmetric gravitational field originating from a central mass M, more details can be found in appendix B.1. Matching the Lemaître with the LTB solution we obviously need to consider the case $E(\sigma_r) = 0$. This is often referred to as the marginally bound case. Furthermore, in the vacuum case, $\epsilon(\sigma_r)$ is zero and as a consequence F' vanishes. Hence, in this case F is just equal to the constant $\frac{1}{2}M(0)$ introduced in (5.7). Using this we obtain $R(\sigma_r, 0) = \left[3/2\sqrt{\frac{1}{2}M(0)}(\beta(\sigma_r))\right]^{2/3}$ whereas $R_{\mathrm{Lem}}(\sigma_r, 0) = [3/2\sqrt{R_s}(\sigma_r)]^{2/3}$. Thus, we can match these two radii by simply choosing $\beta(\sigma_r) = \sigma_r$ and identifying $M(0) = 4\,M$ because $R_s = 2M = 1/2M(0)$.

5.1. Newtonian limit

Let us have a look at the Newtonian limit of this spacetime. In [24] it was argued that $R(\tau, \sigma_r)$ is the quantity that should actually be identified with 'Newtonian distance', for instance in the context of luminosity distance. Having this in mind let us look at equation (5.8) again. By differentiating this one with respect to τ we obtain

$$\ddot{R} = -\frac{F}{2R^2}, \tag{5.14}$$

which coincides formally with Newton's equation of motion for a point particle under the influence of a central mass. The crucial difference is that the effective gravitating mass $F/2$

is less than the central mass $M = R_s/2$. So the gravitational field is weaker than one would expect without having the dust around.

There is one remark necessary concerning the falloff behaviour of $R(\tau, \sigma_r)$. One might wonder whether the boundary conditions discussed in appendix A are not violated by the explicit form of R as in (5.12). The solution to this puzzle lies in the fact that in the derivation of the falloff conditions (A.3) we assumed a coordinate system which approaches a flat spherical Minkowskian one in the asymptotic limit. In contrast, the coordinate system we are using here is the coordinate system of a freely falling observer so one cannot expect (5.12) to hold automatically. Nevertheless, we assumed that $\epsilon(\sigma_r) \to 0$ for $\sigma_r \to \infty$, so in the limit of vanishing dust density and in the marginally bound case (that is $E = 0$) the LTB coordinates merge into Lemaître coordinates which themselves can be transformed to Schwarzschild coordinates having the correct Newtonian limit (see appendix B.1).

We have seen above that the exact Schwarzschild solution is obtained for $F =$ const. and $E = 0$. However, it is possible to obtain the Schwarzschild solution also for certain $E \neq 0$ and $F =$ const. The idea is to transform the LTB coordinates first into generalized Painlevé–Gullstrand coordinates (GPG), introduced in [28] and then to Schwarzschild coordinates. For later comparison, let us first transform the Lemaître solution from Lemaître coordinates (τ, σ_r) to the ordinary Painlevé–Gullstrand (PG) coordinates (τ, R) corresponding to observers moving on radial time-like geodesics labelled by Schwarzschild radial coordinate R. This yields a non-diagonal metric whose line element is of the form [29, 30]

$$\mathrm{d}s^2 = -\,\mathrm{d}\tau^2 + \left(\mathrm{d}R + \sqrt{\frac{R_s}{R}}\,\mathrm{d}\tau\right)^2 + R^2\,\mathrm{d}\Omega^2, \tag{5.15}$$

where we used τ for the Lemaître and PG time coordinate. We present in appendix B.2 how to transform from Schwarzschild to PG coordinates. For a pedagogical introduction to PG coordinates and more details, see for instance [31]. Often one uses Kruskal coordinates to obtain a maximally extended Schwarzschild spacetime. The PG coordinates do not cover the total Kruskal manifold. In the form given in equation (5.15) they cover the black hole horizon and the black hole singularity at $R = 0$, hence the physically interesting part of the manifold for our discussion. If we choose the sign of $\dot{R} = \pm\sqrt{E + F/R}$ to be negative then we obtain another set of coordinates with a line element similar to that in (5.15) but with a minus sign in front of the off-diagonal term. These PG-coordinates will then describe the so-called 'white hole region' of the Kruskal manifold. One of the properties of Kruskal coordinates is that R is given only implicitly; therefore, for our purpose of discussing the Newtonian limit GPG coordinates are more appropriate. One of the interesting properties of the ordinary PG metric shown in (5.15) is that the spatial hypersurfaces (that is $\mathrm{d}\tau = 0$) are flat since then $\mathrm{d}s^2 = \mathrm{d}R^2 + R^2\,\mathrm{d}\Omega^2$ and all information about the curvature is encoded in the shift vector given by $\vec{N} = (N^R := \sqrt{R_s/R}, 0, 0)$. Furthermore, asymptotically ($R \to \infty$) the PG metric coincides with the Minkowski metric.

Now we transform the LTB coordinates $(\tau, \sigma_r, \theta, \phi)$ into GPG coordinates (τ, R, θ, ϕ). Following [28] this yields the metric

$$\mathrm{d}s^2 = -\,\mathrm{d}\tau^2 + \frac{\left(\mathrm{d}R + \sqrt{E(\tau, R) + \frac{F(\tau,R)}{R}}\,\mathrm{d}\tau^2\right)}{1 + E(\tau, R)} + R^2\,\mathrm{d}\Omega^2 \tag{5.16}$$

subject to $E(\tau, R) > -1$. The functions E and F that are functions of σ_r only in LTB coordinates now become functions depending on τ and R in GPG coordinates. Therefore, the time dependence of E and F in these coordinates is restricted by the partial differential equations

$$\frac{\partial E(\tau,R)}{\partial\tau}-\sqrt{E(\tau,R)+\frac{F(\tau,R)}{R}}\frac{\partial E(\tau,R)}{\partial R}=0$$
$$\frac{\partial F(\tau,R)}{\partial\tau}-\sqrt{E(\tau,R)+\frac{F(\tau,R)}{R}}\frac{\partial F(\tau,R)}{\partial R}=0 \tag{5.17}$$

More details about the transformation from LTB to GPG coordinates can be found in appendix B.4. We will not comment on the full class of solutions of this system of PDEs but just remark that, e.g., (local) analytic solutions can be found by providing analytic initial data $F(0,R)$, $E(0,R)$, and determining the coefficients of the Taylor expansion (in terms of τ) of $E(\tau,R)$, $F(\tau,R)$ by taking higher derivatives of (5.17) at $\tau=0$ (Kovalevskaja method). For instance, a trivial solution is given by taking $E=$ const. and $F=$ const. In the special case $E(\tau,R)=0$ and $F(\tau,R)=R_s=$ const., the GPG metric coincides with the PG metric justifying the name generalized Painlevé–Gullstrand metric. In contrast to the PG solution in general, the spatial hypersurfaces of the GPG metric are no longer flat due to the $1/(1+E)$ factor in front of $\mathrm{d}R^2$. So one could think that in the limit of vanishing dust density $\epsilon(t,R)\to 0$, a transformation to Schwarzschild coordinates is only possible in the marginally bound case $E=0$. As discussed in [28] this is not the case and we summarize their discussion in the following.

Let us consider the case of vanishing dust density $\epsilon(t,R)=0$. Since $\epsilon(\tau,R)\sim F'$, it follows $F=R_s$. The equation for F in (5.17) is thus trivially satisfied. For each function E satisfying the equation in (5.17), there exists a transformation from GPG coordinates (τ,R,θ,ϕ) to Schwarzschild coordinates (T,R,θ,ϕ) given by

$$\left(\frac{\partial\tau}{\partial T}\right)^2=1+E \qquad \frac{\partial\tau}{\partial R}=\frac{\sqrt{E+\frac{R_s}{R}}}{1-\frac{R_s}{R}}. \tag{5.18}$$

Thus one obtains a family of coordinate transformations parametrized by the functions E. Indeed, the integrability condition for the system (5.18) leads back to the first condition in (5.17) (note that one has to write $E(T,R)\equiv E(\tau(T,R),R)$ in order to derive it).

Therefore, by first transforming the LTB to GPG coordinates, it is possible to transform to Schwarzschild coordinates in the limit of vanishing dust density for all (allowed) functions E. Thus we can transform to Schwarzschild coordinates not only in the marginally bound case $E=0$ but also in the elliptic ($E<0$) and hyperbolic ($E>0$) cases provided $F=$ const. Consequently, the Newtonian limit is correctly implemented for all values of E and $F=$ const.

6. Semistatic properties

Now we want to analyse whether the physical spacetime (M,G) under consideration is semistatic in a certain sense or not. We already know that the physical metric $G_{\mu\nu}$ takes the form of a Lemaître metric in the limit $\epsilon(\sigma_r)\to 0$ which in turn is just a Schwarzschild spacetime written in comoving coordinates (see appendix B.1). So one would expect to recover its static properties in some limit.

Let us have a look at the Killing equation

$$(\mathcal{L}_{\vec{\xi}}G_{\mu\nu})\stackrel{!}{=}0 \tag{6.1}$$

for the observable metric $G_{\mu\nu}$ and examine whether there exists a time-like Killing vector field $\vec{\xi}$. For the Lemaître metric, such a Killing vector field is given by $\vec{\xi}=[1,1,0,0]^T$, at least outside the event horizon, so we would expect to find something similar here. We will start with the ansatz

$$\vec{\xi} = [\xi^\tau(\tau,\sigma_r), \xi^{\sigma_r}(\tau,\sigma_r), 0, 0]^T. \tag{6.2}$$

For the physical metric described above (6.1) reduces to

$$\dot{\xi}^\tau = 0 \tag{6.3}$$

$$\xi^\tau \dot{R}' + \xi^{\sigma_r} R'' + R'(\xi^{\sigma_r})' = 0 \tag{6.4}$$

$$\xi^\tau \dot{R} + \xi^{\sigma_r} R' = 0 \tag{6.5}$$

$$-(\xi^\tau)' + R'^2 \dot{\xi}^{\sigma_r} = 0. \tag{6.6}$$

From (6.5) we get

$$\xi^{\sigma_r} = -\frac{\dot{R}}{R'}\xi^\tau, \tag{6.7}$$

and using (6.3) we obtain

$$\dot{\xi}^{\sigma_r} = \left[-\frac{\ddot{R}}{R'} + \frac{\dot{R}\dot{R}'}{R'^2}\right]\xi^\tau. \tag{6.8}$$

Equation (6.6) can be solved for

$$(\xi^\tau)' = [-R'\ddot{R} + \dot{R}\dot{R}']\xi^\tau, \tag{6.9}$$

and using this in (6.7) we see that

$$(\xi^{\sigma_r})' = \left[-\frac{\dot{R}'}{R'} + \frac{\dot{R}R''}{R'^2} + \dot{R}\ddot{R} - \frac{\dot{R}^2\dot{R}'}{R'}\right]\xi^\tau. \tag{6.10}$$

Putting all this together we obtain a condition on the solution

$$R'\ddot{R} - \dot{R}\dot{R}' \overset{!}{=} 0, \tag{6.11}$$

and using the explicit form of the solution (5.12) we see that this holds only if

$$F'(\sigma_r) \overset{!}{=} 0. \tag{6.12}$$

This means that there exists a time-like Killing vector field $\vec{\xi}$ only if $\epsilon(\sigma_r) = 0$, that is, if the dust matter fields are vanishing.

Nevertheless, as one can easily check, the metric is invariant under the action of the vector field

$$\vec{\xi}_0 := \left[1, -\frac{\dot{R}}{R'}, 0, 0\right]^T \tag{6.13}$$

up to terms of at least $\mathcal{O}(\epsilon(\sigma_r))$. So with

$$(\mathcal{L}_{\vec{\xi}_0} G)_{\mu\nu} = \mathcal{O}(\epsilon(\sigma_r)) \tag{6.14}$$

there is a precise sense in which the physical metric $G_{\mu\nu}$ can be called 'semistatic'. For vanishing $\epsilon(\sigma_r)$ this would be an exact symmetry and the metric would be isomorphic to the standard Schwarzschild metric. $\epsilon(\sigma_r) = 0$ would be inconsistent because we use real dust fields as dynamical clocks rather than ideal ones and vanishing $\epsilon(\sigma_r)$ would mean that the clocks do not carry any energy at all[11]. Nevertheless, $\epsilon(\sigma_r)$ is a free function in the theory and we can choose it arbitrarily small[12]. So by tuning $\epsilon(\sigma_r)$ in the right way we can get as close to a static spacetime as we wish.

[11] The dust part of the Hamiltonian constraint would vanish.

[12] There are certain restrictions on $\epsilon(\sigma_r)$ which have to be fulfilled in order to avoid singularities; see section 7 for further comments on this issue.

7. Discussion of singularities

In this section we will discuss the possible singularities which might appear for the metric given in (5.9) and displayed again below:

$$ds^2 = -\,d\tau^2 + \frac{R'^2}{1+E(\sigma_r)}\,d\sigma_r{}^2 + R^2\,d\Omega^2. \tag{7.1}$$

As is well known from earlier studies concerning LTB spacetimes (see for example [26, 32]) there appear two different kinds of singularities. Recalling that the dust density was given by $\rho_{\mathrm{dust}} = \frac{F'}{R'R^2}$, there are potentially two cases when it can diverge. The first one, if $R(\tau,\sigma_r) = 0$, is called a *collapse singularity* in the literature, and the second, if $R'(\tau,\sigma_r) = 0$, is known as a *shell-crossing singularity*. It was pointed out by [33] for the first time that in the latter case an appropriate decay behaviour for F' when $\sigma_r \to \infty$ can avoid the divergence of ρ_{dust} in the shell-crossing case. From the line element above we can read off that in the case of a collapse singularity the metric components $Q_{\theta\theta}$ and $Q_{\phi\phi}$ vanish while for a shell-crossing singularity only $Q_{\sigma_r\sigma_r}$ is zero.

7.1. Shell-crossing singularities

For standard LTB spacetimes with positive dust densities, one can always choose the arbitrary functions $E(\sigma_r)$, $F(\sigma_r)$ in such a way that shell-crossing singularities will not appear [32]. This holds not only for the marginally bound case ($E(\sigma_r) = 0$) but also for the hyperbolic ($E(\sigma_r) > 0$) and elliptic ($-1 < E(\sigma_r) < 0$) case.

In our framework the dust energy density enters with a negative sign. Therefore, $F'(\sigma)$ has a negative sign. Furthermore, we have the requirement $F/R + E > 0$. Consequently we arrive at slightly different conditions on $E(\sigma_r)$, $F(\sigma_r)$ in order to avoid shell-crossing singularities.

The analysis for $E(\sigma_r) \neq 0$ involves the parametric solutions for $R(\tau,\sigma_r)$ shown in (5.10) and (5.11). A singularity discussion becomes more involved in these two cases and we will restrict ourselves to the marginally bound case ($E(\sigma_r) = 0$) here; similar results can also be obtained for the hyperbolic and elliptic case.

If $E = 0$ then $Q_{\sigma_r\sigma_r}$ is given by

$$R'^2(\tau,\sigma_r) = \frac{1}{4R(\tau,\sigma_r)F(\sigma_r)}[F'(\sigma_r)(\sigma_r - \tau) + 2F(\sigma_r)]^2. \tag{7.2}$$

Considering the constraints $F > 0$, $F' < 0$, $\sigma_r - \tau \geqslant 0$, in order to avoid shell crossing, i.e. $R' \neq 0$ for all τ, σ_r we obviously must have

$$0 < -F' < \frac{2F}{\sigma_r - \tau} \qquad \text{or} \qquad -F' > \frac{2F}{\sigma_r - \tau} > 0 \tag{7.3}$$

for all σ_r, τ. Since F only depends on σ_r, for any given σ_r we can choose $\sigma_r - \tau$ arbitrarily small so that for any choice of F the second possibility can be violated. Hence the only dynamically stable condition is

$$0 < -F' < \frac{2F}{\sigma_r - \tau}. \tag{7.4}$$

Since $\sigma_r \geqslant \sigma_r - \tau \geqslant 0$, this condition is certainly implied by

$$0 < -F' < \frac{2F}{\sigma_r} \quad \Leftrightarrow \quad 0 < \left[\ln\left(F\sigma_r{}^2\right)\right]'. \tag{7.5}$$

Since the logarithm is an increasing function, we obtain that $F\sigma_r{}^2$ should be an increasing function while F should be a decreasing function. This leaves us with a large class of possible

F, for instance $F(\sigma_r) = \frac{R_s}{1+\sigma_r}$ which also fulfils $F \leqslant R_s$. Inserting this form of F into R and R' yields the following density for the dust:

$$\rho_{\text{dust}} = -\frac{2}{3}\frac{1}{\left(1 + \frac{1}{2}(\sigma_r + \tau)\right)(\sigma_r - \tau)}. \tag{7.6}$$

The dust density diverges at $\tau = \sigma_r$ as expected because this is exactly the singularity at $R = 0$ that cannot be avoided; see also the discussion in the next section. If we take the limit $\sigma_r \to \infty$ for fixed values of τ then $\rho_{\text{dust}} \to 0$. Hence, ρ_{dust} shows a physically reasonable behaviour.

7.2. Collapse singularities

The second kind of singularities are those where $R = 0$ and therefore besides the dust density ρ_{dust} also the expression for $\dot{R}^2$ in (5.9) becomes singular. These singularities occur for all three cases—marginally bound, hyperbolic and elliptic—when $\tau \to \sigma_r$. Having collapsing dust shells in mind, this is exactly the moment when the dust shell labelled by σ_r reaches the singularity. Note that close to $R = 0$ the expression for $\dot{R}^2$ is dominated by the $E = 0$ regime and thus we will again restrict our discussion to the marginally bound case. Looking at the explicit form of $R(\tau, \sigma_r)$ written down in (5.12), we can analyse how the different sign of the dust alters the behaviour during the collapse. In contrast to the shell-crossing singularity there is no way to get around this collapse singularity for the system gravity plus pressureless dust. As long as we do not introduce additional matter fields and take their non-gravitational interactions into account[13] we end up with a singularity after $\tau = \sigma_r$, just as in the gauge-variant framework using standard LTB–models.

Finally we want to summarize all the physical selection criteria for $\epsilon(\sigma_r)$ in the marginally bound case we have encountered so far. First, due to the special role of the dust fields in our formalism, we demand $F' < 0$ or equivalently $\epsilon > 0$. Second we need to ensure that $0 < F(\sigma_r) \leqslant R_s$, otherwise (5.12) is not a solution anymore; in terms of ϵ this means that $\int_0^\infty \mathrm{d}\sigma_r\, \epsilon(\sigma_r) < 2R_s$. Third, we need to choose $\epsilon(\sigma_r)$ such that $|F'(\sigma_r)| = -F'(\sigma_r) < \frac{2F(\sigma_r)}{\sigma_r}$ is fulfilled in order to not run into any shell-crossing singularities. Given these conditions the physical metric $G_{\mu\nu}$ is well defined until $\tau \to \sigma_r$ when a free-falling observer approaches the central singularity.

8. Conclusions

The task of finding Dirac observables for general relativity involves the construction of quantities that commute with all constraints of general relativity. Dirac observables are thus associated with gauge-invariant objects for general relativity where the gauge group is closely related to $\text{Diff}(M)$, the group of diffeomorphism of the underlying manifold M. The reason why it is more complicated to construct gauge-invariant quantities in the framework of general relativity is that the mathematical structure of $\text{Diff}(M)$ is richer than the structures of the gauge groups used for instance in the standard model of particle physics. In the context of the relational formalism one can at least formally construct Dirac observables, with respect to chosen clocks, one for each occurring constraint. Choosing these clocks

[13] In phenomenological matter models this is usually done by introducing matter with nonvanishing pressure. This pressure is mandatory in order to describe stable configurations such as stars. In the gauge-invariant formalism discussed in this paper, one would essentially follow the same steps, but one has to take into account the additional component ρ_{dust} when setting up the equation of state. For non-collapsing configurations, ρ_{dust} could be tuned arbitrarily small.

means choosing an observer that is dynamically coupled to the system. Considering the case of vacuum gravity these clocks must necessarily be four components of the 4-metric $g_{\mu\nu}$. If one applies the techniques developed in the relational framework and tries to compute (Dirac) observables for pure GR then one realizes that the physical time evolution of those observables does not resemble the Einstein equations. This is because their dynamics is described by an observer who sits in a laboratory whose motion through spacetime is defined through the dynamics of these 4-metric components. Of course, theoretically there always exist a coordinate transformation from this observer to, for instance, a free-falling observer; however, practically such a transformation will be hard to find. Fortunately, for certain types of matter coupled to gravity, the constraints, consisting of the gravitational and the matter contribution, can be rewritten in (partially) deparametrized form. Hence, these enlarged systems fall into the class of *deparametrizable theories* for which the question of observables can be addressed technically easier than in the general case. This idea was used in [19, 20] where instead of considering pure gravity (i.e. the Einstein–Hilbert action) one considers gravity plus pressureless dust-matter fields (i.e. Einstein–Hilbert plus dust action). The dust fields become the clocks of the system and correspond to a dynamically coupled free-falling observer. One obtains *physical equations of motion* for the *observable 3-metric* Q_{jk} and *observable momenta* P^{jk} in the dynamical reference frame defined through the dust fields. Of course, when introducing a dynamically coupled observer, we have to ensure that the occurring fingerprints of the observer are still in agreement with experimental data.

That this is the case for the cosmological sector was already shown in [19, 20] and in this paper we demonstrated that this framework also describes gravitational physics in the spherically symmetric sector to arbitrary precision. We showed that within this sector the dynamical evolution of Q_{ij}, P^{ij}, generated by a physical Hamiltonian $\mathbf{H}_{\rm phys}$, is in one-to-one correspondence with the class of Lemaitre–Tolman–Bondi (LTB) solutions for standard GR. Interpreted in our language, the choice of lapse and shift considered for the LTB solutions corresponds to a gauge fixing of the (spherically symmetric) spacetime diffeomorphism invariance whose gauge-invariant extension is precisely induced by our choice of clocks. In addition, while in the usual LTB framework one starts with a spherically symmetric, pressure free, perfect fluid ansatz for the energy–momentum tensor whose dynamics is then derived from the Bianchi identity; here we start with a fully covariant matter Lagrangian whose equations of motion then reduce to the usual ones. In other words, our framework provides a Lagrangian underpinning of the usual LTB framework. That this works so well is nontrivial as we discuss in appendix C.

The LTB class of solutions has been carefully investigated in the literature and its dynamics *in the comoving frame* is well understood. One might object that the LTB class does not contain *static* solutions, at least for non-vanishing dust energy which is mandatory if one wants to use the dust fields as clocks. But this had to be expected when taking into account the influence of a realistic observer (i.e. an observer which is *dynamically* coupled to the system as opposed to a mere test observer). In a sense one has to abandon the idealization of a static (vacuum[14]) spacetime when describing physics in terms of observable quantities.

Nevertheless, we showed that there exists a well-defined notion of a *semistatic spacetime* and one can get as close to the standard Schwarzschild solution as desired by appropriately choosing certain constants of motion (which are related to the dust energy density). Additionally, we discussed that by transforming the LTB system to generalized Painlevé–

[14] These considerations do not necessarily hold for more realistic models with additional matter fields, i.e. equilibrium states for complex systems such as stars still exist, just the point of equilibrium will be slightly shifted due to the influence of the dust.

Gullstrand coordinates and considering the limit of vanishing dust density, there exists a well-defined coordinate transformation to Schwarzschild coordinates for the elliptic, hyperbolic and marginally bound case. Consequently, in the limit where the dust density can be neglected the Newtonian limit is correctly implemented. Finally, although we consider phantom dust rather than usual dust, there exists a range of solutions for which shell-crossing singularities are avoided while the collapse singularity is unavoidable.

To conclude, the framework presented in [19, 20] seems to be compatible with observations, at least in the cosmological and spherically symmetric sectors which are the most relevant analytically solvable ones when it comes to phenomenological applications. This framework might also be useful when it comes to quantizing general relativity. The fact that the constraints have already been solved at the classical level opens the door for a reduced phase space quantization as opposed to the Dirac programme which is usually employed in loop quantum gravity [22, 23]. First steps into this direction have already been performed in [21] and a detailed analysis of this framework in the context of spherical symmetry is the subject of future work.

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Appendix A. Boundary conditions

We are dealing with asymptotically flat spacetimes and hence must impose suitable boundary conditions. For the full theory, these boundary conditions for the gauge-invariant observables as obtained via Brown–Kuchař dust reduction were discussed extensively in [19]. Here we need their reduction to spherical symmetry. Consider, as in the main text, a spherically symmetric coordinate system as $(\sigma_r, \sigma_\theta, \sigma_\phi)$. Tensor indices on the dust manifold $\mathcal{S}$ are denoted by $i, j, k, \ldots = 1, 2, 3$. The reduced observables are $Q_{ij}(\sigma)$ and their canonically conjugate momenta $P^{ij}(\sigma)$, both of which are *observables* in the sense described above. As usual, spherical symmetry constrains these fields to the following nonvanishing components and coordinate dependence, respectively:

$$Q_{ij}(\sigma_r) = \mathrm{diag}\left[\Lambda^2(\sigma_r), \quad R^2(\sigma_r), \quad R^2(\sigma_r)\sin^2\sigma_\theta\right] \tag{A.1}$$

$$P^{ij}(\sigma_r) = \sin\sigma_\theta\,\mathrm{diag}\left[\frac{P_\Lambda(\sigma_r)}{2\Lambda(\sigma_r)}, \quad \frac{P_R(\sigma_r)}{4R(\sigma_r)}, \quad \frac{P_R(\sigma_r)}{4R(\sigma_r)}\sin^{-2}\sigma_\theta,\right]. \tag{A.2}$$

Following [34], in particular the parity conditions derived there, the following decay behaviour is sufficient to guarantee a well-defined symplectic structure

$$\begin{aligned} \Lambda &\to 1 + \frac{\mu}{\sigma_r} + \mathcal{O}({\sigma_r}^{-(1+\epsilon)}) \\ R &\to \sigma_r + \mathcal{O}({\sigma_r}^{-(1+\epsilon)}) \\ P_\Lambda &\to \mathcal{O}({\sigma_r}^{-\epsilon}) \\ P_R &\to \mathcal{O}({\sigma_r}^{-(1+\epsilon)}). \end{aligned} \tag{A.3}$$

In order to derive well-defined equations of motion one should also make sure that the Hamiltonian is finite and functionally differentiable. In contrast to the gauge-variant framework, it is not the canonical Hamiltonian (with independent lapse and shift fields) but rather the physical Hamiltonian (with prescribed, field-dependent lapse and shift) that one has to consider. While their variations are algebraically almost identical, the field dependence of lapse and shift prevents one from adding the usual ADM counterterms under the usual decay behaviour.

In [19] it was already emphasized that due to the dynamical nature of lapse and shift in this framework (see (2.24)), the (geometry parts of the) diffeomorphism constraints C_i must fall off strictly faster than the (geometry parts of the) Hamiltonian constraint C in order to accommodate asymptotically flat spacetimes:

$$\lim_{\sigma_r\to\infty}\frac{C_i}{C}\to 0. \tag{A.4}$$

So one has to be careful in choosing elementary fields Q_{ij}, P^{ij} such that these conditions hold. As we have seen in the last section, the standard conditions are generally not strong enough for this purpose, they only guarantee that $C\to\mathcal{O}(\sigma_r{}^{-(1+\epsilon)})$ and $C_{\sigma_r}\to\mathcal{O}(\sigma_r{}^{-(1+\epsilon)})$. In this work we did satisfy this stronger fall-off behaviour by demanding that the shift (or equivalently the momentum density) $N^i = -Q^{ij}C_j/H = 0$ vanishes. It is then sufficient to add to the physical Hamiltonian the ADM mass term

$$E_{\mathrm{ADM}} = \frac{1}{\kappa}8\pi\lim_{r\to\infty}\left[\Lambda^2\sigma_r + \frac{R^2}{\sigma_r} - 2RR'\right] = \mu. \tag{A.5}$$

In the general case one has to choose dynamical fields Λ, R, P_Λ, P_R such that, in addition to (A.3), also (A.4) holds. This means that the physical Hamiltonian decays as $H\to\mathcal{O}\big(\sigma_r{}^{-(1+\epsilon)}\big)$ and asymptotically lapse and shift behave as

$$N := \frac{C}{H}\to 1 \tag{A.6}$$

$$N^i := -\frac{Q^{ij}C_j}{H}\to 0. \tag{A.7}$$

Appendix B. Spherically symmetric coordinate systems

In the main text we have worked with various presentations of spherically symmetric metrics in various coordinate systems. For the benefit of the reader we recall here how these coordinate systems are related with each other. Our notation is as follows. We call (τ,σ_r) the Lemaître time and radial coordinate which coincide with our dust time and radial coordinate. Schwarzschild coordinates are denoted by (T, R). The (generalized) Painlevé–Gullstrand hybrid coordinates are (τ, R). The Lemaître, Schwarzschild and strict Painlevé–Gullstrand solutions are nothing else than coordinate transformations of the static vacuum Schwarzschild solution into comoving coordinates (on restricted patches of the fully extended Kruskal spacetime). The LTB family are not vacuum solutions and are expressed most easily in Lemaître coordinates. Transforming the nontrivial LTB solutions (i.e. nonvanishing dust energy density) into Schwarzschild coordinates is not possible without picking up a nonvanishing shift since these spacetimes are neither stationary nor static. However, there is a notion of semistaticity as elaborated on in the main text.

B.1. Lemaître solution

Starting from the usual Schwarzschild solution for vacuum spacetimes one can perform a coordinate transformation into comoving coordinates and arrive at what is known as the Lemaître solution [27].

Let us start with the Schwarzschild solution in a spherical coordinate chart (T, R, θ, ϕ), where T and R approach the usual Minkowskian temporal and radial coordinates of an observer located at spatial infinity. In this coordinate system the line element can be written as

$$ds^2 = -\left(1 - \frac{R_s}{R}\right) dT^2 + \frac{1}{1 - \frac{R_s}{R}} dR^2 + R^2\, d\Omega^2, \tag{B.1}$$

where $R_s = 2MG/c^2$ is the Schwarzschild radius, M the central mass and $d\Omega^2 := d\theta^2 + \sin^2\theta\, d\phi^2$ the area element on the unit sphere. That means the metric components are given by

$$g_{\mu\nu}^{\text{Schw}} := \text{diag}\left[-\left(1 - \frac{R_s}{R}\right), \quad \frac{1}{1 - \frac{R_s}{R}}, \quad R^2, \quad R^2 \sin\theta\right]. \tag{B.2}$$

Then we perform the following coordinate transformation:

$$d\tau := dT + \sqrt{\frac{R_s}{R}}\left(\frac{1}{1 - \frac{R_s}{R}}\right) dR \tag{B.3}$$

$$d\sigma_r := dT + \sqrt{\frac{R}{R_s}}\left(\frac{1}{1 - \frac{R_s}{R}}\right) dR, \tag{B.4}$$

or in the integrated form

$$\tau(T, R) = T + 2\sqrt{R_s R} - R_s \log\left[\frac{1 + \sqrt{\frac{R}{R_s}}}{\left|1 - \sqrt{\frac{R}{R_s}}\right|}\right] \tag{B.5}$$

$$\sigma_r(\tau, R) = t(T, R) + \frac{2}{3}\sqrt{\frac{R}{R_s}} R. \tag{B.6}$$

Obviously this coordinate transformation is only valid for $R \neq R_s$ where the Schwarzschild coordinates are not well defined. The inverse coordinate transformation is given by

$$R(\tau, \sigma_r) = [3/2\sqrt{R_s}(\sigma_r - \tau)]^{2/3} \tag{B.7}$$

$$T(\tau, \sigma_r) = \tau - 2\sqrt{R_s R(\tau, \sigma_r)} + \log\left(1 + \sqrt{\frac{R(\tau, \sigma_r)}{R_s}}\right) - \log\left|1 - \sqrt{\frac{R(\tau, \sigma_r)}{R_s}}\right|, \tag{B.8}$$

or in the differential form

$$dT = \frac{1}{1 - \frac{R_s}{R}} d\tau + \frac{1}{1 - \frac{R_s}{R}} d\sigma_r \tag{B.9}$$

$$dR = -\sqrt{\frac{R_s}{R}} d\tau + \sqrt{\frac{R_s}{R}} d\sigma_r. \tag{B.10}$$

This leads to the following line element for a vacuum spacetime in the coordinate chart: $(\tau, \sigma_r, \theta, \phi)$

$$ds^2 = -\,d\tau^2 + \frac{R_s}{R(\tau,\sigma_r)}\,d\sigma_r{}^2 + R^2(\tau,\sigma_r)\,d\Omega^2. \tag{B.11}$$

This solution is known as the Lemaître metric and as one can easily read the metric components in this coordinate chart:

$$g_{\mu\nu}^{Lem} = \text{diag}\left[-1, \quad \frac{R_s}{R(\tau,\sigma_r)}, \quad R^2(\tau,\sigma_r), \quad R^2(\tau,\sigma_r)\sin^2\theta\right]. \tag{B.12}$$

This metric describes the local coordinate system of an observer who is freely falling under the influence of the central mass. It it not obvious (as for the Schwarzschild solution) right from the beginning that this solution describes a static spacetime, but one can easily calculate that for $R > R_s$ there exists a time-like, hypersurface-orthogonal Killing vector $\vec{\xi}^K \propto [1, 1, 0, 0]^T$.

B.2. Painlevé–Gullstrand solution

In order to transform the Schwarzschild solution shown in equations (B.1) and (B.2) respectively to Painlevé–Gullstrand coordinates, we introduce an observer that moves along ingoing radial, time-like geodesics of the Schwarzschild spacetime. Thus the observer's time is identical with the Lemaître time τ introduced in the last section in equation (B.3) in the differential form and in equation (B.5) in the integrated form. In contrast to the Lemaître metric, the Painlevé–Gullstrand solution [29, 30] has as the radial component the Schwarzschild R. Hence, we want to perform a transformation from the Schwarzschild coordinates (T, R, θ, ϕ) to the Painlevé–Gullstrand coordinates (τ, R, θ, ϕ). This coordinate transformation has the following differential form:

$$d\tau := dT + \sqrt{\frac{R_s}{R}}\left(\frac{1}{1-\frac{R_s}{R}}\right)dR. \tag{B.13}$$

The vacuum spacetime in the coordinate chart (τ, R, θ, ϕ) is then given by the following line element [29, 30]:

$$ds^2 = -\,d\tau^2 + \left(dR + \sqrt{\frac{R_s}{R}}\,d\tau\right)^2 + R^2\,d\Omega^2. \tag{B.14}$$

The components of the Painlevé–Gullstrand metric are given by

$$g_{\mu\nu}^{PG} = \begin{pmatrix} -1 & \sqrt{\frac{R_s}{R}} & 0 & 0 \\ \sqrt{\frac{R_s}{R}} & 1 & 0 & 0 \\ 0 & 0 & R^2 & 0 \\ 0 & 0 & 0 & R^2\sin^2\theta \end{pmatrix}. \tag{B.15}$$

This metric is no longer diagonal but still has a simple form. In particular the spatial hypersurfaces associated with this spacetime are flat because for $dt = 0$ we get $ds^2 = dR^2 + R^2\,d\Omega^2$. All information about the curvature is encoded in the shift vector $\vec{N} = (N^R = \sqrt{R_s/R}, 0, 0)$. As in the case of the Lemaître solution the coordinate transformation from Schwarzschild to Painlevé–Gullstrand coordinates can only be performed when $R \neq R_s$. This coordinate transformation corresponds to a negative $\dot{R}$ in Schwarzschild coordinates (in general $\dot{R} = \pm\sqrt{\frac{R_s}{R}}$); R decreases in time since we are considering observers moving along ingoing geodesics. Choosing the opposite sign for $\dot{R}$ yields the transformation

$$d\tau := dT - \sqrt{\frac{R_s}{R}}\left(\frac{1}{1-\frac{R_s}{R}}\right)dR. \tag{B.16}$$

For this form of dt we end up with the following line element:

$$ds^2 = -d\tau^2 + \left(dR - \sqrt{\frac{R_s}{R}}\,dt\right)^2 + R^2\,d\Omega^2. \tag{B.17}$$

This line element corresponds to the so-called 'white hole region' of the extended Schwarzschild spacetime.

B.3. Lemaître–Tolman–Bondi solutions

The Lemaître–Tolman–Bondi solution (LTB) is a family of exact solutions to Einstein's field equations (see e.g. [24, 27, 35]) that describe dynamics of a spherically symmetric spacetime filled with inhomogeneous, pressureless dust with the energy–momentum tensor $T_{\mu\nu} = \rho_D U_\mu U_\nu$, where ρ_D is the dust's energy density and $U_\mu = [1, 0, 0, 0]^T$ its velocity vector field in comoving coordinates[15].

A general solution to this problem (in a spherical coordinate chart $(\tau, \sigma_r, \theta, \phi)$) is given by

$$\begin{aligned} ds^2 &= -d\tau^2 + \frac{R'^2(\tau,\sigma_r)}{1+E(\sigma_r)}\,d\sigma_r{}^2 + R^2(\tau,\sigma_r)\,d\Omega^2 \\ \rho_D(\tau,\sigma_r) &= \frac{2F'(\sigma_r)}{R^2(\tau,\sigma_r)R'(\tau,\sigma_r)} \\ \dot{R}(\tau,\sigma_r) &= \pm\sqrt{E(\sigma_r) + \frac{F(\sigma_r)}{R(\tau,\sigma_r)}}, \end{aligned} \tag{B.18}$$

where slash and dot denote derivatives with respect to σ_r and τ, respectively.

So one can characterize a particular model by choosing particular σ_r-dependent functions $E(\sigma_r)$ and $F(\sigma_r)$. Then the physical radius $R(\tau, \sigma_r)$ is fixed up to an arbitrary function $\beta(\sigma_r)$ which characterizes different initial conditions for $R(\sigma_r, 0)$. The quantity $E(\sigma_r)$ determines the evolution of R as well as the local geometry and $F(\sigma_r)$ is related to the mass inside a shell with the radial label σ_r, see [24] for a more detailed explanation of the physical relevance of this model.

For different values of $E(\sigma_r)$, one obtains different solutions, which using the notation of [25] can be given in the parametric form as follows:

- $E(\sigma_r) > 0$:

$$\begin{aligned} R(\tau,\sigma_r) &= \frac{F(\sigma_r)}{2E(\sigma_r)}(\cosh\eta - 1) \\ (\sinh\eta - \eta) &= \frac{2E^{3/2}(\sigma_r)(\beta(\sigma_r)-\tau)}{F(\sigma_r)} \end{aligned} \tag{B.19}$$

- $E(\sigma_r) = 0$:

$$R(\tau,\sigma_r) = \left[\tfrac{3}{2}\sqrt{F(\sigma_r)}(\beta(\sigma_r)-\tau)\right]^{2/3} \tag{B.20}$$

[15] See appendix C for a covariant derivation of the equations of motion for this model.

- $E(\sigma_r) < 0$:

$$\begin{aligned} R(\tau,\sigma_r) &= \frac{F(\sigma_r)}{2(-E(\sigma_r))}(1-\cos\eta) \\ (\eta - \sin\eta) &= \frac{2(-E(\sigma_r))^{3/2}(\beta(\sigma_r)-\tau)}{F(\sigma_r)}. \end{aligned} \tag{B.21}$$

The LTB-family of solutions is widely used to describe astrophysical situations, for example, it can be used to model the gravitational collapse of a (only gravitationally interacting) matter cloud. Furthermore it can be applied to cosmological models which go beyond the standard assumptions of homogeneity in ordinary FRW-evolution.

B.4. Generalized Painlevé–Gullstrand solutions

Following [28] we present in this section how the LTB metric can be transformed into a metric expressed in terms of generalized Painlevé–Gullstrand coordinates (GPG). As for the ordinary Painlevé–Gullstrand coordinates we want the LTB-time and the GPG-time to coincide and for the radial component we take the function $R(\tau,\sigma_r)$ occurring in front of $\mathrm{d}\Omega^2$ in the LTB line element in equation (B.18). Hence, the transformation from LTB-coordinates $(\tau,\sigma_r,\theta,\phi)$ to GPG-coordinates denoted by (τ,R,θ,ϕ) is in the differential form given by

$$\mathrm{d}R = \frac{\partial R}{\partial\tau}\,\mathrm{d}\tau + \frac{\partial R}{\partial\sigma_r}\,\mathrm{d}\sigma_r = \dot{R}\,\mathrm{d}\tau + R'\,\mathrm{d}\sigma_r. \tag{B.22}$$

Let us consider the following general ansatz for the line element

$$\mathrm{d}s^2 = -X\,\mathrm{d}\tau^2 + Y\,\mathrm{d}R^2 + Z\,\mathrm{d}\tau\,\mathrm{d}R + R^2\,\mathrm{d}\Omega^2, \tag{B.23}$$

where X, Y, Z are functions of τ and R. Considering the explicit form of $\mathrm{d}R$ in equation (B.22) and comparing with the LTB line element in equation (B.18), we obtain the following conditions for the functions X, Y and Z:

$$X - Y\dot{R}^2 - Z\dot{R} = 1 \tag{B.24}$$

$$YR'^2 = \frac{R'^2}{1+E} \tag{B.25}$$

$$ZR' + 2Y\dot{R}R' = 0. \tag{B.26}$$

This system of equations has the following solutions for X, Y and Z:

$$X(\tau,R) = 1 - \frac{\dot{R}^2}{1+E(\tau,R)}, \qquad Y(t,R) = \frac{1}{1+E(\tau,R)}, \qquad Z(\tau,R) = -\frac{2\dot{R}}{1+E(\tau,R)}, \tag{B.27}$$

where substitution for $\dot{R}$ via (D.1) is being understood. The function $E(\sigma_r)$ occurring in the LTB solution becomes a function of $E(\tau,R)$ when expressing σ_r in terms of τ, R. The equation for $\dot{R}^2$ in LTB coordinates given in equation (B.18) has the following expression in the GPG coordinates:

$$\dot{R}^2 = E(\tau,R) + \frac{F(\tau,R)}{R} \quad \Leftrightarrow \quad \dot{R} = \pm\sqrt{E(\tau,R) + \frac{F(\tau,R)}{R}}. \tag{B.28}$$

Since we want to cover the black hole region of the extended Schwarzschild spacetime, we choose similar to the case of the ordinary GP coordinates a negative square root for $\dot{R}$ and obtain the following line element:

$$\mathrm{d}s^2 = -\,\mathrm{d}\tau^2 + \frac{\left(\mathrm{d}R + \sqrt{E(\tau,R) + \frac{F(\tau,R)}{R}}\,\mathrm{d}\tau\right)^2}{1+E(\tau,R)} + R^2\,\mathrm{d}\Omega^2. \tag{B.29}$$

This line element is well defined for all values of τ, R for which $E(\tau, R) > -1$ and $E(\tau, R) + F(\tau, R)/R \geqslant 0$. The opposite sign choice for $\dot{R}$ leads to a minus sign in front of the dt-term in the bracket of the second term and describes, as before, the generalized 'white hole region' of the spacetime.

Now when solving the ADM equations that lead to the GPG metric, one obtains an evolution equation for the shift vector $\vec{N}(\tau, R) = (N^R(\tau, R), 0, 0)$ given by

$$\mathcal{L}_{\vec{n}}\left((N^R(\tau, R))^2 - \frac{F(\tau, R)}{R}\right) = 0 \qquad \text{with} \quad n^\mu = (1, -N^R, 0, 0) \tag{B.30}$$

being the unit vector normal to the spatial hypersurfaces of that spacetime. This is a second-order equation for N^R because as shown in [28] the term $F(\tau, R)/R$ is related to the shift vector component by $2R\mathcal{L}_{\vec{n}}(N^R)(\tau, R) = F(\tau, R)/R$. See [28] for more details. From equation (B.29) we can easily read off the explicit form of the shift vector

$$N^R(\tau, R) = \sqrt{E(\tau, R) + \frac{F(\tau, R)}{R}}. \tag{B.31}$$

Consequently, the equation for the lapse carries over to an equation for $E(\tau, R)$ given by

$$\mathcal{L}_{\vec{n}}\,(E)\,(\tau, R) = 0 \quad \Leftrightarrow \quad \frac{\partial E(\tau, R)}{\partial \tau} - \sqrt{E(\tau, R) + \frac{F(\tau, R)}{R}}\,\frac{\partial E(\tau, R)}{\partial R} = 0. \tag{B.32}$$

Furthermore, by applying the Lie derivative $\mathcal{L}_{\vec{n}}$ onto $(N^R)^2$ and using its explicit expression in terms of E, F and R shown in equation (B.31), we obtain

$$\mathcal{L}_{\vec{n}}(F) = N^R\left(2R\mathcal{L}_{\vec{n}}(N^R)(\tau, R) - \frac{F(\tau, R)}{R}\right) = 0 \tag{B.33}$$

which yields a partial differential equation for $F(\tau, R)$:

$$\frac{\partial F(\tau, R)}{\partial \tau} - \sqrt{E(\tau, R) + \frac{F(\tau, R)}{R}}\,\frac{\partial F(\tau, R)}{\partial R} = 0. \tag{B.34}$$

Consequently, for the GPG metric only those functions E and F are allowed that satisfy the partial differential equations in (B.32) and (B.34).

Appendix C. Covariant analysis of spherically symmetric gravity plus pressureless dust

For the sake of completeness, we analyse the dynamics of pressureless dust-matter coupled to gravity in a spherically symmetric setting using the usual covariant framework of GR. The metric $g_{\mu\nu}$ is *a priori* not a gauge-invariant object and the 'dynamics' generated by Einstein's equations has to be interpreted as gauge transformations in the strict sense. In order to make the framework physically meaningful, we must fix the spacetime diffeomorphism freedom. We therefore fix a comoving coordinate system with respect to which the lapse is unity and the shift vanishes. In the Hamiltonian language, this completely fixes the gauge freedom generated by the radial diffeomorphism and Hamiltonian constraints, respectively. The comoving coordinates will be denoted by (t, r) in order to emphasize that they are measuring proper time t along *ideal test observer* geodesics labelled by r. In the usual LTB framework, and in contrast to the Brown–Kuchař framework, there is no 'observer Lagrangian' that actually models these observers and their gravitational interaction and whose proper time and geodesic label we have denoted by (τ, σ_r) throughout the text. Rather, one constructs a spherically symmetric, pressure free, perfect fluid energy–momentum tensor whose Lagrangian origin remains obscure and whose dynamics is simply induced by the Bianchi identity of the Einstein equations.

In what follows we will see that one ends up with an exact mathematical match between the two frameworks although their conceptual starting points are quite different, upon identifying $(\tau, \sigma_r) := (t, r)$. At first sight this may seem mathematically not too surprising because the energy–momentum tensor of the Brown–Kuchař Lagrangian has a perfect fluid form whose pressure is constrained[16] to vanish. However, since the Brown–Kuchař Lagrangian involves altogether eight dust fields to begin with and displays a complicated gauge symmetry involving first- and second-class constraints, it is after all not straightforward to see that one obtains a perfect match. In particular, the velocity field of the Brown–Kuchař Lagrangian is a complicated aggregate composed out of dust fields and *a priori* cannot be prescribed to take a distinguished form.

We will assume spherical symmetry; therefore, we can make the following ansatz for the line-element $\mathrm{d}s^2$:

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + \Lambda^2(t,r)\,\mathrm{d}r^2 + R^2(t,r)\,\mathrm{d}\theta^2 + R^2(t,r)\sin^2\theta\,\mathrm{d}\phi^2. \tag{C.1}$$

This means the metric components are given by

$$g_{\mu\nu} = \mathrm{diag}\big[-1, \quad \Lambda^2(t,r), \quad R^2(t,r), \quad R^2(t,r)\sin^2\theta\big]. \tag{C.2}$$

Now we compute the Christoffel-symbols $\Gamma^\lambda_{\mu\nu} := \frac{1}{2}g^{\lambda\sigma}[g_{\sigma\mu,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma}]$ of $g_{\mu\nu}$ where we used the abbreviation $g_{\mu\nu,\sigma} := \frac{\partial}{\partial x^\sigma}g_{\mu\nu}$.

The Christoffel-symbols are given by

$$\Gamma^t = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \Lambda\dot{\Lambda} & 0 & 0 \\ 0 & 0 & R\dot{R} & 0 \\ 0 & 0 & 0 & R\dot{R}\sin^2\theta \end{pmatrix} \tag{C.3}$$

$$\Gamma^r = \begin{pmatrix} 0 & \frac{\dot{\Lambda}}{\Lambda} & 0 & 0 \\ \frac{\dot{\Lambda}}{\Lambda} & \frac{\Lambda'}{\Lambda} & 0 & 0 \\ 0 & 0 & -\frac{RR'}{\Lambda^2} & 0 \\ 0 & 0 & 0 & -\frac{RR'}{\Lambda^2}\sin^2\theta \end{pmatrix} \tag{C.4}$$

$$\Gamma^\theta = \begin{pmatrix} 0 & 0 & \frac{\dot{R}}{R} & 0 \\ 0 & 0 & \frac{R'}{R} & 0 \\ \frac{\dot{R}}{R} & \frac{R'}{R} & 0 & 0 \\ 0 & 0 & 0 & -\sin\theta\cos\theta \end{pmatrix} \tag{C.5}$$

$$\Gamma^\phi = \begin{pmatrix} 0 & 0 & 0 & \frac{\dot{R}}{R} \\ 0 & 0 & 0 & \frac{R'}{R} \\ 0 & 0 & 0 & \cot\theta \\ \frac{\dot{R}}{R} & \frac{R'}{R} & \cot\theta & 0 \end{pmatrix}. \tag{C.6}$$

This leads to the following non-vanishing components of the Ricci-curvature tensor $R_{\mu\nu} = \Gamma^\rho_{\mu\nu,\rho} - \Gamma^\rho_{\mu\rho,\nu} + \Gamma^\rho_{\mu\nu}\Gamma^\sigma_{\rho\sigma} - \Gamma^\sigma_{\mu\rho}\Gamma^\rho_{\nu\sigma}$:

$$R_{tt} = -\frac{\ddot{\Lambda}}{\Lambda} - 2\frac{\ddot{R}}{R}$$

$$R_{rr} = \Lambda\ddot{\Lambda} - 2\frac{R''}{R} + \frac{2\Lambda\dot{\Lambda}\dot{R}}{R} + \frac{2\Lambda' R'}{\Lambda R}$$

[16] That is, the vanishing of the pressure is an equation of motion and not put in by hand.

$$R_{\theta\theta} = R\ddot{R} - \frac{RR''}{\Lambda^2} + \dot{R}^2 - \frac{R'^2}{\Lambda^2} + \frac{R\Lambda' R'}{\Lambda^3} + \frac{R\dot{\Lambda}\dot{R}}{\Lambda} + 1$$

$$R_{\phi\phi} = \sin^2\theta R_{\theta\theta}$$

$$R_{tr} = -2\frac{\dot{R}'}{R} + 2\frac{\dot{\Lambda}R'}{\Lambda R}. \tag{C.7}$$

The Ricci-scalar $R := g^{\mu\nu}R_{\mu\nu}$ then takes the form

$$R = 2\frac{\ddot{\Lambda}}{\Lambda} + 4\frac{\ddot{R}}{R} - 4\frac{R''}{\Lambda^2 R} + 2\left(\frac{\dot{R}}{R}\right)^2 - \frac{2}{\Lambda^2}\left(\frac{R'}{R}\right)^2 + 4\frac{\dot{\Lambda}\dot{R}}{\Lambda R} + 4\frac{\Lambda' R'}{\Lambda^3 R} + \frac{2}{R^2}. \tag{C.8}$$

Now we want to couple matter to the system; more precisely we will use inhomogeneous pressureless dust-matter. Assuming spherical symmetry, the pressure-free dust stress–energy tensor is given by

$$T_{\mu\nu} = \rho_{\rm D}(t,r)U_\mu U_\nu, \tag{C.9}$$

where $\rho_{\rm D}(t,r)$ can be interpreted as the dust energy density and U^μ is the dust velocity vector field which in comoving coordinates takes the form $U^\mu = \delta^\mu_t$.

Having collected all ingredients we can now write down Einstein's equations $R_{\mu\nu} - 1/2\,{}^{(4)}Rg_{\mu\nu} = \kappa/2T_{\mu\nu}$ with $\kappa = 16\pi G/c^4$ for the system under consideration:

$$\begin{aligned}
&-\frac{2R''}{\Lambda^2 R} - \frac{1}{\Lambda^2}\left(\frac{R'}{R}\right)^2 + \left(\frac{\dot{R}}{R}\right)^2 + 2\frac{\Lambda' R'}{\Lambda^3 R} + 2\frac{\dot{\Lambda}\dot{R}}{\Lambda R} + \frac{1}{R^2} = \kappa\frac{\rho_D}{2} \qquad tt\text{-comp.}\\
&-2\frac{\ddot{R}\Lambda^2}{R} - \Lambda^2\left(\frac{\dot{R}}{R}\right)^2 + \left(\frac{R'}{R}\right)^2 - \frac{\Lambda^2}{R^2} = 0 \qquad rr\text{-comp.}\\
&-R\ddot{R} + \frac{RR''}{\Lambda^2} - \frac{R^2\ddot{\Lambda}}{\Lambda} - \frac{R\Lambda' R'}{\Lambda^3} - \frac{R\dot{\Lambda}\dot{R}}{\Lambda} = 0 \qquad \theta\theta\text{- and } \phi\phi\text{-comp.}\\
&\dot{R}' - \frac{\dot{\Lambda}R'}{\Lambda} = 0 \qquad tr\text{-comp.}
\end{aligned} \tag{C.10}$$

Imposing the Bianchi identity (energy–momentum conservation), we find

$$\nabla^\mu T_{\mu\nu} = -\delta^t \rho_{\rm D}\frac{\partial}{\partial t}[\ln(\rho_{\rm D}\Lambda R^2)] = 0 \tag{C.11}$$

which obviously constrains ρ_D to have the form $-\epsilon(r)/[\Lambda(t,r)R^2(t,r)]$ for some free function ϵ of r only.

One can easily see that this system of partial differential equations coincides formally with the one obtained in section 4, equations (4.9)–(4.12). So we can map the problem of finding spherically symmetric solutions in the framework of gauge-invariant observables to the problem of finding spherically symmetric solutions for gravity coupled to pressureless dust-matter in the usual framework of Einstein's general relativity in the comoving gauge $N = 1, N^r = 0$.

The system of solutions to this system of partial differential equations is the so-called Lemaître–Tolman–Bondi family (see appendix B.3).

References

[1] Arnowitt R, Deser S and Misner C 1962 The dynamics of general relativity *Gravitation: An Introduction to Current Research* ed L Witten (New York: Wiley) pp 227–65

[2] Dirac P 1967 *Lectures on Quantum Mechanics* (New York: Yeshiva University)

[3] Henneaux M and Teitelboim C 1992 *Quantization of Gauge Systems* (Princeton: Princeton University Press)

[4] Bergmann P G 1961 'Gauge-invariant' variables in general relativity *Phys. Rev.* **124** 274–78

[5] Kuchar K V 1991 Time and interpretations of quantum gravity *General Relativity and Relativistic Astrophysics Proceedings (Winnipeg)* pp 211–314

[6] Bergmann P G 1961 Observables in general relativity *Rev. Mod. Phys.* **33** 510–4

[7] Komar A 1958 Construction of a complete set of independent observables in the general theory of relativity *Phys. Rev.* **111** 1182–87

[8] Bergmann P G and Komar A B 1960 Poisson brackets between locally defined observables in general relativity *Phys. Rev. Lett.* **4** 432–3

[9] Rovelli C 1991 What is observable in classical and quantum gravity? *Class. Quantum Grav.* **8** 297–316

[10] Rovelli C 2002 Partial observables *Phys. Rev.* D **65** 124013 (arXiv:gr-qc/0110035)

[11] Vytheeswaran A 1994 Gauge unfixing in second class constrained systems *Ann. Phys.* **236** 297

[12] Dittrich B 2007 Partial and complete observables for Hamiltonian constrained systems *Gen. Rel. Grav.* **39** 1891–927 (arXiv:gr-qc/0411013)

[13] Dittrich B 2006 Partial and complete observables for canonical general relativity *Class. Quantum Grav.* **23** 6155–84 (arXiv:gr-qc/0507106)

[14] Dittrich B and Tambornino J 2007 A perturbative approach to Dirac observables and their space-time algebra *Class. Quantum Grav.* **24** 757–84 (arXiv:gr-qc/0610060)

[15] Dittrich B and Tambornino J 2007 Gauge invariant perturbations around symmetry reduced sectors of general relativity: applications to cosmology *Class. Quantum Grav.* **24** 4543–86 (arXiv:gr-qc/0702093)

[16] Thiemann T 2006 Reduced phase space quantization and Dirac observables *Class. Quantum Grav.* **23** 1163–80 (arXiv:gr-qc/0411031)

[17] Thiemann T 2006 Solving the problem of time in general relativity and cosmology with phantoms and k-essence arXiv:astro-ph/0607380

[18] Brown J D and Kuchar K V 1995 Dust as a standard of space and time in canonical quantum gravity *Phys. Rev.* D **51** 5600–29 (arXiv:gr-qc/9409001)

[19] Giesel K, Hofmann S, Thiemann T and Winkler O 2010 Manifestly gauge-invariant general relativistic perturbation theory: I. Foundations *Class. Quantum Grav.* **27** 055005 (arXiv:0711.0115 [gr-qc])

[20] Giesel K, Hofmann S, Thiemann T and Winkler O 2010 Manifestly gauge-invariant general relativistic perturbation theory: II. FRW background and first order *Class. Quantum Grav.* **27** 055006 (arXiv:0711.0117 [gr-qc])

[21] Giesel K and Thiemann T 2007 Algebraic quantum gravity (AQG): IV. Reduced phase space quantisation of loop quantum gravity arXiv:0711.0119 [gr-qc]

[22] Rovelli C 2004 *Quantum Gravity* (Cambridge: Cambridge University Press)

[23] Thiemann T 2007 *Modern Canonical Quantum General Relativity* (Cambridge: Cambridge University Press)

[24] Bondi H 1947 Spherically symmetrical models in general relativity *Mon. Not. R. Astron. Soc.* **107** 410–25

[25] Landau L and Lifshitz E 1987 *The Classical Theory of Fields* 4th edn, vol 2 (Oxford: Butterworth-Heinemann)

[26] Szekeres P and Lun A 1999 What is a shell crossing singularity? *J. Aust. Math. Soc.* B **41** 167–79

[27] Lemaître G 1933 L'univers en expansion *Ann. Soc. Sci. Brux.* A **53** 51–85

[28] Lasky P D, Lun A W C and Burston R B 2007 Initial value formalism for dust collapse *ANZIAM J.* **49** (arXiv:gr-qc/0606003)

[29] Painlevé P 1921 La Mecanique classique et la theorie de la relativite *C. R. Acad. Sci. (Paris)* **173** 677–80

[30] Gullstrand A 1922 Allgemeine Lösung des statischen Einkörperproblems in der Einsteinschen Gravitationstheorie *Arkiv. Mat. Astron. Fys.* **16** 1–15

[31] Martel K and Poisson E 2001 Regular coordinate systems for Schwarzschild and other spherical spacetimes *Am. J. Phys.* **69** 476–80 (arXiv:gr-qc/0001069)

[32] Hellaby C and Lake K 1985 Shell crossings and the Tolman model *Astrophys. J.* **290** 381

[33] Grishchuk L and Zel'dovich Y B 1984 Structure and future of the new universe *Mon. Not. R. Astron. Soc.* **207** 23–8

[34] Beig R and Ó Murchadha N 1987 The Poincaré group as the symmetry group of canonical general relativity *Ann. Phys. (NY)* **174** 463–98

[35] Tolman R C 1934 Effect of inhomogeneity on cosmological models *Proc. Natl Acad. Sci.* **20** 169–76