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ASYMPTOTICS OF LINEARIZED COSMOLOGICAL PERTURBATIONS

PAUL T. ALLEN*,† and ALAN D. RENDALL*

**Max-Planck-Institut für Gravitationsphysik*
Albert-Einstein-Institut, Am Mühlenberg 1
14476 Potsdam, Germany

†*Department of Mathematical Sciences*
Lewis and Clark College
0615 S. W. Palatine Hill Road
Portland, OR 97219, USA

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Abstract. In cosmology an important role is played by homogeneous and isotropic solutions of the Einstein–Euler equations and linearized perturbations of these. This paper proves results on the asymptotic behavior of scalar perturbations both in the approach to the initial singularity of the background model and at late times. The main equation of interest is a linear hyperbolic equation whose coefficients depend only on time. Expansions for the solutions are obtained in both asymptotic regimes. In both cases, it is shown how general solutions with a linear equation of state can be parametrized by certain functions which are coefficients in the asymptotic expansion. For some nonlinear equations of state, it is found that the late-time asymptotic behavior is qualitatively different from that in the linear case.

Keywords: Cosmology; hyperbolic equations; energy estimates; asymptotic behavior.

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1. Introduction

Astronomical observations allow information to be collected about the distribution of matter in the universe. This distribution contains structures on many different scales. Astrophysicists would like to provide a theoretical account of how these structures formed. In particular, cosmologists would like to do this for structures on the largest scales which can be observed. This means for instance giving an explanation of the way in which galaxies cluster. The most powerful influence on the dynamics of the matter distribution on very large scales is gravity. The most appropriate description of the gravitational field in this context is given by the Einstein equations of general relativity. It is also necessary to choose a model of the matter which generates the gravitational field. A frequent choice for this is a perfect fluid satisfying the Euler equations. Thus, from a mathematical point of view, the basic object of study is the Einstein–Euler system describing the evolution of a self-gravitating fluid. This is a system of quasilinear hyperbolic equations.

The standard cosmological models are the Friedmann–Lemaitre–Robertson–Walker (FLRW) models which are homogeneous and isotropic. This means in particular that the unknowns in the Einstein–Euler system depend only on time and the partial differential equations reduce to ordinary differential equations. With appropriate assumptions on the fluid these ODE's can be solved explicitly or, at least, the qualitative behavior of their solutions can be determined in great detail. When it comes to the study of inhomogeneous structures, however, the FLRW models are by definition not sufficient. Since fully inhomogeneous solutions of the Einstein–Euler system are difficult to understand a typical strategy is to linearize the system about a background FLRW model. Under favorable conditions the linearized perturbations could give information about the evolution under the Einstein–Euler system of initial data which are small but finite perturbations of those for the FLRW background.

Linearization about a highly symmetric solution is a classical practice in applied mathematics. For some examples, see [2, 5, 12]. It should be noted, however, that there is an unusual feature in the case of the Einstein–Euler system which has to do with the fact that these equations are invariant under diffeomorphisms. This is related to the fact that the only thing that is of physical significance are equivalence classes of solutions under diffeomorphisms. Since it is not known how to develop PDE theory in a manifestly diffeomorphism-invariant way this leads to difficulties. There is a corresponding equivalence relation on linearized solutions. Different linearized solutions are related by the linearizations of one-parameter families of diffeomorphisms, which are known in the literature on cosmology as gauge transformations. In the end what is interesting is not the vector space of solutions of the linearized equations but its quotient by gauge transformations. It is useful to represent this quotient space by a subspace. This is what is known in the literature on cosmology as gauge-invariant perturbation theory. This subject would no doubt benefit from closer mathematical scrutiny but that task will not be attempted in the present paper.

Instead the following pragmatic approach will be adopted: take an equation from the astrophysical literature on cosmological perturbation theory and analyze the properties of its solutions. As a basic source the book of Mukhanov [8] will be used. The notation in the following will generally agree with that of [8]. It is standard to classify cosmological perturbations into scalar, vector and tensor perturbations. These terms will not be defined here. It should be noted that scalar perturbations play a central role in the analysis of structure formation. This motivates the fact that the results of this paper are concerned with that case. After a suitable gauge choice scalar perturbations are described by solutions of a scalar wave equation for a function Φ which corresponds, roughly speaking, to the Newtonian gravitational potential. In order to get definite expressions for the Einstein–Euler system and its linearization about an FLRW model, it is necessary to choose an equation of state $p = f(\epsilon)$ for the fluid. Here ϵ is the energy density of the fluid and p its pressure. A case which is particularly simple analytically is that of a linear equation of state $p = w\epsilon$ where w is a constant. For physical reasons w is chosen to belong to the interval $[0, 1]$. In fact the condition $w \geq 0$ is necessary in order to make the Euler equations hyperbolic. The case $w = 0$, known as dust, is somewhat exceptional and does not always fit well with the general arguments in the sequel. Since, however, dust frequently comes up in the literature on cosmology it is important to include it. In those cases where the general argument fails for dust this will be pointed out.

For a linear equation of state as just described the equation for Φ is

$$\Phi'' + \frac{6(1+w)}{1+3w}\frac{1}{\eta}\Phi' = w\Delta\Phi. \tag{1.1}$$

Here a prime stands for $\frac{d}{d\eta}$. The time coordinate η belongs to the interval $(0, \infty)$. The spatial variables, which will be denoted collectively by x, are supposed to belong to the torus T^3. Thus, periodic boundary conditions are imposed. The Laplacian is that of a fixed flat metric on the torus. Its expression in adapted coordinates agrees with that for the usual Laplacian on $\mathbf{R}^3$. As a consequence of standard theory for linear hyperbolic equations this equation has a unique solution on the whole time interval $(0, \infty)$ for appropriate initial data given at a fixed time $\eta = \eta_0 > 0$. These are the restrictions of Φ and Φ' to $\eta = \eta_0$.

In the following, after some background and notation has been collected in Sec. 2, the asymptotics of solutions of Eq. (1.1) is studied in the regimes $\eta \to 0$ and $\eta \to \infty$. Theorems and proofs for the first of these cases are given in Sec. 3 (Theorems 3.1 and 3.2) and for the second in Sec. 4 (Theorem 4.2). It is shown how all solutions can be parametrized by asymptotic data in either of these regimes. These are alternatives to the usual parametrization of solutions by Cauchy data. An interesting feature of the expanding direction $\eta \to \infty$ is that the main part of the asymptotic data is a solution of the flat space wave equation $W'' = w\Delta W$. Many of these results can be extended to more general equations of state. This is the subject of Theorem 3.3 of Sec. 3 (limit $\eta \to 0$) and Sec. 5. It is found that for equations of state with power law behavior $p \sim \epsilon^{1+\sigma}$ at low density there is a bifurcation with a fundamental change in the asymptotic behavior at $\sigma = \frac{1}{3}$.

2. Preliminaries

As outlined above, we study perturbations of FLRW cosmological models which are spatially flat and have T^3 spatial topology. The spacetime being perturbed, which we refer to as the background, is described by a metric of the form

$$a^2(-d\eta^2 + dx^2) \tag{2.1}$$

on $(0,\infty)\times T^3$. Here dx^2 indicates the flat metric on T^3 and the scale factor $a = a(\eta)$ is a non-decreasing function of the conformal time η. We use x to indicate points on T^3. The signature used here is the opposite of that used by Mukhanov [8] but all the equations required in the following are unaffected by this change.

We make use of the perfect fluid matter model, described by the pressure p and energy density ϵ of the fluid. In order to specify the matter model completely, one must provide an equation of state $p = f(\epsilon)$. Under this ansatz, the Einstein–Euler equations reduce to a coupled system of ODEs for a and ϵ:

$$a'' = \frac{4\pi G}{3}(\epsilon - 3f(\epsilon))a^3 \tag{2.2}$$

$$\epsilon' = -3\mathcal{H}(\epsilon + f(\epsilon)). \tag{2.3}$$

As mentioned in the introduction, $(\ )'$ indicates a derivative with respect to η. Here G is Newton's gravitational constant and $\mathcal{H}$ is the conformal Hubble parameter, given by $\mathcal{H} = a^{-1}a'$. We note the following useful relation (known as the Hamiltonian constraint)

$$\mathcal{H}^2 = \frac{8\pi G}{3}a^2\epsilon. \tag{2.4}$$

For a linear equation of state $f(\epsilon) = w\epsilon$, solutions $a(\eta)$ of (2.2) are explicitly given by

$$\frac{a(\eta)}{a(\eta_0)} = \left(\frac{\eta}{\eta_0}\right)^{2/(1+3w)}, \tag{2.5}$$

for some arbitrarily fixed $\eta_0 \in (0,\infty)$. As the scale factor a vanishes as $\eta \to 0$, the spacetime develops a curvature singularity in that limit, which is known as a "big-bang" type singularity and is viewed as being in the past of η_0. Likewise the limit as $\eta \to \infty$ is referred to as "late times" as it corresponds to the distant future of η_0. Note that spacetimes described by these models are expanding, in the sense that the scale factor is an increasing function of η. Note also that, since ϵ' is negative, large values of η correspond to small values of ϵ and vice-versa.

We study behavior near the singularity and at late times for those perturbations to the metric (2.1) which are of the type usually referred to as scalar perturbations. They satisfy evolution equations obtained by linearizing the Einstein equations about the FLRW background. For the perfect fluid matter model all such perturbations can be described, up to gauge freedom, by a single function $\Phi(\eta, x)$. Using a certain gauge, the conformal-Newtonian gauge, the metric takes the form

$$a^2[-(1+2\lambda\Phi)d\eta^2 + (1-2\lambda\Phi)dx^2] \tag{2.6}$$

up to an error which is quadratic in the expansion parameter λ. The first order perturbation satisfies the linearized Einstein–Euler equations provided

$$\Phi'' + 3(1+f'(\epsilon))\mathcal{H}\Phi' + 3\left(f'(\epsilon) - \frac{f(\epsilon)}{\epsilon}\right)\mathcal{H}^2\Phi - f'(\epsilon)\Delta\Phi = 0, \tag{2.7}$$

where Δ is the Laplacian for the flat metric on T^3. For a derivation of this equation, we refer the reader to [8, Sec. 7.2]. The corresponding perturbations to the energy density, denoted by $\delta\epsilon$, are determined by

$$\delta\epsilon = \frac{1}{4\pi G a^2}(-3\mathcal{H}\Phi' - 3\mathcal{H}^2\Phi + \Delta\Phi) \tag{2.8}$$

and thus can be computed once (2.7) is understood.

The quantity $f'(\epsilon)$ represents the square of the speed of sound for the fluid. For physical reasons we require that f' always take values in the interval $[0, 1]$, i.e. that the speed of sound be real and not exceed the speed of light. A special case of particular interest is that of a linear equation of state $p = w\epsilon$. In this situation, the speed of sound is constant and Eq. (2.7) reduces to (1.1). Before the asymptotics of solutions of (2.7) can reasonably be studied a prerequisite is a theorem which guarantees global existence of solutions on the interval $(0, \infty)$. In order to get this from the standard theory of hyperbolic equations, it is necessary to assume that f' never vanishes. In the following, it is always assumed that this holds except in the special case of dust which is discussed separately.

Our analysis below relies on establishing a number of energy-type estimates for solutions to (2.7). As the coefficients of this linear equation depend only on η, any spatial derivative of Φ satisfies the same equation. Thus, any estimate we obtain for Φ, Φ', or $\nabla\Phi$ (the gradient of Φ with respect to the flat metric on T^3) holds also for all spatial derivatives of those quantities. One may then make use of the Sobolev embedding theorem in order to establish pointwise estimates. We also make use of the Poincaré estimate which implies that quantities having zero (spatial) mean value are controlled in L^2 by the norm of their (spatial) gradient.

Each of these norms is defined on the η-constant "spatial" slices of $(0, \infty) \times T^3$ with respect to the flat (η-independent) metric induced on T^3 by viewing T^3 as a quotient of Euclidean space. All integration on T^3 is done with respect to the corresponding volume element which we suppress in our notation. We generally suppress dependence of functions on the spatial variable x, except in situations where the inclusion of such dependence provides additional clarity. When necessary, we denote Cartesian coordinates on T^3 by $x = (x^i)$; the corresponding derivatives are denoted ∂_i.

3. Asymptotics in the Approach to the Singularity

The purpose of this section is to analyse the asymptotics of solutions of (1.1) in the limit $\eta \to 0$ and to give some extensions of these results to more general equations of state which need not be linear. Define $\nu = \frac{1}{2}(\frac{5+3w}{1+3w})$. Note that ν belongs to the interval $[1, 5/2]$.

Theorem 3.1. *Let $\Phi(\eta)$ be a smooth solution of (1.1) on $(0, \infty) \times T^3$. Then, there are coefficients $\Phi_{k,l}$ with $k \geq -2\nu$ belonging to an increasing sequence of real numbers tending to infinity and $l \in \{0, 1\}$, smooth functions on T^3, such that the formal* *series* $\sum_k(\Phi_{k,0}+\Phi_{k,1}\log\eta)\eta^k$ *is asymptotic to* $\Phi(\eta)$ *in the limit* $\eta\to 0$ *in the sense of uniform convergence of the function and its spatial derivatives of all orders. All coefficients can be expressed as linear combinations of* $\Phi_{-2\nu,0},\Phi_{0,0}$ *and their spatial derivatives. If* ν *is not an integer, then all coefficients with* $l=1$ *vanish. For any value of* w *the coefficients* $\Phi_{k,l}$ *with* $l=1$ *and* $k<0$ *vanish.*

In more detail, $\Phi_{k,0}$ *may only be nonzero when* k *is of the form* $-2\nu+2i$ *or* $2i$ *for a non-negative integer* i *while* $\Phi_{k,1}$ *may only be nonzero for* k *of the form* $2i$ *with* i *a non-negative integer. These coefficients are related by the following equations*:

$$k(k+2\nu)\Phi_{k,0}=w\Delta\Phi_{k-2,0}-(2k+2\nu)\Phi_{k,1} \tag{3.1}$$

and

$$k(k+2\nu)\Phi_{k,1}=w\Delta\Phi_{k-2,1}. \tag{3.2}$$

Proof. The basic tool which allows the solutions to be controlled is provided by energy estimates. Let

$$E_1(\eta)=\frac{1}{2}\int_{T^3}|\Phi'(\eta)|^2+w|\nabla\Phi(\eta)|^2. \tag{3.3}$$

It satisfies the identity

$$\frac{d}{d\eta}[\eta^{2(2\nu+1)}E_1(\eta)]=(2\nu+1)\eta^{4\nu+1}\int_{T^3}w|\nabla\Phi(\eta)|^2. \tag{3.4}$$

Since the right-hand side is manifestly non-negative it can be concluded that if an initial time η_0 is given then $\eta^{2(2\nu+1)}E_1(\eta)$ is bounded for $\eta\le\eta_0$. Any spatial derivative of Φ satisfies the same equation as Φ. Thus, corresponding bounds can be obtained for the L^2 norms of all spatial derivatives. Applying the Sobolev embedding theorem then provides pointwise bounds for Φ and its spatial derivatives of all orders in the past of a fixed Cauchy surface. These estimates can now be put back into the equation to obtain further information about the asymptotics. To do this, it is convenient to write (1.1) in the form

$$\frac{d}{d\eta}[\eta^{2\nu+1}\Phi'(\eta)]=\eta^{2\nu+1}w\Delta\Phi(\eta). \tag{3.5}$$

It can be deduced that

$$\Phi'(\eta)=\eta^{-2\nu-1}\left[\eta_0^{2\nu+1}\Phi'(\eta_0)-w\int_0^{\eta_0}\zeta^{2\nu+1}\Delta\Phi(\zeta)d\zeta+w\int_0^{\eta}\zeta^{2\nu+1}\Delta\Phi(\zeta)d\zeta\right]. \tag{3.6}$$

The bounds already obtained guarantee the convergence of the integrals. This formula allows the asymptotic expansions to be derived inductively. Using the fact that the second integral is $O(\eta^2)$ already gives a one-term expansion for Φ' and this can be integrated to give a one-term expansion for Φ. Analogous expansions can be obtained for all spatial derivatives of Φ in the same way using the corresponding spatial derivatives of (3.6). When an asymptotic expansion with a finite number of explicit terms is substituted into the right-hand side of (3.6), an expansion for Φ' (and thus by integration for Φ) with additional explicit terms is obtained. If the last explicit term in the input is a multiple of η^p with $p < -2$, then there is one new term in the output and it is a multiple of η^{p+2}. If the last explicit term is a multiple of η^{-2}, there is one new term and it is a multiple of $\log \eta$. If the last explicit term is a multiple of $\log \eta$, then there are two new terms, one a multiple of $\eta^2 \log \eta$ and one a constant. If the last explicit term is η^p or $\eta^p \log \eta$ with $p > -2$, then there is one new term and it is a multiple of η^{p+2} or $\eta^{p+2} \log \eta$ respectively. These statements rely on the fact that when any of the terms in the asymptotic expansion is substituted into the last integral in (3.6) the power -1 never arises. These remarks suffice to prove the first part of the theorem. The resulting series is by construction a formal series solution of the original equation. Comparing coefficients gives the rest of the theorem. □

Note that the only two values of w in the range of interest where logarithmic terms occur in the expansions of the theorem are $w = \frac{1}{9}$ and $w = 1$ corresponding to $\nu = 2$ and $\nu = 1$ respectively. The two cases of most physical interest, $w = 0$ (dust) and $w = \frac{1}{3}$ (radiation), are free of logarithms. In the case $w = 0$, most of the expansion coefficients vanish and the two non-vanishing terms define an explicit solution which is a linear combination of two powers of η.

The relative density perturbation is given by

$$\frac{\delta\epsilon}{\epsilon} = -2\Phi - 2\mathcal{H}^{-1}\Phi' + \frac{2}{3}\mathcal{H}^{-2}\Delta\Phi. \tag{3.7}$$

Now $\mathcal{H} = \frac{2}{(1+3w)\eta}$. Substituting this relation and the asymptotic expansion for Φ into the expression for the density perturbation gives:

$$\frac{\delta\epsilon}{\epsilon} = \sum_k \bigg[-(k(1+3w)+2)\Phi_{k,0} - (1+3w)\Phi_{k,1} + \frac{1}{6}(1+3w)^2\Delta\Phi_{k-2,0} + (-(k(1+3w)+2)\Phi_{k,1} + \frac{1}{6}(1+3w)^2\Delta\Phi_{k-2,1}\log\eta \bigg] \eta^k. \tag{3.8}$$

The relations in Theorem 3.1 place no restrictions on the coefficients $\Phi_{-2\nu,0}$ and $\Phi_{0,0}$ and so it is natural to ask if these can be prescribed freely. In other words, if two smooth functions on T^3 are given, is there a smooth solution of the equations in whose asymptotic expansion for $\eta \to 0$ precisely these functions occur as the coefficients $\Phi_{-2\nu,0}$ and $\Phi_{0,0}$? The next theorem answers this question in the affirmative. Since the proof is closely analogous to arguments which are already in the literature, it will only be sketched.

Theorem 3.2. *Let Ψ_1 and Ψ_2 be smooth functions on T^3. Then, there exists a unique solution of* (1.1) *of the type considered in Theorem* 3.1 *with $\Phi_{-2\nu,0} = \Psi_1$ and $\Phi_{0,0} = \Psi_2$.*

Proof (sketch). The proof of this theorem uses Fuchsian techniques. It implements the strategy applied in [9] to prove theorems on the existence of solutions of the vacuum Einstein equations belonging to the Gowdy class with prescribed singularity structure. In the present situation, some simplifications arise in comparison to the argument for Gowdy due to the fact that the equation being considered is linear. The procedure is to first treat the case of analytic data and then use the resulting analytic solutions to handle the smooth case. To reduce the equation to Fuchsian form, the following new variables are introduced. First, define a function $v(\eta, x)$ by the relation

$$\Phi(\eta) = \Psi_1\eta^{-2\nu} + \sum_{-2\nu<k<0} \Phi_{k,0}\eta^k + \Phi_{0,1}\log\eta + \Psi_2 + v(\eta). \tag{3.9}$$

Here it is assumed that the consistency relations (3.1) hold for $-2\nu \le k \le 0$. As a consequence of these relations and the original equation, v satisfies

$$v'' + \frac{2\nu+1}{\eta}v' - w\Delta v = w\Delta\Phi_{0,1}\log\eta + w\Delta\Psi_2 + w\sum_{0<k<2}\Delta\Phi_{k,0}\eta^k. \tag{3.10}$$

Note that the last sum will contain one non-vanishing term for ν not an integer and none for ν an integer. Denote the right-hand side of (3.10) by Q. This equation can be reduced to a first order system by introducing new variables $v^0 = \eta v'$ and $v^i = \eta\partial_i v$. Let V be the vector-valued unknown with components (v, v^0, v^i). Then, the first order system is

$$\eta\partial_\eta V + NV = \eta^\zeta f(\eta, V, DV) \tag{3.11}$$

where

$$N = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 2\nu & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad f = \begin{bmatrix} 0 \\ \eta^{1-\zeta}w\partial_i v^i + \eta^{2-\zeta}Q \\ \eta^{1-\zeta}\partial_i(v^0+v) \end{bmatrix}. \tag{3.12}$$

Here ζ is any positive real number less than one and DV denotes the collection of spatial derivatives of V. It will be shown that this equation has a unique solution v which converges to zero as $\eta \to 0$. Initially we assume that the functions Ψ_1 and Ψ_2 are analytic. Then results proved in [7] can be applied. See also [1, Sec. 4] for some further information on these ideas. One of the hypotheses required is that f is regular in the analytic sense defined there. What this means is that f and all its derivatives with respect to any argument other than η are real analytic for $\eta > 0$ and extend continuously to $\eta = 0$. The other hypothesis is that the matrix exponential σ^N should be uniformly bounded for all $0 < \sigma < 1$. This follows from the fact that N is diagonalizable with non-negative eigenvalues.

To extend this result to the smooth case more work is necessary. The basic idea is to approximate the smooth functions Ψ_1 and Ψ_2 by sequences of analytic functions $(\Psi_1)_n$ and $(\Psi_2)_n$, apply the analytic existence theorem just discussed to get a sequence of solutions V_n of the Fuchsian system and then show that V_n tends to a limit V as $n \to \infty$. The function V is then the solution of the problem with smooth data. To show the convergence of V_n, suitable estimates are required and in order to obtain these the Fuchsian equation is written in an alternative form which is symmetric hyperbolic. This rewriting is only possible for $w \neq 0$ but for $w = 0$ the system, being an ODE, is already symmetric hyperbolic and so the extra step is not required. In general, a simplification of the system is achieved by introducing a new time variable by $t = \eta^\zeta$ and rescaling f by a factor ζ^{-1}. Then, the system can be written as

$$tA^0\partial_t V + tA^j\partial_j V + MV = tg(t, V, DV), \tag{3.13}$$

where

$$M = \begin{bmatrix} 0 & -1 & 0 \\ 0 & \dfrac{2\nu}{\zeta w} & 0 \\ 0 & 0 & -\dfrac{1+\zeta}{\zeta} I \end{bmatrix}, \quad g = \begin{bmatrix} 0 \\ t^{\frac{2-\zeta}{\zeta}} Q \\ 0 \end{bmatrix} \tag{3.14}$$

and the other coefficient matrices are given by $A^0 = \mathrm{diag}(1, \frac{1}{w}, I)$ and

$$A^j = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\zeta^{-1} t^{\frac{1-\zeta}{\zeta}} e_j \\ 0 & -\zeta^{-1} t^{\frac{1-\zeta}{\zeta}} e_j & 0 \end{bmatrix} \tag{3.15}$$

with e_j the jth standard basis vector in R^3. This is a symmetric hyperbolic system. A disadvantage is that in passing from N to M positivity is lost.

The fact that M has a negative eigenvalue can be overcome by subtracting an approximate solution from v to obtain a new unknown. Expressing the equation in terms of the new unknown leads to a system which is similar to that for v but with M replaced by $M + nI$ for an integer n. For n sufficiently large, this means that the replacement for M is positive definite. With this choice the system is both in Fuchsian form and symmetric hyperbolic. The necessary approximate solution can be taken to be a formal solution of sufficently high order as introduced in [9, Sec. 2]. The fact that the system is symmetric hyperbolic leads to energy estimates which can be used to prove the convergence of the sequence of analytic solutions to a solution corresponding to the smooth initial data, thus completing the proof of the existence part of the theorem. Uniqueness can be proved using an energy estimate as has been worked out in [9]. □

It would presumably be possible to extend the above results to the case that the data are only assumed to belong to a suitable Sobolev space. An alternative approach to doing so would be try to apply ideas in the paper [6] of Kichenassamy.

The proofs just presented have been strongly influenced by work on Gowdy spacetimes. For a special class of these, the polarized Gowdy spacetimes, the basic field equation is $P_{tt}+t^{-1}P_t = P_{xx}$. Evidently this is closely related to (1.1) although they are not identical for any choice of w, even if attention is restricted to solutions of (1.1) depending on only one space variable. The energy arguments above were inspired by those applied to the polarized Gowdy equation in [3]. The following analogue of Theorem 3.2 is a special case of a result in [9]. If smooth periodic functions $k(x)$ and $\omega(x)$ are given with k everywhere positive, there is a smooth solution of the polarized Gowdy equations which satisfies

$$P(t,x) = k(x)\log t + \omega(x) + o(1) \tag{3.16}$$

as $t \to 0$. It is plausible that the positivity restriction on k, while very important for general (non-polarized) Gowdy spacetimes, should be irrelevant in the polarized case. It turns out that following the arguments used above to analyze (1.1) allows this intuition to be proved correct.

One way of attempting to reduce the polarized Gowdy equation to Fuchsian form is to mimic (3.9) and write $P = k\log t + \omega + v$. This fails because the analogue of the matrix N has ν replaced by zero. Thus, the matrix has all eigenvalues zero and includes a nontrivial Jordan block. To access the Fuchsian theory in the analytic case, the expansion for P may be replaced by

$$P = k\log t + \omega + t^{\delta} v \tag{3.17}$$

for a small positive δ. With this modification the reduction procedure applied to (1.1) gives a Fuchsian system. It can be concluded that k and ω can be prescribed in the case that they are analytic. Once this has been achieved the smooth case can be handled just as in the proof of Theorem 3.2.

It will now be shown that some of the results which have been proved for a linear equation of state can be extended to more general equations of state. In the discussion which follows, it will be convenient to exclude the case of a linear equation of state which has been treated already. This in particular excludes dust so that by our general assumptions f' never vanishes. In this case, we consider solutions to (2.7) rather than (1.1). Choose an initial time η_0 and for a given background solution let $\epsilon(\eta_0) = \epsilon_0$. From the condition that $f'(\epsilon) \le 1$, it follows that

$$\Lambda := \sup_{(\epsilon_0,\infty)} \left| f'(\epsilon) - \frac{f(\epsilon)}{\epsilon} \right|^{1/2} \tag{3.18}$$

is strictly positive and finite. It will be assumed in addition that the equation of state satisfies the condition

$$\sup_{(\epsilon_0,\infty)} \left| \left(\frac{\epsilon + f(\epsilon)}{f'(\epsilon)} \right) \frac{d^2 f}{d\epsilon^2} \right| < \infty. \tag{3.19}$$

Using the fact that $\Lambda > 0$, it follows that there exists a positive number λ satisfying the following three inequalities:

$$2\lambda \frac{df}{d\epsilon} \ge 3(\epsilon + f(\epsilon)) \frac{d^2 f}{d\epsilon^2}, \tag{3.20}$$

$$4\lambda^2 - 2\left[6(1+f'(\epsilon)) + \left(1+\frac{3f(\epsilon)}{\epsilon}\right)\right]\lambda$$
$$+6(1+f'(\epsilon))\left(1+\frac{3f(\epsilon)}{\epsilon}\right) - \Lambda^{-2}\left|\Lambda^2 - 3\left(f'(\epsilon)-\frac{f(\epsilon)}{\epsilon}\right)\right|^2 \geq 0 \qquad (3.21)$$

and

$$\lambda \geq 3(1+f'(\epsilon)). \qquad (3.22)$$

That (3.20) can be satisfied follows from (3.19). The fact that $f'(\epsilon)$ and $f(\epsilon)/\epsilon$ are bounded means that the first term in the expression on the left-hand side of (3.21) dominates the other terms for λ sufficiently large and so the second condition on λ can also be satisfied. The constant λ can be chosen to satisfy (3.22) since the right-hand side of that inequality is bounded. Note for comparison that for a linear equation of state $\Lambda = 0$. In that case λ can be taken to be the larger root of the expression obtained from the left-hand side of (3.21) by omitting the term containing Λ. This root is $3(1+w)$. Define the following generalization of the energy functional (3.3):

$$E_2(\eta) = \frac{1}{2}\int_{T^3} |\Phi'(\eta)|^2 + f'(\epsilon)|\nabla\Phi(\eta)|^2 + \Lambda\mathcal{H}^2|\Phi(\eta)|^2. \qquad (3.23)$$

(Note that we suppress the dependence of ϵ and $\mathcal{H}$ on η.) A computation shows that if a denotes the scale factor, then due to the inequalities (3.20)–(3.22)

$$\frac{d}{d\eta}[a^{2\lambda}E_2(\eta)] \geq 0. \qquad (3.24)$$

In more detail, computing the time derivative of $a^{2\lambda}E_2$ and using Eq. (2.7) along with the equations satisfied by the background quantities ϵ and $\mathcal{H}$ gives an integral where the integrand is a sum of terms each of which has a factor $\Phi^2, |\Phi'|^2, \Phi\Phi'$ or $|\nabla\Phi|^2$. The aim is to show that the sum of these terms is non-negative. To do this, it is first assumed that the coefficient of $|\nabla\Phi|^2$ is non-negative. This leads to the condition (3.20). Next, it is shown that the quadratic form in Φ and Φ' is positive semidefinite. This can be done by using the inequality

$$|\Lambda\mathcal{H}\Phi\Phi'| \leq \frac{\delta}{2}\Lambda^2\mathcal{H}^2\Phi^2 + \frac{1}{2\delta}(\Phi')^2, \qquad (3.25)$$

which holds for any $\delta > 0$, to estimate the quadratic form from below by the following sum of a term containing Φ^2 and one containing $|\Phi'|^2$:

$$\frac{1}{2}\Lambda^2\mathcal{H}^2\left[2\lambda - \left(1+\frac{3f(\epsilon)}{\epsilon}\right) - \delta\Lambda^{-1}\left|\Lambda^2 - 3\left(f'(\epsilon)-\frac{f(\epsilon)}{\epsilon}\right)\right|\right]\Phi^2$$
$$+\frac{1}{2}\left[2\lambda - 6(1+f'(\epsilon)) - \frac{\Lambda^{-1}}{\delta}\left|\Lambda^2 - 3\left(f'(\epsilon)-\frac{f(\epsilon)}{\epsilon}\right)\right|\right]|\Phi'|^2. \qquad (3.26)$$

It remains to ensure that the coefficients of these terms are non-negative and this follows from (3.21) and (3.22), choosing δ sufficiently small. It can be concluded from (3.24) that $E_2(\eta) = O(a(\eta)^{-2\lambda})$ as $\eta \to 0$. As in the case of a linear equation of state, corresponding estimates hold for spatial derivatives and pointwise estimates follow by Sobolev embedding. An integral formula for Φ' can be obtained as in the case of a linear equation of state. It reads (with some arguments suppressed; recall ϵ and $\mathcal{H}$ depend on η)

$$\Phi'(\eta) = (f(\epsilon)+\epsilon)\left[\frac{\Phi'(\eta_0)}{f(\epsilon_0)+\epsilon_0} - \int_{\eta_0}^{\eta} \frac{1}{f(\epsilon)+\epsilon}\left(f'(\epsilon)\Delta\Phi + 3\left(f'(\epsilon) - \frac{f(\epsilon)}{\epsilon}\right)\mathcal{H}^2\Phi\right)\right]. \tag{3.27}$$

If no further assumptions are made on the equation of state, then using the known boundedness statements and repeatedly substituting into the right-hand side of (3.27) would lead to unwieldy expressions involving iterated integrals. Simpler results can be obtained if it is assumed that in the limit $\epsilon \to \infty$ the function f is linear in leading order with lower powers as corrections. In other words, for this assume that f admits an asymptotic expansion of the form

$$f(\epsilon) \sim w\epsilon + \sum_{j=1}^{\infty} f_j \epsilon^{a_j} \tag{3.28}$$

as $\epsilon \to \infty$. Here the f_j are constants while $\{a_j\}$ is a decreasing sequence of real numbers all of which are less than one and which tend to $-\infty$ as $j \to \infty$. Assume further that the relation obtained by differentiating this expansion term by term any number of times is also a valid asymptotic expansion. To have a concrete example, consider the polytropic equation of state which is given parametrically by the relations

$$\epsilon = m + Knm^{\frac{n+1}{n}}, \quad p = Km^{\frac{n+1}{n}} \tag{3.29}$$

with constants K and n satisfying $0 < K < 1$ and $n > 1$. In this case, the asymptotic expansion is of the form

$$f(\epsilon) = n^{-1}\epsilon + n^{-1}(Kn)^{\frac{1}{n+1}}\epsilon^{\frac{n}{n+1}} + \cdots . \tag{3.30}$$

Returning to the more general case (3.28), define a quantity m by

$$m(\epsilon) = \exp\left\{\int_1^{\epsilon} (\xi + f(\xi))^{-1} d\xi\right\}. \tag{3.31}$$

Substituting the asymptotic expression (3.28) into (3.31) gives a corresponding asymptotic expansion for the function $m(\epsilon)$ as a sum of powers of ϵ with the leading term being proportional to $\epsilon^{\frac{1}{w+1}}$. It follows from the continuity equation (2.3) for the fluid that m is proportional to a^{-3}. This leads to an asymptotic expansion for ϵ in terms of a. Equation (2.4) implies that $a' = \sqrt{8\pi G/3}\epsilon^{1/2}a^2$; substituting for ϵ in terms of a gives rise to a relation which can be integrated to give an asymptotic expansion for a in terms of η in the limit $\eta \to 0$. The leading term is proportional to $\eta^{\frac{2}{3w+1}}$. Substituting this back in leads to an asymptotic expansion for a' from which an asymptotic expansion for $\mathcal{H}$ can be obtained. An asymptotic expression for ϵ in terms of η can also be derived. Thus in the end, there are expansions for all the important quantities in the background solution in terms of η. In all cases, the leading term in the expansion agrees with that in the case of a linear equation of state. The result is an integral equation which can be written in the form

$$\Phi'(\eta) = h_1(\eta)\left[C - \int_0^{\eta_0} h_2(\zeta)\Delta\Phi(\zeta) - h_3(\zeta)\Phi(\zeta)d\zeta + \int_0^{\eta} h_2(\zeta)\Delta\Phi(\zeta) + h_3(\zeta)\Phi(\zeta)d\zeta\right], \tag{3.32}$$

where C is a constant depending only on the data at time η_0 and asymptotic expansions are available for the functions h_1, h_2 and h_3. The leading terms in h_1 and h_2 are constant multiples of the corresponding powers of η for a linear equation of state. To see the leading order behavior of h_3 recall that in (3.32), the coefficient of Φ is

$$3\left(f'(\epsilon) - \frac{f(\epsilon)}{\epsilon}\right)\mathcal{H}^2. \tag{3.33}$$

Hence if f_j is the first non-vanishing coefficient in the expansion (3.30), then the leading order power in h_3 is less than that in h_2 by $\alpha = 2 - \frac{6(1+w)(1-a_j)}{1+3w}$. To obtain estimates close to $\eta = 0$, the estimate for the energy can be applied starting from η very small. In other words, ϵ_0 can be chosen as large as desired. Then, all the coefficients in the left-hand side of (3.21) not involving Λ are as close as desired to those for the corresponding linear equation of state. Since Λ is arbitrarily small, the coefficient involving Λ is also arbitrarily small. It follows that λ can be chosen to have any value strictly greater than $3(1+w)$. Hence, E can be bounded by any power greater than the power in the corresponding linear case. This is enough to proceed as in the proof of Theorem 3.1 to obtain an asymptotic expansion for Φ where each invidual term is a constant multiple of an expression of the form $\eta^k(\log\eta)^l$ with $l = 0$ or $l = 1$ and the leading term is just as in Theorem 3.1 with the corresponding value of w. The key thing that makes this work is that $\alpha < 2$ so that no logarithms are generated when evaluating the integral in (3.32) in the course of the iteration. The results of this discussion can be summed up as follows.

Theorem 3.3. *Let Φ be a smooth solution of (2.7) on $(0,\infty)\times T^3$. Suppose that the equation of state has an asymptotic expansion of the form (3.28). Then, there are coefficients $\Phi_{k,l}$ with $k \geq -2\nu$ belonging to an increasing sequence of real numbers tending to infinity and $l \in \{0,1\}$, smooth functions on T^3, such that the formal series $\sum_k \Phi_{k,l}(\log\eta)^l\eta^k$ is asymptotic to Φ in the limit $\eta \to 0$. All coefficients in the expansion are determined uniquely by $\Phi_{-2\nu,0}$ and $\Phi_{0,0}$.*

4. Late-Time Asymptotics for a Linear Equation of State

In this section, information is obtained about the asymptotics of solutions of Eq. (1.1) in the limit $\eta \to \infty$; some extensions of these results to more general equations of state are derived in Sec. 5. Once again energy estimates play a fundamental role. In this case, it is convenient to treat homogeneous solutions separately. By a homogeneous solution, we mean one which does not depend on the spatial coordinates. These can be characterized as the solutions whose initial data on a given spacelike hypersurface do not depend on the spatial coordinates. For this class of Eq. (1.1) solutions can be solved explicitly with the result that $\Phi = A + B\eta^{-2\nu}$ for constants A and B. A general solution can be written as the sum of a homogeneous solution and a solution such that Φ has zero mean on any hypersurface of constant conformal time. Call solutions of the latter type zero-mean solutions. Then, in order to determine the late-time asymptotics for general solutions, it suffices to do so for zero-mean solutions. In this case, define $\psi(\eta) = \eta^{\nu+\frac{1}{2}}\Phi(\eta)$. Then, ψ satisfies the equation

$$\psi'' = w\Delta\psi + \left(\nu^2 - \frac{1}{4}\right)\eta^{-2}\psi. \tag{4.1}$$

Define an energy by

$$E_3(\eta) = \frac{1}{2}\int_{T^3} |\psi'(\eta)|^2 + w|\nabla\psi(\eta)|^2. \tag{4.2}$$

Then

$$E_3'(\eta) = 2\left(\nu^2 - \frac{1}{4}\right)\eta^{-2}\int_{T^3}\psi(\eta)\psi'(\eta). \tag{4.3}$$

The integral on the right-hand side of this equation can be bounded, using the Cauchy–Schwarz inequality, in terms of the L^2-norms of ψ' and ψ. The first of these can be bounded in terms of the energy and due to the fact that the mean value of ψ is zero, the same is true of the second. Thus, $E_3'(\eta) \leq C\eta^{-2}E_3(\eta)$ for a constant C. By Gronwall's inequality, it follows that E_3 is globally bounded in the future. These arguments apply equally well to spatial derivatives of ψ of any order. By the Sobolev embedding theorem, it can be concluded that ψ and its spatial derivatives of any order are bounded. The energy bounds and the basic equation then imply that all spacetime derivatives of any order are uniformly bounded in time.

Let η_j be a sequence of times tending to infinity and consider the translates defined by $\psi_j(\eta) = \psi(\eta + \eta_j)$. The sequence ψ_j satisfies uniform C^∞ bounds. Consider the restriction of this sequence to an interval $[\eta_0, \eta_1]$. By the Arzelà–Ascoli theorem, the sequence of restrictions has a uniformly convergent subsequence. By passing to further subsequences and diagonalization, it can be shown that ψ and its spacetime derivatives of all orders converge uniformly on compact subsets to a limit W. Passing to the limit in the evolution equation for ψ along one of these sequences shows that W satisfies the flat-space wave equation $W'' = w\Delta W$. Note that *a priori* the function W could depend on the sequence of times chosen. This issue is examined more closely below.

Given a smooth solution of (4.1), it is possible to do a Fourier transform in space to get the equation

$$\hat{\psi}'' = -w|k|^2\hat{\psi} + \left(\nu^2 - \frac{1}{4}\right)\eta^{-2}\hat{\psi} \tag{4.4}$$

which is referred to below as the mode equation. Here k is a vector. The restriction to zero-mean solutions implies that the case $k = 0$ of (4.4) can be ignored.

Lemma 4.1. *Any solution $\hat{\phi}$ of Eq. (4.4) has an asymptotic expansion of the form*

$$\hat{\phi}(\eta) = \bar{W}_k \cos(\sqrt{w}|k|(\eta - \bar{\eta}_k)) + O(\eta^{-1}), \tag{4.5}$$

for constants $\bar{\eta}_k$ and $\bar{W}_k$, in the limit $\eta \to \infty$.

Proof. To prove the lemma, it is convenient to introduce polar coordinates associated to the variables $\hat{\psi}$ and $\frac{1}{\sqrt{w}|k|}\hat{\psi}'$. Thus, $\hat{\psi} = r\cos\theta$ and $\frac{1}{\sqrt{w}|k|}\hat{\psi}' = r\sin\theta$. This leads to the equations:

$$r' = \frac{1}{\sqrt{w}|k|}\left(\nu^2 - \frac{1}{4}\right) r\eta^{-2}\sin\theta\cos\theta, \tag{4.6}$$

$$\theta' = -\sqrt{w}|k| + \frac{1}{\sqrt{w}|k|}\left(\nu^2 - \frac{1}{4}\right)\eta^{-2}\cos^2\theta. \tag{4.7}$$

It follows from (4.7) that

$$\theta(\eta) = -\sqrt{w}|k|(\eta - \bar{\eta}_k) + O(\eta^{-1}) \tag{4.8}$$

for a constant $\bar{\eta}_k$. From (4.6), it follows that

$$r(\eta) = \bar{W}_k(1 + O(\eta^{-1})) \tag{4.9}$$

for a constant $\bar{W}_k$. As a consequence of (4.8), we have

$$\cos(\eta(\theta)) = \cos(\sqrt{w}|k|(\eta - \bar{\eta}_k)) + O(\eta^{-1}). \tag{4.10}$$

Together with (4.9) this gives the conclusion of the lemma. □

Consider a zero-mean solution of the type considered before. Let a function W be defined by taking the sequence η_j used above to consist of integer multiples of 2π. We now show that the function $\psi - W$ tends to zero as $\eta \to \infty$. In order to do this, it suffices to show that it does so along a subsequence of an arbitrary sequence of values ζ_j of η tending to infinity. By passing to a subsequence, as before it can be arranged that the translates by the amounts ζ_j converge uniformly on compact subsets as $j \to \infty$. Call the limit Y. The aim is to prove that $Y = 0$. If not there must be some mode $\hat{Y}$ which is nonzero. It can be obtained as the limit of some $\hat{\psi} - \hat{W}$. From Lemma 4.1, it can be seen that $\hat{W} = \bar{W}_k \cos(\sqrt{w}|k|(\eta - \eta_k))$. Hence, $\hat{\psi} - \hat{W} = O(\eta^{-1})$ and so $\hat{Y} = 0$, a contradiction. Convergence of derivatives can be obtained in a corresponding way. Thus any solution can be written as

$\Phi(\eta, x) = \eta^{-\nu-\frac{1}{2}}(W(\eta, x) + o(1))$. A similar result for the polarized Gowdy equation with a sharper estimate on the error term was proved in [4].

A late-time asymptotic expansion has now been derived which involves a solution W of the flat-space wave equation. Comparing with the results on parametrizing solutions by the coefficients in an asymptotic expansion near the singularity it is natural to ask if the function W can be prescribed freely. It will now be shown that this is the case by following the proof of an analogous result for the polarized Gowdy equation due to Ringström [11]. Write an arbitrary zero-mean solution in the form

$$\Phi(\eta, x) = \eta^{-\nu-\frac{1}{2}} W(\eta, x) + \omega(\eta, x). \tag{4.11}$$

Then, ω satisfies the equation

$$\omega'' + \eta^{-1}\omega' - w\Delta\omega = \left(\nu^2 - \frac{1}{4}\right)\eta^{-\nu-\frac{5}{2}} W. \tag{4.12}$$

Define

$$H(\eta) = \frac{1}{2}\int_{T^3} |\omega'(\eta)|^2 + w|\nabla\omega(\eta)|^2 \tag{4.13}$$

and

$$\Gamma(\eta) = \frac{1}{2\eta}\int_{T^3} \omega(\eta)\omega'(\eta). \tag{4.14}$$

The aim is to study late times and attention will be restricted to the region where $\eta \geq w^{-1}$. At this point it is necessary to assume that $w > 0$. The following inequalities show the equivalence of H and $H + \Gamma$ as norms of $(\omega', \nabla\omega)$:

$$|\Gamma(\eta)| \leq \frac{1}{2w\eta} H(\eta), \quad \frac{1}{2}H \leq H + \Gamma \leq \frac{3}{2}H. \tag{4.15}$$

Now

$$\frac{d}{d\eta}[H + \Gamma] = -\frac{1}{\eta}(H + \Gamma) - \frac{4\nu + 3}{2\eta}\Gamma + \left(\nu^2 - \frac{1}{4}\right)\eta^{-\nu-\frac{5}{2}}\int_{T^3} \omega' W + \frac{1}{2}\left(\nu^2 - \frac{1}{4}\right)\eta^{-\nu-\frac{7}{2}}\int_{T^3} \omega W. \tag{4.16}$$

Using the equivalence of $H + \Gamma$ and H this can be used to derive the following differential inequality

$$\frac{d}{d\eta}[H + \Gamma] \geq -\left(\frac{1}{\eta} + \frac{4\nu + 3}{2w\eta^2}\right)(H + \Gamma) - \eta^{-\nu-\frac{5}{2}}\|W\|_{L^2}\left(\nu^2 - \frac{1}{4}\right)\left(\frac{2 + \sqrt{w}}{\sqrt{2}}\right)(H + \Gamma)^{1/2}. \tag{4.17}$$

By analogy with [11, Eq. (3.7)] define

$$E_4(\eta) = \eta e^{\frac{4\nu+3}{2\eta w}}(H(\eta) + \Gamma(\eta)). \tag{4.18}$$

This quantity satisfies an inequality of the form

$$E_4'(\eta) \geq -C\eta^{-\nu-2}\|W(\eta)\|_{L^2}E_4(\eta)^{1/2} \tag{4.19}$$

for a positive constant C depending on w. Since $\eta^{-\nu-2}$ is integrable at infinity this inequality can be used in just the same way as the corresponding inequality in [11]. In this way, it can be proved that given a solution W of the flat space wave equation there is a corresponding solution Φ of (1.1). It follows from the proof that $E_4(\eta) = O(\eta^{-2\nu-2})$. Hence, $H(\eta) = O(\eta^{-2\nu-3})$ and the solution decays like $\eta^{-3/2-\nu}$.

The information obtained concerning the asymptotics of the solutions constructed starting from a solution W of the wave equation is stronger that what was proved about general solutions of (1.1) up to this point. This can be improved on as follows. Given a solution Φ of (1.1) a solution W of the flat space wave equation is obtained. From there a solution $\tilde{\Phi}$ of (1.1) is obtained with stronger information on the asymptotics. The aim is now to show that $\tilde{\Phi} = \Phi$. To do this, it is enough to show that each Fourier mode agrees. This means showing that a solution $\hat{\psi}$ of (4.4) vanishes if it tends to zero as $\eta \to \infty$. That the latter statement holds follows easily from (4.6). What has been proved can be summed up in the following theorem.

Theorem 4.2. *Let Φ be a global smooth solution of* (1.1). *Then, there exist constants A and B and a smooth solution W of the equation $W'' = w\Delta W$ with zero spatial average such that*

$$\Phi(\eta, x) = A + W(\eta, x)\eta^{-\nu-\frac{1}{2}} + B\eta^{-2\nu} + O(\eta^{-\nu-\frac{3}{2}}). \tag{4.20}$$

This asymptotic expansion may be differentiated term by term in space as often as desired.

Note that the third explicit term in this asymptotic expansion is often no larger than the error term. The function W can be prescribed freely.

5. Late-Time Asymptotics for a General Equation of State

It will now be investigated how the results of the previous section can be extended to the case of a more general equation of state. The class of equations of state which will be treated is defined by requiring that they admit an asymptotic expansion of the form

$$f(\epsilon) \sim w\epsilon + \sum_{j=1}^{\infty} f_j \epsilon^{a_j} \tag{5.1}$$

for $\epsilon \to 0$. Here $w \geq 0$, the coefficients a_j are all greater than one and form an increasing sequence. To ensure the positivity of f', it is assumed that if $w = 0$ the coefficient f_1 is positive. This form of the equation of state may be compared with that of (3.28). It is further assumed that this expansion retains its validity when differentiated term by term as often as desired. An example is given by the polytropic equation of state (3.29). In that case $w = 0, f_1 = K$ and $a_1 = \frac{n+1}{n}$. With this assumption information can be obtained on the leading order asymptotics of the background solution as $\eta \to \infty$. To simplify the notation define $\sigma = a_1 - 1$. It is convenient to use the mass density once more, writing (3.31) in the equivalent form

$$m(\epsilon) = \exp\left\{-\int_\epsilon^1 (\xi + f(\xi))^{-1} d\xi\right\}. \tag{5.2}$$

Then, $m(\epsilon)$ has an expansion about $\epsilon = 0$ where the leading term is proportional to $\epsilon^{\frac{1}{w+1}}$. In particular, when $w = 0$ the leading term is linear. Using the fact that m is proportional to a^{-3} for any equation of state leads to an asymptotic expansion for ϵ in terms of a. Putting this information into (2.4) shows that $a(\eta)$ has an expansion in the limit $\eta \to \infty$ with the leading term proportional to $\eta^{\frac{2}{3w+1}}$. Finally, it follows that ϵ and $\mathcal{H}$ have expansions with leading terms proportional to $\eta^{-\frac{6(1+w)}{1+3w}}$ and η^{-1} respectively. With the leading asymptotics of the background solution having been determined, it is possible to derive asymptotics for the coefficients in the equation for Φ.

As in the case of a linear equation of state, it is convenient to treat homogeneous and zero-mean solutions separately. The homogeneous solutions will be analysed first. This leads to consideration of the equation obtained from (2.7) by omitting the term containing spatial derivatives. It is convenient here to exclude the case of a linear equation of state which was previously analysed so as to ensure that σ is defined uniquely in terms of the equation of state. The coefficients satisfy:

$$3(1 + f'(\epsilon)) = 3(1 + w + (\sigma + 1)f_1\eta^{-\beta}) + o(\eta^{-\beta}) \tag{5.3}$$

and

$$3\left(f'(\epsilon) - \frac{f(\epsilon)}{\epsilon}\right) = 3f_1\sigma\eta^{-\beta} + o(\eta^{-\beta}), \tag{5.4}$$

where $\beta = \frac{6\sigma(1-w)}{1+3w}$. Define

$$F = \frac{1}{2}\Phi'^2 + \alpha\eta^{-2-\beta}\Phi^2, \tag{5.5}$$

where α is a positive constant which needs to be chosen appropriately in what follows. Computing the derivative of F with respect to η and using the equation gives a sum of terms involving Φ'^2, Φ^2 and $\Phi\Phi'$. The aim is to show that F is bounded and to do this it suffices to consider arbitrarily late times. The leading order terms in the coefficients of Φ^2 and Φ'^2 are $-\frac{6(1+w)}{1+3w}\eta^{-1}$ and $-A\eta^{-3-\beta}$ respectively, where A is a positive constant. The coefficient of $\Phi\Phi'$ has a leading term proportional to $\eta^{-2-\beta}$ for a general choice of α. However, if α is chosen to be half the coefficient of the leading order term in the expansion of the coefficient of Φ in (2.7), then a cancellation occurs and the coefficient becomes $o(\eta^{-2-\beta})$. This choice is made here.

The aim is to show that the term containing $\Phi\Phi'$ can be absorbed by the sum of the other two so as to leave a non-positive remainder. To do this the inequality

$$|\eta^{-2-\beta}\Phi\Phi'| \leq \frac{1}{2}(\eta^{-1}\Phi'^2 + \eta^{-3-2\beta}\Phi^2) \tag{5.6}$$

is used. The powers of η which arise from this inequality match those in the leading order terms in the coefficients of the manifestly negative terms in the expression for the derivative of F with respect to η. Thus, at late times the cross-term can be absorbed in the terms with the desired sign. The conclusion is that F is bounded. In fact, this can be improved somewhat. The derivative of F can be estimated above by $-2\gamma\eta^{-1}F$ for any positive constant $\gamma < 2\nu+1$. This means that Φ' decays like $\eta^{-\gamma}$. It can be concluded that Φ is bounded. From the evolution equation for Φ and the boundedness statements already obtained, it follows that $(\eta^{2\nu+1}\Phi')'$ is integrable. Thus, $\Phi = A + B\eta^{-2\nu} + \cdots$ for constants A and B and the leading order behavior is as in the case of a linear equation of state.

It turns out to be useful for the analysis of the zero-mean solutions in the expanding direction to introduce a new time variable τ satisfying the relation $d\tau/d\eta = \sqrt{f'(\epsilon)}$. Substituting the asymptotics of $f'(\epsilon)$ in terms of η into this provides an asymptotic expansion for τ in terms of η. For $w > 0$ a linear relation is obtained in leading order while for $w = 0$ and $\sigma \neq \frac{1}{3}$ the expansion reads

$$\tau = C_1\eta^{1-3\sigma} + \tau_\infty + \cdots \tag{5.7}$$

for constants C_1 and τ_∞. Note that the second term in this expansion is only smaller than the first for $\sigma < \frac{1}{3}$. For $w = 0$ and $\sigma = \frac{1}{3}$, the power in this expression gets replaced by $\log\eta$. From these facts, it can be seen that $\tau \to \infty$ for $\eta \to \infty$ when $w > 0$ or when $w = 0$ and $\sigma \leq \frac{1}{3}$. In contrast τ tends to the finite limit τ_∞ for $\eta \to \infty$ when $w = 0$ and $\sigma > \frac{1}{3}$. This is a symptom of a bifurcation where the asymptotics of the linearized solution undergoes a major change. For convenience we say that the dynamics for an equation of state with an asymptotic expansion of the form (5.1) is underdamped if $w > 0$ or $\sigma < \frac{1}{3}$, critical if $w = 0$ and $\sigma = \frac{1}{3}$ and overdamped if $w = 0$ and $\sigma > \frac{1}{3}$.

Next, the late-time behavior will be analysed for zero-mean solutions with an equation of state corresponding to underdamped dynamics. The first step is to introduce the time variable τ into (2.7) with the result:

$$\Phi_{\tau\tau} + 3Z\tilde{\mathcal{H}}\Phi_\tau + 3Y\tilde{\mathcal{H}}^2\Phi - \Delta\Phi = 0, \tag{5.8}$$

where

$$Y = f'(\epsilon) - \frac{f(\epsilon)}{\epsilon}, \tag{5.9}$$

$$Z = 1 + f'(\epsilon) - \frac{1}{2}\frac{(\epsilon + f(\epsilon))f''(\epsilon)}{f'(\epsilon)} \tag{5.10}$$

and $\tilde{\mathcal{H}} = a^{-1}a_\tau$. Derivatives with respect to τ are denoted by subscripts. Next, the term containing Φ_τ will be eliminated by multiplying Φ by a suitable factor Ω^{-1}.

Choose Ω to satisfy

$$\frac{\Omega_\tau}{\Omega} = -\frac{3}{2} Z\tilde{\mathcal{H}}. \tag{5.11}$$

For all three types the behavior of Ω as a function of a in the limit $\epsilon \to 0$ can be determined. The result is that the leading order term in Ω is proportional to $a^{-\frac{3}{2}(1+w)}$ for $w > 0$ and proportional to $a^{-\frac{3}{2}(1-\frac{\sigma}{2})}$ for $w = 0$. The function $\Psi = \Omega^{-1}\Phi$ satisfies an equation of the form

$$\Psi_{\tau\tau} = A(\epsilon)\tilde{\mathcal{H}}^2\Psi + \Delta\Psi, \tag{5.12}$$

where $A(\epsilon)$ is a rational function of ϵ, $f(\epsilon)$, $f''(\epsilon)$ and $f'''(\epsilon)$. Under the given assumptions on the equation of state it is bounded. Proving this requires examining many terms but is routine. For example, the only term containing the third derivative of f is $\frac{3(\epsilon+f(\epsilon))^2 f'''(\epsilon)}{2f'(\epsilon)}$. The leading order terms in the asymptotic expansions of numerator and denominator are both proportional to ϵ^σ. Note also that the leading order term in the expansion for $\tilde{\mathcal{H}}$ is proportional to τ^{-1} for any $\sigma < \frac{1}{3}$. Note for comparison that $\tilde{\mathcal{H}}$ tends to a constant value as $\tau \to \infty$ in the case $\sigma = \frac{1}{3}$.

Define an energy by

$$E_5(\tau) = \frac{1}{2}\int \Psi_\tau^2 + |\nabla\Psi|^2. \tag{5.13}$$

Then, using the same techniques as in previous energy estimates shows that there is a constant C such that

$$\frac{dE_5}{d\tau} \le C|A|\tilde{\mathcal{H}}^2 E_5. \tag{5.14}$$

Using the information available concerning A and $\tilde{\mathcal{H}}$ shows that E_5 is bounded in the future. Taking derivatives of the equation and using the same arguments as in previous cases shows that Ψ and its derivatives of all orders with respect to x and τ are bounded. It follows that any sequence of translates $\Psi(\tau + \tau_n)$ for a sequence τ_n tending to infinity has a subsequence which converges on compact subsets to a limit W.

Doing a Fourier transform of Eq. (5.12) in space leads to the mode equation

$$\hat{\Psi}_{\tau\tau} = -|k|^2\hat{\Psi} + A\tilde{\mathcal{H}}^2\hat{\Psi}. \tag{5.15}$$

Introducing polar coordinates in the $(\frac{\hat{\Psi}}{|k|}, \frac{\hat{\Psi}_\tau}{|k|})$-plane leads to the system

$$\frac{dr}{d\tau} = \frac{1}{|k|} Ar\tilde{\mathcal{H}}^{-2}\sin\theta\cos\theta, \tag{5.16}$$

$$\frac{d\theta}{d\tau} = -|k| + \frac{1}{|k|} A\tilde{\mathcal{H}}^{-2}\cos^2\theta. \tag{5.17}$$

This implies that

$$\theta(\tau) = -|k|(\tau - \bar{\tau}_k) + O(\tau^{-1}) \tag{5.18}$$

and

$$r(\tau) = \bar{W}_k(1 + O(\tau^{-1})), \tag{5.19}$$

for some constants $\bar{\tau}_k$ and $\bar{W}_k$. Arguing as in the case of a linear equation of state leads to the relation $\Psi(\tau, x) = W(\tau, x) + o(1)$ where W is a solution of the equation $W_{\tau\tau} = \Delta W$. Using the form of the leading order term in Ω as a function of a, it can be shown that the leading order term in Ω as a function of τ is given by $\tau^{-\frac{3(1+w)}{1+3w}} = \tau^{-\nu-\frac{1}{2}}$ and $\tau^{-\frac{3(1-\frac{\sigma}{2})}{(1-3\sigma)}}$ in the cases $w > 0$ and $w = 0$ respectively. Note that the first of these reproduces the result in the case of a linear equation of state. It does not seem to be possible to write the expansion directly in terms of η in such a way that it gives more insight than the expression in terms of τ. The leading asymptotics of a zero-mean solution is obtained by taking a solution of the flat space wave equation, distorting the time variable by a diffeomorphism and multiplying by a power of the time coordinate which has been explicitly computed.

Consider next the case $w = 0, \sigma > \frac{1}{3}$ (overdamped case). The time coordinate τ tends to the finite limit τ_∞ as $\eta \to \infty$. Define $G = \tilde{\mathcal{H}}^{-2}\partial_\tau\tilde{\mathcal{H}}$. The function G tends to the limit $\frac{3}{2}(\sigma - \frac{1}{3})$ as $\tau \to \tau_\infty$. Let

$$E_6 = \int \Phi_\tau^2 + |\nabla\Phi|^2 + \Lambda^2\tilde{\mathcal{H}}^2\Phi^2 \tag{5.20}$$

for a constant Λ which remains to be chosen. For a constant λ computing $\partial_\tau(a^{2\lambda}E_6)$ gives rise to a sum of expressions containing $\Phi^2, \Phi_\tau^2, \Phi\Phi_\tau$ and $|\nabla\Phi|^2$. Using the inequality

$$\tilde{\mathcal{H}}^2\Phi_\tau\Phi \le \frac{1}{2\delta}\tilde{\mathcal{H}}\Phi_\tau^2 + \frac{\delta}{2}\tilde{\mathcal{H}}^3\Phi^2 \tag{5.21}$$

leads to an inequality where the term involving $\Phi\Phi_\tau$ has been eliminated. To obtain some control on the energy by means of the inequality, the coefficients Λ and λ should be chosen in such a way that all terms on the right-hand side are manifestly non-positive. The conditions for this to happen are the inequalities $\lambda \le 0$,

$$\frac{1}{2\delta}|\Lambda^2 - 3Y| \le 3Z - \lambda \tag{5.22}$$

and

$$\frac{\delta}{2}|\Lambda^2 - 3Y| \le -\Lambda^2(\lambda + G). \tag{5.23}$$

Note that these inequalities imply in particular that $\lambda < 0$. Consider now the limit $\tau \to \tau_\infty$ where Y behaves asymptotically like $f_1\sigma\epsilon^\sigma$ and $Z \to 1 - \frac{1}{2}\sigma$. In this limit the inequality (5.23) reduces to $\lambda \le -\frac{3}{2}(\sigma - \frac{1}{3}) - \frac{\delta}{2}$. Suppose therefore that $\lambda < -\frac{3}{2}(\sigma - \frac{1}{3})$. Then by choosing δ sufficiently small, it can be arranged that the limiting inequality is satisfied. In the limit the inequality (5.22) reduces to $\frac{\Lambda^2}{2\delta} \le 3 - \frac{3}{2}\sigma - \lambda$. Choose Λ so that this inequality is satisfied strictly. With these choices both inequalities are satisfied strictly in the limit. For τ sufficiently close to τ_∞ the coefficients in (5.22) and (5.23) are as close as desired to their limiting values. Making them close enough ensures that these two inequalities continue to be satisfied. It follows that with these choices of the parameters $\partial_\tau(a^{2\lambda}E_6)$ is non-positive at late times. It can be concluded that $E_6 = O(a^{-2\lambda})$. This gives a limit on the growth rate of E_6 in terms of that of the scale factor. As in previous cases corresponding estimates can be obtained for derivatives and as a consequence pointwise estimates derived. It follows that $\Phi = O(a^{-\lambda}\tilde{\mathcal{H}}^{-1})$. From what is known about the background solution, it follows that $\tilde{\mathcal{H}}$ is proportional to $a^{\frac{3}{2}(\sigma-\frac{1}{3})}$. Thus, if $\rho = -\frac{3}{2}(\sigma - \frac{1}{3}) - \lambda$ then $\Phi = O(a^\rho)$. This power is positive but may be made as small as desired by choosing λ suitably. By the usual methods, similar bounds can be obtained for spatial derivatives of Φ.

To get more information about the asymptotics as $\tau \to \tau_\infty$, it is convenient to rewrite the equation in terms of the new time variable $s = \tau_\infty - \tau$. The resulting equation is

$$\Phi_{ss} - 3Z\tilde{\mathcal{H}}\Phi_s + 3Y\tilde{\mathcal{H}}^2\Phi - \Delta\Phi = 0. \tag{5.24}$$

As $s \to 0$ the coefficient Z tends to $1 - \frac{\sigma}{2}$ while $\tilde{\mathcal{H}}$ and $Y\tilde{\mathcal{H}}^2$ are proportional in leading order to s^{-1} and $s^{\frac{2}{3(\sigma-1/3)}}$ respectively. The last exponent is positive for any $\sigma > 1/3$ so that the corresponding coefficient tends to zero as $s \to 0$. Let B be a positive solution of the equation $\frac{dB}{ds} = -3Z\tilde{\mathcal{H}}B$. Then, (5.24) implies the following integral equation:

$$\Phi_s = \frac{1}{B}\left(\bar{\Phi}_1 + \int_0^s B(-3Y\mathcal{H}^2\Phi + \Delta\Phi)\right) \tag{5.25}$$

for a function $\bar{\Phi}_1(x)$. Here, the fact has been used that the integral occurring in this equation converges. This follows from the fact that in leading order B is proportional to $s^{-\frac{\sigma-2}{\sigma-1/3}}$ and the bounds already obtained for Φ and its derivatives. When B^{-1} diverges faster than s^{-1} in the limit $s \to 0$, which happens for $\sigma < \frac{7}{6}$, the known bounds on Φ imply that $\bar{\Phi}_1 = 0$. Hence, Φ is bounded in the limit $s \to 0$ in that case. When $\sigma > \frac{7}{6}$ it can also be concluded that Φ is bounded. For $\sigma = \frac{7}{6}$ a logarithmic divergence of Φ is not ruled out. In all cases, the integral equation can be used to obtain an asymptotic expansion for Φ. Schematically this expansion is of the form

$$\Phi(\eta, x) = \sum_i \Phi_i(x)\zeta_i(\eta) \tag{5.26}$$

for some functions ζ_i with $\zeta_{i+1}(\eta) = O(\zeta_i(\eta))$ for each i. This is very different from the expansion in the limit $\eta \to \infty$ obtained when $w > 0$ or $\sigma < \frac{1}{3}$. In the present case, scaling the solution by a suitable function of η gives a result which converges to a function of x as $\eta \to \infty$. In the other case, a similar rescaling can lead to a profile which moves around the torus with constant velocity. (In general, it leads to a superposition of profiles of this kind.) In the latter case, there are waves which continue to propagate at arbitrarily late times. In the case $\sigma > \frac{1}{3}$, the waves "freeze". This is reminiscent of the late-time asymptotics of the gravitational field in spacetimes with positive cosmological constant (cf. [10]).

To make this argument more concrete, consider the special case where the equation of state is $f(\epsilon) = f_1\epsilon^{\sigma+1}$ for some σ between $\frac{1}{3}$ and $\frac{7}{6}$. Using the convergence of the integral, it follows immediately that $\Phi(s,x) = \Phi_0(x) + O(s^2)$ for some function Φ_0. Putting this information back into the integral equation gives $\Phi(s,x) = \Phi_0(x) + \frac{\sigma-1/3}{4-2\sigma}\Delta\Phi_0(x)s^2 + \cdots$.

Consider finally the case $w = 0, \sigma = \frac{1}{3}$ (critical case). Then, $\eta = \eta_0 e^{\frac{\tau}{C_1}} + \cdots$ where η_0 is a constant and C_1 corresponds to the constant appearing in (5.7). The arguments leading to the estimate $\Phi = O(\tau^\rho)$ can be carried out as in the case $\sigma > \frac{1}{3}$. The only difference is that the limit $\tau \to \tau_\infty$ is replaced by $\tau \to \infty$. In the case $\sigma = \frac{1}{3}$, the quantity $\tilde{\mathcal{H}}$ tends to a constant for $\tau \to \infty$ and $Y\tilde{\mathcal{H}}^2$ is proportional to $e^{-\frac{6}{C_1}\tau}$ in leading order. A quantity B can be introduced as before and an integral equation obtained. In this case, B is a decaying exponential. Unfortunately, it does not seem to be possible to use this integral equation to refine the asymptotics in this case and this matter will not be pursued further here.

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