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Recent Developments in (0,2) Mirror Symmetry

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Abstract. Mirror symmetry of the type II string has a beautiful generalization to the heterotic string. This generalization, known as (0,2) mirror symmetry, is a field still largely in its infancy. We describe recent developments including the ideas behind quantum sheaf cohomology, the mirror map for deformations of (2,2) mirrors, the construction of mirror pairs from worldsheet duality, as well as an overview of some of the many open questions. The (0,2) mirrors of Hirzebruch surfaces are presented as a new example.

Key words: mirror symmetry; (0,2) mirror symmetry; quantum sheaf cohomology

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1 Introduction

In the landscape of string compactifications, the corner comprised of heterotic string compactifications is particularly appealing. Only in this corner is there a possibility of a conventional worldsheet description of flux vacua. In addition, there is a real hope of exploring the interplay between low-energy particle physics and cosmology. These are strong motivations to understand heterotic worldsheet theories and their associated mathematics more deeply.

For $\mathcal{N} = 1$ space-time supersymmetry, we are interested in worldsheet theories with (0,2) worldsheet supersymmetry. The special case of models with (2,2) supersymmetry has been heavily studied in both physics and mathematics. These are worldsheet theories that can be used to define type II string compactifications. Perhaps the most studied and most striking discovery in (2,2) theories is mirror symmetry: namely, that topologically distinct target spaces can give rise to isomorphic superconformal field theories. Physically, this identification permits the computation of quantum corrected observables in one model from protected observables in the mirror model. In

particular, classes of Yukawa couplings can be exactly determined this way.

In mathematics, mirror symmetry has played a crucial role in the development of curve counting techniques, quantum cohomology, and modern Gromov-Witten theory. In the the setting of (2,2) models, mirror symmetry is a fairly mature topic. Yet (2,2) theories are a special case of (0,2) models and physically are less appealing. Generic $\mathcal{N} = 1$ supersymmetric compactifications of the critical heterotic string, including the more phenomenologically appealing vacua, are of (0,2) type. Our aim in this review is to describe recent developments in extending the ideas of mirror symmetry and quantum cohomology to (0,2) models.

Although general (0,2) superconformal field theories need not have a geometric interpretation, there is a familiar geometric set-up that leads to such theories: a stable holomorphic bundle $\mathcal{E}$ over a smooth Calabi-Yau manifold X. The chosen bundle must satisfy a basic consistency condition to guarantee freedom from anomalies:

$$\mathrm{ch}_2(\mathcal{E}) \;=\; \mathrm{ch}_2(TX). \tag{1}$$

In this setting, (0,2) mirror symmetry is the assertion that two topologically distinct pairs $(X, \mathcal{E})$ and $(X^\circ, \mathcal{E}^\circ)$ of spaces and bundles can correspond to isomorphic conformal field theories. Physically, having such an isomorphism can shed light on the structure of the conformal field theory and the quantum geometry associated to the classical data $(X, \mathcal{E})$. Mathematically, this isomorphism provides generalizations of curve counting relations and quantum cohomology. Much like the (2,2) case, this isomorphism generalizes beyond conformal models to include massive (0,2) models, including those that describe target spaces with $c_1(T_X) \neq 0$.

The general structure of the correspondence is currently not well understood. However, there is an important special case that has been well-studied by mathematicians and physicists alike. This is the situation where we take the bundle $\mathcal{E}$ to be the tangent bundle over the Calabi-Yau space. In this case, the resulting conformal field theory enjoys (2,2) supersymmetry, and (0,2) mirror symmetry reduces to the assertion that a Calabi-Yau X and its mirror dual X° lead to isomorphic conformal field theories. Of course, this is the celebrated (2,2) mirror symmetry [16, 24].

There are two ways in which one can seek to generalize the familiar mirror symmetry notions from this starting point. The first is to consider the (2,2) conformal field theory on a world-sheet with boundaries. In string theory language, this leads to the study of D-branes on Calabi-Yau manifolds. When we restrict the conformal field theory data to the topological category (i.e. the data associated to certain topological sub-sectors of the full theory), the appropriate mathematical structure is framed by the homological mirror symmetry conjectures [29].

To describe the second, "heterotic" generalization, we observe that the bundle T_X has deformations as a holomorphic bundle over X; infinitesimally these are counted by $H^1(X, \mathrm{End}\, T_X)$, and it is easy to find examples with a large unobstructed moduli space. For instance, in the case of the quintic hypersurface $X \in \mathbb{P}^4$ the tangent bundle has 224 unobstructed deformations. Turning on these deformations reduces the worldsheet supersymmetry to (0,2). As in the D-brane case, one can identify certain (quasi)-topological sub-sectors. In this class of models, (0,2) mirror symmetry

then has two primary concerns:

- how are the deformations of T_X realized in the mirror theory, and what is the map between the two sets of deformations?
- how does the map relate the quasi-topological observables on the two sides of the mirror?

For more general (0,2) theories with bundles $\mathcal{E}$ not necessarily related to T_X or even of the same rank, we need to ask more basic questions like:

- how do we characterize mirror pairs?
- how do we compute the non-perturbative effects which give (0,2) generalizations of curve counting and quantum cohomology?

We should stress that given an isomorphism of the heterotic conformal field theories for (X, T_X) and (X°, T_X°), as well as the existence of unobstructed deformations for the (X, T_X) conformal field theory, we know on general grounds that corresponding deformations must exist on the mirror side, and that the deformed theories must remain isomorphic as (0,2) theories. This is the crucial point that makes the "(2,2) locus" (i.e. the choice $\mathcal{E} = T_X$) a natural starting point for explorations of (0,2) mirror symmetry: we are assured of success, and our primary job is to find the appropriate map.

Just as ordinary mirror symmetry exchanges the topological A and B models, (0,2) mirror symmetry exchanges what are termed the A/2 and B/2 models. We therefore begin in section 2 with a discussion of the A/2 model. If X and X° are (2,2) mirror Calabi-Yau spaces then their Hodge numbers are exchanged,

$$h^{i,j}(X) \;=\; h^{n-i,j}(X^\circ),$$

where n is the dimension of both X and X°. Instead of exchanging Hodge numbers, sheaf cohomology groups are exchanged for (0,2) mirror pairs:

$$h^j\left(X, \wedge^i \mathcal{E}^*\right) \;=\; h^j\left(X^\circ, \wedge^i \mathcal{E}^\circ\right). \tag{2}$$

Non-perturbative effects are therefore encoded in "quantum sheaf cohomology," which generalizes the quantum cohomology ring of (2,2) mirror symmetry. This is physically a ground ring, isomorphic to a deformation of a classical cohomology ring, which is corrected by non-perturbative effects. In section 2, we review the current status of quantum sheaf cohomology.

One of the early successes of the mirror symmetry program was finding a precise map between complex and Kähler moduli for certain mirror pairs, known as the monomial-divisor mirror map [3]. If one considers deformations of T_X for Calabi-Yau spaces defined by reflexively plain polytopes then a (0,2) generalization of the monomial-divisor mirror map exists [37]. The map describes the exchange of complex, Kähler and bundle moduli. Even in this class of models, finding the map

is relatively challenging, and the story is far from complete. However, the last few years has seen some important progress, leading to interesting structures and opening up new routes for further investigation. We describe this progress in section 3.

In section 4, we describe the construction of mirror pairs using worldsheet duality [1]. This generalizes the physical approach taken in [25] to construct (2,2) mirror pairs for non-compact toric spaces. In the (0,2) setting, worldsheet duality leads to mirror descriptions of sigma models with toric target spaces, including non-compact Calabi-Yau spaces and models without a (2,2) locus. As an example, we present the (0,2) mirrors for Hirzebruch surfaces.

In terms of historical development, explorations of (0,2) mirror symmetry go back more than a decade. Early work provided numerical evidence for the existence of a (0,2) duality by computing the dimensions of sheaf cohomology groups in a large class of examples [12]. A visual check showed that most cases came in pairs satisfying the symmetry (2). Later work generalized the original Greene-Plesser construction [21] to a class of (0,2) models, with mirrors generated via orbifolding by a finite symmetry group [11, 13].

The more recent work was initiated by the construction of (0,2) mirrors via worldsheet duality [1], which led to precise definitions of heterotic chiral rings [1, 2], and to the notion of quantum sheaf cohomology [18, 19, 28, 42, 43]. Most of our presentation of mirror maps for deformations of T_X is based on [30, 34, 37]. Perhaps the most important point to stress is how many of the central questions remain wide open. The field of (0,2) mirror symmetry and (0,2) string compactifications is still truly in its infancy. In section 5, we end by overviewing some of the current and future directions of investigation.

2 Quantum Sheaf Cohomology

We begin by describing computations of non-perturbative corrections in (0,2) theories, which are encapsulated in the notion of "quantum sheaf cohomology." This is a generalization of ordinary quantum cohomology. Instead of giving a quantum deformation of ordinary cohomology rings, quantum sheaf cohomology gives a deformation of sheaf cohomology rings. Specifically, let X be a complex Kähler manifold and $\mathcal{E} \to X$ a holomorphic vector bundle satisfying the following two conditions:

$$\wedge^{\text{top}}\mathcal{E}^* \cong K_X, \qquad \text{ch}_2(\mathcal{E}) = \text{ch}_2(TX) \tag{3}$$

A pair $(X, \mathcal{E})$ satisfying the conditions (3) is sometimes known as "omalous," which is a shortening of "non-anomalous." These conditions are slightly stronger than the basic conditions needed for a consistent (0,2) theory, and arise from demanding that the A/2 theory, to be reviewed shortly, be well-defined.

Quantum sheaf cohomology is then a deformation of the classical ring generated by

$$H^*(X, \wedge^*\mathcal{E}^*)$$

In the special case that $\mathcal{E} = TX$, quantum sheaf cohomology reduces to ordinary quantum cohomology.

Historically, the existence of quantum sheaf cohomology was first proposed in [1]. The paper [28] worked out sufficient details to mathematically compute in examples. Questions concerning existence of OPE ring structures in theories with only (0,2) supersymmetry were discussed in [2]. The subject has since been further developed in a number of works including [18, 19, 22, 23, 30, 32–36, 38, 42, 43, 47, 48].

In principle, quantum sheaf cohomology should exist for any omalous pair $(X, \mathcal{E})$; that said, at the moment, computational techniques only exist in more limited cases. Specifically, the special case that X is a toric variety and $\mathcal{E}$ is a deformation of the tangent bundle is well-understood. For hypersurfaces in toric varieties, there is a "quantum restriction" proposal [34] that generalizes an old technique of Kontsevich, though more work on hypersurfaces is certainly desirable.

Ordinary quantum cohomology arises physically as a description of correlation functions in the A model topological field theory,

$$S_A = \frac{1}{\alpha'} \int_\Sigma d^2 z \left((g_{\mu\nu} + iB_{\mu\nu}) \partial\phi^\mu \overline{\partial} \phi^\nu \; + \; \frac{i}{2} g_{\mu\nu} \psi_+^\mu D_{\overline{z}} \psi_+^\nu \; + \; \frac{i}{2} g_{\mu\nu} \psi_-^\mu D_z \psi_-^\nu \; + \; R_{i\overline{j}k\overline{l}} \psi_+^i \psi_+^{\overline{j}} \psi_-^k \psi_-^{\overline{l}} \right),$$

where $\phi : \Sigma \to X$ is a map from the worldsheet Σ into the space X in which the string propagates, and the $\psi_\pm^\mu$ are fermionic superpartners of the coordinates ϕ^μ on X. There is a nilpotent scalar operator Q, known as the BRST operator, whose action on the fields above is schematically as follows:

$$\delta\phi^i \; \propto \; \chi^i, \qquad \delta\phi^{\overline{\imath}} \; \propto \; \chi^{\overline{\imath}}, \qquad \delta\chi^i \; = \; 0 \; = \; \delta\chi^{\overline{\imath}}, \qquad \delta\psi_{\overline{z}}^{\overline{\imath}} \; \neq \; 0, \qquad \delta\psi_z^i \; \neq \; 0.$$

The states of the theory are BRST-closed dimension zero operators (modulo BRST-exact operators), which constrains them to be of the form

$$b_{i_1 \cdots i_p \overline{\imath}_1 \cdots \overline{\imath}_q}(\phi) \chi^{i_1} \cdots \chi^{i_p} \chi^{\overline{\imath}_1} \cdots \chi^{\overline{\imath}_q}. \tag{4}$$

Witten observed that the states of (4) are in one-to-one correspondence with differential forms [49],

$$b_{i_1 \cdots i_p \overline{\imath}_1 \cdots \overline{\imath}_q}(\phi) dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\overline{z}^{\overline{\imath}_1} \wedge \cdots \wedge d\overline{z}^{\overline{\imath}_q},$$

and the BRST operator Q with the exterior derivative d, hence the states are in one-to-one correspondence with elements of $H^{p,q}(X)$.

The A/2 model is defined by the action:

$$S_{A/2} = \frac{1}{\alpha'} \int_\Sigma d^2 z \left((g_{\mu\nu} + iB_{\mu\nu}) \partial\phi^\mu \overline{\partial} \phi^\nu \; + \; \frac{i}{2} g_{\mu\nu} \psi_+^\mu D_{\overline{z}} \psi_+^\nu \; + \; \frac{i}{2} h_{\alpha\beta} \lambda_-^\alpha D_z \lambda_-^\beta \; + \; F_{i\overline{j}a\overline{b}} \psi_+^i \psi_+^{\overline{j}} \lambda_-^a \lambda_-^{\overline{b}} \right).$$

In the special case of $\mathcal{E} = TX$, the A/2 model becomes the A model. Anomaly cancellation in the

A/2 model requires

$$\wedge^{\rm top}\mathcal{E}^* \cong K_X, \qquad {\rm ch}_2(\mathcal{E}) = {\rm ch}_2(TX).$$

The first statement is a condition specific to the A/2 model, an analogue of the condition that the closed string B model can only propagate on spaces X such that $K_X^{\otimes 2} \cong \mathcal{O}_X$ [43, 49]. The second statement is commonly known as the "Green-Schwarz anomaly cancellation condition," and is generic to all heterotic theories.

There is a BRST operator Q in the A/2 model, which acts as follows:

$$\delta\phi^i = 0, \qquad \delta\phi^{\overline{\imath}} \propto \psi_+^{\overline{\imath}}, \qquad \delta\psi_+^{\overline{\imath}} = 0 = \delta\lambda_-^a, \qquad \delta\psi_+^i \neq 0, \qquad \delta\lambda_-^{\overline{a}} \neq 0.$$

The states of the A/2 model generalizing the A model states are of the form

$$b_{\overline{\imath}_1\cdots\overline{\imath}_q a_1\cdots a_p}\psi_+^{\overline{\imath}_1}\cdots\psi_+^{\overline{\imath}_q}\lambda_-^{a_1}\cdots\lambda_-^{a_p}. \tag{5}$$

Proceeding in an analogous fashion, we identify the BRST operator Q with $\overline{\partial}$, and the states of (5) with elements of sheaf cohomology $H^q(X, \wedge^p\mathcal{E}^*)$.

In addition to the A/2 model which provides the (0,2) version of the A model, there also exists a B/2 model which provides a (0,2) version of the B model topological field theory. As one might guess, (0,2) mirror symmetry exchanges A/2 and B/2 model correlation functions, just as ordinary mirror symmetry exchanges A and B model correlation functions. There are also some surprising new symmetries; for example, the B/2 model on X with bundle $\mathcal{E}$ is equivalent, at least classically, to the A/2 model on X with bundle $\mathcal{E}^*$. We do not have space here to discuss the B/2 model separately (and indeed, given the symmetry just mentioned, little discussion is really needed); see instead [43] for further information.

2.1 Formal computations

Quantum sheaf cohomology is determined by correlation function computations in the A/2 model. In this section we will briefly outline how such computations are defined, at least at a formal level.

First consider the case with no ($\psi_+^{\overline{\imath}}$, λ_-^a) zero modes (no "excess intersection"). In this case, a correlation function will have the form,

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = \sum_\beta \int_{\mathcal{M}_\beta} \omega_1 \wedge \cdots \omega_n, \tag{6}$$

where $\mathcal{M}_\beta$ is a moduli space of curves of degree β, and ω_i is an element of $H^q(\mathcal{M}_\beta, \wedge^p\mathcal{F}^*)$ induced by the element of $H^q(X, \wedge^p\mathcal{E}^*)$ corresponding to $\mathcal{O}_i$. The sheaf $\mathcal{F}$ is induced from $\mathcal{E}$. For example, if the moduli space $\mathcal{M}$ admits a universal instanton α, then $\mathcal{F} = R^0\pi_*\alpha^*\mathcal{E}$.

The integrand above is an element of,

$$H^{\sum q_i}\left(\mathcal{M}_\beta, \wedge^{\sum p_i}\mathcal{F}^*\right),$$

and so will vanish unless

$$\sum q_i \;=\; \dim \mathcal{M}_\beta, \qquad \sum p_i \;=\; \mathrm{rank}\,\mathcal{F}.$$

Furthermore, Grothendieck-Riemann-Roch tells us that the conditions for $(X, \mathcal{E})$ to be omalous, namely

$$\wedge^{\mathrm{top}}\mathcal{E}^* \cong K_X, \qquad \mathrm{ch}_2(\mathcal{E}) = \mathrm{ch}_2(TX),$$

imply that, at least formally, $\wedge^{\mathrm{top}}\mathcal{F}^* \cong K_{\mathcal{M}_\beta}$, which guarantees that the integrand of (6) determines a number.

Next, let us briefly consider the case of excess intersection where there are $(\psi_+^{\bar{\imath}}, \lambda_-^a)$ zero modes. In the ordinary A model, this would involve adding an Euler class factor, corresponding physically to using four-fermi terms to soak up extra zero modes. In the A/2 model, four-fermi terms are interpreted as generating elements of

$$H^1\left(\mathcal{M}_\beta, \mathcal{F}^* \otimes \mathcal{F}_1 \otimes (\mathrm{Obs})^*\right).$$

Taking into account those four-fermi terms, the integrand is then an element of

$$H^{\mathrm{top}}\left(\mathcal{M}_\beta, \wedge^{\mathrm{top}}\mathcal{F}^* \otimes \wedge^{\mathrm{top}}\mathcal{F}_1 \otimes \wedge^{\mathrm{top}}\mathrm{Obs}^*\right).$$

As before, Grothendieck-Riemann-Roch and the anomaly cancellation conditions imply that the integrand determines a number.

2.2 Linear sigma model compactifications

In order to do actual computations, we need to pick a compactification of the moduli space of curves, and describe how to extend the induced sheaves $\mathcal{F}$, $\mathcal{F}_1$ over that compactification. We will work with toric varieties and linear sigma model compactifications of moduli spaces. These compactifications are well-known, so we shall be brief.

Schematically, a linear sigma model compactification of a moduli space of curves in a toric variety is built by expressing the toric variety as a $\mathbb{C}^\times$ quotient, expanding the homogeneous coordinates in zero modes, and then taking those zero modes to be homogeneous coordinates on the moduli space (with the same $\mathbb{C}^\times$ quotient and weightings as the original homogeneous coordinates, and exceptional set determined by that of the original space). For example, consider $\mathbb{P}^{N-1}$, which is described as a $\mathbb{C}^\times$ quotient of N homogeneous coordinates, each of weight 1. For a moduli space of maps $\mathbb{P}^1 \to \mathbb{P}^{N-1}$ of degree d, we expand each homogeneous coordinate in a basis of sections of

$\phi^* \mathcal{O}(1) = \mathcal{O}(d)$, and interpret coefficients as homogeneous coordinates on the moduli space. Since the space of sections of $\mathcal{O}(d)$ has dimension $d+1$, that means the moduli space is a $\mathbb{C}^\times$ quotient of $\mathbf{C}^{N(d+1)}$, which naturally leads to $\mathbb{P}^{N(d+1)-1}$.

The construction of induced sheaves is described in [18,19,28]. Schematically, it works as follows: in the original physical theory, the bundle $\mathcal{E}$ is built from kernels and cokernels of maps between direct sums of line bundles, *i.e.* sums of powers of the universal sub-bundle S and analogues thereof. Briefly, we lift each such line bundle on the original toric variety to a line bundle on $\mathbb{P}^1 \times \mathcal{M}$, in such a way that sums and powers of universal sub-bundles are preserved. After lifting, these are then pushed forward to $\mathcal{M}$.

For example, consider the completely reducible bundle,

$$\mathcal{E} \ = \ \oplus_a \mathcal{O}(n_a),$$

on $\mathbb{P}^{N-1}$. Corresponding to the universal sub-bundle,

$$S := \mathcal{O}(-1) \ \longrightarrow \ \mathbb{P}^{N-1},$$

is

$$\mathcal{S} \ = \ \pi_1^* \mathcal{O}_{\mathbb{P}^1}(-d) \otimes \pi_2^* \mathcal{O}_{\mathcal{M}}(-1) \ \longrightarrow \ \mathbb{P}^1 \times \mathcal{M}.$$

The lift of $\mathcal{E}$ is

$$\oplus_a \mathcal{S}^{\otimes -n_a} \ \longrightarrow \ \mathbb{P}^1 \times \mathcal{M}$$

which pushes forward to

$$\begin{aligned} \mathcal{F} &= \oplus_a H^0\left(\mathbb{P}^1, \mathcal{O}(n_a d)\right) \otimes_{\mathbb{C}} \mathcal{O}(n_a), \\ \mathcal{F}_1 &= \oplus_a H^1\left(\mathbb{P}^1, \mathcal{O}(n_a d)\right) \otimes_{\mathbb{C}} \mathcal{O}(n_a). \end{aligned}$$

This generalizes to other toric varieties as well as to Grassmannians. Physically, this is equivalent to expanding worldsheet fermions in a basis of zero modes, and identifying each basis element with a line bundle of the same $\mathbb{C}^\times$ weights as the original line bundle.

The example above illustrates what happens for gauge bundles that are sums of line bundles. Next, let us consider a cokernel of a map between sums of line bundles:

$$0 \ \longrightarrow \ \mathcal{O}^{\oplus k} \ \longrightarrow \ \oplus_i \mathcal{O}(\vec{q}_i) \ \longrightarrow \ \mathcal{E} \ \longrightarrow \ 0,$$

over some toric variety X. Lifting to $\mathbb{P}^1 \times \mathcal{M}$ and pushing forward gives the long exact sequence

$$\begin{aligned} 0 \ &\longrightarrow \ \oplus_k H^0(\mathcal{O}) \otimes \mathcal{O} \ \longrightarrow \ \oplus_i H^0(\mathcal{O}(\vec{q}_i \cdot \vec{d})) \otimes \mathcal{O}(\vec{q}_i) \ \longrightarrow \ \mathcal{F} \\ &\longrightarrow \ \oplus_k H^1(\mathcal{O}) \otimes \mathcal{O} \ \longrightarrow \ \oplus_i H^1(\mathcal{O}(\vec{q}_i \cdot \vec{d})) \otimes \mathcal{O}(\vec{q}_i) \ \longrightarrow \ \mathcal{F}_1 \ \longrightarrow \ 0, \end{aligned}$$

which simplifies to the statements

$$0 \longrightarrow \mathcal{O}^{\oplus k} \longrightarrow \oplus_i H^0(\mathcal{O}(\vec{q}_i \cdot \vec{d})) \otimes \mathcal{O}(\vec{q}_i) \longrightarrow \mathcal{F} \longrightarrow 0,$$

$$\mathcal{F}_1 \cong \oplus_i H^1(\mathcal{O}(\vec{q}_i \cdot \vec{d})) \otimes \mathcal{O}(\vec{q}_i).$$

It can be shown that if $\mathcal{E}$ is locally-free, then $\mathcal{F}$ will also be locally-free.

As a consistency check, let us examine the special case that $\mathcal{E} = TX$. The tangent bundle of a (compact, smooth) toric variety can be expressed as a cokernel

$$0 \longrightarrow \mathcal{O}^{\oplus k} \longrightarrow \oplus_i \mathcal{O}(\vec{q}_i) \longrightarrow TX \longrightarrow 0.$$

Applying the previous ansatz, we have

$$0 \longrightarrow \mathcal{O}^{\oplus k} \longrightarrow \oplus_i H^0(\mathcal{O}(\vec{q}_i \cdot \vec{d})) \otimes \mathcal{O}(\vec{q}_i) \longrightarrow \mathcal{F} \longrightarrow 0,$$

$$\mathcal{F}_1 \cong \oplus_i H^1(\mathcal{O}(\vec{q}_i \cdot \vec{d})) \otimes \mathcal{O}(\vec{q}_i).$$

In this case, for $\mathcal{E} = TX$, we expect $\mathcal{F} = T\mathcal{M}$ and $\mathcal{F}_1$ to be the obstruction sheaf in the sense of [4,39]. The sequences above have precisely the right form, and it can be shown that the induced maps are also correct.

2.3 Results

Consider a deformation of the tangent bundle of a toric variety X defined as a cokernel

$$0 \longrightarrow \mathcal{O}_X \otimes_{\mathbb{C}} W^* \xrightarrow{E} \bigoplus_\rho \mathcal{O}_X(D_\rho) \longrightarrow \mathcal{E} \longrightarrow 0, \tag{7}$$

where D_ρ parametrizes the toric divisors, and $W = H^2(X, \mathbb{C})$. The components of E are a collection of W-valued sections E_ρ of $\mathcal{O}_X(D_\rho)$. We write,

$$E_\rho = \sum_{\rho'} a_{\rho\rho'} x_{\rho'} + \ldots,$$

where $a_{\rho\rho'} \in W$, $x_{\rho'}$ is the homogeneous coordinate in the Cox ring associated to ρ' (*i.e.* $x_\rho \in H^0(X, \mathcal{O}_X(D_\rho))$). Nonlinear terms in x's have been omitted.

For each linear equivalence class c of toric divisors, define a matrix A_c to be the $|c| \times |c|$ matrix whose entries are $a_{\rho\rho'}$, where ρ, ρ' are within the same linear equivalence class c. Define:

$$Q_c = \det A_c. \tag{8}$$

Recall that a collection of edges K of the fan is called a primitive collection if K does not span any cone in the fan, but every proper sub-collection of K does. Equivalently, the intersection of all of the divisors in K is empty, but the intersection of any sub-collection is non-empty.

It was shown in [18] that any primitive collection K is a union of linear equivalence classes. With that in mind, define Q_K to be the product of all Q_c for c a linear equivalence class contained in K.

Define:

$$SR(X,\mathcal{E}) \;=\; \{Q_K \,|\, \text{K a primitive collection}\}.$$

It was shown in [18] that the classical product structure on $\oplus H^*(X, \wedge^* \mathcal{E}^*)$ is encoded in the statement,

$$\oplus H^*(X, \wedge^* \mathcal{E}^*) \;=\; \mathrm{Sym}^* W / SR(X,\mathcal{E}).$$

Because all linear sigma model moduli spaces for toric varieties are also toric, the classical sheaf cohomology ring in each separate worldsheet instanton sector has the same form:

$$\oplus H^*\left(\mathcal{M}_\beta, \wedge^* \mathcal{F}_\beta^*\right) \;=\; \mathrm{Sym}^* W / \widehat{SR}\left(\mathcal{M}_\beta, \mathcal{F}_\beta\right),$$

where

$$\widehat{SR}\left(\mathcal{M}_\beta, \mathcal{F}_\beta\right) \;=\; \{Q_{K_\beta} \,|\, \text{K a primitive collection}\}$$

and

$$Q_{K_\beta} \;=\; \prod_{c \in [K]} Q_c^{h^0(D_c \cdot \beta)},$$

where $[K]$ denotes the set of linear equivalence classes of the D_ρ with $\rho \in K$, for K a primitive collection ($D_c \cdot \beta$ means $D_\rho \cdot \beta$ for any ρ in the linear equivalence class c).

To generate the quantum sheaf cohomology relations, one must find relations between correlation functions in different instanton sectors. For the sake of brevity, we omit the details here, and instead refer the interested reader to [18, 19].

To define the final result, we must define a set K^-, and a class $\beta_K \in H_2(X, \mathbb{Z})$. For any primitive collection K consider the element,

$$v \;=\; \sum_{\rho \in K} v_\rho,$$

of the toric lattice. Then v lies in the relative interior of a unique cone σ. Let K^- denote the set of edges of σ. Then one can write

$$v \;=\; \sum_{\rho \in K^-} c_\rho v_\rho,$$

where each $c_\rho > 0$, hence

$$\sum_{\rho\in K} v_\rho \;=\; \sum_{\rho\in K^-} c_\rho v_\rho,$$

or equivalently

$$\sum a_\rho v_\rho \;=\; 0. \tag{9}$$

Dualizing the sequence

$$0 \;\longrightarrow\; M \;\longrightarrow\; \mathbb{Z}^{\Sigma(1)} \;\longrightarrow\; \mathrm{Pic}(X) \;\longrightarrow\; 0,$$

we see that equation (9) can be induced by intersection with elements of $H_2(X,\mathbb{Z})$. Hence, for each primitive collection K, there is a class $\beta_K \in H_2(X,\mathbb{Z})$ such that $D_\rho \cdot \beta_K = a_\rho$; see also [7, 18] for more information on this notation.

Finally, [18, 19] show that the quantum sheaf cohomology relations are given by

$$\prod_{c\in[K]} Q_c \;=\; q^{\beta_K} \prod_{c\in[K^-]} Q_c^{-D_c\cdot\beta_K},$$

for quantum parameters q^{β_K}.

In this section we have outlined a mathematical description of quantum sheaf cohomology. The same results (at least for purely linear maps E) can also be obtained physically from one-loop effective action arguments on Coulomb branches of gauged linear sigma models, in analogy with [39]; see, for example, [33, 34].

2.4 Example: $\mathbb{P}^1 \times \mathbb{P}^1$

Let $X = \mathbb{P}^1 \times \mathbb{P}^1$, and consider the vector bundle $\mathcal{E}$ given as the cokernel

$$0 \;\longrightarrow\; \mathcal{O}\oplus\mathcal{O} \;\stackrel{E}{\longrightarrow}\; \mathcal{O}(1,0)^2 \oplus \mathcal{O}(0,1) \;\longrightarrow\; \mathcal{E} \;\longrightarrow\; 0,$$

where

$$E \;=\; \begin{bmatrix} Ax & Bx \\ C\tilde{x} & D\tilde{x} \end{bmatrix},$$

where A, B, C, D are 2×2 matrices and

$$x \;=\; \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \tilde{x} \;=\; \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

are arrays of homogeneous coordinates on the two $\mathbb{P}^1$ factors. The bundle $\mathcal{E}$ is a deformation of the tangent bundle, which itself corresponds to the special case $A = D = I_{2\times 2}$, $B = C = 0$). In general $\mathcal{E}$ is not isomorphic to the tangent bundle. It can be shown that $H^1(X, \mathcal{E}^*)$ is two-dimensional.

In the language of the previous section, there are two primitive collections. The two Q_K's corresponding to each of those primitive collections are

$$\det\left(A\psi + B\tilde{\psi}\right), \qquad \det\left(C\psi + D\tilde{\psi}\right).$$

For both of those primitive collections, K^- is empty.

It can be shown [18,19,33,34] that the quantum sheaf cohomology ring for this case is given by $\mathbb{C}[\psi, \tilde{\psi}]$ modulo the relations,

$$\det\left(A\psi + B\tilde{\psi}\right) \;=\; q, \qquad \det\left(C\psi + D\tilde{\psi}\right) \;=\; \tilde{q},$$

where ψ, $\tilde{\psi}$ form a basis for $H^1(X, \mathcal{E}^*)$. As a consistency check, note that in the special case that $\mathcal{E} = TX$, the relations above reduce to

$$\psi^2 \;=\; q, \;\; \tilde{\psi}^2 \;=\; \tilde{q}$$

duplicating the ordinary quantum cohomology ring of $\mathbb{P}^1 \times \mathbb{P}^1$.

2.5 General Hirzebruch surfaces

Let us now outline a similar computation for more general Hirzebruch surfaces $\mathbb{F}_n$. Describe such a surface as a quotient of the homogeneous coordinates u, v, s, t (corresponding to the four toric divisors) by two $\mathbf{C}^\times$ actions with the following weights:

u	v	s	t
1	1	0	n
0	0	1	1

Without loss of generality, we assume $n \geq 0$. Describe a deformation of the tangent bundle as the cokernel

$$0 \;\longrightarrow\; \mathcal{O} \oplus \mathcal{O} \;\stackrel{E}{\longrightarrow}\; \mathcal{O}(1,0)^{\oplus 2} \oplus \mathcal{O}(0,1) \oplus \mathcal{O}(n,1) \;\longrightarrow\; \mathcal{E} \;\longrightarrow\; 0,$$

where

$$E \;=\; \begin{bmatrix} Ax & Bx \\ \gamma_1 s & \gamma_2 s \\ \alpha_1 t + s f_1(u,v) & \alpha_2 t + s f_2(u,v) \end{bmatrix},$$

with

$$x \equiv \begin{bmatrix} u \\ v \end{bmatrix},$$

A, B constant 2×2 matrices, γ_1, γ_2, α_1, α_2 constants, and $f_{1,2}(u,v)$ homogeneous polynomials of degree n. In particular when $n > 0$, the polynomials f_i define "nonlinear" deformations, in the sense that they define nonlinear entries in E.

Let us define:

$$Q_{K1} = \det\left(\psi A + \tilde{\psi} B\right), \qquad Q_s = \psi\gamma_1 + \tilde{\psi}\gamma_2, \qquad Q_t = \psi\alpha_1 + \tilde{\psi}\alpha_2, \tag{10}$$

following the nomenclature used in [18, 19]. The Hirzebruch surface has two primitive collections, with corresponding Q's: Q_{K1} and Q_sQ_t. Following the general procedure described earlier, the quantum sheaf cohomology ring is given by $\mathbb{C}[\psi, \tilde{\psi}]$ modulo the relations,

$$Q_{K1} = q_1 Q_s^n, \qquad Q_sQ_t = q_2. \tag{11}$$

Although the quantum sheaf cohomology relations above depend upon the "linear" parameters $(A, B, \gamma_{1,2}, \alpha_{1,2})$, they do not depend on the nonlinear contributions from the polynomials $f_{1,2}(u,v)$. In fact, this is a general feature of these toric quantum sheaf cohomology relations; namely that nonlinear deformations drop out of the quantum sheaf cohomology.

For completeness, let us also consider the special case where $\mathcal{E} = TX$ which is the (2,2) locus. This special case is described by

$$A = I, \qquad B = 0, \qquad \gamma_1 = 0, \qquad \gamma_2 = 1, \qquad \alpha_1 = n, \qquad \alpha_2 = 1, \qquad f_1 = f_2 = 0,$$

so that

$$Q_{K1} = \psi^2, \quad Q_s = \tilde{\psi}, \quad Q_t = n\psi + \tilde{\psi}.$$

In this special case, if we identify $D_u = \psi$, $D_s = \tilde{\psi}$, then the quantum sheaf cohomology ring reduces to

$$D_u^2 = q_1 D_s^n, \qquad D_s(nD_u + D_s) = q_2,$$

which, for example, reduces to the classical cohomology ring relations when $q_1, q_2 \to 0$.

3 Mirror Maps

In the preceding section, we described the A/2 model and the (0,2) analogue of curve counting encoded in quantum sheaf cohomology. With those preliminaries in place, we turn to the mirror

map for deformations of the tangent bundle.

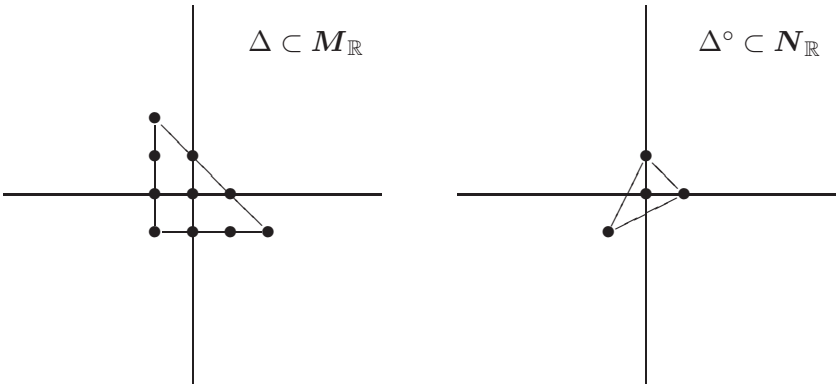
Since (2,2) mirror symmetry is a basic tool in our current understanding of more general (0,2) phenomena, we begin with a brief review of the (2,2) case. As we will see, many of the subtleties of the (0,2) story are already familiar from the better-understood (2,2) setting; however, there are also important complications that are intrinsically (0,2) in nature.

3.1 (2,2) mirror symmetry à la Batyrev

We begin by reviewing what is perhaps the best-known general construction of mirror pairs X and X° [8, 16]. Consider a d-dimensional lattice polytope Δ containing the origin in a d-dimensional lattice $\boldsymbol{M} \subset \boldsymbol{M}_{\mathbb{R}} \sim \mathbb{R}^d$. Let $\boldsymbol{N} \subset \boldsymbol{N}_{\mathbb{R}}$ be the dual lattice, and denote the natural pairing $\boldsymbol{M}_{\mathbb{R}} \times \boldsymbol{N}_{\mathbb{R}}$ by $\langle \cdot, \cdot \rangle$. The dual polytope $\Delta^\circ \subset \boldsymbol{N}_{\mathbb{R}}$ is defined by

$$\Delta^\circ = \{ y \in \boldsymbol{N}_{\mathbb{R}} | \langle x, y \rangle \geq -1 \ \forall x \in \Delta \}.$$

Δ is reflexive iff Δ° is also a lattice polytope; note that Δ° is reflexive iff Δ is reflexive. A familiar $d = 2$ example is given in (12)



(12)

The polytope Δ may be thought of as the Newton polytope for a hypersurface $\{P = 0\} \in (\mathbb{C}^*)^d$ which has a natural compactification to a subvariety $X = \overline{\{P = 0\}}$ in a toric variety V with fan $\Sigma_V \subset \boldsymbol{N}_{\mathbb{R}}$, given by taking cones over faces of Δ°. When Δ is reflexive then $X \subset V$ is a Calabi-Yau hypersurface with "suitably mild" Gorenstein singularities; for instance, when $d = 4$ the generic hypersurface is smooth.

This construction has two notable virtues. First, it gives a combinatorial condition (i.e. reflexivity) for constructing many examples of smooth Calabi-Yau three-folds; this was used to great effect in [31] to produce the largest know set of such manifolds. Second, since the reflexive polytopes occur in pairs, there is a simple conjecture to produce the mirror of X: we just construct the dual hypersurface $X^\circ \subset V^\circ$ by interpreting Δ° as the Newton polytope for the hypersuface and Δ as defining the toric variety V°. Mirror symmetry predicts the equality of Hodge numbers $h^{1,1}(X) = h^{1,2}(X^\circ)$ and $h^{1,2}(X) = h^{1,1}(X^\circ)$, and the conjectured pairing passes this important test.

The equality of the Hodge numbers is a first step in matching the moduli spaces of deformations of the Calabi-Yau manifolds and the corresponding (2,2) conformal field theories. Recall that at a generic point, the moduli space associated to X is a product of two special Kähler manifolds, the complex structure moduli space $\mathcal{M}_{cx}(X)$ and the complexified Kähler moduli space $\mathcal{M}_{cK}(X)$. The tangent spaces of these manifolds are canonically identified with infinitesimal deformations:

$$T_{\mathcal{M}_{cx}} \simeq H^1(X, T_X), \qquad T_{\mathcal{M}_{cK}} \simeq H^1(X, T_X^*).$$

The special Kähler metrics are determined from two holomorphic prepotentials $\mathcal{F}_{cx}(X)$ and $\mathcal{G}_{cK}(X)$, and the mirror map $\mathcal{M}_{cK}(X) \to \mathcal{M}_{cx}(X^\circ)$ is an isomorphism of the moduli spaces as special Kähler manifolds. The resulting relation $\mathcal{G}_{cK}(X) = \mu^* \mathcal{F}_{cx}(X^\circ)$ leads to the celebrated relations between Gromov-Witten invariants of X encoded in $\mathcal{G}_{cK}(X)$, and the variations of Hodge structure on the mirror X° encoded by $\mathcal{F}_{cx}(X^\circ)$.

The monomial-divisor mirror map

The equality of Hodge numbers and the mirror isomorphism of the moduli spaces receives an important refinement in the context of Batyrev mirror pairs $X \subset V$ and $X^\circ \subset V^\circ$.

A simple set of complex structure deformations of X is obtained by considering variations of the defining hypersurface modulo automorphisms of the ambient toric variety. The resulting subspace of "polynomial" deformations has complex dimension

$$h^{1,2}_{\text{poly}} = \ell(\Delta) - d - 1 - \sum_{\varphi} \ell^*(\varphi),$$

where ℓ counts the number of lattice points contained in a closed subset, φ is a facet of Δ, and ℓ^* counts the number of lattice points in the relative interior of the indicated closed subset. We can understand this number as follows: $\ell(\Delta)$ is the number of monomials μ in the defining polynomial P; the group of connected automorphisms of V contains the $(\mathbb{C}^*)^d$ action which can be used to rescale d of the coefficients to, say, 1, and of course an overall rescaling of P does not affect X; finally, there are additional automorphisms of V that can be used to set the coefficients of monomials $\mu \in \text{relint}(\phi)$ to zero.

Similarly, there is a simple way to obtain a subset of complexified Kähler deformations by taking the classes dual to the "toric divisors" on X, i.e. those obtained by pulling back toric divisors from the ambient space V. These are counted by

$$h^{1,1}_{\text{toric}} = \ell(\Delta^\circ) - d - 1 - \sum_{\varphi^\circ} \ell^*(\varphi). \tag{13}$$

This count also has a simple interpretation: the first three terms count the toric divisors on V if we take the one-dimensional cones to be all lattice points in $\Delta \backslash \{0\}$; we subtract the last term since a toric divisor in relint φ° does not intersect a generic hypersurface.

In general, in addition to these "simple" deformations, a variety $X \subset V$ has both non-polynomial complex structure deformations, and non-toric deformations of complexified Kähler structure. Remarkably, however, $h^{1,1}_{\text{toric}}(X) = h^{1,2}_{\text{poly}}(X^\circ)$! It is then natural to conjecture a restricted isomorphism

$$\mathcal{M}^{\text{toric}}_{cK}(X) = \mathcal{M}^{\text{poly}}_{cx}(X^\circ).$$

As we will review in more detail below, this leads to the Monomial-Divisor Mirror Map (MDMM) [3].

The (2,2) gauged linear sigma model

The MDMM isomorphism is particularly natural in the context of the Gauged Linear Sigma Model (GLSM) construction [39, 50]. A (2,2) GSLM is a two-dimensional abelian gauge theory with (2,2) supersymmetry that can be constructed from the data of $X \subset V$. The theory naturally incorporates the two sets of deformations into two holomorphic superpotentials W and $\widetilde{W}$; the former encodes the complex coefficients of P, while the latter encodes the complexified Kähler deformations of the ambient space V.

While the GLSM is not a conformal field theory, it is believed to reduce to an appropriate (2,2) conformal field theory at low energies. Of course not all naive parameters contained in W and $\widetilde{W}$ correspond to genuine deformations of the conformal field theory. In fact, holomorphic field redefinitions that do not affect the low energy theory can be used to reduce the deformations in W to the $h^{1,2}_{\text{poly}}(M)$ deformations described above [30]. These GLSM parameters yield algebraic (as opposed to special Kähler) coordinates on the moduli space; in terms of these algebraic coordinates the MDMM takes a canonical form.

Furthermore, the GLSM admits the A- and B-topological twists which correspond to the A- and B-model topological subsectors of the conformal field theory [49]. By using the methods of toric residues, combined with summing the instantons in GLSMs it is possible to show that the MDMM indeed exchanges the observables of the A- and B-models [9, 39, 46]. This is one of the most convincing and important tests of the mirror correspondence.[1]

Correlators and the discriminant locus for the quintic

To illustrate some salient features, we review the GLSM presentation [39] of the original mirror computation for the quintic in algebraic coordinates [15]. In this case $h^{1,1}(X) = 1$, and the A-model depends on one complexified Kähler parameter q. The GLSM contains an operator σ that corresponds to the infinitesimal deformation $H^1(X, T^*_X)$, and the single A-model correlator is given by

$$\langle \sigma^3 \rangle_A : \text{Sym}^3 H^1(X, T^*_X) \to \mathbb{C} \qquad \langle \sigma^3 \rangle_A = \frac{5}{1 + 5^5 q}. \tag{14}$$

[1] These results have also been generalized to complete intersections in toric varieties [14, 27].

The mirror quintic is defined is a hypersuface in $\mathbb{P}^4/\mathbb{Z}_5^3$ with

$$P^\circ = Z_0^5 + Z_1^5 + Z_2^5 + Z_3^5 + Z_4^5 - 5\psi Z_0 Z_1 Z_2 Z_3 Z_4.$$

We used the $(\mathbb{C}^*)^5$ rescalings to bring P° into a canonical form; the parameter ψ is a coordinate on (a five-fold cover) of the complex structure moduli space $\mathcal{M}_{cx}(X^\circ)$, and the mirror GLSM contains an operator μ that corresponds to the infinitesimal deformation $H^1(X^\circ, T_{X^\circ})$. Up to a ψ-independent constant, the B-model correlator is given by

$$\langle \mu^3 \rangle_B : \mathrm{Sym}^3 H^1(X^\circ, T_{X^\circ}) \to \mathbb{C} \qquad \langle \mu^3 \rangle_B = \frac{5^4 \psi^2}{1-\psi^5}. \tag{15}$$

The monomial-divisor mirror map in this case reads $q = (-5\psi)^{-5}$, and it exchanges the two correlators provided we identify the deformations as $\sigma \sim q\frac{\partial}{\partial q}$ and $\mu \sim \psi\frac{\partial}{\partial \psi}$.[2] These transformations are a consequence of topological descent in the A and B models [17]. Note also that the correlators diverge at $q = -5^{-5}$ or, equivalently, at $\psi^5 = 1$. The B-model divergence at $\psi^5 = 1$ is easy to see: this is exactly the discriminant locus where the hypersurface $P^\circ = 0$ is singular. The A-model divergence at $q = -5^{-5}$ is due to a divergent sum over the GLSM instantons. In more general theories the discriminant locus has many components, but one can show that the MDMM exchanges the discriminant loci of a pair of mirror theories.

Redundant deformations and plain polytopes

The presentation of the toric and polynomial deformations is complicated by the "redundant" monomials corresponding to points in relint(φ) and the "redundant" divisors corresponding to points in relint(φ°). In fact, in the context of topological field theory and GLSM computations, while it is relatively easy to see that field redefinitions can be used to eliminate the redundant monomials, it is not so simple to understand the decoupling of deformations corresponding to the redundant divisors.[3] Thus, computations in the (2,2) setting become simpler if these redundant monomials and divisors are absent. To quantify this absence, we say a polytope Δ is plain if none of its facets contains an interior lattice point. Thus, in (12) Δ° is plain, while Δ is not plain. A reflexive polytope Δ is reflexively plain if both it and its dual are plain. In two dimensions there is a single self-dual reflexively plain polytope:

The 473,800,776 reflexive polytopes in $d = 4$ have 6,677,743 reflexively plain non-self-dual pairs and 5,518 self-dual reflexively plain polytopes [30]. A simple example of a $d = 4$ reflexively plain

[2] This exchange is again up to an overall independent constant; the ambiguity can be resolved by working with special coordinates and physical Yukawa couplings [15].

[3] It is trivial at the level of classical geometry; however, showing it at the level of GLSM gauge instantons is more involved [30].

pair has vertices

$$\Delta : \begin{pmatrix} 1 & 0 & 2 & 3 & -6 \\ 0 & 1 & 4 & 3 & -8 \\ 0 & 0 & 5 & 0 & -5 \\ 0 & 0 & 0 & 5 & -5 \end{pmatrix}, \qquad \Delta^\circ : \begin{pmatrix} -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 2 \\ -1 & -1 & 2 & 1 \\ -1 & 4 & -3 & -2 \\ 4 & -1 & -1 & -2 \end{pmatrix}. \tag{16}$$

Δ has a total of 26 lattice points, while Δ° has no additional non-zero lattice points. The corresponding three-fold is the $\mathbb{Z}_5$ quotient of the quintic in $\mathbb{CP}^4$ with

$$h^{1,1}(X) = h^{1,1}_{\text{toric}}(X) = 1 \quad \text{and} \quad h^{1,2}(X) = h^{1,2}_{\text{poly}}(X) = 21.$$

3.2 (0,2) Gauged linear sigma models and a mirror map

Having reviewed some basic structure of (2,2) mirror symmetry, we are now ready to discuss the (0,2) deformations and a possible mirror map. The first naive guess is to start with the pair (X, T_X) and (X°, T_{X°) and try to match deformations of T_X and T_{X° as holomorphic bundles. This runs into two problems. The first is familiar from classical algebraic geometry. Unlike the first order deformations $H^1(X, T_X)$ and $H^1(X, T_X^*)$, which can be integrated to finite deformations of complex and Kähler structure, respectively, the infinitesimal deformations of the bundle, characterized by $H^1(X, \operatorname{End} T_X)$ can have higher order obstructions. The second issue has to do with quantum obstructions. In general, a (0,2) supersymmetry-preserving deformation of a (2,2) theory need not preserve conformal invariance, so that turning on a classically unobstructed deformation in $H^1(X, \operatorname{End} T_X)$ can ruin the structure of the conformal field theory. For instance, in the case of the famous "Z-manifold," i.e. the resolution of $T^6/\mathbb{Z}_3$ with $h^{1,1} = 36$ and $h^{1,2} = 0$ it is known that $h^1(X, \operatorname{End} T_X) = 208$, but 108 of these are obstructed at first order by worldsheet instanton effects [5].

(0,2) GLSM deformations

Fortunately, the GLSM construction helps with both the classical and quantum obstructions. The original observation, going back to [50], is that the (2,2) GLSM Lagrangian, viewed as a (0,2) theory, has holomorphic (0,2) deformation parameters encoded in the following complex of sheaves on X.

$$0 \longrightarrow \mathcal{O}^r|_X \xrightarrow{E} \oplus_\rho \mathcal{O}(D_\rho)|_X \xrightarrow{J} \mathcal{O}(\textstyle\sum_\rho D_\rho)|_X \longrightarrow 0\,. \tag{17}$$

Here the ambient toric variety V is presented as a holomorphic quotient $\{\mathbb{C}^n \setminus F\}/(\mathbb{C}^*)^r$, and the D_ρ are the toric divisors on V. The cohomology of this complex, $\mathcal{E} = \ker J/\operatorname{im} E$, defines a rank $d-1$ holomorphic bundle over X that is a deformation of the tangent bundle T_X. The maps E and J have a simple form on the (2,2) (i.e. $\mathcal{E} = T_X$) locus. The J are the differentials of the defining

equation, dP, while E is the familiar map from the Euler sequence for the tangent bundle of the toric variety:

$$0 \longrightarrow \mathcal{O}^r \xrightarrow{E} \oplus_\rho \mathcal{O}(D_\rho) \longrightarrow T_V \longrightarrow 0 \ . \tag{18}$$

Just as the complex coefficients of the defining equation over-parametrize the space of polynomial complex structure deformations, so do the coefficients in E and J over-parametrize the space of "monadic" bundle deformations because of various automorphisms of the complex [30]. However, these can be taken into account and a combinatorial formula, akin to (13), gives the total number of monadic deformations.

For instance, for the Fermat quintic in $\mathbb{CP}^4$ we have

$$r = 1, \qquad E = (Z_0, Z_1, Z_2, Z_3, Z_4)^T, \qquad J = (Z_0^4, Z_1^4, Z_2^4, Z_3^4, Z_4^4).$$

Since $\sum_\rho E^\rho J_\rho = 5P$ this is indeed a complex of sheaves on X. By identifying the automorphisms of the complex, we find that the monadic deformations yield a 224-dimensional space of deformations. Since that is exactly $h^1(X, \mathrm{End}\, T_X)$ in this case, we see that all infintesimal deformations of the quintic's tangent bundle are unobstructed.

Given this complex, we have a simple way of obtaining a family of sheaves $\mathcal{E}$ over X by deforming the defining maps E and J in (17) while preserving $J \circ E|_X = 0$. Moreover, there exists a general set of arguments that holomorphic deformations of a GLSM are protected from worldsheet instanton destabilization [6, 10, 44]. In particular, the work of [10] gives a method that can be used to show that toric worldsheet instantons do not destabilize a (0,2) GLSM. It should be stressed that this result has not yet been formulated as a general vanishing theorem, and in principle it must be checked example by example; however, the structure of the argument suggests that a general formulation may be possible. At any rate, for many models the argument is indeed sufficient.

In general the holomorphic parameters of the (0,2) GLSM cannot describe all of the bundle deformations; this should not surprise us, since even on the (2,2) locus the GLSM only captures the toric Kähler and polynomial complex structure deformations. The "monadic" bundle deformations encoded in (17) do, however, offer a simple parametrization of a subset of unobstructed (0,2) deformations. They have been the focus (and the secret of success) for much recent work in (0,2) theories.

Note that the bundle complex depends on the complex structure of X. This illustrates an important point: in general there is no invariant way to split (0,2) deformations into the sort of canonical form familiar from the (2,2) context. While it is known that the full (0,2) moduli space must be Kähler [41], it need not have any canonical product form akin to that familiar from the (2,2) context, nor does it need to admit a special Kähler metric.

The quintic once again provides a simple example of the general structure. As we mentioned above, $h^1(X, \mathrm{End}\, T_X) = 224$ for the quintic, and all of these deformations are classically unobstructed. Moreover, all of them can be represented by deforming the E and J maps, and the

method of [10] immediately shows that there are no quantum obstructions either. Thus the full (0,2) quintic moduli space has dimension

$$h^1(X, T_X^*) + h^1(X, T_X) + h^1(X, \mathrm{End}\, T_X) = 326.$$

In the context of mirror symmetry it is natural to ask how to represent these deformations in the GLSM for the mirror quintic. Are the 224 (0,2) deformations representable by deforming the mirror complex? Unfortunately, the answer is no. The mirror of (17) only yields 164 deformations, so that 60 additional deformations remain unaccounted [30].

Of course this is not a failure of mirror symmetry but merely a difficulty in presenting the deformations in a useful fashion. Although to our knowledge the computation of $h^1(X^\circ, \mathrm{End}\, T_{X^\circ})$ has not been carried out, the conformal field theory mirror isomorphism on the (2,2) locus implies that this should be 224 as well; moreover, all of these should be unobstructed, both at the classical and quantum levels. However, studying these deformations and any possible mirror map abstractly appears to be rather difficult.

It was observed in [30] that this mismatch of (0,2) GLSM deformations for Calabi-Yau hypersurfaces in toric varieties is a rather general feature among the reflexive polytope pairs. This should be contrasted with the situation on the (2,2) locus, where the matching of GLSM parameters to those of the mirror GLSM corresponds to matching $\mathcal{M}_{cK}^{\mathrm{toric}}(X)$ to $\mathcal{M}_{cx}^{\mathrm{poly}}(X^\circ)$. However, it was also observed that precisely for the case of reflexively plain polytope pairs the number of GLSM (0,2) deformations matched as well. For instance, in the case of the hypersurface in (16) there are 39 (0,2) "monadic" deformations, giving a deformation space of dimension 66 on both sides of the mirror.

A (0,2) mirror map

A pair of GLSMs associated to a reflexively plain polytope and its dual is a natural candidate for finding an explicit mirror map for the full GLSM moduli space. The inspiration for this comes directly from the monomial-divisor mirror map [3], where the equality $h^{1,1}_{\mathrm{toric}}(X) = h^{1,2}_{\mathrm{poly}}(X^\circ)$, combined with the action of the toric automorphism group $\mathrm{Aut}(V)$ as encoded by the combinatorial structure of Δ, yielded a simple Ansatz for the map. The analogue (0,2) analysis was performed in [37], yielding a concrete proposal for a (0,2) mirror map. We will now review that result, but to present it we will need to introduce a little more combinatorial structure.

Let Δ, Δ° be a d-dimensional reflexively plain pair as above. Let Σ_V be a maximal projective subdivision, so that one-dimensional cones $\rho \in \Sigma_V(1)$ are in one-to-one correspondence with the n non-zero lattice points in Δ°. Let $\mathbb{C}[Z_1, \ldots, Z_n]$ denote the homogeneous Cox ring for V, with the

$(\mathbb{C}^*)^r$ action given by[4]

$$(t_1,\ldots,t_r)\cdot Z_\rho \to Z_\rho \times \prod_{a=1}^{r} t_a^{Q_\rho^a}. \tag{19}$$

In terms of these homogeneous coordinates, the hypersurface takes the form

$$P = \sum_{m\in\Delta\cap\boldsymbol{M}} \alpha_m \mathsf{M}_m, \quad \text{with} \qquad \mathsf{M}_m \equiv \prod_\rho Z_\rho^{\langle m,\rho\rangle+1},$$

where α_m are the complex coefficients of the monomials M_m. To simplify some manipulations we will take $\alpha_m \neq 0$. It is useful to introduce the rank d matrix

$$\pi_{m\rho} \equiv \langle m,\rho\rangle \quad \text{for} \quad m\in\{\Delta\cap\boldsymbol{M}\backslash 0\}.$$

The charges Q_ρ^a form an integral basis for the r-dimensional kernel of π; similarly, the integral basis $\widehat{Q}_m^{\widehat{a}}$ for the cokernel of π define the $(\mathbb{C}^*)^{r^\circ}$ action in the mirror toric variety

$$V^\circ = \{C^{n^\circ}\backslash F^\circ\}/(\mathbb{C}^*)^{r^\circ}.$$

Since we will assume that Σ_{V° is also a maximal projective subdivision, $n^\circ = \ell(\Delta)-1$ and $r^\circ = \ell(\Delta)-1-d$ in the reflexively plain case.

The maps of (17) are given by

$$E^{a\rho} = e^{a\rho} Z_\rho, \qquad Z_\rho J_\rho \;=\; \sum_{m\in\Delta\cap\boldsymbol{M}} j_{m\rho}\mathsf{M}_m$$

with $j_{m\rho}=0$ whenever $\langle m,\rho\rangle = -1$. On the (2,2) locus we have

$$E^{a\rho} = Q_\rho^a \quad \text{and} \quad j_{m\rho} \;=\; (\langle m,\rho\rangle+1)\alpha_m.$$

In order for (17) to be a complex we require,

$$\sum_\rho J_\rho(Z)E^{a\rho}(Z) + \delta^a P(Z) = 0, \tag{20}$$

for some complex coefficients δ^a. Note that since P transforms equivariantly under the torus action (19), the constraint holds automatically on the (2,2) locus with $\delta^a = -\sum_\rho Q_\rho^a$.

In addition to these parameters, the (0,2) GLSM also depends on the q_a — the r complexified Kähler parameters. As we described in the discussion of (2,2) theories, these parameters occur in orbits corresponding to holomorphic field redefinitions of the GLSM; with the exception of the action on the q_a (a feature that only shows up in (0,2) theories), the action on the parameters

[4] In general $V = \{\mathbb{C}^n \;\backslash\; F\}/G$, where $G = (\mathbb{C}^*)^r \times H$ and H is a discrete abelian group; as we will focus on automorphisms connected to the identity, H will not play an important role in what follows.

arises via automorphisms of the toric variety and induced automorphisms on (17). By examining the action of redefinitions on the parameter space, we can construct a natural set of invariants from the q_a, α_m, $j_{m\rho}$, $e^{a\rho}$ and δ^a that satisfy the constraint (20):

$$\kappa_a \equiv q_a \prod_\rho \left(\frac{j_{0\rho}}{\alpha_0}\right)^{Q_\rho^a}, \quad \widehat{\kappa}_{\widehat{a}} \equiv \prod_{m \in \{\Delta \cap \boldsymbol{M} \setminus 0\}} \left(\frac{\alpha_m}{\alpha_0}\right)^{\widehat{Q}_m^{\widehat{a}}}, \quad b_{m\rho} \equiv \frac{\alpha_0 j_{m\rho}}{\alpha_m j_{0\rho}} - 1, \quad \text{for} \quad m \neq 0. \tag{21}$$

In order for this data to define a non-singular theory, $b_{m\rho}$ must have rank exactly d.

On the (2,2) locus $b_{m\rho} = \pi_{m\rho}$, $\kappa_a = q_a$, and the $\widehat{\kappa}_{\widehat{a}}$ are the usual invariant algebraic coordinates on $\mathcal{M}_{cx}^{\rm poly}(X)$. The MDMM then takes a simple form: we repeat the construction of invariants for $X^\circ = \{P^\circ = 0\} \subset V^\circ$, which leads to coordinates $\kappa_{\widehat{a}}^\circ$ and $\widehat{\kappa}_a^\circ$. The map is then

$$\kappa_a = \widehat{\kappa}_a^\circ, \qquad \widehat{\kappa}_{\widehat{a}} = \kappa_{\widehat{a}}^\circ. \tag{22}$$

The proposed (0,2) extension is simple: the rank d matrix $b_{\rho m}^\circ$ of the mirror theory is just the transpose of b.

Testing the proposal

How can we test the proposed (0,2) mirror map? The most direct way would be to study the analogue of the A- and B- model correlators as in (14) and (15). As discussed in section 2, the analogous correlators certainly exist, see *e.g.* [1, 2, 18, 19, 28, 34, 42] in the A/2 and B/2 theories; when T_X is deformed to $\mathcal{E}$, they correspond to multilinear maps

$$A/2 \ : \ \mathrm{Sym}^{d-1} H^1(X, \mathcal{E}^*) \to \mathbb{C}, \qquad B/2 \ : \ \mathrm{Sym}^{d-1} H^1(X, \mathcal{E}) \to \mathbb{C}.$$

However, the computations remain difficult, and general techniques remain fairly undeveloped on the B/2-model side.

Although the full correlators remain out of reach, it turns out to be possible to identify an important component of the singular locus — the subvariety in the moduli space where the theory becomes singular. How do these singularities arise? In the A/2 model they come from diverging sums over the GLSM gauge instantons, while in the B/2 model they have a classical manifestation: as we change parameters in the J maps the sheaf $\mathcal{E}$ defined by (17) can fail to be a vector bundle by developing singularities.

As in the case of singularities in complex structure moduli space [26], where P fails to be transverse in V, the resulting discriminant locus is in general reducible. The principal component of the discriminant locus corresponds to degenerations that occur in $X \cap (\mathbb{C}^*)^d \subset V$, and it is characterized in the following fashion [37]. Let $\widehat{\gamma}_{\widehat{a}}^m$ parametrize the cokernel of $b_{m\rho}$ and let $\widehat{\delta}^m \equiv -\sum_{\widehat{a}} \widehat{\gamma}_{\widehat{a}}^m$. The bundle $\mathcal{E}$ is then singular at some point in $X \cap (\mathbb{C}^*)^d \subset V$ if and only if there exists

a vector $\widehat{\sigma} \in \mathbb{C}^{r^\circ}$ satisfying

$$\prod_{m \in \{\Delta \cap \boldsymbol{M} \setminus 0\}} \left[\frac{\widehat{\sigma} \cdot \widehat{\gamma}^m}{\widehat{\sigma} \cdot \widehat{\delta}} \right]^{\widehat{Q}^{\widehat{a}}_m} = \widehat{\kappa}_{\widehat{a}},$$

where $\widehat{\sigma} \cdot \widehat{\gamma}^m \equiv \sum_{\widehat{a}} \widehat{\sigma}^{\widehat{a}} \widehat{\gamma}^m_{\widehat{a}}$. For the A/2 model the principal component of the discriminant locus was computed in [34]. The result, recast in the invariant coordinates, is that there must exist a vector $\sigma \in \mathbb{C}^r$ satisfying

$$\prod_{\rho} \left[\frac{\sigma \cdot \gamma^\rho}{\sigma \cdot \delta} \right]^{Q^a_\rho} = \kappa_a,$$

where γ^ρ_a form a basis for the kernel of $b_{m\rho}$ and $\delta^\rho \equiv -\sum_a \gamma^\rho_a$. It is now evident that under the proposed mirror map the principal component of the discriminant of the A/2 (B/2) model will be mapped to the principal component of the discriminant of the mirror B$^\circ$/2 (A$^\circ$/2) model. This non-trivial check constitutes the best to-date evidence for the (0,2) mirror map in reflexively plain examples.

4 Worldsheet Duality

In the previous section, we decribed a proposal for a (0,2) monomial-divisor mirror map, at least for reflexively plain polytopes and bundles that are deformations of the tangent bundle. However, we did not discuss (0,2) mirrors for non-Calabi-Yau spaces or models without a (2,2) locus, nor did we give any attempt at a physical derivation.

For (2,2) mirror symmetry, there have been very interesting attempts to derive mirror symmetry from a GLSM construction in [40], and from worldsheet duality in [25]. The latter approach provides a nice construction of a mirror description of a GLSM with a toric target space, including non-compact toric Calabi-Yau spaces. It is one of the major open questions of both (2,2) and (0,2) mirror symmetry to find a derivation that applies to compact Calabi-Yau spaces.

Reducing the worldsheet supersymmetry from (2,2) to (0,2) greatly enriches both the physical space of theories and their associated mathematical structures, but at the expense of reduced control over the quantum dynamics of these models. Fortunately, $(0,2)$ supersymmetry still provides sufficient control that we can define and compute rings of observables. These rings define quantum sheaf cohomology discussed in section 2. In this section, we will explain how to construct mirror pairs of $(0,2)$ models using worldsheet duality [1], generalizing the approach used in the (2,2) setting by [25].

4.1 The basic idea

We will consider $(0,2)$ gauged linear sigma models discussed in section 3. We again restrict to abelian gauge groups, but we will allow bundles which are not necessarily deformations of the tangent bundle of the target space toric variety V. Indeed, even the rank of $\mathcal{E}$ can differ from the rank of T_V. Since the construction is a worldsheet duality, we will need to introduce some notions

from superspace to describe the worldsheet theories. Our conventions can be found in Appendix A. The basic ingredients of such a theory are a collection of $U(1)^k$ gauge-fields and coupled charged matter. The gauge-fields reside in fermionic superfields Υ^a with $a = 1, \ldots, k$, while the matter resides in chiral multiplets Φ^i with charges Q_i^a. Under a gauge transformation with parameters Λ^a,

$$\Phi^i \to e^{i\Lambda^a Q_i^a} \Phi^i. \tag{23}$$

In the most commonly studied $(0,2)$ models, this data is sufficient to determine the target space geometry which is the toric variety V.

The basic idea of abelian duality is to implement T-duality along a $U(1)$ isometry direction of the target space. We can see how this duality works in a simple case. Consider a free theory consisting of a circle-valued scalar field ϕ with period 2π. Instead of writing the action in the most straightforward way, introduce a Lagrange multiplier 1-form A and consider the two-dimensional action:

$$S = \frac{1}{4\pi R^2} \int A \wedge *A - \frac{i}{2\pi} \int \phi \, dA. \tag{24}$$

Integrating out A in a quadratic theory like this amounts to solving the classical A equation of motion,

$$A = -iR^2 * d\phi, \tag{25}$$

which gives the free action for a scalar field on a circle of size R:

$$S = \frac{R^2}{4\pi} \int (\partial\phi)^2. \tag{26}$$

On the other hand, integrating out ϕ imposes the constraint $dA = 0$, which we can solve via $A = d\widetilde{\phi}$ with $\widetilde{\phi}$ periodic with period 2π. The dual action is therefore,

$$S = \frac{1}{4\pi R^2} \int (\partial\widetilde{\phi})^2, \tag{27}$$

which describes a scalar field on a circle of size $1/R$. Note that there is no local relation between ϕ and $\widetilde{\phi}$. The map between the two descriptions is non-local and involves an exchange of momentum and winding modes.

We will slightly generalize this procedure and separately dualize the $U(1)$ action acting on the phase of each chiral superfield Φ^i. The $U(1)$ action acting on the phase is not free. We will also gauge k combinations of these $U(1)$ actions. The existence of a fixed point for each $U(1)$ is reflected in the dual description in two ways: there is a non-perturbative term in the superpotential as well as a non-trivial dilaton field. This dualization procedure is a kind of worldsheet analogue of the SYZ proposal for constructing mirror Calabi-Yau spaces [45], except it can be applied to both

conformal and massive models with toric target spaces V.

Now let us introduce additional data determining the target space gauge bundle. Abelian duality is best understood in models without a tree-level superpotential so we set $J = 0$.[5] As noted above, implementing duality with $J \neq 0$ is perhaps the central question in understanding mirror symmetry for compact Calabi-Yau spaces. In this setting, the gauge bundle is encoded in a choice of left-moving fermionic chiral superfields Γ^A satisfying

$$\bar{\mathfrak{D}}_+\Gamma^A = \sqrt{2}E^A(\Phi), \tag{28}$$

with gauge charges Q_A^a. Freedom from worldsheet gauge anomalies imposes a quadratic relation on the gauge charges,

$$\sum_i Q_i^a Q_i^b - \sum_A Q_A^a Q_A^b = 0,$$

for each (a, b). The additional condition,

$$\sum_i Q_i^a = 0,$$

is required for conformal invariance at one-loop. These conditions guarantee that the infrared non-linear theory satisfies the anomaly cancelation condition (1) stated in the introduction.

The holomorphic E^A couplings define the maps which determine the holomorphic bundle over the toric variety V. For a $(2, 2)$ model, there is a single Fermi multiplet Γ for each chiral multiplet Φ. The associated bundle is the tangent bundle defined by the Euler sequence (18) if we choose

$$E^i = Q_i^a \Sigma^a \Phi^i$$

with each Σ^a a neutral chiral multiplet. For more general models with $E^A = \Sigma^a \widehat{E}^{Aa}(\Phi_i)$, the left-moving fermions define a monad bundle via the exact sequence

$$0 \longrightarrow \mathcal{O}^r \xrightarrow{E} \oplus_A \mathcal{O}(D_A) \longrightarrow \mathcal{E} \longrightarrow 0\ . \tag{29}$$

Note that even these choices of E^A are quite special from the perspective of a general $(0, 2)$ model. From a $(0, 2)$ perspective, there is no reason for a distinguished Σ^a multiplet at all! The Σ^a multiplet is really a vestige of $(2, 2)$ supersymmetry and the structure of a $(2, 2)$ vector multiplet, yet there is a rich class of theories of this type as we have already seen.

The starting data is therefore a collection of charged fields (Φ^i, Γ^A), a choice of complexified Kähler parameters $t^a = ir^a + \frac{\theta^a}{2\pi}$, and a choice of holomorphic bundle specified by the E^A. There are also gauge supermultiplets, Υ, and neutral chiral superfields, Σ, which come along for the ride.

[5] To be more precise: duality can currently be implemented to some degree with $E \neq 0$ or $J \neq 0$, but not with both couplings non-vanishing. The case of a compact Calabi-Yau space with a stable holomorphic bundle requires both sets of couplings.

The result of the abelian dualization procedure is a theory expressed in terms of neutral fields (Y^i, F^A) where

$$\mathrm{Im}(Y^i) \sim \mathrm{Im}(Y^i) + 2\pi, \qquad \mathrm{Re}(Y^i) \geq 0,$$

with Y^i a chiral superfield and F^A a neutral Fermi superfield. The dual theory is a $(0,2)$ Landau-Ginzburg theory for which holomorphic quantities are controlled by a fermionic superpotential:

$$L_W = \left(\int d\theta^+ W + \mathrm{h.c.} \right). \tag{30}$$

We will focus on the structure of this superpotential; the full Lagrangian can be found in [1]. Since W is fermionic, it takes the form

$$W = \Gamma \cdot J, \tag{31}$$

where Γ denotes a collection of Fermi superfields and J denotes a collection of functions of chiral superfields. Supersymmetric vacua are found by solving $J = 0$.

Fortunately, supersymmetry combined with R-symmetry is sufficient to determine the general form of the non-perturbative superpotential in models both with and without a $(2,2)$ locus. The exact dual superpotential is given by,

$$W = \sum_a \left(-\frac{i\Upsilon^a}{4} \left(\sum_i Q_i^a Y^i + i t^a \right) + \frac{\Sigma^a}{\sqrt{2}} \sum_A Q_A^a F^A \right) + \mu \sum_{iA} \beta_{iA} F^A e^{-Y^i}, \tag{32}$$

where we have made the mass scale μ explicit. The neutral Y^i fields, dual to the charged fields Φ^i, are axially coupled to the gauge-field. This first term of (32) proportional to $\Upsilon^a Q_i^a Y^i$ corresponds to a dynamical theta-angle, which reflects this axial coupling. The last term in (32) is an instanton-induced coupling. The dependence on the Kähler parameters t^a is explicit. The parameters β_{iA} are the interesting quantities corresponding to the choice of holomorphic gauge bundle.

Determining the map between the β_{iA} and the original E^A parameters is the most challenging step. The way the map has been understood so far is by working on the Coulomb branch of both the original and dual models where Σ^a has an expectation value, rather than the Higgs branch where the Φ^i have expectation values. In simple cases, matching the expressions for the effective potential for $W(\Sigma, \Upsilon)$ in both the original and dual descriptions determines the relation between some of the bundle deformation parameters and the β-parameters.

4.2 The duals of Hirzebruch surfaces

As a new example, let us consider a linear model with target space a Hirzebruch surface $\mathbb{F}_n$. The gauged linear sigma model exists for any n; however, a non-linear sigma model with target $\mathbb{F}_n$ is only asymptotically free for $n < 3$. The gauge theory with $\mathbb{F}_n$ as a moduli space has a $U(1) \times U(1)$

gauge group. We introduce four chiral superfields with charges $(1,0)$, $(0,1)$, $(1,0)$ and $(n,1)$. We can leave the holomorphic bundle unspecified for the moment. It is specified by a choice of Γ^A Fermi superfields in the gauge theory.

The mirror Landau-Ginzburg theory is then described in terms of four Y^i and a collection of neutral Fermi superfields F^A. The mirror superpotential is of the form given in (32). The ground state structure is determined as follows: first, we need to solve the two Υ constraints,

$$Y^1 + Y^3 + nY^4 + it^1 = 0, \qquad Y^2 + Y^4 + it^2 = 0, \tag{33}$$

which follow from setting $(\sum_i Q_i^a Y_i + it^a) = 0$ with $a = 1, 2$. The solution space is two-dimensional, and we can choose a convenient integral basis spanned by the two vectors $(1, 0, -1, 0)$, $(0, -1, -n, 1)$. In terms of this basis, we solve (33) in terms of $(Y, \widetilde{Y})$ by setting:

$$Y^1 = Y, \qquad Y^2 = -\widetilde{Y} - it^2, \qquad Y^3 = -n\widetilde{Y} - Y - it^1, \qquad Y^4 = \widetilde{Y}. \tag{34}$$

Integrating out the two massive Σ fields imposes two linear relations on the Fermi superfields F^A.[6] For simplicity, let us assume the gauge bundle is a deformation of the tangent bundle. In this case, we solve the constraints

$$F^1 + F^3 + nF^4 = 0, \qquad F^2 + F^4 = 0, \tag{35}$$

in terms of

$$F^1 = F, \qquad F^2 = -\widetilde{F}, \qquad F^3 = -n\widetilde{F} - F, \qquad F^4 = \widetilde{F}, \tag{36}$$

in parallel with the Y^i discussion. On the $(2,2)$ locus, $\beta_{iA} = -\frac{1}{\sqrt{2}}\delta_{iA}$. We will deform around this point. Expressed in terms of these fields, the resulting superpotential takes the form

$$W = \frac{\mu}{\sqrt{2}} F \left[e^{it^1 + Y + n\widetilde{Y}} - e^{-Y} \right] + \frac{\mu}{\sqrt{2}} \widetilde{F} \left[e^{it^2 + \widetilde{Y}} + n e^{it^1 + Y + n\widetilde{Y}} - e^{-\widetilde{Y}} \right] + \widetilde{W}, \tag{37}$$

where deformations away from the $(2,2)$ locus are captured in $\widetilde{W}$ with parameters $\widetilde{\alpha}$:

$$\begin{aligned} \widetilde{W} &= \frac{\mu}{\sqrt{2}} F \left[\widetilde{\alpha}_{11} e^{-Y} + \widetilde{\alpha}_{12} e^{it^1 + Y + n\widetilde{Y}} + \widetilde{\alpha}_{13} e^{-\widetilde{Y}} + \widetilde{\alpha}_{14} e^{it^2 + \widetilde{Y}} \right] \\ &+ \frac{\mu}{\sqrt{2}} \widetilde{F} \left[\widetilde{\alpha}_{21} e^{-Y} + \widetilde{\alpha}_{22} e^{it^1 + Y + n\widetilde{Y}} + \widetilde{\alpha}_{23} e^{-\widetilde{Y}} + \widetilde{\alpha}_{24} e^{it^2 + \widetilde{Y}} \right]. \end{aligned} \tag{38}$$

If all $\widetilde{\alpha} = 0$, we are on the $(2,2)$ locus. It is important to stress that we could have considered models with no connection to the tangent bundle. As long as the rank of the chosen holomorphic bundle, given by the sequence (29), is at least as large as the dimension of the target space, the

[6] It is worth noting that the Σ fields are not always massive. By tuning the E-couplings appropriately, one can find new branches in which combinations of the Σ fields becomes massless. The locus where such a branch meets a conventional Higgs branch usually corresponds to a bundle singularity with the rank of the bundle changing. There is a rich array of physical phenomena that can happen at these loci.

structure of the dual theory is still determined by a superpotential of the form (32). The form of the dual theory for models with $\mathrm{rk}(\mathcal{E}) < \mathrm{rk}(T_V)$ will be described in section 4.3.

To obtain the quantum sheaf cohomology ring for this case, let us introduce $\mathbb{C}$-valued fields:

$$Z = e^{-Y}, \qquad \widetilde{Z} = e^{-\widetilde{Y}}. \tag{39}$$

Solving the F and $\widetilde{F}$ constraints of (37) in terms of these variables gives the ring relations. If all $\widetilde{\alpha} = 0$, we find the quantum cohomology ring:

$$Z^2 \widetilde{Z}^n = e^{it^1}, \qquad \widetilde{Z}^2 - nZ\widetilde{Z} = e^{it^2}. \tag{40}$$

As a basic check on this structure, we can see if this ring reduces to the classical cohomology ring of a Hirzebruch surface. The classical limit is obtained by taking the Kähler classes of both fiber and base to infinity. This corresponds to the limit $e^{it^a} \to 0$ from which we see the classical cohomology ring of $\mathbb{F}_n$ emerge.

The more general quantum sheaf cohomology ring is given by solving the F and $\widetilde{F}$ constraints, including a non-vanishing $\widetilde{W}$ in (37). To make life simpler, let us set to zero the diagonal $\widetilde{\alpha}$ coefficients that correspond to rescalings of the $(2,2)$ couplings which already appear in (37):

$$\widetilde{\alpha}_{11} = \widetilde{\alpha}_{12} = 0, \qquad \widetilde{\alpha}_{22} = \widetilde{\alpha}_{23} = \widetilde{\alpha}_{24} = 0. \tag{41}$$

This leaves three deformation coefficients. The quantum sheaf cohomology ring takes the form,

$$Z^2 \widetilde{Z}^n - \widetilde{\alpha}_{13} Z \widetilde{Z}^{n+1} - \widetilde{\alpha}_{14} Z \widetilde{Z}^{n-1} e^{it^2} = e^{it^1}, \qquad \widetilde{Z}^2 - nZ\widetilde{Z} - \widetilde{\alpha}_{21} Z\widetilde{Z} = e^{it^2}. \tag{42}$$

This includes both a classical deformation of both relations which survives $e^{it^a} \to 0$, as well as a possible quantum deformation of the first relation proportional to e^{it^2}. These results should be contrasted with the analysis presented in section 2.5, noting that $q^a = e^{it^a}$.

4.3 Key differences between $(0,2)$ and $(2,2)$ models

There are significant differences in the structure of the mirror depending on how the rank of the bundle $\mathcal{E}$ compares with the rank of the tangent bundle of the target space V. These are uniquely $(0,2)$ phenomena with no analogue in the $(2,2)$ setting.

Models with $\mathrm{rk}(\mathcal{E}) < \mathrm{rk}(T_V)$

In these cases, the dual theory is not a Landau-Ginzburg theory at all! This can be seen from counting constraints. Imagine a dual model with N Y^i superfields and M F^A Fermi superfields

and r abelian gauge-fields. The vacuum structure is determined by solving the constraints

$$\sum_{i=1}^{N} Q_i^a Y^i = -it^a, \qquad \sum_{A=1}^{M} Q_A^a F^A = 0, \tag{43}$$

with $N > M$. We are left with $N - r$ Y variables, and $M - r$ Fermi superfields. A generic non-perturbative superpotential of the form $\mu \sum_{iA} \beta_{iA} F^A e^{-Y^i}$ imposes a further $M - r$ constraints on the Y fields. However, the potential now has flat directions corresponding to the excitations of the massless $N - M$ Y fields. The low-energy theory is not a Landau-Ginzburg theory with isolated vacua, but a non-linear sigma model with the vacuum manifold as a target space.

On examining the sigma model metric on this space, we find a finite distance singularity at $Y = 0$. This singularity signals a breakdown of worldsheet perturbation theory. On the singular locus, the dilaton of the dual theory diverges. This same phenomena is not uncommon in dualities relating minimal models and sigma models; for examples, see [20]. The physics of models in this class has yet to be deeply explored.

Models with $\mathrm{rk}(\mathcal{E}) \geq \mathrm{rk}(T_V)$

For the purpose of constructing realistic models of particle physics, these cases are the most interesting. The case of $\mathrm{rk}(\mathcal{E}) = \mathrm{rk}(T_V)$ is the most heavily studied class of examples; particularly the case of deformations of T_V. For models with $\mathrm{rk}(\mathcal{E}) > \mathrm{rk}(T_V)$, the dual theory is again a Landau-Ginzburg theory with a superpotential given by (32). However, counting constraints as above leads to the conclusion that there are generically no supersymmetric vacua, or only vacua at $Y^i \to \infty$ for all i. For models with all positive charges, there are generically no vacua at all. Supersymmetry is spontaneously broken in these models!

Dynamical supersymmetry breaking is a phenomenon that can happen in (0,2) models even with simple target spaces like $\mathbb{P}^N$. In the original frame, the physics leading to supersymmetry breaking is non-perturbative and is typically seen in a large N expansion. This is really just an illustration of the rich physics waiting to be understood in (0,2) field theories. In terms of field theory dynamics, (0,2) models are quite akin to $\mathcal{N} = 1$ four-dimensional gauge theories, while (2,2) models are similar to $\mathcal{N} = 2$ four-dimensional gauge theories. Supersymmetry breaking is just one example of this analogy. Of course, one can tune the E-couplings in the original model so that the dual theory has supersymmetric vacua. It would be very interesting to characterize the quite special bundles which actually preserve supersymmetry.

5 Current and Future Directions

The class of (0,2) theories we described offer a rich set of structures of interest to both mathematicians and physicists. Mirror symmetry in the (0,2) arena is a part of a larger story which we expect will provide new insights into the nature of quantum geometry. The study of tangent bundle

deformations offers a tractable entry point into this mysterious and tantalizing world, where we can usefully apply many lessons from conventional (2,2) mirror symmetry. Clearly the most important and exciting direction is to generalize these studies to more general bundles, but there are also important questions remaining even for tangent bundle deformations. We end by summarizing a few of those questions. The list is by no means complete, but we hope it will whet the reader's appetite for a deeper look at (0,2) theories.

1. Vanishing theorems for quantum obstructions. We discussed the issue of stability of (0,2) deformations, citing the advantages of the GLSM deformations and results of [6, 10, 44]. Using those results, can we prove, say for the class of reflexively plain models, that (0,2) GLSM deformations are not quantum obstructed?

2. Quantum sheaf cohomology for hypersurfaces and complete intersections. Although quantum sheaf cohomology should exist for much more general cases, computations are currently well-understood only for toric varieties. For hypersurfaces in toric varieties, there is a "quantum restriction" formula [34], but more work needs to be done to test and generalize this proposal.

3. Testing the conjectured (0,2) mirror map. It would be nice to develop techniques to test the (0,2) mirror map proposal further. A first step might be to check that other (non-principal) components of the discriminant locus are also exchanged by the map. A more ambitious step would be to show the equality of A/2 and B/2 correlators. This will require a (0,2) generalization of the theory of toric residues, computations of quantum sheaf cohomology for hypersurfaces, as well as properly understanding the operator map in the absence of topological descent.

4. Extending the (0,2) mirror map. The restriction to reflexively plain polytopes, although perhaps elegant, is clearly a computational crutch. Can we develop a more general notion of the map that applies to a wider class of GLSMs? As suggested in [37], a useful way to pursue this might be to study "mirror subfamilies" along the lines of [40].

5. "Quantum bundles" and quantum geometry. A tantalizing feature of (0,2) theories, already apparent from the results on singular loci, is that a sheaf corresponding to a classically singular bundle can nevertheless lead to a perfectly smooth conformal field theory. Can we develop a precise mathematical characterization of which sheaves have this property? What exactly are their mirrors?

6. Deriving the (0,2) mirror map. Can the derivation of mirror pairs from worldsheet duality be extended to compact Calabi-Yau spaces? Any such extension would provide a proof of the mirror map for tangent bundle deformations, and hopefully, provide a generalization to a wider class of bundles.

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A Superspace and Superfield Conventions

A.1 Chiral and Fermi superfields

In this appendix, we summarize our notation and conventions used primarily in section 4. For a nice review of $(0,2)$ theories, see [32]. Throughout this appendix, we will use the language of $(0,2)$ superspace with coordinates $(x^+, x^-, \theta^+, \bar{\theta}^+)$. The worldsheet coordinates are defined by $x^\pm = \frac{1}{2}(x^0 \pm x^1)$ so the corresponding derivatives $\partial_\pm = \partial_0 \pm \partial_1$ satisfy $\partial_\pm x^\pm = 1$. We define the measure for Grassman integration so that $\mathrm{d}^2\theta^+ = \mathrm{d}\bar{\theta}^+\mathrm{d}\theta^+$ and $\int \mathrm{d}^2\theta^+ \, \theta^+\bar{\theta}^+ = 1$. The $(0,2)$ super-derivatives

$$D_+ \;=\; \partial_{\theta^+} \;-\; i\bar{\theta}^+\partial_+, \qquad \bar{D}_+ \;=\; -\partial_{\bar{\theta}^+} \;+\; i\theta^+\partial_+,$$

satisfy the usual anti-commutation relations

$$\{D_+, D_+\} \;=\; \{\bar{D}_+, \bar{D}_+\} \;=\; 0, \qquad \{\bar{D}_+, D_+\} \;=\; 2i\partial_+.$$

In the absence of gauge fields, $(0,2)$ sigma models involve two sets of superfields: chiral superfields annihilated by the $\bar{D}_+$ operator,

$$\bar{D}_+\Phi^i \;=\; 0,$$

and Fermi superfields Γ^α which satisfy,

$$\bar{D}_+\Gamma^\alpha \;=\; \sqrt{2}E^\alpha,$$

where E^α is chiral: $\bar{D}_+E^\alpha = 0$. These superfields have the following component expansions:

$$\Phi^i \;=\; \psi^i \;+\; \sqrt{2}\theta^+\psi^i_+ \;-\; i\theta^+\bar{\theta}^+\partial_+\psi^i, \tag{44}$$

$$\Gamma^\alpha \;=\; \gamma^\alpha \;+\; \sqrt{2}\theta^+F^\alpha \;-\; \sqrt{2}\bar{\theta}^+E^\alpha \;-\; i\theta^+\bar{\theta}^+\partial_+\gamma^\alpha. \tag{45}$$

If we omit superpotential couplings, the most general Lorentz invariant $(0,2)$ supersymmetric action involving only chiral and Fermi superfields and their complex conjugates takes the form,

$$\mathcal{L} \;=\; -\frac{1}{2}\int \mathrm{d}^2\theta^+ \left[\frac{i}{2}K_i\partial_-\Phi^i \;-\; \frac{i}{2}K_{\bar{\imath}}\partial_-\bar{\Phi}^{\bar{\imath}} \;+\; h_{\alpha\bar{\beta}}\bar{\Gamma}^{\bar{\beta}}\Gamma^\alpha \;+\; h_{\alpha\beta}\Gamma^\alpha\Gamma^\beta \;+\; h_{\bar{\alpha}\bar{\beta}}\bar{\Gamma}^{\bar{\alpha}}\bar{\Gamma}^{\bar{\beta}}\right]. \tag{46}$$

The one-forms K_i determine the metric; the functions $h_{\alpha\beta}$ and $h_{\alpha\bar{\beta}}$ determine the bundle metric.

A.2 Gauged linear sigma models

We now introduce gauge fields. For a general $U(1)^n$ abelian gauge theory, we require a pair $(0,2)$ gauge superfields A^a and V_-^a for each abelian factor, $a = 1, \ldots, n$. Let us restrict to $n = 1$ for now.

Under a super-gauge transformation, the vector superfields transform as follows,

$$\begin{aligned} \delta A &= i(\bar{\Lambda} - \Lambda)/2, \\ \delta V_- &= -\partial_-(\Lambda + \bar{\Lambda})/2, \end{aligned}$$

where the gauge parameter Λ is a chiral superfield: $\bar{D}_+\Lambda = 0$. In Wess-Zumino gauge, the gauge superfields take the form

$$\begin{aligned} A &= \theta^+\bar{\theta}^+ A_+, \\ V_- &= A_- - 2i\theta^+\bar{\lambda}_- - 2i\bar{\theta}^+\lambda_- + 2\theta^+\bar{\theta}^+ D, \end{aligned}$$

where $A_\pm = A_0 \pm A_1$ are the components of the gauge field. We will denote the gauge covariant derivatives by

$$\mathcal{D}_\pm = \partial_\pm + iQA_\pm$$

when acting on a field of charge Q. This allows us to replace our usual superderivatives $D_+, \bar{D}_+$ with gauge covariant ones

$$\mathfrak{D}_+ = \partial_{\theta^+} - i\bar{\theta}^+\mathcal{D}_+, \qquad \bar{\mathfrak{D}}_+ = -\partial_{\bar{\theta}^+} + i\, i\theta^+\mathcal{D}_+$$

which now satisfy the modified algebra

$$\{\mathfrak{D}_+, \mathfrak{D}_+\} = \{\bar{\mathfrak{D}}_+, \bar{\mathfrak{D}}_+\} = 0, \qquad \{\bar{\mathfrak{D}}_+, \mathfrak{D}_+\} = 2i\mathcal{D}_+.$$

We must also introduce the supersymmetric gauge covariant derivative,

$$\nabla_- = \partial_- + iQV_-,$$

which contains $\mathcal{D}_-$ as its lowest component. The gauge invariant Fermi multiplet containing the field strength is defined as follows,

$$\Upsilon = [\bar{\mathfrak{D}}_+, \nabla_-] = \bar{D}_+(\partial_- A + iV_-) = -2\big(\lambda_- - i\theta^+(D - iF_{01}) - i\theta^+\bar{\theta}^+\partial_+\lambda_-\big).$$

Kinetic terms for the gauge field are given by

$$\mathcal{L} = \frac{1}{8e^2}\int \mathrm{d}^2\theta^+\, \bar{\Upsilon}\Upsilon = \frac{1}{e^2}\left(\frac{1}{2}F_{01}^2 + i\bar{\lambda}_-\partial_+\lambda_- + \frac{1}{2}D^2\right). \tag{47}$$

Since we are considering abelian gauge groups, we can also introduce an FI term with complex coefficient $t = ir + \frac{\theta}{2\pi}$:

$$\frac{t}{4}\int \mathrm{d}\theta^+\Upsilon\Big|_{\bar{\theta}^+=0} + c.c. = -rD + \frac{\theta}{2\pi}F_{01}. \tag{48}$$

In order to charge our chiral fields under the gauge action, we should ensure that they satisfy the covariant chiral constraint $\bar{\mathfrak{D}}_+\Phi = 0$. Since $\bar{\mathfrak{D}}_+ = e^{QA}\bar{D}_+e^{-QA}$ it follows that $e^{QA}\Phi_0$ is a chiral field of charge Q, where Φ_0 is the neutral chiral field appearing in (44). In components,

$$\Phi \;=\; \phi \;+\; \sqrt{2}\psi \;-\; i\theta^+\bar{\theta}^+\mathcal{D}_+\phi$$

The standard kinetic terms for charged chirals in $(0,2)$ gauged linear sigma models (GLSMs) are

$$\begin{aligned}\mathcal{L} \;&=\; \frac{-i}{2}\int \mathrm{d}^2\theta^+\;\bar{\Phi}^i\nabla_-\Phi^i, \qquad (49)\\ &=\; \left(-\left|\mathcal{D}_\mu\phi^i\right|^2 \;+\; \bar{\psi}_+ i\mathcal{D}_-\psi^i_+ \;-\; \sqrt{2}iQ_i\bar{\phi}^i\lambda_-\psi^i_+ \;+\; \sqrt{2}iQ_i\phi^i\bar{\psi}^i_+\bar{\lambda}_- \;+\; DQ_i\left|\phi^i\right|^2\right).\end{aligned}$$

Fermi superfields are treated similarly. We promote them to charged fields by defining $\Gamma = e^{QA}\Gamma_0$ so that in components

$$\Gamma \;=\; \gamma \;+\; \sqrt{2}\theta^+F \;+\; \sqrt{2}\bar{\theta}^+E \;-\; i\theta^+\bar{\theta}^+\mathcal{D}_+\gamma,$$

where we have introduced a non-vanishing E again. If we make the standard assumption that E is a holomorphic function of the Φ^i then the kinetic terms for the Fermi fields are:

$$\begin{aligned}\mathcal{L} \;&=\; -\frac{1}{2}\int \mathrm{d}^2\theta^+\,\bar{\Gamma}^\alpha\Gamma^\alpha, \qquad (50)\\ &=\; \left(i\bar{\gamma}^\alpha\mathcal{D}_+\gamma^\alpha \;+\; \left|F^\alpha\right|^2 \;-\; \left|E^\alpha\right|^2 \;-\; \bar{\gamma}^\alpha\partial_i E^\alpha\psi^i_+ \;-\; \bar{\psi}^i_+\partial_{\bar{i}}\bar{E}^\alpha\gamma^\alpha\right).\end{aligned}$$

A.3 Superpotential couplings

We can introduce superpotential couplings,

$$S_J \;=\; -\frac{1}{\sqrt{2}}\int \mathrm{d}^2x\mathrm{d}\theta^+\,\Gamma\cdot J(\Phi) \;+\; \text{c.c.}, \qquad (51)$$

supersymmetric if $E\cdot J = 0$, which give a total bosonic potential

$$V \;=\; |E|^2 \;+\; |J|^2.$$

The action consisting of the terms (47), (48), (49), (50) and (51) comprises the standard $(0,2)$ GLSM.

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