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Small Surfaces of Willmore Type in Riemannian Manifolds

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In this paper, we investigate the properties of small surfaces of Willmore type in three-dimensional Riemannian manifolds. By *small* surfaces, we mean topological spheres contained in a geodesic ball of small enough radius. In particular, we show that if there exist such surfaces with positive mean curvature in the geodesic ball $B_r(p)$ for arbitrarily small radius r around a point p in the Riemannian manifold, then the scalar curvature must have a critical point at p. As a byproduct of our estimates, we obtain a strengthened version of the non-existence result of Mondino [9] that implies the non-existence of certain critical points of the Willmore functional in regions where the scalar curvature is non-zero.

1 Introduction

In a previous paper [7], Willmore type surfaces were introduced, and foliations of asymptotically flat manifolds by such surfaces were studied. In this paper, we turn to the local situation and consider Willmore type surfaces in small geodesic balls in three-dimensional Riemannian manifolds. The focus is on a priori estimates for such surfaces under the assumption of positive mean curvature and a growth condition for the

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Lagrange parameter. As an application of these estimates, we derive a necessary condition for the existence of such surfaces.

By surfaces of Willmore type, we mean surfaces Σ that are critical for the Willmore functional

$$\mathcal{W}(\Sigma) = \frac{1}{2}\int_\Sigma H^2 \, \mathrm{d}\mu$$

subject to an area constraint $|\Sigma| = a$, where a is some fixed constant. These surfaces are solutions of the Euler–Lagrange equation

$$\Delta H + H|\overset{\circ}{A}|^2 + H\,\mathrm{Ric}(\nu,\nu) + \lambda H = 0, \tag{1.1}$$

where $\lambda \in \mathbf{R}$ is the Lagrange parameter, H the mean curvature of Σ, $\overset{\circ}{A}$ denotes the traceless part of the second fundamental form, Ric refers to the Ricci curvature of the ambient manifold, ν is the normal of Σ, and Δ denotes the Laplace–Beltrami operator on Σ. In particular, these surfaces are a generalization of Willmore surfaces that are critical for $\mathcal{W}$ without constraint and therefore satisfy the equation

$$\Delta H + H|\overset{\circ}{A}|^2 + H\,\mathrm{Ric}(\nu,\nu) = 0.$$

We note here that there are other functionals that can be considered as generalizations of the Willmore functional in Riemannian manifolds, for example, the functional $\mathcal{U}$ introduced in Section 2 could be used (see [13]).

The precise statement of the main result of this paper is the following:

Theorem 1.1. Assume that (M, g) is a three-dimensional Riemannian manifold such that the curvature and the first two derivatives of the curvature are bounded. Then there exist $\varepsilon_0 > 0$ and $C < \infty$, depending only on these bounds, with the following properties.

Given p in M and assume that there is $r_0 > 0$ such that for each $r \in (0, r_0]$, there exists a surface Σ_r of Willmore type in $B_r(p)$, that is, on Σ_r, we have

$$\Delta H + H|\overset{\circ}{A}|^2 + H\,\mathrm{Ric}(\nu,\nu) + H\lambda_r = 0,$$

such that, in addition, the following conditions are satisfied for some $\varepsilon < \varepsilon_0$:

1. Σ_r is a topological sphere,
2. $\lambda_r \geq -\varepsilon/|\Sigma_r|$, and
3. $H > 0$ on Σ_r.

Then

$$\lim_{r\to 0} |\lambda_r + \frac{1}{3}\,\mathrm{Sc}(p)| = 0.$$

Here, $\mathrm{Sc}(p)$ denotes the Scalar curvature of M at the point p. Furthermore,

$$\nabla\,\mathrm{Sc}(p) = 0,$$

where $\nabla\,\mathrm{Sc}(p)$ denotes the gradient of the scalar curvature of M at p. □

The first claim is proved in Section 3 as a consequence of the a priori estimates for surfaces of Willmore type derived there. Section 4 is devoted to the proof of the second claim.

For surfaces of constant mean curvature (CMC), that is, surfaces satisfying $H = \mathrm{const}$, analogous properties have been derived. Ye showed that if there locally exists a regular foliation by CMC surfaces near a point p, then p is necessarily a critical point of the scalar curvature. For the detailed statement including the technical conditions, we refer to [14, Theorem 2.1] (see also [15]). There are further results in this direction by Druet [3] and Nardulli [10] where the expansion of the isoperimetric profile of a Riemannian manifold is computed. This computation shows that isoperimetric surfaces also concentrate near critical points of the scalar curvature.

Indeed, it has been shown by Ye in [14] that near the non-degenerate critical points of the scalar curvature, there exist spherical surfaces with arbitrarily large mean curvature or, equivalently, arbitrarily small area. We expect that a similar statement is true for surfaces of Willmore type, namely that near a non-degenerate critical point of the scalar curvature, there exist surfaces of Willmore type with arbitrarily small area. We will address this elsewhere.

An immediate corollary of Theorem 1.1 is the following strengthened version of the non-existence result of Mondino [9, Theorem 1.3] for Willmore surfaces. These surfaces are of Willmore type with multiplier $\lambda = 0$, and thus, the previous theorem is applicable.

Corollary 1.2. Let (M, g) be a three-dimensional Riemannian manifold as in Theorem 1.1, and let $p \in M$. If $\mathrm{Sc}(p) \neq 0$ or $\nabla\,\mathrm{Sc}(p) \neq 0$, then there exists $r > 0$ such that $B_r(p)$ does not contain spherical Willmore surfaces with positive mean curvature. □

We conclude the paper with Section 5, where the a priori estimates are used to calculate the expansion of the Willmore functional on surfaces as in Theorem 1.1. More precisely, we show that for these surfaces, we have

$$\mathcal{W}(\Sigma) = 8\pi - \frac{|\Sigma|}{3}\operatorname{Sc}(p) + O(r|\Sigma|). \tag{1.2}$$

This is analogous to the expansion derived by Mondino [9, Proposition 3.1] for perturbed spheres.

2 Preliminaries

In this section, we describe our notation, and we provide some tools in order to analyze small surfaces in a Riemannian manifold.

2.1 Notation

We consider surfaces Σ in a three-dimensional Riemannian manifold (M, g), where g denotes the metric on M. We denote by ∇ the induced Levi–Civita connection, by Ric its Ricci curvature, and by Sc its scalar curvature.

If $p \in M$ and $\rho < \operatorname{inj}(M, g, p)$, the injectivity radius of (M, g) at p, we can introduce Riemannian normal coordinates on $B_\rho(p)$, the geodesic ball of radius ρ around p. These are given by the map

$$\Phi : B^E_\rho(0) \to B_\rho(p) : x \mapsto \exp_p(x),$$

where $B^E_\rho(0)$ is the Euclidean ball of radius ρ in $\mathbf{R}^3 \cong T_pM$. In these coordinates, the metric g satisfies

$$g = g^E + h, \tag{2.1}$$

where g^E denotes the Euclidean metric and h satisfies

$$|x|^{-2}|h| + |x|^{-1}|\partial h| + |\partial^2 h| \le h_0 \tag{2.2}$$

for all $x \in B^E_\rho(0)$. Here, h_0 is a constant depending only on the maximum of $|\operatorname{Ric}|$, $|\nabla \operatorname{Ric}|$, and $|\nabla^2 \operatorname{Ric}|$ in $B_\rho(p)$. More detailed expansions are not needed here but can be found in [11, Lemma V.3.4]. For our purposes, it is sufficient to consider $M = B^E_\rho(0)$ to be equipped with the two metrics g and g^E. We will denote $B_\rho = B^E_\rho(0)$ in the sequel.

If $\Sigma \subset B_\rho(p)$ is a surface, we denote its normal vector by ν. Later, it will be clear that the surfaces Σ which we consider will bound a compact set in B_ρ and then we choose the outward pointing normal. For the moment, we just assume that one choice is made. The induced metric on Σ is denoted by γ and its second fundamental form by A. Our conventions for the sign of A is that

$$A(X, Y) = \langle \nabla_X, Y \rangle.$$

The mean curvature of Σ is denoted by $H = \operatorname{tr}_\gamma A$ and the traceless part of the second fundamental form by $\overset{\circ}{A} = A - (1/2)H\gamma$. Furthermore, $\mathrm{d}\mu$ denotes the measure on Σ. Note that also the Euclidean metric induces a full set of geometric quantities on Σ, which will be distinguished by the superscript ^{E}, for example, ν^E, A^E, H^E, etc. All geometric quantities which we leave undecorated correspond to the metric g.

2.2 The Willmore functional

Assume that $\Sigma \subset M$. Then we consider the *Willmore functional* on Σ, that is, the functional

$$\mathcal{W}(\Sigma) = \frac{1}{2} \int_\Sigma H^2 \,\mathrm{d}\mu.$$

We say that a surface is of *Willmore type with multiplier* $\lambda \in \mathbf{R}$ if it satisfies the equation

$$\Delta H + H|\overset{\circ}{A}|^2 + H \operatorname{Ric}(\nu, \nu) + \lambda H = 0. \tag{2.3}$$

Here, Δ denotes the Laplace–Beltrami operator on Σ, and Ric refers to the Ricci curvature of the ambient metric g as before. Equation (2.3) arises as the Euler–Lagrange equation for the following variational problem:

$$\begin{cases} \text{Minimize} & \mathcal{W}(\Sigma) \\ \text{subject to} & |\Sigma| = a \end{cases}$$

where a is a given constant. The parameter λ in (2.3) is then just the Lagrange parameter of the critical point. For a derivation of this and further motivation, we refer to [7].

Denoting by ${}^\Sigma\mathrm{Sc}$ the scalar curvature of Σ, the Gauss equation implies that

$${}^\Sigma\mathrm{Sc} = \mathrm{Sc} - 2 \operatorname{Ric}(\nu, \nu) + \frac{1}{2} H^2 - |\overset{\circ}{A}|^2. \tag{2.4}$$

Integrating this equation on Σ yields the identity

$$\mathcal{W}(\Sigma) = 8\pi(1 - q(\Sigma)) + \mathcal{U}(\Sigma) + \mathcal{V}(\Sigma) \tag{2.5}$$

where $q(\Sigma)$ denotes the genus of Σ,

$$\begin{aligned} \mathcal{U}(\Sigma) &= \int_\Sigma |\mathring{A}|^2 \, \mathrm{d}\mu, \qquad \text{and} \\ \mathcal{V}(\Sigma) &= 2\int_\Sigma G(\nu, \nu)\, \mathrm{d}\mu. \end{aligned}$$

Here, $G = \mathrm{Ric} - (1/2)\,\mathrm{Sc}\, g$ denotes the Einstein tensor of M. This splitting was used in [7] to obtain a priori estimates for the position of Willmore-type surfaces in asymptotically flat manifolds and shall also play an important role in Section 4.

2.3 **Small surfaces**

Since we compare the geometry of a surface Σ with respect to the ambient metrics g and g^E, we need the following lemma.

Lemma 2.1. Let $g = g^E + h$ on B_ρ be given. Then there exists a constant C depending only on ρ and h_0 from Eq. (2.2) such that for all surfaces $\Sigma \subset B_r$ with $r < \rho$, we have

$$\begin{aligned} &|\gamma - \gamma^E| \le C|x|^2, \\ &|\,\mathrm{d}\mu - \mathrm{d}\mu^E| \le C|x|^2, \\ &|\nu - \nu^E| \le C|x|^2, \quad \text{and} \\ &|A - A^E| \le C(|x| + |x|^2|A|). \end{aligned}$$

□

In the sequel, we will use the big-O notation. By the statement $f = O(r^\alpha)$, we mean that for any $r_0 > 0$, there exists a constant $C < \infty$ such that $|f| \le Cr^\alpha$ provided that $r < r_0$.

Observe that the area of a surface in B_ρ is bounded in terms of ρ and the Willmore functional. This lemma is a slight generalization of [12, Lemma 1.1].

Lemma 2.2. Let $g = g^E + h$ on B_ρ be given. Then there exist $0 < \rho_0 < \rho$ and a constant C depending only on ρ and h_0 such that for all surfaces $\Sigma \subset B_r$ with $r < \rho_0$, we have

$$|\Sigma| \le Cr^2 \int_\Sigma H^2 \, \mathrm{d}\mu.$$

□

Proof. Let $\Sigma \subset B_r$ be a hypersurface for some $r \leq \rho$. We consider the position vector field x on B_ρ. Then we have

$$\operatorname{div}_\Sigma x = 2 + O(|x|),$$

where $\operatorname{div}_\Sigma$ means the tangential divergence along Σ. Integrating this relation yields

$$2|\Sigma| = \int_\Sigma \operatorname{div}_\Sigma x \, \mathrm{d}\mu + |\Sigma| O(r).$$

Since

$$\left| \int_\Sigma \operatorname{div}_\Sigma x \, \mathrm{d}\mu \right| = \left| \int_\Sigma H \langle x, \nu \rangle \, \mathrm{d}\mu \right| \leq \left(\int_\Sigma H^2 \, \mathrm{d}\mu \right)^{1/2} \left(\int_\Sigma |\langle x, \nu \rangle|^2 \, \mathrm{d}\mu \right)^{1/2}$$

and $|\langle x, \nu \rangle| \leq r$, we find that

$$|\Sigma| \leq Cr |\Sigma|^{1/2} \left(\int_\Sigma H^2 \, \mathrm{d}\mu \right)^{1/2} + Cr|\Sigma|.$$

Now we can fix ρ_0 small so that for all $0 < r < \rho_0$, the second term on the right can be absorbed to the left. This yields the claimed inequality. ■

For the subsequent curvature estimates, we also need a version of the Michael–Simon–Sobolev inequality suitable for our situation. The extension by Hoffman and Spruck [5] of the Euclidean version of the inequality [8] in conjunction with Lemma 2.2 implies the following.

Lemma 2.3. Let $g = g^E + h$ on B_ρ be given. Then there exist $0 < \rho_0 < \rho$ and a constant C depending only on ρ and h_0 such that for all surfaces $\Sigma \subset B_{\rho_0}$ with $\|H\|_{L^2(\Sigma)} < \infty$ and all $f \in C^\infty(\Sigma)$, we have

$$\left(\int_\Sigma f^2 \, \mathrm{d}\mu \right)^{1/2} \leq C \int_\Sigma |\nabla f| + |Hf| \, \mathrm{d}\mu.$$

□

2.4 Almost umbilical surfaces

Subsequently, it is necessary to approximate a given surface Σ by a Euclidean sphere. The main tool will be the following theorem from [1] and [2]. We denote the L^2-norm of the trace-free part of the second fundamental form by

$$\|\overset{\circ}{A}{}^E\|^2_{L^2(\Sigma,\gamma^E)} = \int_\Sigma |\overset{\circ}{A}{}^E|^2_E \,\mathrm{d}\mu^E,$$

where all geometric quantities are with respect to the Euclidean background. In addition, we denote by

$$\|\overset{\circ}{A}\|^2_{L^2(\Sigma,\gamma)} = \int_\Sigma |\overset{\circ}{A}|^2 \,\mathrm{d}\mu$$

the norm of the same tensor, where all geometric quantities are calculated with respect to the background metric g. The following theorem is a purely Euclidean theorem.

Theorem 2.4. There exists a universal constant C with the following properties. Assume that $\Sigma \subset \mathbf{R}^3$ is a surface with $\|\overset{\circ}{A}{}^E\|^2_{L^2(\Sigma,\gamma^E)} < 8\pi$. Let $R^E := \sqrt{|\Sigma|^E/4\pi}$ be the Euclidean area radius of Σ and $a^E := |\Sigma|_E^{-1} \int_\Sigma x \,\mathrm{d}\mu^E$ be the Euclidean center of gravity. Then there exists a conformal map $F : S := S_{R^E}(a^E) \to \Sigma \subset \mathbf{R}^3$ with the following properties. Let γ^S be the standard metric on S, N the Euclidean normal vector field, and ϕ the conformal factor, that is, $F^*\gamma^E = \phi^2\gamma^S$. Then the following estimates hold

$$\begin{aligned} \|H^E - 2/R^E\|_{L^2(\Sigma,\gamma^E)} &\le C\|\overset{\circ}{A}{}^E\|_{L^2(\Sigma,\gamma^E)} \\ \|F - \mathrm{id}_S\|_{L^\infty(S)} &\le C R^E \|\overset{\circ}{A}{}^E\|_{L^2(\Sigma,\gamma^E)} \\ \|\phi^2 - 1\|_{L^\infty(S)} &\le C\|\overset{\circ}{A}{}^E\|_{L^2(\Sigma,\gamma^E)} \\ \|N - \nu^E \circ F\|_{L^2(S)} &\le C R^E \|\overset{\circ}{A}{}^E\|_{L^2(\Sigma,\gamma^E)}. \end{aligned} \qquad \square$$

To apply the previous theorem, we need to estimate $\|\overset{\circ}{A}{}^E\|_{L^2(\Sigma,\gamma^E)}$ in terms of $\|\overset{\circ}{A}\|_{L^2(\Sigma,\gamma)}$. This is the content of the following lemma.

Lemma 2.5. Let $g = g^E + h$ on B_ρ be given. Then there exist $0 < \rho_0 < \rho$ and a constant C depending only on ρ and h_0 such that for all surfaces $\Sigma \subset B_r$ with $r < \rho_0$, we have

$$\|\overset{\circ}{A}{}^E\|^2_{L^2(\Sigma,\gamma^E)} \le C\|\overset{\circ}{A}\|^2_{L^2(\Sigma,\gamma)} + Cr^4\|H\|^2_{L^2(\Sigma,\gamma)} \qquad \square$$

Proof. In a first step, we estimate

$$\begin{aligned}
&\left|\int_\Sigma |\mathring{A}|^2 - |\mathring{A}^E|^2 \, \mathrm{d}\mu\right| \\
&\quad \le \int_\Sigma |\mathring{A}||\mathring{A} - \mathring{A}^E| + |\mathring{A}^E||\mathring{A} - \mathring{A}^E| \, \mathrm{d}\mu \\
&\quad \le \left(\int_\Sigma |\mathring{A}|^2 \, \mathrm{d}\mu\right)^{1/2} \left(\int_\Sigma |\mathring{A} - \mathring{A}^E|^2\right)^{1/2} \\
&\qquad + \left(\int_\Sigma |\mathring{A}^E|^2 \, \mathrm{d}\mu\right)^{1/2} \left(\int_\Sigma |\mathring{A} - \mathring{A}^E|^2 \, \mathrm{d}\mu\right)^{1/2}
\end{aligned}$$

Next, we use the last estimate from Lemma 2.1 and 2.2 to get

$$\begin{aligned}
\int_\Sigma |\mathring{A} - \mathring{A}^E|^2 \, \mathrm{d}\mu &\le C \int_\Sigma |A - A^E|^2 \, \mathrm{d}\mu \\
&\le C \int_\Sigma |x|^2 + |x^4||A|^2 \, \mathrm{d}\mu \\
&\le Cr^4\big(\|H\|^2_{L^2(\Sigma,\gamma)} + \|A\|^2_{L^2(\Sigma,\gamma)}\big) \\
&\le Cr^4\big(\|H\|^2_{L^2(\Sigma,\gamma)} + \|\mathring{A}\|^2_{L^2(\Sigma,\gamma)}\big).
\end{aligned}$$

Putting this together with the first estimate, we find that

$$\|\mathring{A}^E\|^2_{L^2(\Sigma,\gamma)} \le \|\mathring{A}\|^2_{L^2(\Sigma,\gamma)} + Cr^2\big(\|H\|_{L^2(\Sigma,\gamma)} + \|\mathring{A}\|_{L^2(\Sigma,\gamma)}\big)\big(\|\mathring{A}\|_{L^2(\Sigma,\gamma)} + \|\mathring{A}^E\|_{L^2(\Sigma,\gamma)}\big).$$

Via Cauchy–Schwarz, we can absorb the term containing $\mathring{A}^E$ on the right to the left and arrive at

$$\|\mathring{A}^E\|^2_{L^2(\Sigma,\gamma)} \le C(1 + r^2 + r^4)\|\mathring{A}\|^2_{L^2(\Sigma,\gamma)} + Cr^4\|H\|^2_{L^2(\Sigma,\gamma)}.$$

The remainder of the proof consists of changing the metric when taking norms and the area element to the Euclidean version. Indeed, for any tensor T, we have the estimate

$$\int_\Sigma |T|^2_{g^E} \, \mathrm{d}\mu^E \le (1 + Cr^2) \int_\Sigma |T|^2_g \, \mathrm{d}\mu.$$

This then proves the claim. ∎

3 A Priori Estimates

A crucial ingredient in the proof of Theorem 1.1 is an estimate for the L^2-norm of the traceless part of the second fundamental form of Σ. This allows us to control the shape of the surface Σ in view of Theorem 2.4.

Throughout this section, we assume that the metric $g = g^E + h$ is fixed on B_ρ. We furthermore assume that ρ is chosen so small that Lemmas 2.1, 2.2, and 2.3 can be applied to any surface in B_ρ. We allow ρ to shrink as it becomes necessary. All surfaces we consider here are of Willmore type, that is, they satisfy Eq. (1.1) for some λ and are furthermore contained in B_ρ.

Subsequently, all constants C may depend on ρ and h_0 without further notice. In addition, these constants are allowed to change from line to line.

3.1 The initial estimate for $\mathcal{U}$

Lemma 3.1. Let $g = g^E + h$ on B_ρ be given. Then for each $\varepsilon_0 \in [0, 1)$, there exist $0 < \rho_0 < \rho$ and a constant C with the following properties. If $\varepsilon < \varepsilon_0$ and Σ is of Willmore type with multiplier λ in B_r with $r < \rho_0$ and

1. Σ is a topological sphere,
2. $\lambda \geq -\varepsilon/|\Sigma|$,
3. $H > 0$ on Σ.

Then

$$\begin{aligned}
&\int_\Sigma |\overset{\circ}{A}|^2 + |\nabla \log H|^2 \, \mathrm{d}\mu \leq Cr^2 + \varepsilon, \\
&\left| \int_\Sigma H^2 \, \mathrm{d}\mu - 16\pi \right| \leq C(\varepsilon + r^2), \\
&|\Sigma| \leq Cr^2, \quad \text{and} \\
&|\lambda| \leq C(1 + \varepsilon/|\Sigma|).
\end{aligned}$$

□

Note that in Eq. (1.1), the term ΔH scales like $|\Sigma|^{-3/2}$, so that the assumption on λ implies that the negative part of the term λH is of the same order of magnitude as this leading order term but small in comparison. There is no assumption on the positive part of this term.

The proof is similar to the proof of Lemmas 3.1 and 3.3 in [7] although the role of the individual terms is somewhat different.

Proof. Multiplying Eq. (1.1) by H^{-1} and integrating by parts give

$$\int_\Sigma |\nabla \log H|^2 + |\mathring{A}|^2 + \mathrm{Ric}(\nu, \nu) + \lambda \,\mathrm{d}\mu = 0. \tag{3.1}$$

In view of the assumption on λ and the fact that Ric is bounded, this yields the estimate

$$\int_\Sigma |\nabla \log H|^2 + |\mathring{A}|^2 \,\mathrm{d}\mu \leq C|\Sigma| + \varepsilon. \tag{3.2}$$

Inserting this back into (3.1), we get

$$|\lambda| \leq C(1 + \varepsilon|\Sigma|^{-1}),$$

which yields the last claim of the lemma.

Integrating (2.4) over Σ and using (3.2) and the Gauss–Bonnet theorem, we find that

$$\left| \frac{1}{2} \int_\Sigma H^2 \,\mathrm{d}\mu - 8\pi \right| \leq C(|\Sigma| + \varepsilon).$$

In view of the area estimate from Lemma 2.2, this yields an estimate of the form

$$|\Sigma| \leq Cr^2 \int_\Sigma H^2 \,\mathrm{d}\mu \leq Cr^2(1 + |\Sigma|).$$

If $r < \rho_0$ is chosen small enough, we can absorb the term $Cr^2|\Sigma|$ on the right to the left and obtain the estimates

$$\left| \int_\Sigma H^2 \,\mathrm{d}\mu - 16\pi \right| \leq C(\varepsilon + r^2)$$

and

$$|\Sigma| \leq Cr^2$$

which are the second and third claims. Plugging this into estimate (3.2), we obtain the remaining estimate. ■

3.2 An improved estimate for $\mathcal{U}$

The initial estimate from Lemma 3.1 allows to apply the a priori estimates from section 3 in [7] to get higher order estimates and to improve on the initial estimate.

Theorem 3.2. Let $g = g^E + h$ on B_ρ be given. Then there exist $\varepsilon > 0$, $0 < \rho_0 < \rho$ and a constant C with the following properties. If $\varepsilon < \varepsilon_0$ and Σ is of Willmore type with multiplier λ in B_r with $r < \rho_0$ and

1. Σ is a topological sphere,
2. $\lambda \geq -\varepsilon/|\Sigma|$,
3. $H > 0$ on Σ.

Then

$$\int_\Sigma \frac{|\nabla^2 H|^2}{H^2} + |\nabla A|^2 + |A|^2 |\overset{\circ}{A}|^2 \,\mathrm{d}\mu \leq C \int_\Sigma |\omega|^2 + \big(\mathrm{Ric}(\nu,\nu) + \lambda\big)^2 \,\mathrm{d}\mu.$$

Here, $\omega = \mathrm{Ric}(\nu,\cdot)^T$, the 1-form which results from projecting $\mathrm{Ric}(\nu,\cdot)$ to Σ. □

Proof. This is a consequence of the calculation in Section 3 of [7]. Note that the calculation there makes use of the fact that $\|\overset{\circ}{A}\|_{L^2} + \|\nabla \log H\|_{L^2}$ can be made arbitrarily small (cf. Lemma 3.8 there), and this is where the initial estimate from Section 3.1 enters. In particular, the procedure used to prove Theorem 3.9 in [7] implies that in the local situation, we have the following estimate.

$$\begin{aligned}\int_\Sigma \frac{|\nabla^2 H|^2}{H^2} &+ |\nabla A|^2 + |A|^2 |\overset{\circ}{A}|^2 \,\mathrm{d}\mu \\ &\leq C \int_\Sigma |\omega|^2 + \big(\mathrm{Ric}(\nu,\nu) + \lambda\big)^2 \,\mathrm{d}\mu \\ &\quad + C \sup_{B_\rho} |\,\mathrm{Ric}\,| \int_\Sigma |\overset{\circ}{A}|^2 + |\nabla \log H|^2 \,\mathrm{d}\mu. \end{aligned} \tag{3.3}$$

The only difference to [7] is that there, we were able to use the decay of the curvature, which is the origin of the factors $C r_{\min}^{-3}$, which have to be replaced by $C \sup_{B_\rho} |\,\mathrm{Ric}\,|$ here.

Next, we use the Michael–Simon–Sobolev inequality to estimate

$$\int_\Sigma |\overset{\circ}{A}|^2 \,\mathrm{d}\mu \leq C \left(\int_\Sigma |\nabla \overset{\circ}{A}| + H |\overset{\circ}{A}| \,\mathrm{d}\mu \right)^2 \leq C |\Sigma| \int_\Sigma |\nabla A|^2 + H^2 |\overset{\circ}{A}|^2 \,\mathrm{d}\mu. \tag{3.4}$$

Similarly, we get

$$\begin{aligned}\int_\Sigma |\nabla \log H|^2 \,\mathrm{d}\mu &\leq C \left(\int_\Sigma \frac{|\nabla^2 H|}{H} + |\nabla \log H|^2 + |\nabla H| \,\mathrm{d}\mu \right)^2 \\ &\leq C |\Sigma| \int_\Sigma \frac{|\nabla^2 H|^2}{H^2} + |\nabla A|^2 + |\nabla \log H|^4 \,\mathrm{d}\mu. \end{aligned}$$

Applying the Michael–Simon–Sobolev inequality once more, we have

$$\begin{aligned}&\int_\Sigma |\nabla \log H|^4 \,\mathrm{d}\mu\\&\le C\left(\int_\Sigma \frac{|\nabla^2 H|}{H}|\nabla \log H| + |\nabla \log H|^3 + H|\nabla \log H|^2 \,\mathrm{d}\mu\right)^2\\&\le C\|\nabla \log H\|^2_{L^2(\Sigma)}\int_\Sigma \frac{|\nabla^2 H|^2}{H^2} + |\nabla \log H|^4 + |\nabla A|^2 \,\mathrm{d}\mu.\end{aligned}$$

Using Lemma 3.1, we know that for ε and $r < \rho_0$ small enough, we get

$$\int_\Sigma |\nabla \log H|^4 \,\mathrm{d}\mu \le C||\nabla \log H||^2_{L^2(\Sigma)}\int_\Sigma \frac{|\nabla^2 H|^2}{H^2} + |\nabla A|^2 \,\mathrm{d}\mu. \tag{3.5}$$

Inserting this into the above estimate for $\int_\Sigma |\nabla \log H|^2 \,\mathrm{d}\mu$, we conclude

$$\int_\Sigma |\nabla \log H|^2 \,\mathrm{d}\mu \le C|\Sigma|\int_\Sigma \frac{|\nabla^2 H|^2}{H^2} + |\nabla A|^2 \,\mathrm{d}\mu. \tag{3.6}$$

Hence, we see that for $r < \rho_0$ small enough, we can absorb the second term on the right-hand side of (3.3). ■

The remaining task is to estimate the term on the right-hand side in Theorem 3.2. We start with the following calculation.

Lemma 3.3. Assume that the metric $g = g^E + h$ on B_ρ is given. Then there exists a constant C such that for all surfaces $\Sigma \subset B_r$, we have

$$\left|\int_\Sigma \mathrm{Ric}(\nu,\nu)\,\mathrm{d}\mu - \frac{|\Sigma|}{3}\,\mathrm{Sc}(0)\right| \le C|\Sigma|\big(\|\overset{\circ}{A}\|_{L^2(\Sigma)} + r\big)$$

□

Proof. Note that if either $\|\overset{\circ}{A}\|_{L^2(\Sigma)}$ or r is large, the estimate is trivially satisfied, so that it is sufficient to show it in the case where Theorem 2.4 is applicable, and we are furthermore allowed to assume that $0 < r < 1$. We use Theorem 2.4 to approximate Σ by a Euclidean sphere $S = S_{a^E}(R^E)$ with $a^E \in B_r(0)$ and $R^E = \sqrt{|\Sigma|^E/4\pi}$. Since

$$\big||\Sigma|^E - |\Sigma|\big| \le \int_\Sigma |\,\mathrm{d}\mu^E - \mathrm{d}\mu|\,\mathrm{d}\mu \le Cr^2|\Sigma|, \tag{3.7}$$

we infer that

$$|R^E - R| \le CrR,$$

where $R = \sqrt{|\Sigma|/4\pi}$. It is well known that

$$\int_S \mathrm{Ric}_0(N, N)\, \mathrm{d}\mu^E = \frac{|\Sigma|^E}{3}\, \mathrm{Sc}(0)$$

where we use the notation of Theorem 2.4, that is, N is the Euclidean normal of S. Furthermore, Ric_0 denotes the Ricci tensor of M evaluated at the origin.

The first step is to estimate

$$\begin{aligned}\left|\int_S \mathrm{Ric}(N, N)\, \mathrm{d}\mu^E - \int_S \mathrm{Ric}_0(N, N)\, \mathrm{d}\mu^E\right| \\ \le |\Sigma|^E \sup_{p\in S} |\,\mathrm{Ric}_p(N, N) - \mathrm{Ric}_0(N, N)| \le C|\Sigma|^E r\end{aligned}$$

and therefore,

$$\left|\int_S \mathrm{Ric}(N, N)\, \mathrm{d}\mu^E - \frac{|\Sigma|^E}{3}\, \mathrm{Sc}(0)\right| \le Cr|\Sigma|.$$

In the next step, we estimate $\int_\Sigma \mathrm{Ric}(\nu, \nu)$ in terms of $\int_S \mathrm{Ric}(N, N)\, \mathrm{d}\mu^E$. To this end, note that

$$\left|\int_\Sigma \mathrm{Ric}(\nu, \nu)\, \mathrm{d}\mu - \int_\Sigma \mathrm{Ric}(\nu, \nu)\, \mathrm{d}\mu^E\right| \le C|\Sigma|r^2.$$

The resulting integral can be evaluated using the conformal parametrization $F : S \to \Sigma$ from Theorem 2.4. We can express

$$\int_\Sigma \mathrm{Ric}(\nu, \nu)\, \mathrm{d}\mu^E = \int_S \mathrm{Ric} \circ F(\nu \circ F, \nu \circ F)\phi^2\, \mathrm{d}\mu^E.$$

The estimates of Theorem 2.4 and the Cauchy–Schwarz inequality imply that

$$\begin{aligned}
&\left|\int_\Sigma \mathrm{Ric}(\nu,\nu)\,\mathrm{d}\mu^E - \int_S \mathrm{Ric}(N,N)\,\mathrm{d}\mu^E\right| \\
&\quad\le \left|\int_S \big(\mathrm{Ric}\circ F - \mathrm{Ric})(\nu\circ F,\nu\circ F)\phi^2\,\mathrm{d}\mu^E\right| \\
&\qquad + \left|\int_S \mathrm{Ric}(\nu\circ F - N,\nu\circ F)\phi^2\,\mathrm{d}\mu^E\right| \\
&\qquad + \left|\int_S \mathrm{Ric}(N,\nu\circ F - N)\phi^2\,\mathrm{d}\mu^E\right| + \left|\int_S \mathrm{Ric}(N,N)(\phi^2-1)\,\mathrm{d}\mu^E\right| \\
&\quad\le C|\Sigma|\|F-\mathrm{id}\,\|_{L^\infty(S)} + C|\Sigma|^{1/2}\|\nu\circ F - N\|_{L^2(S)} + C|\Sigma|\|\phi^2-1\|_{L^\infty(S)} \\
&\quad\le C|\Sigma|\|\overset{\circ}{A}{}^E\|_{L^2(\Sigma,\gamma^E)}.
\end{aligned}$$

In combination with Lemma 2.5, we infer

$$\left|\int_\Sigma \mathrm{Ric}(\nu,\nu)\,\mathrm{d}\mu^E - \int_S \mathrm{Ric}(N,N)\,\mathrm{d}\mu^E\right| \le C|\Sigma|\big(\|\overset{\circ}{A}\|_{L^2(\Sigma)} + r^2\big).$$

Collecting all the above estimates results in the estimate

$$\left|\int_\Sigma \mathrm{Ric}(\nu,\nu)\,\mathrm{d}\mu - \frac{|\Sigma|}{3}\,\mathrm{Sc}(0)\right| \le C|\Sigma|\big(\|\overset{\circ}{A}\|_{L^2(\Sigma)} + r\big)$$

which is precisely the claim. ■

In the following lemma, we derive an estimate for the Lagrange parameter λ.

Lemma 3.4. Assume that the metric $g = g^E + h$ on B_ρ is given. Then there exist ε_0, $r_0 < \rho$, and a constant C such that all surfaces $\Sigma \subset B_r$ as in the statement of Theorem 3.2 with $\varepsilon < \varepsilon_0$ and $r < r_0$ satisfy

$$\left|\lambda + \frac{1}{3}\,\mathrm{Sc}(0)\right| \le C|\Sigma|^{-1}\big(\|\overset{\circ}{A}\|^2_{L^2(\Sigma)} + \|\nabla \log H\|^2_{L^2(\Sigma)}\big) + Cr.$$

In particular,

$$|\lambda| \le C|\Sigma|^{-1}\big(\|\overset{\circ}{A}\|^2_{L^2(\Sigma)} + \|\nabla \log H\|^2_{L^2(\Sigma)}\big) + C.$$

□

Proof. Equation (3.1) implies that

$$\left|\lambda + \frac{1}{|\Sigma|}\int_\Sigma \operatorname{Ric}(\nu,\nu)\,\mathrm{d}\mu\right| \le |\Sigma|^{-1}\big(\|\mathring{A}\|^2_{L^2(\Sigma)} + \|\nabla \log H\|^2_{L^2(\Sigma)}\big). \tag{3.8}$$

Apply Lemma 3.3 to calculate

$$\left|\frac{1}{|\Sigma|}\int_\Sigma \operatorname{Ric}(\nu,\nu)\,\mathrm{d}\mu - \frac{1}{3}\operatorname{Sc}(0)\right| \le C\big(\|\mathring{A}\|_{L^2(\Sigma)} + r\big),$$

and note that

$$\|\mathring{A}\|_{L^2(\Sigma)} = |\Sigma|^{1/2}|\Sigma|^{-1/2}\|\mathring{A}\|_{L^2(\Sigma)} \le \tfrac{1}{2}|\Sigma| + \tfrac{1}{2}|\Sigma|^{-1}\|\mathring{A}\|^2_{L^2(\Sigma)}.$$

Since $|\Sigma| \le Cr^2$ by Lemma 3.1, this implies the claim in combination with Eq. (3.8). ■

Theorem 3.5. Assume that the metric $g = g^E + h$ on B_ρ is given. Then there exist $\varepsilon_0 > 0$, $r_0 < \rho$, and a constant C such that all surfaces $\Sigma \subset B_r$ as in the statement of Theorem 3.2 with $\varepsilon < \varepsilon_0$ and $r < r_0$ satisfy

$$\int_\Sigma \frac{|\nabla^2 H|^2}{H^2} + |\nabla A|^2 + |A|^2|\mathring{A}|^2\,\mathrm{d}\mu \le C|\Sigma|.$$

□

Proof. In view of Theorem 3.2 and the fact that Ric and ω are bounded, we infer the estimate

$$\int_\Sigma \frac{|\nabla^2 H|^2}{H^2} + |\nabla A|^2 + |A|^2|\mathring{A}|^2\,\mathrm{d}\mu \le C|\Sigma|(1+\lambda^2). \tag{3.9}$$

The crucial term to estimate, thus, is $\lambda^2|\Sigma|$. We use the estimate from Lemma 3.4 to get

$$\lambda^2|\Sigma| \le C|\Sigma|^{-1}\big(\|\mathring{A}\|^4_{L^2(\Sigma)} + \|\nabla \log H\|^4_{L^2(\Sigma)}\big) + C|\Sigma|. \tag{3.10}$$

Combining (3.4) and (3.6) with (3.9) and (3.10), we infer

$$\begin{aligned}\int_\Sigma &\frac{|\nabla^2 H|^2}{H^2} + |\nabla A|^2 + |A|^2|\mathring{A}|^2\,\mathrm{d}\mu \\ &\le C|\Sigma| + C|\Sigma|\left(\int_\Sigma \frac{|\nabla^2 H|^2}{H^2} + |\nabla A|^2 + |A|^2|\mathring{A}|^2\,\mathrm{d}\mu\right)^2. \end{aligned}\tag{3.11}$$

To proceed, note that by Eq. (3.9) and Lemma 3.1, we find that

$$\int_\Sigma \frac{|\nabla^2 H|^2}{H^2} + |\nabla A|^2 + |A|^2|\overset{\circ}{A}|^2 \,\mathrm{d}\mu \le C|\Sigma|(1+\lambda^2) \le C(|\Sigma| + \varepsilon^2|\Sigma|^{-1})$$

Using this in Eq. (3.11) to estimate part of the right-hand side, we get

$$\begin{aligned}&\int_\Sigma \frac{|\nabla^2 H|^2}{H^2} + |\nabla A|^2 + |A|^2|\overset{\circ}{A}|^2 \,\mathrm{d}\mu \\ &\quad \le C|\Sigma| + C(|\Sigma|^2 + \varepsilon^2)\left(\int_\Sigma \frac{|\nabla^2 H|^2}{H^2} + |\nabla A|^2 + |A|^2|\overset{\circ}{A}|^2 \,\mathrm{d}\mu\right).\end{aligned}$$

Thus, choosing $0 < \varepsilon$ and $r < r_0$ small enough, we can absorb the second term on the right to the left and infer the claimed estimate. ■

Corollary 3.6. Assume that the metric $g = g^E + h$ on B_ρ is given. Then there exist $\varepsilon_0 > 0$, $0 < r_0 < \rho$, and a constant C and such that all surfaces $\Sigma \subset B_r$ as in the statement of Theorem 3.2 with $\varepsilon < \varepsilon_0$ and $r < r_0$ satisfy

$$\|\overset{\circ}{A}\|_{L^2(\Sigma)} + \|\nabla \log H\|_{L^2(\Sigma)} \le C|\Sigma|$$

and

$$|\lambda + \tfrac{1}{3}\,\mathrm{Sc}(0)| \le Cr. \qquad \square$$

Proof. The first claim follows from (3.4), (3.6), and Theorem 3.5, whereas the second claim follows from the first one in view of Lemma 3.4. ■

Note that this corollary yields the first claim in Theorem 1.1.

3.3 Estimates in the L^∞-norm

To proceed further, we need an estimate for the size of H^{-1} in the L^∞-norm. To this end, we recall Lemma 4.7 from [7].

Lemma 3.7. Assume that the metric $g = g^E + h$ on B_ρ is given. Then there exist $\varepsilon_0 > 0$, $r_0 < \rho$, and a constant $C < \infty$ such that for all surfaces $\Sigma \subset B_r$ as in the statement of Theorem 3.2 with $\varepsilon < \varepsilon_0$ and $r < r_0$ and for all smooth forms ϕ on Σ, we have

$$\|\phi\|^4_{L^\infty(\Sigma)} \leq C\|\phi\|^2_{L^2(\Sigma)} \int_\Sigma |\nabla^2\phi|^2 + |H|^4|\phi|^2 \,\mathrm{d}\mu. \qquad \square$$

Proof. This lemma is a variant of [6, Lemma 2.8]. The proof from there can be carried over to our situation, since it mainly relies on the Michael–Simon–Sobolev inequality which is also available in this situation, cf. Lemma 2.3. ■

Proposition 3.8. Assume that the metric $g = g^E + h$ on B_ρ is given. Then there exist $\varepsilon_0 > 0$, $r_0 < \rho$, and a constant $C < \infty$ such that for all surfaces $\Sigma \subset B_r$ as in the statement of Theorem 3.2 with $\varepsilon < \varepsilon_0$ and $r < r_0$, we have

$$\|H^{-1}\|_{L^\infty(\Sigma)} \leq C|\Sigma|^{1/2} \qquad \square$$

Proof. The idea is to apply Lemma 3.7 to the function H^{-1}. We thus estimate

$$\|H^{-1}\|^2_{L^2(\Sigma)} \leq |\Sigma|\|H^{-1}\|^2_{L^\infty(\Sigma)} \tag{3.12}$$

and calculate

$$\nabla^2(H^{-1}) = -H^{-2}\nabla^2 H + 2H^{-3}\nabla H \otimes \nabla H.$$

Thus,

$$\begin{aligned}\int_\Sigma |\nabla^2(H^{-1})|^2 \,\mathrm{d}\mu &\leq C\int_\Sigma H^{-6}|\nabla H|^4 + H^{-4}|\nabla^2 H|^2 \,\mathrm{d}\mu \\ &\leq C\|H^{-1}\|^2_{L^\infty(\Sigma)} \int_\Sigma \frac{|\nabla^2 H|^2}{H^2} + |\nabla \log H|^4 \,\mathrm{d}\mu. \end{aligned}\tag{3.13}$$

From (3.5), we get for ε and r small enough

$$\int_\Sigma |\nabla \log H|^4 \,\mathrm{d}\mu \leq C\left(\int_\Sigma |\nabla \log H|^2 \,\mathrm{d}\mu\right)\left(\int_\Sigma \frac{|\nabla^2 H|^2}{H^2} + |\nabla A|^2 \,\mathrm{d}\mu\right).$$

By Theorem 3.5 and Corollary 3.6, we therefore conclude

$$\int_\Sigma |\nabla \log H|^4 \,\mathrm{d}\mu \le C|\Sigma|^3. \tag{3.14}$$

Together with Eq. (3.13) and Theorem 3.5, this yields

$$\int_\Sigma |\nabla^2(H^{-1})|^2 \,\mathrm{d}\mu \le C\|H^{-1}\|^2_{L^\infty(\Sigma)}|\Sigma|. \tag{3.15}$$

Plugging estimates (3.15) and (3.12) into the estimate from Lemma 3.7, we find that

$$\begin{aligned}\|H^{-1}\|^4_{L^\infty(\Sigma)} &\le C|\Sigma|\|H^{-1}\|^2_{L^\infty(\Sigma)}\big(\|H^{-1}\|^2_{L^\infty(\Sigma)}|\Sigma| + \int_\Sigma H^2 \,\mathrm{d}\mu\big)\\ &\le C|\Sigma|^2\|H^{-1}\|^4_{L^\infty(\Sigma)} + C|\Sigma|\|H^{-1}\|^2_{L^\infty(\Sigma)}\\ &\le (C|\Sigma|^2 + \tfrac{1}{2})\|H^{-1}\|^4_{L^\infty(\Sigma)} + C|\Sigma|^2.\end{aligned}$$

If $r < r_0$ and thus $|\Sigma|$ is small, the first term on the right can be absorbed, and the claim follows. ■

4 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. Note that the first claim already follows from Corollary 3.6; hence, it remains to show the second claim.

Throughout this section, we assume that the surface Σ in question is of Willmore type with multiplier λ. We assume further that $H > 0$ on Σ, $\lambda \ge -\varepsilon|\Sigma|^{-1}$, and $\Sigma \subset B_r(0)$ where $\varepsilon < \varepsilon_0$ and $r < r_0$. Here, $\varepsilon_0 > 0$ and $r_0 > 0$ are chosen so that all the estimates from Section 3 are applicable.

To get started, we recall the splitting (2.5) of the Willmore functional:

$$\mathcal{W}(\Sigma) = 8\pi(1 - q(\Sigma)) + \mathcal{U}(\Sigma) + \mathcal{V}(\Sigma). \tag{4.1}$$

Since the first term on the right is a topological constant, we infer that the variation of $\mathcal{W}$, when Σ is varied by the normal vector field $f\nu$ for $f \in C^\infty(\Sigma)$, satisfies

$$\delta_f\mathcal{W}(\Sigma) = \delta_f\mathcal{U}(\Sigma) + \delta_f\mathcal{V}(\Sigma).$$

Equation (1.1) implies that the variation of $\mathcal{W}$ is given by

$$\delta_f\mathcal{W}(\Sigma) = \lambda\int_\Sigma Hf \,\mathrm{d}\mu,$$

whenever Σ is of Willmore type with multiplier λ. Thus, on such a surface, we have

$$\lambda \int_\Sigma Hf \,\mathrm{d}\mu = \delta_f \mathcal{U}(\Sigma) + \delta_f \mathcal{V}(\Sigma). \tag{4.2}$$

We shall evaluate these terms when the normal velocity f of the variation is given by

$$f = H^{-1} g(b, \nu), \tag{4.3}$$

where $b \in \mathbf{R}^3$ is a fixed vector with $|b| = 1$. We start with the left-hand side of Eq. (4.2).

4.1 **The left-hand side of** (4.2)

We have

$$\int_\Sigma Hf \,\mathrm{d}\mu = \int_\Sigma g(b, \nu) \,\mathrm{d}\mu. \tag{4.4}$$

To evaluate this expression, note that since Σ is assumed to be a topological sphere in B_ρ, it must bound a region Ω. We wish to estimate the volume of Ω. To this end, we approximate Σ by a Euclidean sphere $S = S_{R^E}(a^E)$. With x the position vector field in B_ρ, we define the vector field

$$X = x - a^E \tag{4.5}$$

such that

$$\operatorname{div}_{g^E} X = 3, \tag{4.6}$$

in Ω. On Σ, we have

$$\begin{aligned} |X| = |F - a^E| &\le \|F - \mathrm{id}_S\|_{L^\infty(S)} + \|\mathrm{id}_S - a^E\|_{L^\infty(S)} \\ &\le C|\Sigma|^{1/2}\big(1 + \|\overset{\circ}{A}{}^E\|_{L^2(\Sigma, g^E)}\big), \end{aligned}$$

since $\|\mathrm{id}_S - a^E\|_{L^\infty(S)} = R^E$. Here, $\mathrm{id}_S : S \to B_\rho$ denotes the standard embedding of S into B_ρ.

We integrate the relation (4.6) over Ω and use partial integration to conclude that

$$3\,\mathrm{Vol}^E(\Omega) = \int_\Sigma g^E(X, \nu^E) \,\mathrm{d}\mu^E.$$

Replacing the integral over Σ by an integral over S introduces an error of the form

$$\begin{aligned}&\left|\int_\Sigma g^E(X,\nu^E)\,\mathrm{d}\mu^E-\int_S g^E(X,N)\,\mathrm{d}\mu^E\right|\\&\quad\le C\big(|\Sigma|\|F-\mathrm{id}\|_{L^\infty(S)}+|\Sigma|^{3/2}\|\phi^2-1\|_{L^\infty(S)}+|\Sigma|\|\nu^E\circ F-N\|_{L^2(S)}\big)\\&\quad\le C|\Sigma|^{3/2}\|\overset{\circ}{A}{}^E\|_{L^2(\Sigma,g^E)}.\end{aligned}$$

In view of Lemma 2.5, we thus obtain the estimate

$$\left|\mathrm{Vol}^E(\Omega)-\frac{(|\Sigma|^E)^{3/2}}{6\pi^{1/2}}\right|\le C|\Sigma|^{3/2}\big(\|\overset{\circ}{A}\|_{L^2(\Sigma)}+r^2\big).$$

The assumption (2.2) implies that for the volume elements of g and g^E, we have that

$$\left|\mathrm{d}V_g-\mathrm{d}V_{g^E}\right|\le C|x|^2.$$

Combining the last two estimates with (3.7), we get

$$\left|\mathrm{Vol}(\Omega)-\frac{|\Sigma|^{3/2}}{6\pi^{1/2}}\right|\le C|\Sigma|^{3/2}\big(\|\overset{\circ}{A}\|_{L^2(\Sigma)}+r^2\big). \tag{4.7}$$

Using Corollary 3.6, we finally conclude

$$\left|\mathrm{Vol}(\Omega)-\frac{|\Sigma|^{3/2}}{6\pi^{1/2}}\right|\le Cr^2|\Sigma|^{3/2}. \tag{4.8}$$

The right-hand side of (4.4) can be expressed as a volume integral

$$\int_\Sigma g(b,\nu)\,\mathrm{d}\mu=\int_\Omega \mathrm{div}_M\, b\,\mathrm{d}V,$$

and since $|\nabla b|\le Cr$, we estimate

$$\left|\int_\Omega \mathrm{div}_M\, b\mathrm{d}V\right|\le Cr\,\mathrm{Vol}(\Omega)\le Cr|\Sigma|^{3/2}$$

Thus, since λ is bounded in view of Corollary 3.6, we obtain

$$\left|\lambda\int_\Sigma Hf\,\mathrm{d}\mu\right|\le Cr|\Sigma|^{3/2}. \tag{4.9}$$

4.2 The variation of $\mathcal{U}$

A fairly straight forward calculation shows that the variation of $\mathcal{U}$ is given by

$$\delta_f \mathcal{U}(\Sigma) = -\int_\Sigma 2\langle \mathring{A}, \nabla^2 f\rangle + 2f\langle \mathring{A}, \mathrm{Ric}^T\rangle + fH|\mathring{A}|^2 \,\mathrm{d}\mu, \tag{4.10}$$

where Ric^T denotes the tangential projection of the Ricci curvature of M on to Σ. With f as in Eq. (4.3), the second and third term are easily bounded as follows

$$\begin{aligned} &\int_\Sigma 2f\langle \mathring{A}, \mathrm{Ric}^T\rangle + fH|\mathring{A}|^2 \,\mathrm{d}\mu \\ &\quad \le C|\Sigma|^{1/2} \sup_\Sigma |f| \left(\int_\Sigma |\mathring{A}|^2 \,\mathrm{d}\mu\right)^{1/2} + C\int_\Sigma |\mathring{A}|^2 \,\mathrm{d}\mu \le C|\Sigma|^2 \end{aligned} \tag{4.11}$$

where we used the fact that $|g(b,\nu)| \le C$ together with Corollary 3.6 and Proposition 3.8.

To treat the first term in (4.10), we calculate the first and second derivatives of f. Choosing a local ON-frame $\{e_1, e_2\}$ on Σ, we obtain

$$\nabla_{e_i} f = H^{-1} g(\nabla_{e_i} b, \nu) + H^{-1} g(b, e_j) A_i^j - H^{-2} \nabla_{e_i} H g(b, \nu), \tag{4.12}$$

and thus in view of the estimates from Theorem 3.5, Proposition 3.8, and the fact that $|\nabla b| \le Cr$, we find

$$\int_\Sigma |\nabla f|^2 \,\mathrm{d}\mu \le Cr^2.$$

Differentiating Eq. (4.12) once more, we obtain

$$\begin{aligned} \nabla_{e_i}\nabla_{e_j} f = {} & -A_i^k A_{jk} f + 2H^{-3}\nabla_{e_i} H \nabla_{e_j} H g(b,\nu) - H^{-2}\nabla^2_{i,j} H g(b,\nu) \\ & + H^{-1}\big(g(\nabla_{e_i}\nabla_{e_j} b, \nu) + g(\nabla_{e_i} b, e_k) A_j^k + g(\nabla_{e_j} b, e_k) A_i^k + \nabla_{e_j} A_i^k g(b, e_k)\big) \\ & - H^{-2}\big(\nabla_{e_i} H (g(\nabla_{e_j} b, \nu) + g(b, e_k) A_j^k) + \nabla_{e_j} H \big(g(\nabla_{e_i} b, \nu) + g(b, e_k) A_i^k\big)\big). \end{aligned} \tag{4.13}$$

Our goal is to estimate $\int_\Sigma \langle \mathring{A}, \nabla^2 f\rangle \,\mathrm{d}\mu$ so that all we need of $\nabla^2 f$ is its traceless part. Note that the leading order term in expression (4.13) is the first one on the right-hand side, all others decay faster as $r \to 0$. Its contribution consists mainly of the trace part. When removing the trace, we find that we can estimate

$$\begin{aligned} |(\nabla^2 f)^\circ| \le C\big(&|A||\mathring{A}||f| + H^{-1}|\nabla^2 b| + H^{-1}|A||\nabla b| + H^{-1}|\nabla A| \\ &+ H^{-2}|\nabla H||\nabla b| + H^{-2}|\nabla H||A| + H^{-2}|\nabla^2 H| + H^{-3}|\nabla H|^2\big). \end{aligned}$$

In view of the fact that $|\nabla b| \le Cr$ and $|\nabla^2 b| \le C$, and using the estimates from Theorem 3.5, Corollary 3.6, and Proposition 3.8, we infer that

$$\int_\Sigma |(\nabla^2 f)^\circ|^2 \,\mathrm{d}\mu \le Cr^2|\Sigma|, \tag{4.14}$$

Here, we also used that by the estimates from Section 3, (3.14), and the fact that $|A|^2 = (1/2)H^2 + |\mathring{A}|^2$, we have

$$\begin{aligned}\int_\Sigma H^{-4}|\nabla H|^2|A|^2 \,\mathrm{d}\mu &\le C\left(\int_\Sigma |\nabla \log H|^4 \,\mathrm{d}\mu\right)^{1/2}\left(\int_\Sigma H^{-4}(H^4 + |\mathring{A}|^4) \,\mathrm{d}\mu\right)^{1/2}\\ &\le C|\Sigma|^{3/2}\big(|\Sigma|^{1/2} + |\Sigma|^{3/2}\big) \le C|\Sigma|^2.\end{aligned}$$

In view of the Cauchy–Schwarz inequality, Corollary 3.6, and estimate (4.14), we infer that

$$\int_\Sigma \langle \mathring{A}, \nabla^2 f\rangle \,\mathrm{d}\mu \le \left(\int_\Sigma |\mathring{A}|^2 \,\mathrm{d}\mu\right)^{1/2}\left(\int_\Sigma |(\nabla^2 f)^\circ|^2 \,\mathrm{d}\mu\right)^{1/2} \le Cr|\Sigma|^{3/2}. \tag{4.15}$$

The estimates (4.11) and (4.15) imply the desired bound on (4.10), namely

$$|\delta_f \mathcal{U}(\Sigma)| \le Cr|\Sigma|^{3/2}. \tag{4.16}$$

4.3 **The variation of $\mathcal{V}(\Sigma)$**

In Section 5.5 of [7], the following expression for $\delta_f\mathcal{V}(\Sigma)$ was derived:

$$\delta_f\mathcal{V}(\Sigma) = \int_\Sigma -fHG(\nu,\nu) - \frac{1}{2}fH\,\mathrm{Sc} + 2f\langle \mathring{A}, G^T\rangle - 2\omega(\nabla f)\,\mathrm{d}\mu \tag{4.17}$$

where as before, $G = \mathrm{Ric} - (1/2)\,\mathrm{Sc}\, g$ denotes the Einstein tensor of M and $\omega = \mathrm{Ric}(\nu,\cdot)^T$. Recall that we chose $f = H^{-1}g(b,\nu)$ above. In the expression (4.12), we split $A = \mathring{A} + (1/2)H\gamma$ and obtain

$$\nabla_{e_i} f = \frac{1}{2}g(b, e_i) + H^{-1}g(\nabla_{e_i} b, \nu) + H^{-1}\mathring{A}_i^j g(b, e_j) - H^{-2}\nabla H g(b,\nu).$$

Plugging this expression into Eq. (4.17) yields

$$\begin{aligned}\delta_f\mathcal{V}(\Sigma) = \int_\Sigma &-G(b,\nu) - \tfrac{1}{2}g(b,\nu)\,\mathrm{Sc} + 2f\langle \mathring{A}, G^T\rangle\\ &-2w(e_i)\big(H^{-1}g(\nabla_{e_i} b, \nu) + H^{-1}\mathring{A}_i^j g(b, e_j) - H^{-2}\nabla H g(b,\nu)\big)\,\mathrm{d}\mu.\end{aligned} \tag{4.18}$$

Using Theorem 3.5, Corollary 3.6, and Proposition 3.8, we estimate

$$\begin{aligned}\Bigg|\int_\Sigma 2f\langle \mathring{A}, G^T\rangle - 2w(e_i)\big(H^{-1}g(\nabla_{e_i}b,\nu)\\ + H^{-1}\mathring{A}_i^j g(b,e_j) - H^{-2}\nabla H g(b,\nu)\big)\,\mathrm{d}\mu\Bigg| \le Cr|\Sigma|^{3/2}.\end{aligned} \tag{4.19}$$

In [7] the Pohozaev identity was used to estimate the term $\int_\Sigma G(b,\nu)\,\mathrm{d}\mu$. This is not really necessary here; the following simpler approach is sufficient. Recall that the divergence of G with respect to the g-metric is zero due to the Bianchi identity. Define the vector field X by the requirement that $g(X,Y) = G(b,Y)$ for all vector fields Y on B_ρ. Then the fact that G is divergence-free implies that

$$\mathrm{div}_M X = \langle G, \nabla b\rangle. \tag{4.20}$$

In Section 4.1, we used that Σ bounds a region Ω with $\mathrm{Vol}(\Omega) \le C|\Sigma|^{3/2}$. To proceed, we integrate the relation (4.20) over Ω, and after integration by parts, we get (recall that $|\nabla b| \le Cr$)

$$\left|\int_\Sigma G(b,\nu)\,\mathrm{d}\mu\right| = \left|\int_\Omega \mathrm{div}_M X \mathrm{d}V\right| \le Cr\,\mathrm{Vol}(\Omega) \le Cr|\Sigma|^{3/2}.$$

In combination with Eq. (4.18) and estimate (4.19), we infer that

$$\left|\delta_f \mathcal{V}(\Sigma) + \tfrac{1}{2}\int_\Sigma g(b,\nu)\,\mathrm{Sc}\,\,\mathrm{d}\mu\right| \le Cr|\Sigma|^{3/2}.$$

The final task is to estimate $\int_\Sigma g(b,\nu)\,\mathrm{Sc}\,\mathrm{d}\mu$. As before, we express this surface integral as a volume integral. To this end, we consider the vector field $X = \mathrm{Sc}\, b$ and calculate

$$\mathrm{div}_M X = g(b, \nabla\,\mathrm{Sc}) + \mathrm{Sc}\,\mathrm{div}_M b.$$

Since $\nabla\,\mathrm{Sc} = \nabla\,\mathrm{Sc}(0) + O(r)$, $\nabla b = O(r)$, and $g = g^E + O(r^2)$, we infer

$$\mathrm{div}_M X = g^E(b, \nabla\,\mathrm{Sc}(0)) + O(r).$$

Thus, we can calculate

$$\int_\Sigma g(b,\nu)\,\mathrm{Sc}\,\,\mathrm{d}\mu = \int_\Omega \mathrm{div}_M X\mathrm{d}V = \mathrm{Vol}(\Omega) g^E(b, \nabla\,\mathrm{Sc}(0)) + O(r\,\mathrm{Vol}(\Omega)).$$

This finally leaves us with the estimate

$$\left|\delta_f \mathcal{V}(\Sigma) + \frac{1}{2}\operatorname{Vol}(\Omega) g^E(b, \nabla \operatorname{Sc}(0))\right| \leq Cr|\Sigma|^{3/2}. \tag{4.21}$$

4.4 The conclusion

To prove Theorem 1.1, we combine the results from the previous sections. Combining Eq. (4.2) with the estimates (4.9) and (4.16) yields that for f as in (4.3), we have

$$\left|\delta_f \mathcal{V}(\Sigma)\right| \leq Cr|\Sigma|^{3/2}.$$

In combination with (4.21), this gives

$$\left|\operatorname{Vol}(\Omega) g^E(b, \nabla \operatorname{Sc}(0))\right| \leq Cr|\Sigma|^{3/2},$$

and since $\operatorname{Vol}(\Omega) \geq C^{-1}|\Sigma|^{3/2}$, we infer

$$\left|g^E(b, \nabla \operatorname{Sc}(0))\right| \leq Cr.$$

Setting $b = \nabla \operatorname{Sc}(0)/|\nabla \operatorname{Sc}(0)|^E$ finally shows that

$$|\nabla \operatorname{Sc}(0)|^E \leq Cr.$$

Since we can let $r \to 0$ by the assumptions of Theorem 1.1, we infer the claim, namely that $\nabla \operatorname{Sc}(0) = 0$.

5 Expansion of the Willmore Functional

In this section, we calculate the expansion of the Willmore functional on small surfaces using the estimates from Section 3. We wish to emphasize here that similar expansions for the Willmore functional have been computed previously for geodesic spheres in [4, Section 3], where also the subsequent term in the expansion is calculated, and for perturbations of geodesic spheres in [9].

The calculation here has the advantage that it works under much more general conditions. Namely, we have the following theorem which holds in particular for surfaces as in Theorem 1.1 due to the estimates of Lemma 3.1 and Corollary 3.6.

Theorem 5.1. Let $g = g^E + h$ on B_ρ be given, and let $c < \infty$ be a constant. Then there exists a constant C depending only on c, ρ, and h_0 as in Eq. (2.2), such that the following holds.

Let $\Sigma \subset B_r$ be a spherical surface with $r < \rho$ such that

$$\mathcal{U}(\Sigma) \leq cr|\Sigma| \qquad \text{and} \qquad |\Sigma| \leq cr.$$

Then the following estimate holds:

$$\left|\mathcal{W}(\Sigma) - 8\pi + \frac{|\Sigma|}{3}\operatorname{Sc}(0)\right| \leq Cr|\Sigma|.$$

☐

Proof. We use the Gauss equation to express the Willmore functional as in Eq. (2.5)

$$\mathcal{W}(\Sigma) = 8\pi + \mathcal{U}(\Sigma) + \mathcal{V}(\Sigma).$$

By the first assumption, the term $\mathcal{U}(\Sigma)$ is a lower order term and can be neglected. Furthermore,

$$\mathcal{V}(\Sigma) = 2\int_\Sigma G(\nu,\nu)\,\mathrm{d}\mu = \int_\Sigma 2\operatorname{Ric}(\nu,\nu) - \operatorname{Sc}\,\mathrm{d}\mu.$$

In view of the assumptions of the theorem, Lemma 3.3 implies that

$$\left|\int_\Sigma \operatorname{Ric}(\nu,\nu)\,\mathrm{d}\mu - \frac{|\Sigma|}{3}\operatorname{Sc}(0)\right| \leq Cr|\Sigma|.$$

Since $\operatorname{Sc} = \operatorname{Sc}(0) + O(r)$, we furthermore have

$$\int_\Sigma \operatorname{Sc}\,\mathrm{d}\mu = |\Sigma|\operatorname{Sc}(0) + O(r|\Sigma|)$$

so that in combination

$$\mathcal{V}(\Sigma) = -\frac{|\Sigma|}{3}\operatorname{Sc}(0) + O(r|\Sigma|).$$

Altogether, this yields

$$\mathcal{W}(\Sigma) = 8\pi - \frac{|\Sigma|}{3}\operatorname{Sc}(0) + O(r|\Sigma|),$$

which is the desired expansion. ■

Corollary 5.2. Let g be as in Theorem 5.1, and assume that Σ is a spherical surface in B_r satisfying

$$\mathcal{U}(\Sigma) \le c|\Sigma|^2 \qquad \text{and} \qquad |\Sigma| \le cr.$$

If Ω denotes the region bounded by Σ, then the Hawking mass of Σ,

$$m_H(\Sigma) = \frac{|\Sigma|^{1/2}}{(16\pi)^{3/2}}\big(16\pi - 2\mathcal{W}(\Sigma)\big)$$

satisfies

$$\frac{m_H(\Sigma)}{\mathrm{Vol}(\Omega)} = \frac{\mathrm{Sc}(0)}{16\pi} + O(r).$$ □

Proof. This is a simple consequence of the expansion in Theorem 5.1 which holds also under the stronger assumption of the corollary. In addition, we use the fact that the volume of Ω satisfies

$$\left|\mathrm{Vol}(\Omega) - \frac{|\Sigma|^{3/2}}{6\pi^{1/2}}\right| \le Cr|\Sigma|^{3/2}$$

which follows from Eq. (4.7). ■

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