Comment on "Late-time tails of a self-gravitating scalar field revisited" by Bizoń et al: The leading order asymptotics

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In arXiv:0812.4333v2 Bizoń et al discuss the power-law tail in the long-time evolution of a spherically symmetric self-gravitating massless scalar field in odd spatial dimensions. They derive explicit expressions for the leading order asymptotics for solutions with small initial data by using formal series expansions. Unfortunately, this approach lacks insight into the very origin of the leading asymptotics and misses an interesting observation that the final decay rate results from very special cancellations in the nonlinear terms. Here, we show that one can avoid tedious manipulations of series and calculate the leading asymptotics more directly by recognizing the special structure and cancellations already on the level of the wave equation.

Since the works of John [1] and Asakura [2] who studied nonlinear wave equations with power nonlinearities it is known that the late-time asymptotics can be ruled by the nonlinearity even for solutions starting from small initial data (for which one might naively want to ignore the nonlinear terms as causing only "higher order" corrections). For wave equations with nonlinearities containing first derivatives the asymptotics may even depend on the particular linear combination of the terms. Christodoulou [3] and Klainerman [4] discovored special *null structures*, which lead to a faster than generic decay or decide about a global existence of solutions. The underlying mechanism is based on asymptotic cancellations of the leading order terms in these special nonlinear structures (cf. [5] for a detailed analysis of such cancellations). The purpose of this Comment is to demonstrate that the same phenomenon occurs here, in the wave equation for the scalar field, and make use of it to simplify the calculation of the leading asymptotics at late-times.

In the commented Article, later referred to as BCR, evolution of a self-gravitating real massless scalar field ϕ is considered. The Einstein equations for a $d+1$-dimensional metric with odd $d \geq 3$ restricted to spherical symmetry

$$ds^2 = e^{2\alpha(t,r)} \left(-e^{2\beta(t,r)} dt^2 + dr^2 \right) + r^2 d\Omega^2_{d-1}\,, \tag{1}$$

are analyzed, where $d\Omega^2_{d-1}$ is the round metric on the unit $(d-1)$–dimensional sphere. The scalar field satisfies a (quasilinear) wave equation with smooth, and compactly supported initial data $(\phi, \dot\phi)_{t=0} = (\varepsilon f, \varepsilon g)$ where ε is a small number. The functions ϕ, β and $m = (1 - e^{-2\alpha}) r^{d-2}$ are formally expanded in the Taylor series in ε about Minkowski spacetime $m_0 = \beta_0 = \phi_0 = 0$ and substituted into the field equations. This gives an infinite hierarchy of equations on ϕ_i, β_i, m_i, $i = 1, 2, 3, ...$ which can be solved recursively.

At the first order $m_1 = \beta_1 = 0$ while

$$\Box \phi_1 = 0, \qquad (\phi_1, \dot\phi_1)_{t=0} = (f, g) \tag{2}$$

can be solved explicitly. At the second order $\phi_2 = 0$ and

$$m_2' = \kappa\, r^{d-1} \left(\dot\phi_1^2 + \phi_1'^2 \right), \tag{3}$$

$$\dot m_2 = 2\kappa\, r^{d-1} \dot\phi_1 \phi_1'\,, \tag{4}$$

$$\beta_2' = \frac{(d-2) m_2}{r^{d-1}} \tag{5}$$

where $\kappa = \dfrac{8\pi}{d-1}$. And at the third order

$$\Box\phi_3 = 2\beta_2\ddot{\phi}_1 + \dot{\beta}_2\dot{\phi}_1 + \beta_2'\phi_1'. \tag{6}$$

In BCR it is claimed that the late-time asymptotics of ϕ is dominated by that of ϕ_3. In order to calculate it some of the functions ϕ_i, β_i, m_i must additionally be expanded in powers of r^{-1} and then inserted into (6). Here, cancellations of the leading terms occur since this equation has a very special structure of the nonlinear terms originating from the wave equation for ϕ. Hence, next-to-leading order terms become important in the asymptotics.

Below, we present a method which makes use of these cancellations already at the level of the wave equation (cancellation of terms before their evaluation) and thus reduces the amount of necessary asymptotic information about the source functions on the right-hand side. In Section I we regroup the nonlinear terms, eliminate the subdominant ones and calculate the leading asymptotics for ϕ_3 by solely evaluating the dominant nonlinear term. In Section II we prove estimates which give a rigorous background for the term selection in Section I.

I. ANALYSIS OF THE LEADING ORDER ASYMPTOTICS

A. 3+1 dimensions

For $d = 3$ the solution of the free wave equation (2) can be written as (cf. (13)-(14) of BCR with $l = 0$)

$$\phi_1(t,r) = \frac{a(u) - a(v)}{r}\,. \tag{7}$$

where $u = t - r, v = t + r$ and the function a is determined by f and g and has compact support in $[-R, +R]$. The most obvious way of calculating the asymptotics of ϕ is to substitute the above function into (3)-(5), calculate $\dot{\phi}_1, \phi_1', \ddot{\phi}_1, \phi_1'', \dot{\beta}_2, \beta_2'$ and insert into (6) to obtain the desired decay in time for ϕ_3, as was done in Section III of BCR. However, as follows from the rough estimates (cf. those obtained in Section II), the right-hand side of (6) is a function supported in the vicinity of the lightcone $t = r$ and decaying like $1/r^3$. It suggests that ϕ_3 should decay in time like $1/t^2$. As the section III of BCR shows, the true decay is by one power faster. Indeed, there happens a cancellation of leading terms in the asymptotic expansion. Here, we want to explain the cancellation mechanism already on the level of the differential equation by regrouping terms to form special structures. This transformation also allows for a considerable simplification of the calculations.

By introducing null derivatives $\partial_\pm := \frac{1}{2}(\partial_t \pm \partial_r)$ and rearranging terms the equation (6) can be rewritten as

$$\Box\phi_3 = -\frac{1}{r}\beta_2\partial_-\phi_1 + 2\partial_+\beta_2\partial_+\phi_1 + \frac{2}{r}\partial_-(r\beta_2\partial_-\phi_1) + \frac{2}{r}\beta_2\partial_+^2(r\phi_1). \tag{8}$$

The last term is identically zero for $t > R$ because there ϕ_1 is a purely outgoing wave (7). The second last term, in the process of inversion of $\Box = \frac{1}{r}\partial_-\partial_+(r\cdot)$, will turn out to be a complete derivative in the ingoing direction u. Since $r\beta_2\partial_-\phi_1$ has compact support in u this term will vanish after integration. The second term of the above expression, as is explained in Section II, has faster decay in r (by at least one power) than any other combination of null derivatives and together with the compact support in u (localization near to the lightcone) leads to a faster decay in time for ϕ_3. Hence, it is the first term that will determine the asymptotic behavior of ϕ_3 at late times (see Section II for proof). We denote it symbolically

$$\Box\phi_3 \cong -\frac{1}{r}\beta_2\partial_-\phi_1. \tag{9}$$

Hence, all we need to calculate is the leading order behavior of β_2. Substituting (7) into (3) we first find

$$m_2(u,r) \overset{r+u>R}{=} 4\pi \left(2 \int_u^\infty a'^2(x)\, dx - \frac{a^2(u)}{r} \right) \cong 8\pi F(u) + \mathcal{O}(r^{-1}), \tag{10}$$

where $F(u) := I_1^0(u)$ is defined in (28) of BCR. Next, integrating equation (5) and using $F(u) = 0$ for $u > R$ we get for $r > t - R$

$$\begin{aligned} \beta_2(u,r) &\overset{r+u>R}{\cong} 8\pi \int_u^R \frac{1}{(t-u')^2} [F(u') + \mathcal{O}(r^{-1})]\, du' \\ &\cong \frac{8\pi}{t^2} \int_u^\infty F(u')\, du' + \mathcal{O}(r^{-1}t^{-2}) + \mathcal{O}(t^{-3}) = \frac{8\pi}{t^2} G(u) + \mathcal{O}(t^{-3}) \end{aligned} \tag{11}$$

where $G(u) := I_1^1(u)$, otherwise $\beta_2(t,r) = 0$. Substituting this and (7) into (9) and using (21) of BCR (with $l = 0$) we get

$$\phi_3(t,r) = -\frac{2^4\pi}{r} \int_{-\infty}^{+\infty} d\eta \int_{t-r}^{t+r} \frac{d\xi}{(\xi-\eta)(\xi+\eta)^2} \left[G(\eta) a'(\eta) + \mathcal{O}\left(\frac{1}{(\xi+\eta)} \right) \right]. \tag{12}$$

Elementary integration over ξ and by parts over η yields the asymptotic result (34) of BCR

$$\phi_3(t,r) = \frac{\Gamma_0}{t^3} + \mathcal{O}\left(\frac{1}{t^4}\right), \qquad \Gamma_0 := -2^5\pi \int_{-\infty}^{+\infty} F(u) a(u)\, du \tag{13}$$

for late times ($t \gg r$). The cited formula (34) does not really give asymptotics at future null infinity ($t + r \to \infty, t - r = const$, called also *scri*) as is claimed, since it heavily relies on the relation $t \gg r$ implying $t - r \to \infty$. This way of taking limits defines in fact a line only approaching *scri* asymptotically on the way to timelike infinity. Without further analysis it is unclear whether the asymptotic behavior of ϕ_3 along this line agrees with asymptotics calculated along *scri*.

B. Higher dimensions

At present we cannot rigorously prove the leading order asymptotics due to a lack of an optimal decay estimate for the wave equation in d+1 dimensions, but analogously to the the 3+1 case, we are able to regroup and estimate the right-hand side terms of the wave equation for ϕ_3 and so determine the leading order source term (see Section II).

We can rearrange the nonlinear terms in the equation (6) to write them as

$$\Box\phi_3 = \frac{2}{r^{d-1}} \partial_- \left(r^{d-1} \beta_2 \partial_- \phi_1 \right) + \frac{2}{r^{d-1}} \partial_+ \left(r^{d-1} \beta_2 \partial_+ \phi_1 \right) \tag{14}$$

The second source term of the above expression, as is explained in Section II, has faster decay in r (by at least one power) than any other combination of null derivatives and together with the compact support in u leads to a faster decay in time for ϕ_3. Hence, it is the first term that will determine the asymptotic behavior of ϕ_3 at late times. See Section II for a quantitative analysis. We denote it symbolically

$$\Box\phi_3 \cong \frac{2}{r^{d-1}}\partial_-\left(r^{d-1}\beta_2\partial_-\phi_1\right). \tag{15}$$

Using (21) of BCR we get

$$\phi_3(t,r) \;=\; \frac{1}{2^{l+2}r^{l+1}}\int_{t-r}^{t+r} d\xi \int_{-\infty}^{+\infty} d\eta\, \frac{P_l(\mu)}{(\xi-\eta)^{l+1}}\partial_\eta\left[(\xi-\eta)^{2l+2}\beta_2\partial_\eta\phi_1\right] \tag{16}$$

where $l=(d-3)/2$. The inner integral over η can be integrated by parts (it produces no boundary terms since the integrand has compact support in η)

$$\phi_3(t,r) \;=\; -\frac{1}{2^{l+2}r^{l+1}}\int_{t-r}^{t+r} d\xi \int_{-\infty}^{+\infty} d\eta\; \partial_\eta\left[\frac{P_l(\mu)}{(\xi-\eta)^{l+1}}\right](\xi-\eta)^{2l+2}\beta_2\partial_\eta\phi_1. \tag{17}$$

Now we only need to find the asymptotic form of the expression $\beta_2\partial_-\phi_1$. According to (13)-(14) of BCR we have

$$\phi_1(t,r) \cong \frac{a^{(l)}(u)}{r^{1+l}} + \mathcal{O}(r^{-2-l}), \quad \dot\phi_1(t,r) \cong \frac{a^{(l+1)}(u)}{r^{1+l}} + \mathcal{O}(r^{-2-l}), \quad \phi_1'(t,r) \cong -\frac{a^{(l+1)}(u)}{r^{1+l}} + \mathcal{O}(r^{-2-l}), \tag{18}$$

and

$$m_2(u,r) \overset{r+u>R}{\cong} 2\kappa\int_u^\infty [a^{(l+1)}(x)]^2\,dx + \mathcal{O}(r^{-1}) =: 2\kappa F_l(u) + \mathcal{O}(r^{-1}), \tag{19}$$

where $F_l(u) := I^1_{l+1}(u)$. Next, integrating equation (5) and using $F_l(u)=0$ for $u>R$ we get for $r>t-R$

$$\begin{aligned}\beta_2(u,r) &\overset{r+u>R}{\cong} (d-2)2\kappa\int_u^R \frac{1}{(t-u')^{2+2l}}\left[F_l(u') + \mathcal{O}(r^{-1})\right]du' \\ &\cong \frac{(2l+1)}{(2l+2)}\frac{16\pi}{t^{2+2l}}\int_u^\infty F_l(u')\,du' + \mathcal{O}(r^{-1}t^{-2-2l}) + \mathcal{O}(t^{-3-2l}) = \frac{(2l+1)}{(2l+2)}\frac{16\pi}{t^{2+2l}}G_l(u) + \mathcal{O}(t^{-3-2l})\end{aligned} \tag{20}$$

where $G_l(u) := I^1_{l+1}(u)$, otherwise $\beta_2(t,r)=0$. Substituting this and (18) into (17) we obtain

$$\phi_3(t,r) \;=\; -\frac{(2l+1)}{(2l+2)}\frac{2^{2l+5}\pi}{r^{l+1}}\int_{t-r}^{t+r} d\xi \int_{-\infty}^{+\infty} d\eta\;\partial_\eta\left[\frac{P_l(\mu)}{(\xi-\eta)^{l+1}}\right]\frac{(\xi-\eta)^{l+1}}{(\xi+\eta)^{2+2l}}G_l(\eta)a^{(l+1)}(\eta). \tag{21}$$

Analogously to (22) or BCR it holds

$$\int_{t-r}^{t+r} d\xi\,\partial_\eta\left[\frac{P_l(\mu)}{(\xi-\eta)^{l+1}}\right]\frac{(\xi-\eta)^{l+1}}{(\xi+\eta)^{2+2l}} \cong (-1)^l 2^l\frac{(2l+2)}{(2l+1)}\frac{r^{l+1}}{t^{3+3l}}\left[1+\mathcal{O}\left(\frac{1}{t}\right)\right], \tag{22}$$

which applied to the above integral yields the asymptotic result (44) of BCR

$$\phi_3(t,r) = \frac{\Gamma_l}{t^{3l+3}} + \mathcal{O}\left(\frac{1}{t^{3l+4}}\right), \qquad \Gamma_l := (-1)^{l+1}2^{3l+5}\pi\int_{-\infty}^{+\infty} F_l(\eta)a^{(l)}(\eta)\,d\eta, \tag{23}$$

for late times ($t \gg r$), where in Γ_l we integrated by parts over η.

II. ESTIMATES

A. 3+1 dimensions

In this section we will prove decay estimates for m_2, β_2 and ϕ_3. They will lose information about the exact amplitudes but will be helpful in separating the leading asymptotics from the subleading corrections decaying faster.

From (7) we know that ϕ_1 is supported in the strip $0 < t - R \leq r \leq t + R$ and can estimate there its derivatives

$$|\phi_1(t,r)|, |\dot{\phi}_1(t,r)|, |\phi_1'(t,r)| \lesssim \frac{1}{\langle r\rangle}, \qquad |\dot{\phi}_1 + \phi_1'| \lesssim \frac{1}{\langle r\rangle^2}, \tag{24}$$

where $\langle x\rangle := 1 + |x|$ and "$\lesssim$" means "less or equal than" up to some multiplicative constant which we skip for brevity. Observe that for functions supported in the strip $|t-r| < R$ estimates by powers of $\langle r\rangle$ and $\langle t\rangle$ are equivalent, since there exist constants C_1, C_2 such that $C_1\langle t\rangle \leq \langle r\rangle \leq C_2\langle t\rangle$.

From (3) we have $m_2(t,r) = 0$ for $r < t - R$ and for $0 < t - R \leq r \leq t + R$ we estimate

$$|m_2(t,r)| \leq \int_0^r r'^2 \left(|\dot{\phi}_1|^2 + |\phi_1'|^2\right)\, dr' \lesssim \int_{t-R}^r \frac{r'^2}{\langle r'\rangle^2} dr' \leq C \tag{25}$$

where C depends only on R. Actually, there is a universal bound $|m_2(t,r)| \leq M$ where M is the total energy ("ADM mass") of ϕ_1. Moreover, for small r, say $r < 1$, we immediately see that $|m_2(t,r)| \lesssim r^3$. These two facts can be put together to give

$$|m_2(t,r)| \lesssim \frac{r^3}{\langle r\rangle^3} \tag{26}$$

Next, from (5) we get $\beta_2(t,r) = 0$ for $r < t - R$ and for $0 < t - R \leq r \leq t + R$ we again estimate

$$|\beta_2(t,r)| \leq \int_0^r \frac{m_2(t,r')}{r'^2}\, dr' \lesssim \int_{t-R}^{t+R} \frac{1}{\langle r'\rangle^2} dr' \leq \frac{2R}{\langle t-R\rangle\langle t+R\rangle} \lesssim \frac{1}{\langle t\rangle^2} \tag{27}$$

We will also need a similar estimate for the outgoing derivative of β_2. Therefore we combine the equations (3)-(5) to get the identity

$$\partial_+\beta_2 = \dot{\beta}_2 + \beta_2' = \int_0^r \left(\dot{\phi}_1 + \phi_1'\right)^2 dr' - 2\int_0^r \frac{m_2}{r'^3} dr'. \tag{28}$$

Using the above estimates on $\dot{\phi}_1 + \phi_1'$ and m_2 we find

$$|\partial_+\beta_2| \lesssim \int_{t-R}^{t+R} \frac{1}{\langle r'\rangle^4} dr' + \int_{t-R}^{t+R} \frac{1}{\langle r'\rangle^3} dr' \lesssim \frac{1}{\langle t\rangle^3} \tag{29}$$

for $0 < t - R \leq r \leq t + R$ as well as $\partial_+\beta_2(t,r) = 0$ for $r < t - R$.

Finally, we analyze the various source terms in the wave equation (6). Let us split the solution into four components introduced in (8)

$$\Box\phi_{3A} = -\frac{1}{r}\beta_2\partial_-\phi_1, \qquad \Box\phi_{3B} = 2\partial_+\beta_2\partial_+\phi_1, \tag{30}$$

$$\Box\phi_{3C} = \frac{2}{r}\partial_-(r\beta_2\partial_-\phi_1), \qquad \Box\phi_{3D} = \frac{2}{r}\beta_2\partial_+^2(r\phi_1), \tag{31}$$

with $\phi_3 = \phi_{3A} + \phi_{3B} + \phi_{3C} + \phi_{3D}$. All terms on the right hand side are supported in $|t-r| < R$. By the above bounds we can estimate the first two components

$$\left|\frac{1}{r}\beta_2\partial_-\phi_1\right| \lesssim \frac{1}{r\langle r\rangle\langle t\rangle^2} \lesssim \frac{1}{r\langle r\rangle^3}, \qquad |\partial_+\beta_2\partial_+\phi_1| \lesssim \frac{1}{\langle t\rangle^3\langle r\rangle^2} \lesssim \frac{1}{\langle r\rangle^5} \tag{32}$$

Now, e.g. from [6], we find

$$|\phi_{3A}(t,r)| \lesssim \frac{1}{\langle t+r\rangle\langle t-r\rangle^2}, \qquad |\phi_{3B}(t,r)| \lesssim \frac{1}{\langle t+r\rangle\langle t-r\rangle^3}, \tag{33}$$

so we see that ϕ_{3B} becomes subdominant to ϕ_{3A} regarding the late time asymptotics.

Next, in spherical symmetry $\Box\phi_{3C} \equiv \frac{4}{r}\partial_-\partial_+(r\phi_{3C})$ and hence

$$\phi_{3C}(t,r) = \frac{1}{4r}\int_{t-r}^{t+r} du \int_{-(t+r)}^{t-r} dv\; \partial_v(r\beta_2\partial_v\phi_1). \tag{34}$$

The inner integral is an integral of a total derivative of a compactly supported function of v. The integration range is bigger than its support $[-R, R]$ so the integral vanishes. This gives $\phi_{3C} = 0$ for $t > r + R$.

The last component $\phi_{3D} = 0$ for $t > r + R$ because the source vanishes identically in the integration region when inverting the wave operator as we did above.

Finally, for late times, in the region $t > r + R$,

$$|\phi_3(t,r)| \cong |\phi_{3A}(t,r)| \lesssim \frac{1}{\langle t+r\rangle\langle t-r\rangle^2} \tag{35}$$

and other components of ϕ can be a priori estimated to be subdominant. Therefore, in the asymptotic analysis, concerned solely with the leading order behavior, it is sufficient to keep only the first source term in (8).

B. Higher dimensions

Here, we want to show that the second source term in (14) is asymptotically subdominant with respect to the first one. In dimension d+1, the estimates for ϕ_1, m_2 and β_2 and their derivatives can be obtained analogously to the 3+1 case and read for $0 < t - R \le r \le t + R$

$$|\phi_1(t,r)|, |\dot{\phi}_1(t,r)|, |\phi_1'(t,r)| \lesssim \frac{1}{\langle r\rangle^{1+l}}, \qquad |\partial_+^k\phi_1| \lesssim \frac{1}{\langle r\rangle^{1+l+k}}, \qquad |\partial_-^k\phi_1| \lesssim \frac{1}{\langle r\rangle^{1+l}}, \tag{36}$$

$$|m_2(t,r)| \lesssim C\frac{r^d}{\langle r\rangle^d}, \qquad |\beta_2(t,r)| \lesssim \frac{1}{\langle t\rangle^{d-1}}, \qquad |\partial_+\beta_2| \lesssim \frac{1}{\langle t\rangle^d}, \qquad |\partial_-\beta_2| \lesssim \frac{1}{\langle t\rangle^{d-1}} \tag{37}$$

while all these functions vanish for $r < t - R$. It allows us to control both source terms in (14)

$$\left|\frac{1}{r^{d-1}}\partial_-\left(r^{d-1}\beta_2\partial_-\phi_1\right)\right| \le |\partial_-\beta_2\partial_-\phi_1| + \left|\beta_2\partial_-^2\phi_1\right| + \left|\frac{(d-1)}{r}\beta_2\partial_-\phi_1\right| \tag{38}$$

$$\lesssim \frac{1}{\langle t\rangle^{d-1}\langle r\rangle^{1+l}} + \frac{1}{\langle t\rangle^{d-1}\langle r\rangle^{1+l}}. + \frac{1}{r\langle t\rangle^{d-1}\langle r\rangle^{1+l}} \lesssim \frac{1}{\langle r\rangle^{3+3l}}$$

$$\left|\frac{1}{r^{d-1}}\partial_+\left(r^{d-1}\beta_2\partial_+\phi_1\right)\right| \le |\partial_+\beta_2\partial_+\phi_1| + \left|\beta_2\partial_+^2\phi_1\right| + \left|\frac{(d-1)}{r}\beta_2\partial_+\phi_1\right| \tag{39}$$

$$\lesssim \frac{1}{\langle t\rangle^{d}\langle r\rangle^{2+l}} + \frac{1}{\langle t\rangle^{d-1}\langle r\rangle^{3+l}}. + \frac{1}{r\langle t\rangle^{d-1}\langle r\rangle^{2+l}} \lesssim \frac{1}{\langle r\rangle^{5+3l}}.$$

Both estimates are optimal (c.f. the explicit asymptotic expressions in Section IV of BCR), hence the second therm is indeed subdominant what justifies neglecting it in the leading order calculations.

However, there is a problem in d+1 dimensions which is absent in 3+1. We lack an optimal decay estimate for the wave equation in d+1 dimensions, given a source with prescribed decay. The presently best known estimates (c.f. [7, 8]) lose l powers in the late-time decay relative to what is optimal. Therefore, for the above sources we are able to show rigorously only the decay

$$|\phi_3(t,r)| \lesssim \frac{1}{\langle t+r\rangle\langle t-r\rangle^{2+2l}}, \tag{40}$$

while $1/t^{3+3l}$ is optimal for late times. However, it does not change the fact that in the asymptotic analysis it is the first source term in (14) that dominates the asymptotics of ϕ_3 at late times.

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