Decay estimates for the quadratic tilt-excess of integral varifolds

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Abstract

This paper concerns integral varifolds of arbitrary dimension in an open subset of Euclidean space with its first variation given by either a Radon measure or a function in some Lebesgue space. Pointwise decay results for the quadratic tilt-excess are established for those varifolds. The results are optimal in terms of the dimension of the varifold and the exponent of the Lebesgue space in most cases, for example if the varifold is not two-dimensional.

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Introduction

Overview This paper investigates pointwise regularity properties of integral varifolds satisfying integrability conditions on its generalised mean curvature where pointwise regularity is measured by the decay of the quadratic tilt-excess. As classical regularity may fail on a set of positive measure, see Allard [All72, 8.1 (2)] and Brakke [Bra78, 6.1], the notion of tilt-excess decay serves as a weak measure of regularity suitable for studying regularity near almost every point of a varifold. In fact, aside from being used as an intermediate step to classical regularity, see Allard [All72], decay estimates have been employed as a tool for both perpendicularity of mean curvature in Brakke [Bra78] and locality of mean curvature in Schätzle [Sch09, Sch04, Sch01].

In the present paper it is established that there is a qualitative change in the nature of the results obtainable when the Sobolev exponent corresponding to the integrability exponent of the mean curvature drops below 2. The core of the proof of the pointwise results relies on the harmonic approximation procedure introduced by de Giorgi in [DG61] (see also [DG06, p. 231–263]) and Almgren in [Alm68] and used in the present setting by Allard in [All72] and Brakke in [Bra78]. Additionally, to obtain the present pointwise results, a new coercive estimate is proven, the Sobolev Poincaré type estimates of [Men09b] are adapted and a new iteration procedure is introduced.

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Known results The notation follows Federer [Fed69] and, concerning varifolds, Allard [All72], see Section 1.

Hypotheses. Suppose m and n are positive integers, $m < n$, $1 \leq p \leq \infty$, U is an open subset of $\mathbf{R}^n$, $V \in \mathbf{IV}_m(U)$, $\|\delta V\|$ is a Radon measure and, if $p > 1$,

$$(\delta V)(g) = -\textstyle\int g(z) \bullet \mathbf{h}(V; z) \, \mathrm{d}\|V\|(z) \quad \text{whenever } g \in \mathscr{D}(U, \mathbf{R}^n),$$
$$\mathbf{h}(V; \cdot) \in \mathbf{L}_p(\|V\| \llcorner K, \mathbf{R}^n) \quad \text{whenever } K \text{ is a compact subset of } U.$$

Consider the question for which $0 < \alpha \leq 1$ the given hypotheses imply

$$\limsup_{r \to 0+} r^{-\alpha - m/2} \big(\textstyle\int_{\mathbf{U}(a,r) \times \mathbf{G}(n,m)} |S_\natural - T_\natural|^2 \, \mathrm{d}V(z, S) \big)^{1/2} < \infty$$

for V almost all $(a, T) \in U \times \mathbf{G}(n, m)$. Brakke has shown that one can take any $0 < \alpha < 1$ in case $p = 2$ and $\alpha = 1/2$ with "$< \infty$" replaced by "$= 0$" in case $p = 1$ in [Bra78, 5.5, 7]. Schätzle [Sch04] has used results on viscosity solutions from Caffarelli [Caf89] and Trudinger [Tru89] to establish several regularity results, in particular that if $p > m$, $p \geq 2$ and $n - m = 1$ then one can take $\alpha = 1$, see also Schätzle [Sch01] for a special case. Moreover, Schätzle showed in [Sch09, Theorem 3.1] that if $p = 2$ then the key to the general case is to prove existence of an approximate second order structure of the varifold. Namely, if $p = 2$ and there exists a countable collection C of m dimensional submanifolds of $\mathbf{R}^n$ of class 2 with $\|V\|(U \sim \bigcup C) = 0$ then one can take $\alpha = 1$.

Whereas consideration of varifolds associated to submanifolds of class 2 clearly shows that $\alpha = 1$ is the largest α possibly having this property, in case $\sup\{2, p\} < m$ and $\frac{mp}{m-p} < 2$ it can be seen from the examples in [Men09a, 1.2] that one cannot take $\alpha > \frac{mp}{2(m-p)}$. Comparing this to Brakke's results, little is known for the case $1 < p < 2$ and also in case $p = 1$ and $m > 2$ there is a gap between known positive results for $\alpha \leq 1/2$ and known counterexamples for $\alpha > \frac{m}{2(m-1)}$.

Results of the present paper In case $2 < m < p$ and $\frac{mp}{m-p} < 2$ these gaps are closed by the following corollary.

8.6 Corollary. *Suppose m, n, p, U, and V are as in the preceding hypotheses, and either $m = 2$ and $0 < \tau < 1$ or $\sup\{2, p\} < m$ and $\tau = \frac{mp}{2(m-p)} < 1$.*

Then there holds for V almost all $(a, T) \in U \times \mathbf{G}(n, m)$ that

$$\limsup_{r \to 0+} r^{-\tau - m/2} \big(\textstyle\int_{\mathbf{U}(a,r) \times \mathbf{G}(n,m)} |S_\natural - T_\natural|^2 \, \mathrm{d}V(z, S) \big)^{1/2} < \infty.$$

From the afore-mentioned examples it follows that τ cannot be replaced by any larger number if $m > 2$, see 8.7. However, it will be shown in [Men09c] that "$< \infty$" can be replaced by "$= 0$". The corollary is a direct consequence of the following pointwise result.

8.3 Theorem. *Suppose m, n, and p are as in the preceding hypotheses, Q is a positive integer, either $p = m = 1$ or $1 \leq p < m$, $0 < \delta \leq 1$, $0 < \alpha \leq 1$, $0 < \tau \leq 1$, and $\tau = 1$ if $m = 1$, $p/2 \leq \tau < \frac{mp}{2(m-p)}$ if $m = 2$ and $\tau = \frac{mp}{2(m-p)}$ if $m > 2$.*

Then there exist positive, finite numbers ε and Γ with the following property.

If $a \in \mathbf{R}^n$, $0 < r < \infty$, $V \in \mathbf{IV}_m(\mathbf{U}(a,r))$, *$V$ is related to p as in the preceding hypotheses, ψ is the measure defined by* $\psi = \|\delta V\|$ *if* $p = 1$ *and* $\psi = |\mathbf{h}(V;\cdot)|^p \|V\|$ *if* $p > 1$, $T \in \mathbf{G}(n,m)$, $\omega : \mathbf{R} \cap \{t : 0 < t \leq 1\} \to \mathbf{R}$ *with* $\omega(t) = t^{\alpha\tau}$ *if* $\alpha\tau < 1$ *and* $\omega(t) = t(1 + \log(1/t))$ *if* $\alpha\tau = 1$ *whenever* $0 < t \leq 1$, *and* $0 < \gamma \leq \varepsilon$,

$$\mathbf{\Theta}^{*m}(\|V\|, a) \geq Q - 1 + \delta, \quad \|V\| \, \mathbf{U}(a,r) \leq (Q + 1 - \delta)\boldsymbol{\alpha}(m) r^m,$$
$$\left(r^{-m} \textstyle\int |S_\natural - T_\natural|^2 \, \mathrm{d}V(z,S)\right)^{1/2} \leq \gamma,$$
$$\|V\|(\mathbf{B}(a,\varrho) \cap \{z : \mathbf{\Theta}^m(\|V\|, z) \leq Q - 1\}) \leq \varepsilon \boldsymbol{\alpha}(m) \varrho^m \quad \text{for } 0 < \varrho < r,$$
$$\varrho^{1-m/p} \psi(\mathbf{B}(a,\varrho))^{1/p} \leq \gamma^{1/\tau} (\varrho/r)^\alpha \quad \text{for } 0 < \varrho < r,$$

then $\mathbf{\Theta}^m(\|V\|, a) = Q$, $R = \operatorname{Tan}^m(\|V\|, a) \in \mathbf{G}(n,m)$ *and*

$$\left(\varrho^{-m} \textstyle\int_{\mathbf{U}(a,\varrho) \times \mathbf{G}(n,m)} |S_\natural - R_\natural|^2 \, \mathrm{d}V(z,S)\right)^{1/2} \leq \Gamma \gamma \omega(\varrho/r) \quad \text{whenever } 0 < \varrho \leq r.$$

In order to explain this theorem, assume $m > 2$.

In case $\frac{mp}{m-p} = 2$, the theorem states that if the mean curvature expressed in terms of ψ decays with power $\alpha < 1$ so does the tilt-excess of the varifold provided essentially that the tilt-excess is initially small and the density, restricted to the complement of a set with small density at a, is lower semicontinuous at a. If $\alpha = 1$, the modulus of continuity ω obtained is optimal as demonstrated by an example in 8.5, in particular one cannot take $\omega(t) = t$. Moreover, this sharp result seems not to be obtainable using classical excess decay methods as will be explained below.

In the case $\frac{mp}{m-p} < 2$, the situation is different. Comparing it to the case of a weakly differentiable function $u : \mathbf{R}^m \to \mathbf{R}^{n-m}$ whose distributional Laplacian is given by a function locally in $\mathbf{L}_p(\mathscr{L}^m, \mathbf{R}^{n-m})$, the analogous quantity to prove decay for would be

$$\left(\varrho^{-m} \textstyle\int_{\mathbf{U}(c,\varrho)} |\mathbf{D}u(x) - \mathbf{D}u(c)|^2 \, \mathrm{d}\mathscr{L}^m x\right)^{1/2}$$

for $c \in \mathbf{R}^m$, $0 < \varrho < \infty$. However, this quantity not even needs to be finite. Still, in the varifold case, decay of the mean curvature with power α implies, under the same assumptions as before, decay of the tilt-excess with some smaller power $\alpha\tau$ with $\tau = \frac{mp}{2(m-p)}$. This number τ cannot be replaced by any larger number, see 8.4.

Overview of proof As indicated above the main tool in the pointwise regularity proof is the harmonic approximation procedure introduced by de Giorgi and Almgren, see [DG61, DG06, Alm68]. It requires the varifold to be weakly close to a plane with density Q and strongly close to a varifold with density at least Q. Initially, the latter condition was phrased as $\mathbf{\Theta}^m(\|V\|, z) \geq Q$ for $\|V\|$ almost all $z \in \mathbf{U}(a,r)$ in Allard [All72, §8], however the set of points a not satisfying this condition for suitable Q and r may have positive $\|V\|$ measure even if the hypotheses are satisfied with $p = \infty$, see Allard [All72, 8.1 (2)] and Brakke [Bra78, 6.1]. Replacing the condition by the requirement on $\mathbf{\Theta}^m(\|V\|, \cdot)$ to be $\|V\|$ approximately (lower semi) continuous, Brakke was able to treat almost all points with $p = 2$ using an approximation by Almgren's "Q-valued" functions, i.e. functions with values in $\mathbf{Q}_Q(\mathbf{R}^{n-m})$, see below. Additionally, Brakke established a coercive estimate which allowed him to obtain partial results also for the case $p = 1$.

Taking this as a starting point, it will be described, firstly, the new ingredient needed to obtain the optimal modulus of continuity for the case $p = 2$, secondly, the new ingredient needed to obtain optimal results in case $p < 2$ and, thirdly, how these new ingredients can be implemented within the known framework of a (partial or pointwise) regularity proof.

Obtaining the optimal modulus of continuity for $p = 2$ For this purpose a new iteration procedure is introduced which is now presented in the simple case of the Laplace operator. Suppose $c \in \mathbf{R}^m$, $u \in \mathbf{W}^{1,2}(\mathbf{U}(c,1), \mathbf{R}^{n-m})$, $T \in \mathscr{D}'(\mathbf{U}(c,1), \mathbf{R}^{n-m})$,

$$T(\theta) = -\textstyle\int_{\mathbf{U}(c,1)} D\theta \bullet \mathbf{D}u \,\mathrm{d}\mathscr{L}^m \quad \text{for } \theta \in \mathscr{D}(\mathbf{U}(c,1), \mathbf{R}^{n-m}),$$

i.e. T is the distributional Laplace of u, and assume for some $0 \leq \gamma < \infty$ and $0 < \alpha \leq 1$ that

$$\varrho^{-m/2} |T(\theta)| \leq \gamma \varrho^\alpha |D\theta|_{2;c,\varrho}$$

whenever $\theta \in \mathscr{D}(\mathbf{U}(c,1), \mathbf{R}^{n-m})$ with $\operatorname{spt}\theta \subset \mathbf{U}(c,\varrho)$ and $0 < \varrho \leq 1$, where $|f|_{p;c,\varrho}$ denotes the seminorm of $|f| \in \mathbf{L}_p(\mathscr{L}^m \llcorner \mathbf{U}(c,\varrho))$. Define $J = \mathbf{R} \cap \{r : 0 < \varrho \leq 1\}$, for each $\varrho \in J$ choose $u_\varrho : \mathbf{U}(c,\varrho) \to \mathbf{R}^{n-m}$ harmonic with boundary values given by u, i.e.

$$u_\varrho \in \mathscr{E}(\mathbf{U}(c,\varrho), \mathbf{R}^{n-m}) \quad \text{with} \quad \operatorname{Lap} u_\varrho = 0,$$
$$u - u_\varrho \in \mathbf{W}_0^{1,2}(\mathbf{U}(c,\varrho), \mathbf{R}^{n-m}),$$

define $\phi_1 : J \to \mathbf{R}$ and $\phi_2 : J \times \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m}) \to \mathbf{R}$ by

$$\phi_1(\varrho) = |D^2 u_\varrho|_{\infty;c,\varrho/2}, \quad \phi_2(\varrho,\sigma) = \varrho^{-m/2} |\mathbf{D}(u - \sigma)|_{2;c,\varrho}$$

for $(\varrho,\sigma) \in J \times \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m})$ and choose $\sigma_\varrho \in \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m})$ such that

$$\phi_2(\varrho, \sigma_\varrho) \leq \phi_2(\varrho, \sigma) \quad \text{whenever } \sigma \in \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m}),\ \varrho \in J.$$

Using a priori estimates, see [GT01, Theorems 7.26 (ii), 8.10, 9.11], one estimates

$$\begin{aligned}\phi_1(\varrho/4) - \phi_1(\varrho) &\leq |D^2(u_\varrho - u_{\varrho/4})|_{\infty;c,\varrho/8} \leq \Delta \varrho^{-1-m/2} |D(u_\varrho - u_{\varrho/4})|_{2;c,\varrho/4} \\ &\leq \Delta \varrho^{-1-m/2} \big(|\mathbf{D}(u - u_{\varrho/4})|_{2;c,\varrho/4} + |\mathbf{D}(u - u_\varrho)|_{2;c,\varrho} \big) \leq 2\Delta\gamma\varrho^{\alpha-1}\end{aligned}$$

for some positive, finite number Δ depending only on n and

$$\begin{aligned}\phi_2(\varrho, \sigma_\varrho) &\leq \varrho^{-m/2} \big(|\mathbf{D}(u - u_\varrho)|_{2;c,\varrho} + |D(u_\varrho - Du_\varrho(c))|_{2;c,\varrho} \big) \\ &\leq \gamma\varrho^\alpha + \boldsymbol{\alpha}(m)^{1/2} \varrho \phi_1(\varrho),\end{aligned}$$

hence obtains the two *iteration inequalities*

$$\phi_1(\varrho/4) \leq \phi_1(\varrho) + \Gamma\gamma\varrho^{\alpha-1}, \quad \phi_2(\varrho, \sigma_\varrho) \leq \Gamma\big(\varrho\phi_1(\varrho) + \gamma\varrho^\alpha\big)$$

for $\varrho \in J$ where $\Gamma = \sup\{2\Delta, 1, \boldsymbol{\alpha}(m)^{1/2}\}$.

Now, if $0 \leq \gamma_1 < \infty$, $\phi_1(\varrho) \leq \gamma_1 \varrho^{\alpha-1}$ and $\alpha < 1$ then

$$\phi_1(\varrho/4) \leq (\varrho/4)^{\alpha-1}\big(4^{\alpha-1}\gamma_1 + \Gamma\gamma\big) \leq \gamma_1(\varrho/4)^{\alpha-1}$$

provided $\gamma_1 \geq (1-4^{\alpha-1})^{-1}\Gamma\gamma$, noting $4^{\alpha-1} < 1$. Similarly, if $0 \leq \gamma_1 < \infty$, $\phi_1(\varrho) \leq \gamma_1(1+\log(1/\varrho))$ and $\alpha = 1$ then

$$\phi_1(\varrho/4) \leq \gamma_1(1+\log(4/\varrho)) - (\log 4)\gamma_1 + \Gamma\gamma \leq \gamma_1(1+\log(4/\varrho))$$

provided $\gamma_1 \geq \Gamma\gamma(\log 4)^{-1}$. In both cases it has been used crucially that the factor in front of $\phi_1(\varrho)$ in the first iteration inequality is 1. This is the reason for choosing ϕ_1 rather than ϕ_2 as leading iteration quantity. The decay of $\phi_2(\varrho, \sigma_\varrho)$ in terms of ϱ then follows.

Classically, an excess decay inequality of type

$$\phi_2(\lambda\varrho, \sigma_{\lambda\varrho}) \leq \Gamma_1\lambda\phi_2(\varrho, \sigma_\varrho) + \Gamma_2\gamma\varrho^\alpha \quad \text{for } 0 < \lambda \leq 1/2,\ 0 < \varrho \leq 1$$

where $1 \leq \Gamma_i < \infty$ for $i \in \{1,2\}$ is used, see e.g. [Fed69, 5.3.13] or Duzaar and Steffen [DS02, (5.14)]. Sometimes, Γ_2 additionally depends on λ. However, concerning the case $\alpha = 1$, the optimal modulus of continuity cannot be deduced from such an inequality since if $1 < \Gamma_1 < \infty$ and $1/e < \Gamma_2 < \infty$ then it does not exclude that $\phi_2(\varrho, \sigma_\varrho)$ may equal $\gamma\varrho(1+\log(1/\varrho))^s$ for some $s > 1$ with $2^{s-1} \leq \Gamma_1$ and $(2s/e)^s \leq 2\Gamma_2$.

Treating the case $p < 2$ The second new ingredient in the regularity proof will be described focusing on the case $m > 2$. In doing so, a quantity of type

$$\varrho^{-1-m/q}\big(\textstyle\int_{\mathbf{B}(a,\varrho)} \operatorname{dist}(z-a, T)^q \,\mathrm{d}\|V\|z\big)^{1/q}$$

for U and V as in the hypotheses with $a \in \mathbf{R}^n$, $0 < \varrho < \infty$, $\mathbf{B}(a,\varrho) \subset U$, $T \in \mathbf{G}(n,m)$ and $1 \leq q < \infty$ will be referred to as q-height. To derive sharp results with respect to the integrability of the mean curvature two observations will be essential. Firstly, the dependence on the mean curvature in Brakke's coercive estimate, see [Bra78, 5.5], can be improved at the price of using the q-height with $q = \frac{2m}{m-2}$ instead of the 2-height, see 3.13. Secondly, in order to control the q-height, the Sobolev Poincaré type estimates of [Men09b] are adapted. However, a subtlety arises. The mentioned estimates are in full strength only available for the q-height on the set H of points satisfying a smallness condition on the mean curvature, see also the discussion in [Men09b, 3.6]. As estimating the q-height on the complement of H by mean curvature would be insufficient for the present purpose, the coercive estimate of Brakke has to be improved a second time by showing the q-height on H, mean curvature and 2-height are actually sufficient to control the tilt-excess, see 3.9. This is accomplished by constructing a possibly noncontinuous cut-off function with properties reminiscent of a weakly differentiable function, including a partial integration formula, Sobolev embedding and approximate differentiability, see 3.6 and 3.7. These properties are deduced directly from the construction rather than from a general theory.

Implementation of proof Finally, it will be indicated briefly how the previously described pieces fit into the well known pattern of a partial regularity proof. As usual, one assumes the varifold to be close to Q parallel planes with respect to mass, tilt-excess and first variation. Fixing a suitable orthogonal coordinate system, one approximates the varifold by a Lipschitzian $\mathbf{Q}_Q(\mathbf{R}^{n-m})$ valued function f. Recall that $\mathbf{Q}_Q(\mathbf{R}^{n-m})$ may be described as the Q fold product of $\mathbf{R}^{n-m}$ divided by the action of the group of permutations of $\{1, \ldots, Q\}$. The accuracy of this approximation is controlled by tilt-excess and mean curvature. To obtain the comparison functions u_ϱ, one considers the Dirichlet problem with the linear elliptic system with constant coefficients given by a suitable linearisation of the nonparametric area integrand and boundary values given by the "average" g of f. This is somewhat different from the usual procedure where the comparison functions are often constructed either within contradiction arguments (see e.g. Allard [All72, 8.16] or Brakke [Bra78, 5.6]) or by an "A-harmonic approximation lemma" which confines the contradiction argument to the situation of linear systems with constant coefficients (see e.g. Simon [Sim83, 21.1] or Duzaar and Steffen [DS02, 3.3]); however see also Schoen and Simon [SS82] for a different approach. The distributional right hand side for $g - u_\varrho$ can be estimated by mean curvature and a small multiple of the tilt-excess provided a suitable weak norm is employed, namely a norm dual to the norm mapping a smooth function with compact support to the $\mathbf{L}_\infty(\mathscr{L}^m, \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m}))$ norm of its derivatives. This only yields smallness of $g - u_\varrho$ in Lebesgue spaces with exponent below $\frac{m}{m-1}$ if $m > 1$, e.g. in $\mathbf{L}_1(\mathscr{L}^m \llcorner \mathbf{U}(c, \varrho), \mathbf{R}^{n-m})$, here $c \in \mathbf{R}^m$ corresponds to $a \in \mathbf{R}^n$, see 7.4 (7). However, assuming that the set of points with density strictly below Q is small with respect to $\|V\|$, the graph of g coincides with the varifold on a large set, hence using interpolation (Section 5) and estimates for the approximation by f (see Section 4), one can ultimately convert $\mathbf{L}_1(\mathscr{L}^m \llcorner \mathbf{U}(c, \varrho), \mathbf{R}^{n-m})$ closeness of g to an affine function via the coercive estimate to control of the tilt-excess of the varifold with respect to the corresponding plane.

From these estimates one readily obtains modified versions of the iteration inequalities which – upon simultaneous iteration – yield the result.

Organisation of the paper Section 1 introduces the notation and is followed by Section 2 where definitions and basic properties of $\mathbf{Q}_Q(V)$ valued functions are given. In Section 3 the coercive estimate is established. In Section 4 an approximation of an integral varifold by $\mathbf{Q}_Q(\mathbf{R}^{n-m})$ valued functions is carried out, in particular providing the required adaptions of [Men09b]. In Section 5 an interpolation inequality allowing to neglect certain small exceptional sets is proven. In Section 6 some standard estimates for linear elliptic equations are gathered. In Section 7 the core estimates and the iteration procedure are carried out. Section 8 provides the pointwise regularity theorem and its corollary.

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1 Notation

The notation follows [Fed69], see the list of symbols on pp. 669–671 therein. In particular, recall the following maybe less common symbols: $\mathscr{P}$ denoting the positive integers, $\mathbf{U}(a,r)$ and $\mathbf{B}(a,r)$ denoting respectively the open and closed ball with centre a and radius r, $\bigodot^i(V,W)$ and $\bigodot^i V$ denoting the vector space of all i linear symmetric functions (forms) mapping V^i into W and $\mathbf{R}$ respectively, and the seminorms $\phi_{(p)}$ for $1 \leq p \leq \infty$ corresponding to the Lebesgue spaces

$$\phi_{(p)}(f) = \left(\textstyle\int |f|^p \,\mathrm{d}\phi\right)^{1/p} \quad \text{in case } 1 \leq p < \infty,$$
$$\phi_{(\infty)}(f) = \inf(\mathbf{R} \cap \{t : \phi(X \cap \{x : |f(x)| > t\}) = 0\})$$

whenever ϕ measures X, Y is a Banach space, and $f : X \to Y$ is ϕ measurable, see [Fed69, 2.2.6, 2.8.1, 1.10.1, 2.4.12]. The notation for the Lebesgue seminorms is particularly convenient when longer expressions replace the measure ϕ as will repeatedly be the case in 4.8 (8).

Additionally, the following symbols are taken from Allard [All72, 2.3, 2.5, 3.1, 3.5, 4.2]: $T^\perp$, $\mathbf{G}_m$, $\mathbf{V}_m$, $\mathbf{RV}_m$, $\mathbf{IV}_m$, $\|V\|$, δV, and $\|\delta V\|$. Moreover, the following slight modifications and additions apply. (For the convenience of the reader in this section for nearly every symbol the appropriate reference to its definition in [Fed69] is given at its first occurrence.)

Following [Kel55, p. 8], one defines $f[A] = \{y : (x,y) \in f \text{ for some } y \in A\}$ whenever f is a relation and A is a set.

Following Almgren [Alm00, T.1 (9)], for $m, n \in \mathscr{P}$, $m \leq n$, $T \in \mathbf{G}(n,m)$, $T_\natural$ is characterised by, see [Fed69, 2.2.6, 1.6.2],

$$T_\natural \in \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n), \quad T_\natural = T_\natural^*, \quad T_\natural \circ T_\natural = T_\natural, \quad \operatorname{im} T_\natural = T.$$

Similar to Allard's definition in [All72, 8.10], the *closed cuboid* $\mathbf{C}(T,a,r,h)$ is defined by

$$\mathbf{C}(T,a,r,h) = \mathbf{R}^n \cap \{z : |T_\natural(z-a)| \leq r \text{ and } |T_\natural^\perp(z-a)| \leq h\}$$

whenever $m, n \in \mathscr{P}$, $m < n$, $T \in \mathbf{G}(n,m)$, $a \in \mathbf{R}^n$, $0 < r < \infty$, and $0 < h \leq \infty$. One abbreviates $\mathbf{C}(T,a,r,\infty) = \mathbf{C}(T,a,r)$.[1]

Also, following Almgren [Alm86, p. 464], whenever $n \in \mathscr{P}$ the number $\boldsymbol{\beta}(n)$ denotes the least positive integer with the following property, see [Fed69, 2.8.14]: If F is a family of closed balls in $\mathbf{R}^n$ with $\sup\{\operatorname{diam} S : S \in F\} < \infty$ then there exist disjointed subfamilies $F_1, \ldots, F_{\boldsymbol{\beta}(n)}$ of F such that, see [Fed69, 2.8.8, 2.8.1],

$$\{z : \mathbf{B}(z,r) \in F \text{ for some } 0 < r < \infty\} \subset \bigcup \bigcup \{F_i : i = 1, \ldots, \boldsymbol{\beta}(n)\}.$$

As in [Men09a, 2.3] whenever $m \in \mathscr{P}$ the smallest number with the following property will be denoted by $\boldsymbol{\gamma}(m)$: If $n \in \mathscr{P}$, $m \leq n$, $V \in \mathbf{RV}_m(\mathbf{R}^n)$, $\|V\|(\mathbf{R}^n) < \infty$, and $\|\delta V\|(\mathbf{R}^n) < \infty$, then

$$\|V\|(\mathbf{R}^n \cap \{z : \boldsymbol{\Theta}^m(\|V\|, z) \geq 1\}) \leq \boldsymbol{\gamma}(m) \|V\|(\mathbf{R}^n)^{1/m} \|\delta V\|(\mathbf{R}^n).$$

Whenever $m, n \in \mathscr{P}$, $m \leq n$, U is an open subset of $\mathbf{R}^n$, $V \in \mathbf{V}_m(U)$, and $\|\delta V\|$ is a Radon measure, the *generalised mean curvature vector of V at z* is defined to be the unique $\mathbf{h}(V;z) \in \mathbf{R}^n$ such that

$$\mathbf{h}(V;z) \bullet v = - \lim_{r \to 0+} \frac{(\delta V)(b_{z,r} \cdot v)}{\|\delta V\| \, \mathbf{B}(z,r)} \quad \text{for } v \in \mathbf{R}^n$$

[1] The symbol $\mathbf{C}(T,a,r)$ is used by Allard in [All72, 8.10] to denote $\mathbf{R}^n \cap \{z : |T_\natural(z-a)| < r\}$.

where $b_{z,r}$ is the characteristic function of $\mathbf{B}(z,r)$, see [Fed69, 1.7.1]; hence $z \in \operatorname{dmn} \mathbf{h}(V;\cdot)$ if and only if the above limit exists for every $v \in \mathbf{R}^n$. This definition is adapted from Allard [All72, 4.3] in the spirit of [Fed69, 4.1.7].

Whenever ϕ measures X, $0 < \phi(A) < \infty$, Y is a Banach space, and $f \in \mathbf{L}_1(\phi \llcorner A, Y)$ the symbol $\fint_A f \,\mathrm{d}\phi$ denotes $\phi(A)^{-1} \int_A f \,\mathrm{d}\phi$, see [Fed69, 2.4.12].

Suppose $m \in \mathscr{P}$, U is an open subset of $\mathbf{R}^m$, $e_1, \ldots, e_m$ denote the standard base of $\mathbf{R}^m$, Y is a finite dimensional Hilbert space, k is a nonnegative integer, and u is an $\mathscr{L}^m \llcorner U$ measurable function with values in Y. Then u is called *k times weakly differentiable in U* if and only if

(1) $u \in \mathbf{L}_1(\mathscr{L}^m \llcorner K, Y)$ for every compact subset K of U,

(2) defining $T \in \mathscr{D}'(U,Y)$ by $T(\theta) = \int_U \theta \bullet u \,\mathrm{d}\mathscr{L}^m$ for $\theta \in \mathscr{D}(U,Y)$, the distributions $D^\alpha T$ corresponding to all $\alpha \in \Xi(m,i)$ and $i = 0, \ldots, k$ are representable by integration and the measures $\|D^\alpha T\|$ are absolutely continuous with respect to $\mathscr{L}^m \llcorner U$, see [Fed69, 1.9.2, 1.10.1, 2.9.2, 4.1.1, 4.1.5], (α is sometimes called "multi-index of length i").

In this case for $i = 0, \ldots, k$ the $\mathscr{L}^m \llcorner U$ measurable functions $\mathbf{D}^i u$ with values in $\bigodot^i(\mathbf{R}^m, Y)$ are characterised by the following two conditions (here and in the following $\bigodot^i(\mathbf{R}^m, Y)$ is equipped with an inner product as in [Fed69, 1.10.6]):

(3) $D^\alpha T(\theta) = \int_U \theta(x) \bullet \langle e^\alpha, \mathbf{D}^i u(x) \rangle \,\mathrm{d}\mathscr{L}^m x$ whenever $\theta \in \mathscr{D}(U,Y)$ and $\alpha \in \Xi(m,i)$ where $e^\alpha = (e_1)^{\alpha_1} \odot \cdots \odot (e_m)^{\alpha_m}$ is constructed from the standard base $e_1, \ldots, e_m$ of $\mathbf{R}^m$, see [Fed69, 1.9.2, 1.10.1]; in particular $\mathbf{D}^i u$ is 0 times weakly differentiable in U.

(4) $\mathbf{D}^i u(a) = \lim_{r \to 0+} \fint_{\mathbf{B}(a,r)} \mathbf{D}^i u \,\mathrm{d}\mathscr{L}^m$ whenever $a \in U$; hence $a \in \operatorname{dmn} \mathbf{D}^i u$ if and only if the preceding limit exists.

Also, 1 times weakly differentiable in U is abbreviated to *weakly differentiable in U* and $\mathbf{D}^1 u$ to $\mathbf{D}u$. In particular, the symbols $\mathbf{D}^i$, $\mathbf{D}$ will not be used in the sense of [Fed69, 1.5.2, 2.9.1, 4.1.6]. $\mathbf{W}^{k,p}(U,Y)$ denotes the *Sobolev space* of all k times weakly differentiable functions in U with values in Y such that $\mathbf{D}^i u \in \mathbf{L}_p\big(\mathscr{L}^m \llcorner U, \bigodot^i(\mathbf{R}^m, Y)\big)$ whenever $i = 0, \ldots, k$; the corresponding seminorm of u is given by $\sum_{i=0}^k (\mathscr{L}^m \llcorner U)_{(p)}(\mathbf{D}^i u)$, see [Fed69, 2.4.12]. $\mathbf{W}_0^{k,p}(U,Y)$ denotes the closure of $\mathscr{D}(U,Y)$ in $\mathbf{W}^{k,p}(U,Y)$. Note that in these definitions neither in the Sobolev spaces nor in the Lebesgue spaces functions agreeing $\mathscr{L}^m \llcorner U$ almost everywhere are treated as single elements; instead condition (4) is employed.

If $m \in \mathscr{P}$, U is an open subset $\mathbf{R}^m$, Y is a separable Hilbert space, $1 \leq p \leq \infty$, A is an $\mathscr{L}^m \llcorner U$ measurable set, and u and v are $\mathscr{L}^m \llcorner U$ measurable functions with values in Y then $|u|_{p;A} = (\mathscr{L}^m \llcorner A)_{(p)}(u)$ and, provided $\int_A |u(x) \bullet v(x)| \,\mathrm{d}\mathscr{L}^m x < \infty$, $(u,v)_A = \int_A u(x) \bullet v(x) \,\mathrm{d}\mathscr{L}^m x$. Moreover, $|u|_{p;a,r} = |u|_{p;\mathbf{U}(a,r)}$ and $(u,v)_{a,r} = (u,v)_{\mathbf{U}(a,r)}$ whenever $a \in \mathbf{R}^m$, $0 < r < \infty$ with $\mathbf{U}(a,r) \subset U$, see [Fed69, 2.8.1]. These notions extend [Fed69, 5.2.1]. If additionally, i is a nonpositive integer, $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, $1/p + 1/q = 1$, T is a real valued linear functional on $\mathscr{D}(U,Y)$, and V is an open subset of U, then

$$|T|_{i,p;V} = \sup T\big[\mathscr{D}(U,Y) \cap \{\theta : |D^{-i}\theta|_{q;U} \leq 1 \text{ and } \operatorname{spt}\theta \subset V\}\big]$$

and $|T|_{i,p;a,r} = |T|_{i,p;\mathbf{U}(a,r)}$ whenever $a \in \mathbf{R}^m$, $0 < r < \infty$ with $\mathbf{U}(a,r) \subset U$.

The notation for functions with values in $\mathbf{Q}_Q(\mathbf{R}^{n-m})$ for $m, n, Q \in \mathscr{P}$ with $m < n$ which originate from Almgren's work in [Alm00] will be introduced in Section 2 together with basic properties.

Finally, each statement asserting the existence of a positive, finite number, small (ε) or large (Γ), will give rise to a function depending on the listed parameters whose "name" is $\varepsilon_{\mathrm{x.y}}$ or $\Gamma_{\mathrm{x.y}}$ where x.y denotes the number of the statement.

2 Basic facts for $\mathbf{Q}_Q(V)$ valued functions

This section provides some basic definitions for $\mathbf{Q}_Q(V)$ valued functions mainly taken from Almgren [Alm00] in 2.1, 2.2 and 2.4 and a proposition from [Men09b] in 2.3. Finally, the first variation for the varifold associated to the "graph" of a $\mathbf{Q}_Q(\mathbf{R}^{n-m})$ valued functions is given in 2.5 and 2.6.

2.1 (cf. [Alm00, 1.1 (1) (3), 2.3 (2)])**.** Suppose $Q \in \mathscr{P}$ and V is a finite dimensional Euclidean vector space.

$\mathbf{Q}_Q(V)$ is defined to be the set of all 0 dimensional integral currents R such that $R = \sum_{i=1}^{Q} [\![x_i]\!]$ for some $x_1, \ldots, x_Q \in V$. A metric $\mathscr{G}$ on $\mathbf{Q}_Q(V)$ is defined such that

$$\mathscr{G}\big(\textstyle\sum_{i=1}^{Q} [\![x_i]\!], \sum_{i=1}^{Q} [\![y_i]\!]\big) = \inf \Big\{ \big(\sum_{i=1}^{Q} |x_i - y_{\pi(i)}|^2 \big)^{1/2} : \pi \in P(Q) \Big\}$$

whenever $x_1, \ldots, x_Q, y_1, \ldots, y_Q \in V$ where $P(Q)$ denotes the set of permutations of $\{1, \ldots, Q\}$. The function $\boldsymbol{\eta}_Q : \mathbf{Q}_Q(V) \to V$ is defined by

$$\boldsymbol{\eta}_Q(R) = Q^{-1} \textstyle\int x \, \mathrm{d} \|R\|(x) \quad \text{whenever } R \in \mathbf{Q}_Q(V).$$

If $R = \sum_{i=1}^{Q} [\![x_i]\!]$ for some $x_1, \ldots, x_Q \in V$, then $\boldsymbol{\eta}_Q(R) = \frac{1}{Q} \sum_{i=1}^{Q} x_i$. $\operatorname{Lip} \boldsymbol{\eta}_Q = Q^{-1/2}$.

Whenever $f : X \to \mathbf{Q}_Q(V)$ one defines

$$\operatorname{graph}_Q f = (X \times V) \cap \{ (x, v) : v \in \operatorname{spt} f(x) \}$$

and with $g : X \to V$ also $f \,(+)\, g : X \to \mathbf{Q}_Q(V)$ by

$$(f \,(+)\, g)(x) = (\boldsymbol{\tau}_{g(x)})_\#(f(x)) \quad \text{whenever } x \in X.$$

2.2 (cf. [Alm00, 1.1 (9) (10)])**.** Suppose $m, n, Q \in \mathscr{P}$ and $m < n$.

A function $f : \mathbf{R}^m \to \mathbf{Q}_Q(\mathbf{R}^{n-m})$ is called *affine* if and only if there exist affine functions $f_i : \mathbf{R}^m \to \mathbf{R}^{n-m}$, $i = 1, \ldots, Q$ such that

$$f(x) = \textstyle\sum_{i=1}^{Q} [\![f_i(x)]\!] \quad \text{whenever } x \in \mathbf{R}^m.$$

$f_1, \ldots, f_Q$ are uniquely determined up to order. Moreover, one defines

$$|f| = \big(\textstyle\sum_{i=1}^{Q} |Df_i(0)|^2 \big)^{1/2}.$$

Let $a \in A \subset \mathbf{R}^m$ and $f : A \to \mathbf{Q}_Q(\mathbf{R}^{n-m})$. f is called *affinely approximable at a* if and only if $a \in \operatorname{Int} A$ and there exists an affine function $g : \mathbf{R}^m \to \mathbf{Q}_Q(\mathbf{R}^{n-m})$ such that

$$\lim_{x \to a} \mathscr{G}(f(x), g(x)) / |x - a| = 0.$$

The function g is unique and denoted by $Af(a)$. f is called *strongly affinely approximable at* a if and only if $Af(a)$ has the following property: If $Af(a)(x) = \sum_{i=1}^{Q} [\![g_i(x)]\!]$ for some affine functions $g_i : \mathbf{R}^m \to \mathbf{R}^{n-m}$ and $g_i(a) = g_j(a)$ for some i and j, then $Dg_i(a) = Dg_j(a)$. The concepts of *approximate affine approximability* and *approximate strong affine approximability* are obtained through omission of the condition $a \in \operatorname{Int} A$ and replacement of lim by ap lim. The corresponding affine function is denoted by $\operatorname{ap} Af(a)$.

2.3. The following proposition, see [Men09b, 1.3, 6], will be used for calculations involving Lipschitzian $\mathbf{Q}_Q(\mathbf{R}^{n-m})$ valued functions.

If $m, n, Q \in \mathscr{P}$, $m < n$, A is $\mathscr{L}^m$ measurable, $f : A \to \mathbf{Q}_Q(\mathbf{R}^{n-m})$ is Lipschitzian, I is countable, and to each $i \in I$ there corresponds a function $f_i \subset \operatorname{graph}_Q f$ with $\mathscr{L}^m$ measurable domain and $\operatorname{Lip} f_i \leq \operatorname{Lip} f$ such that

$$\operatorname{card}\{i : f_i(x) = y\} = \mathbf{\Theta}^0(\|f(x)\|, y) \quad \text{whenever } (x, y) \in A \times \mathbf{R}^{n-m},$$

then f is approximately strongly affinely approximable with

$$\operatorname{ap} Af(a)(v) = \textstyle\sum_{i \in I(a)} [\![f_i(x) + \langle v, \operatorname{ap} Df_i(x)\rangle]\!] \quad \text{whenever } v \in \mathbf{R}^m$$

at $\mathscr{L}^m$ almost all $a \in A$ where $I(a) = I \cap \{i : a \in \operatorname{dmn} \operatorname{ap} Df_i\}$. Moreover, such functions f_i do exist whenever m, n, Q, A, and f are as above, in particular $\operatorname{graph}_Q f$ is countably m rectifiable. If A is open, then $\operatorname{ap} Af$ may be replaced by Af.

2.4 Definition. Suppose $m, n, Q \in \mathscr{P}$, $m < n$, $A \subset B \subset \mathbf{R}^m$, A is $\mathscr{L}^m$ measurable and $f : B \to \mathbf{Q}_Q(\mathbf{R}^{n-m})$ is Lipschitzian, $C_1 = \operatorname{dmn} \operatorname{ap} Af$, $C_2 = \operatorname{dmn} Af$, and $g : B \to \mathbf{R}$ and $h_i : C_i \to \mathbf{R}$ for $i \in \{1, 2\}$ are defined by

$$g(x) = \mathscr{G}(f(x), Q[\![0]\!]) \quad \text{for } x \in B,$$
$$h_1(x) = |\operatorname{ap} Af(x)| \quad \text{for } x \in C_1, \quad h_2(x) = |Af(x)| \quad \text{for } x \in C_2.$$

Then one defines for $1 \leq p \leq \infty$, noting 2.3,

$$|f|_{p;A} = |g|_{p;A}, \quad |\operatorname{ap} Af|_{p;A} = |h_1|_{p;A},$$
$$|Af|_{p;A} = |h_2|_{p;A} \quad \text{if } A \text{ is open.}$$

Moreover, if $\mathbf{U}(a, r) \subset B$ for some $a \in \mathbf{R}^m$, $0 < r < \infty$, then

$$|f|_{p;a,r} = |f|_{p;\mathbf{U}(a,r)}, \quad |\operatorname{ap} Af|_{p;a,r} = |\operatorname{ap} Af|_{p;\mathbf{U}(a,r)},$$
$$|Af|_{p;a,r} = |Af|_{p;\mathbf{U}(a,r)}.$$

2.5. Suppose U is an open subset of $\mathbf{R}^m$, Y is a Banach space and $T \in \mathscr{D}'(U, Y)$. Then T has a unique extension S to $\mathscr{E}(U, Y) \cap \{\theta : \operatorname{spt} \theta \cap \operatorname{spt} T \text{ is compact}\}$ characterised by the requirement

$$S(\theta) = S(\eta) \quad \text{whenever } \operatorname{spt} T \subset \operatorname{Int}\{x : \theta(x) = \eta(x)\}.$$

The extension will usually be denoted by the same symbol T.

2.6. Suppose $m, n, Q \in \mathscr{P}$ with $m < n$.

Following [Fed69, 5.1.9], the projections $\mathbf{p} \in \mathbf{O}^*(n, m)$, $\mathbf{q} \in \mathbf{O}^*(n, n-m)$ are defined by

$$\mathbf{p}(z) = (z_1, \ldots, z_m), \quad \mathbf{q}(z) = (z_{m+1}, \ldots, z_n)$$

whenever $z = (z_1, \ldots, z_n) \in \mathbf{R}^n$. In case

$$z = \mathbf{p}^*(x) + \mathbf{q}^*(y) = (x_1, \ldots, x_m, y_1, \ldots, y_{n-m}) \quad \text{for } x \in \mathbf{R}^m,\ y \in \mathbf{R}^{n-m}$$

sometimes (x, y) will be written instead of z, $f(x, y)$ instead of $f(z)$ for functions f with $\operatorname{dmn} f \subset \mathbf{R}^n$ and $\mathbf{G}(n, m)$ instead of $\mathbf{G}_m(\mathbf{R}^m \times \mathbf{R}^{n-m})$.

If U is an open subset of $\mathbf{R}^m$, A is an $\mathscr{L}^m$ measurable subset of U, $f : A \to \mathbf{Q}_Q(\mathbf{R}^{n-m})$ is Lipschitzian, and f_i for $i \in I$ are as in 2.3, then defining $V \in \mathbf{IV}_m(\mathbf{p}^{-1}[U])$ by the requirement

$$\|V\|(Z) = \textstyle\int_{Z \cap \mathbf{p}^{-1}[A]} \mathbf{\Theta}^0(\|f(\mathbf{p}(z))\|, \mathbf{q}(z)) \,\mathrm{d}\mathscr{H}^m z$$

for every Borel subset Z of $\mathbf{p}^{-1}[U]$, a simple calculation shows

$$(\delta V)(\mathbf{q}^* \circ \theta \circ \mathbf{p}) = \textstyle\sum_{i \in I} \int_{\operatorname{dmn} f_i} \big\langle D\theta(x), D\Psi_0^\S(\operatorname{ap} Df_i(x)) \big\rangle \,\mathrm{d}\mathscr{L}^m x$$

whenever $\theta \in \mathscr{D}(U, \mathbf{R}^{n-m})$; here $\Psi_0^\S$ denotes the nonparametric integrand at 0 associated with the area integrand Ψ, i.e. $\Psi_0^\S : \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m}) \to \mathbf{R}$ with

$$\Psi_0^\S(\sigma) = \big(\textstyle\sum_{i=0}^m |\Lambda_i \sigma|^2\big)^{1/2} \quad \text{for } \sigma \in \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m}),$$

see [Fed69, 5.1.9], and the convention 2.5 is used.

3 A coercive estimate

In the present section two improved versions of Brakke's coercive estimate in [Bra78, 5.5] are derived in 3.9 and 3.13. First, some computations for the catenoid are carried out in 3.2 which are used in 3.12 to rule out a certain generalisation of the coercive estimate. Then, some basic facts about approximate differentiability with respect to the weight measure of a varifold are given in 3.5 which are needed to construct a cut-off function in 3.6. Finally, the coercive estimate for rectifiable varifolds satisfying a lower bound on the density is proven in 3.9 and a simpler version for general varifolds is indicated in 3.13.

3.1. Frequently, the following estimates from Allard [All72, 8.9 (5)] will be used:

Suppose $m, n \in \mathscr{P}$, $m < n$, $T \in \mathbf{G}(n, m)$ and $\eta_1, \eta_2 \in \operatorname{Hom}(S, S^\perp)$. If

$$S_i = \mathbf{R}^n \cap \{z : z + \eta_i(z) : z \in S\} \quad \textit{for } i = 1, 2,$$

then

$$\|(S_1)_\natural - (S_2)_\natural\| \le \|\eta_1 - \eta_2\|,$$
$$\big(1 - \|(S_1)_\natural - S_\natural\|^2\big)\|\eta_1 - \eta_2\|^2 \le \big(1 + \|\eta_2\|^2\big)\|(S_1)_\natural - (S_2)_\natural\|^2.$$

3.2 Example. Suppose $m = 2$, $n = 3$, and $f : \mathbf{R} \cap \{t : 1 \le t < \infty\} \to \mathbf{R}$ as well as N, T, and P_R are defined by

$$f(t) = \log\big(t + (t^2-1)^{1/2}\big) \quad \text{for } 1 \le t < \infty,$$
$$N = \mathbf{R}^3 \cap \{z : |\mathbf{q}(z)| = f(|\mathbf{p}(z)|)\}, \quad T = \operatorname{im} \mathbf{p}^*,$$
$$P_R = \mathbf{R}^3 \cap \{z : |\mathbf{q}(z)| = \log(2R)\} \quad \text{for } 2 \le R < \infty.$$

Then there exists a universal, positive, finite number Γ with the following two properties:

(1) $\int_{\mathbf{R}^3 \cap \mathbf{B}(0,R)} |\operatorname{dist}(z, P_R)|^2 \,\mathrm{d}(\mathscr{H}^2 \llcorner N) z \le \Gamma R^2$ for $2 \le R < \infty$.

(2) $\int_{\mathbf{R}^3 \cap \mathbf{B}(0,R)} |\operatorname{Tan}(N,z)_\natural - T_\natural|^2 \,\mathrm{d}(\mathscr{H}^2 \llcorner N) z \ge \Gamma^{-1} \log R$ for $2 \le R < \infty$.

Construction of example. First, note

$$f'(t) = \frac{1}{t + (t^2-1)^{1/2}} \cdot \left(1 + \frac{t}{(t^2-1)^{1/2}}\right) \quad \text{for } 1 < t < \infty,$$

hence $(\Gamma_1)^{-1} t^{-1} \le f'(t) \le \Gamma_1 t^{-1}$ for $2 \le t < \infty$ and some universal, positive, finite number Γ_1, in particular $\operatorname{Lip} f | \mathbf{R} \cap \{s : s \ge 2\} < \infty$.

To prove (1), one estimates

$$\int_{\mathbf{C}(T,0,R) \sim \mathbf{C}(T,0,2)} \operatorname{dist}(z, P_R)^2 \,\mathrm{d}(\mathscr{H}^2 \llcorner N) z \le \Gamma_2 (a_1 + a_2)$$

where Γ_2 is a universal, positive, finite number and

$$a_1 = \int_{\mathbf{B}(0,R) \sim \mathbf{B}(0,2)} |\log(2R) - \log(2|x|)|^2 \,\mathrm{d}\mathscr{L}^2 x,$$
$$a_2 = \int_{\mathbf{B}(0,R) \sim \mathbf{B}(0,2)} |\log(2|x|) - f(|x|)|^2 \,\mathrm{d}\mathscr{L}^2 x.$$

Concerning a_1, note

$$a_1 = 2\pi \int_2^R |\log(t/R)|^2 t \,\mathrm{d}\mathscr{L}^1 t \le 2\pi R^2 \int_0^1 |\log(t)|^2 t \,\mathrm{d}\mathscr{L}^1 t < \infty.$$

To estimate a_2, define $h : \mathbf{R} \cap \{t : t > 0\} \to \mathbf{R}$ by $h(t) = t^{1/2}$ and note for $2 \le t < \infty$

$$|\log(2t) - \log(t + (t^2-1)^{1/2})| \le \operatorname{Lip}(\log | \mathbf{R} \cap \{s : s \ge t\}) |t - (t^2-1)^{1/2}|$$
$$\le t^{-1} \operatorname{Lip}(h | \mathbf{R} \cap \{s : s \ge (t^2-1)\}) \le t^{-1} 2^{-1} (t^2-1)^{-1/2} \le 2^{-1/2} t^{-2},$$

hence $a_2 \le \pi \int_2^R t^{-3} \,\mathrm{d}\mathscr{L}^1 t \le \pi/8$. Together, the estimates for a_1 and a_2 yield (1). By 3.1, it follows

$$\| \operatorname{Tan}(N,z)_\natural - T_\natural \| \le f'(|\mathbf{p}(z)|) \le \Gamma_1 |\mathbf{p}(z)|^{-1}$$

for $z \in N \sim \mathbf{C}(T,0,2)$, hence by 3.1 with S, S_1, S_2 replaced by T, $\operatorname{Tan}(N,z)$, T,

$$|\operatorname{Tan}(N,z)_\natural - T_\natural| \ge \| \operatorname{Tan}(N,z)_\natural - T_\natural \| \ge f'(|\mathbf{p}(z)|)/2 \ge (2\Gamma_1)^{-1} |\mathbf{p}(z)|^{-1}$$

for $z \in N \sim \mathbf{C}(T,0,2\Gamma_1)$. Noting for $2 \le R < \infty$

$$f(t) \le f(R) \le 2R \quad \text{for } 1 \le t \le R, \qquad N \cap \mathbf{C}(T,0,R) \subset \mathbf{R}^3 \cap \mathbf{B}(0,3R),$$

this implies for $2\sup\{\Gamma_1, 1\} \leq R < \infty$ that

$$\begin{aligned}
&\textstyle\int_{\mathbf{R}^3 \cap \mathbf{B}(0,3R)} |\operatorname{Tan}(N,z)_\natural - T_\natural|^2 \,\mathrm{d}(\mathscr{H}^2 \llcorner N) z \\
&\quad \geq \textstyle\int_{\mathbf{C}(T,0,R) \sim \mathbf{C}(T,0,2\Gamma_1)} |\operatorname{Tan}(N,z)_\natural - T_\natural|^2 \,\mathrm{d}(\mathscr{H}^2 \llcorner N) z \\
&\quad \geq (2\Gamma_1)^{-2} \textstyle\int_{2\Gamma_1}^{R} t^{-1} \,\mathrm{d}\mathscr{L}^1 t = (2\Gamma_1)^{-2} \log(R/(2\Gamma_1)).
\end{aligned}$$

Since $\int_{\mathbf{R}^3 \cap \mathbf{B}(0,2)} |\operatorname{Tan}(N,z)_\natural - T_\natural|^2 \,\mathrm{d}(\mathscr{H}^2 \llcorner N) z > 0$, one infers (2). □

3.3. The following situation will be studied: $m, n \in \mathscr{P}$, $m < n$, $1 \leq p \leq \infty$, U is an open subset of $\mathbf{R}^n$, $V \in \mathbf{V}_m(U)$, $\|\delta V\|$ is a Radon measure and, if $p > 1$,

$$\begin{aligned}
(\delta V)(g) &= -\textstyle\int g(z) \bullet \mathbf{h}(V;z) \,\mathrm{d}\|V\|(z) \quad \text{whenever } g \in \mathscr{D}(U, \mathbf{R}^n), \\
\mathbf{h}(V;\cdot) &\in \mathbf{L}_p(\|V\| \llcorner K, \mathbf{R}^n) \quad \text{whenever } K \text{ is a compact subset of } U.
\end{aligned}$$

If $p < \infty$ then the measure ψ is defined by

$$\psi = \|\delta V\| \quad \text{if } p = 1, \qquad \psi = |\mathbf{h}(V;\cdot)|^p \|V\| \quad \text{if } p > 1.$$

3.4. Suppose m, n, $p = 1$, U and V are as in 3.3. Then $\delta V \in \mathscr{D}'(U, \mathbf{R}^n)$ will be extended to $\mathbf{L}_1(\|\delta V\|, \mathbf{R}^n)$ by continuity with respect to $\|\delta V\|_{(1)}$ and $(\delta V)(g)$ will be used to denote this extension for $g \in \mathbf{L}_1(\|\delta V\|, \mathbf{R}^n)$ as in [Fed69, 4.1.5].

3.5 Lemma. *Suppose $m, n \in \mathscr{P}$, $m \leq n$, U is an open subset of $\mathbf{R}^n$, and $V \in \mathbf{RV}_m(U)$.*

Then the following four statements hold:

(1) If $f : U \to \mathbf{R}$ is $\|V\|$ measurable and A denotes the set of all $z \in U$ such that f is $(\|V\|, m)$ approximately differentiable at z, then A is $\|V\|$ measurable and $(\|V\|, m) \operatorname{ap} Df(z) \circ \operatorname{Tan}^m(\|V\|, z)_\natural$ depends $\|V\| \llcorner A$ measurably on z.

(2) If $f : U \to \mathbf{R}$ is Lipschitzian, then f is $(\|V\|, m)$ approximately differentiable at $\|V\|$ almost all z.

(3) If $f_i : U \to \mathbf{R}$ is a sequence of functions converging locally uniformly to $f : U \to \mathbf{R}$ and $\sup\{\operatorname{Lip} f_i : i \in \mathscr{P}\} < \infty$, then

$$\textstyle\int \langle g(z), (\|V\|, m) \operatorname{ap} Df_i(z) \rangle \,\mathrm{d}\|V\| z \to \int \langle g(z), (\|V\|, m) \operatorname{ap} Df(z) \rangle \,\mathrm{d}\|V\| z$$

as $i \to \infty$ whenever $g \in \mathbf{L}_1(\|V\|, \mathbf{R}^n)$ with $g(z) \in \operatorname{Tan}^m(\|V\|, z)$ for $\|V\|$ almost all z.

(4) If $f : U \to \mathbf{R}^n$ is a Lipschitzian function with compact support in U and $\|\delta V\|$ is a Radon measure, then (see 3.4)

$$\delta V(f) = \textstyle\int S_\natural \bullet ((\|V\|, m) \operatorname{ap} Df(z) \circ S_\natural) \,\mathrm{d}V(z, S).$$

Proof of (1) *and* (2). Since $\|V\|(U \cap \{z : \boldsymbol{\Theta}^{*m}(\|V\|, z) = \infty\}) = 0$, a set B is $\|V\|$ measurable if and only if $B \cap \{z : \boldsymbol{\Theta}^{*m}(\|V\|, z) > 0\}$ is $\mathscr{H}^m$ measurable by [Fed69, 2.10.19 (1) (3)]. Hence (1) and (2) follow from [Fed69, 3.2.17–19, 3.1.4, 2.10.19 (4), 2.9.9]. □

Proof of (3). Clearly, the assertion needs only to be verified for elements g of some subset X of $\mathbf{L}_1(\|V\|, \mathbf{R}^n)$ whose span is $\|V\|_{(1)}$ dense in $\mathbf{L}_1(\|V\|, \mathbf{R}^n) \cap \{g : g(z) \in \operatorname{Tan}^m(\|V\|, z) \text{ for } z \in U\}$. Therefore one may first assume $\|V\| = \mathscr{H}^m \llcorner W$ for some $(\mathscr{H}^m, m)$ rectifiable and $\mathscr{H}^m$ measurable subset of U by [Fed69, 3.2.19, 2.10.19 (4), 2.9.9] and then $m = n$, $\|V\| = \mathscr{L}^m$ by [Fed69, 3.2.17–20, 3.1.5, 2.9.11]. This case can be treated with $X = \mathscr{D}(\mathbf{R}^m, \mathbf{R}^m)$ using partial integration. □

Proof of (4). (3) readily implies (4) by means of convolution. □

3.6 Lemma. *Suppose m, n, p, U, V, and ψ are as in 3.3, $p < m$, $V \in \mathbf{RV}_m(U)$, $\mathbf{\Theta}^m(\|V\|, z) \geq 1$ for $\|V\|$ almost all z, K is a compact subset of U, $0 < \delta \leq \frac{1}{40}$, and H is the set of all $z \in \operatorname{spt}\|V\|$ such that*

$$\|V\|\,\mathbf{B}(z, r) \geq \delta^m (\boldsymbol{\gamma}(m)m)^{-m} r^m \quad \textit{whenever } 0 < r < \infty,\ \mathbf{B}(z, r) \subset K.$$

Then there exists a Baire function $f : U \to \mathbf{R} \cap \{t : 0 \leq t \leq 1\}$ satisfying for $g \in \mathscr{D}(U, \mathbf{R}^n)$

$$\begin{gathered}\mathbf{R}^n \cap \{z : f(z) \neq 0\} \subset K, \quad \|V\|(U \cap \{z : f(z) \neq 1\} \sim H) = 0,\\ f \textit{ is } (\|V\|, m) \textit{ approximately differentiable at } \|V\| \textit{ almost all } z,\\ \textstyle\int S_\natural \bullet Dg(z) f(z)\, \mathrm{d}V(z, S) = \delta V(fg) - \int \langle S_\natural(g(z)), \operatorname{ap} Df(z)\rangle\, \mathrm{d}V(z, S),\\ \|V\|_{(p)}(|\operatorname{ap} Df|) \leq \delta (400)^m\, \psi(K)^{1/p},\\ \|V\|(U \cap \{z : f(z) \neq 0\}) \leq \Gamma\, \psi(K)^{m/(m-p)}\end{gathered}$$

(see 3.4) where $\Gamma = ((400)^m \boldsymbol{\gamma}(m) m)^{mp/(m-p)}$.

Proof. Let $B = (U \sim H) \cap \{z : \mathbf{\Theta}^m_*(\|V\|, z) \geq 1\}$ and assume $B \neq \emptyset$. First, the following assertion will be shown: *Whenever $z \in B$ there exists $0 < t < \infty$ such that $\mathbf{B}(z, 10t) \subset K$ and*

$$\begin{gathered}t^{-1} \|V\|\mathbf{B}(z, 10t))^{1/p} \leq \delta (400)^m\, \psi(\mathbf{B}(z, t))^{1/p},\\ \|V\|\,\mathbf{B}(z, 10t) \leq \Gamma\, \psi(\mathbf{B}(z, t))^{m/(m-p)}.\end{gathered}$$

For this purpose choose $0 < r < \infty$ with $\mathbf{B}(z, r) \subset K$ and

$$\|V\|\,\mathbf{B}(z, r) \leq \delta^m (\boldsymbol{\gamma}(m)m)^{-m} r^m,$$

let P denote the set of all $0 < t \leq r$ such that

$$\|V\|\,\mathbf{B}(z, t) \leq (20\delta)^m (\boldsymbol{\gamma}(m)m)^{-m} t^m$$

and Q the set of all $0 < t \leq \frac{r}{20}$ such that $\{s : t \leq s \leq 20t\} \subset P$. One notes for $\frac{r}{20} \leq s \leq r$

$$s^{-m} \|V\|\,\mathbf{B}(z, s) \leq (20)^m r^{-m} \|V\|\,\mathbf{B}(z, r) \leq (20\delta)^m (\boldsymbol{\gamma}(m)m)^{-m},$$

hence $\frac{r}{20} \in Q$. Let $\varrho = \inf Q$ and note $\varrho > 0$ since $20\delta < 1$ and $(\boldsymbol{\gamma}(m)m)^{-m} \leq \boldsymbol{\alpha}(m)$, cf. e.g. [Men09a, 2.4]. Clearly, $\{s : \varrho \leq s \leq 20\varrho\} \subset P$. Also, whenever $\varrho \leq s \leq 20\varrho$

$$s^{-m} \|V\|\,\mathbf{B}(z, s) \geq (20)^{-m} \varrho^{-m} \|V\|\,\mathbf{B}(z, \varrho) = \delta^m (\boldsymbol{\gamma}(m)m)^{-m}$$

because $\varrho \in \operatorname{Clos}(\{s : s < \varrho\} \sim P\})$.

Define $\alpha : \{s : 0 < s < r\} \to \mathbf{R}$ and $\beta : \{s : 0 < s < r\} \to \mathbf{R}$ by

$$\alpha(s) = \|V\| \, \mathbf{B}(z,s), \quad \beta(s) = \psi(\mathbf{B}(z,s))^{1/p}$$

whenever $0 < s < r$. Then by [Men08, A.7][2]

$$\boldsymbol{\gamma}(m)^{-1} \leq \alpha(s)^{1/m-1}(\|\delta V\| \, \mathbf{B}(z,s) + \alpha'(s))$$

for $\mathscr{L}^1$ almost every $0 < s < r$, hence by Hölder's inequality

$$(m\boldsymbol{\gamma}(m))^{-1} \leq \alpha(s)^{1/m-1/p}\beta(s) + (\alpha^{1/m})'(s)$$

for $\mathscr{L}^1$ almost every $0 < s < r$. This inequality implies the existence of $\varrho < t < 2\varrho$ satisfying

$$t^{-1}\alpha(10t)^{1/p} \leq \delta(400)^m \beta(t);$$

in fact if this were not the case, then for $\mathscr{L}^1$ almost all $\varrho < s < 2\varrho$, recalling $\{s, 10s\} \subset P$,

$$\begin{aligned}
(\boldsymbol{\gamma}(m)m)^{-1} - (\alpha^{1/m})'(s) &< \alpha(s)^{1/m-1/p}(400)^{-m}\delta^{-1}s^{-1}\alpha(10s)^{1/p} \\
&\leq (1/2)(\boldsymbol{\gamma}(m)m)^{-1}, \\
(20\delta)(\boldsymbol{\gamma}(m)m)^{-1} \leq (1/2)(\boldsymbol{\gamma}(m)m)^{-1} &< (\alpha^{1/m})'(s),
\end{aligned}$$

hence, using $\alpha^{1/m}(\varrho) = (20\delta)(\boldsymbol{\gamma}(m)m)^{-1}\varrho$ and [Fed69, 2.9.19], one would obtain for $\varrho < s < 2\varrho$

$$\alpha^{1/m}(s) > \alpha^{1/m}(\varrho) + \textstyle\int_\varrho^s (\alpha^{1/m})'(t) \, \mathrm{d}\mathscr{L}^1 t \geq (20\delta)(\boldsymbol{\gamma}(m)m)^{-1}s, \quad s \notin P.$$

The second part of the assertion now follows, noting $10t \leq 20\varrho$, from

$$\begin{aligned}
\|V\|(\mathbf{B}(z,10t))^{1/p-1/m} &\leq t^{-1}\delta^{-1}\boldsymbol{\gamma}(m)m \, \|V\|(\mathbf{B}(z,10t))^{1/p} \\
&\leq (400)^m \boldsymbol{\gamma}(m)m \, \psi(\mathbf{B}(z,t))^{1/p}.
\end{aligned}$$

By the preceding assertion and [Fed69, 2.8.5] there exist a nonempty, countable set I and $z_i \in B$, $0 < t_i < \infty$ and $u_i : U \to \mathbf{R}$ for $i \in I$ such that

$$\begin{gathered}
u_i(z) = \sup\{0, 1 - \operatorname{dist}(z, \mathbf{B}(z_i, 5t_i))/t_i\} \quad \text{for } z \in U, \, i \in I, \\
\operatorname{spt} u_i \subset \mathbf{B}(z_i, 10t_i) \subset K \quad \text{for } i \in I, \\
\mathbf{B}(z_i, t_i) \cap \mathbf{B}(z_j, t_j) = \emptyset \quad \text{whenever } i, j \in I, \, i \neq j, \\
\|V\|_{(p)}(|\operatorname{ap} Du_i|) \leq \delta(400)^m \, \psi(\mathbf{B}(z_i, t_i))^{1/p}, \\
\|V\| \, \mathbf{B}(z_i, 10t_i) \leq \Gamma \, \psi(\mathbf{B}(z_i, t_i))^{m/(m-p)}, \\
B \subset \bigcup\{\mathbf{B}(z_i, 5t_i) : i \in I\}.
\end{gathered}$$

Define $v_J : U \to \mathbf{R}$ by

$$v_J(z) = \sup(\{0\} \cup \{u_j(z) : j \in J\}) \quad \text{for } z \in U$$

[2] A similar statement can be found in Leonardi and Masnou [LM09, Proposition 3.1].

whenever $J \subset I$, and $f = v_I$. Note $0 \le f \le 1$ and

$$u_i(z) = 1 \quad \text{whenever } z \in \mathbf{B}(z_i, 5t_i),\ i \in I, \qquad f(z) = 1 \quad \text{for } z \in B.$$

Noting 3.5 (2) and defining $g = \sup\{|\operatorname{ap} Du_i| : i \in I\}$, one estimates for $J \subset I$

$$\begin{aligned}
\|V\|_{(p)}(g)^p &\le \textstyle\sum_{i\in I} \|V\|_{(p)}(|\operatorname{ap} Du_i|)^p \\
&\le \delta^p (400)^{mp} \textstyle\sum_{i\in I} \psi(\mathbf{B}(z_i, t_i)) \le \delta^p (400)^{mp} \psi(K), \\
\|V\|(U \cap \{z : f(z) > v_J(z)\}) & \\
&\le \textstyle\sum_{i\in I \sim J} \|V\| \,\mathbf{B}(z_i, 10t_i) \le \Gamma \textstyle\sum_{i \in I \sim J} \psi(\mathbf{B}(z_i, t_i))^{m/(m-p)} \\
&\le \Gamma \left(\textstyle\sum_{i\in I\sim J} \psi\, \mathbf{B}(z_i, t_i) \right)^{m/(m-p)} \le \Gamma\, \psi(K)^{m/(m-p)}.
\end{aligned}$$

Choose a sequence $J(k)$ with $J(k) \subset J(k+1) \subset I$, $\operatorname{card} J(k) < \infty$ for $k \in \mathscr{P}$ and $\bigcup\{J(k) : k \in \mathscr{P}\} = I$. Then

$$\|V\| \left(U \cap \bigcap \left\{ \{z : f(z) > v_{J(k)}(z)\} : k \in \mathscr{P} \right\} \right) = 0,$$

hence f is $(\|V\|, m)$ approximately differentiable at $\|V\|$ almost all z and

$$\begin{gathered}
\sup\{|\operatorname{ap} Dv_{J(k)}(z)|, |\operatorname{ap} Df(z)|\} \le g(z) \quad \text{for } \|V\| \text{ almost all } z, \\
\|V\|_{(p)}(|\operatorname{ap} Dv_{J(k)} - \operatorname{ap} Df|) \to 0 \quad \text{as } k \to \infty
\end{gathered}$$

by [Fed69, 2.10.19 (4)] and 3.5 (1). The integral formula holds with f replaced by $v_{J(k)}$ for $k \in \mathscr{P}$ by 3.5 (4), hence, taking the limit $k \to \infty$, also for f. □

3.7 Remark. The function f cannot be required to be continuous at $\|V\|$ almost all z. To prove this let $mp/(m-p) < \eta < \infty$, $n = m+1$, $U = \mathbf{R}^n$, apply [Men09a, 1.2] with $\alpha_1 q_1 = \alpha_2 q_2 = \eta$ to obtain μ and T and define V by the requirement $\|V\| = \mu$. Take $\xi \in T$ with $\boldsymbol{\Theta}^m(\psi, \xi) = 0$; the existence of such ξ follows from [Fed69, 2.10.19 (4)] as $\psi(T) = 0$. (Alternately, it follows from the estimates in [Men09a, 1.2] that one can take any $\xi \in T$.) Let $0 < r \le 1$ and $K = \mathbf{B}(\xi, 2r)$. One verifies the existence of $\varepsilon > 0$ depending only on V, δ, η, and m such that

$$\mathbf{B}(\xi, r) \cap \{z : 0 < \operatorname{dist}(z, T) \le \varepsilon\} \cap H = \emptyset.$$

Therefore any such function f would have to satisfy $f(z) = 1$ for $\|V\|$ almost all $z \in T \cap \mathbf{U}(\xi, r)$, hence

$$\|V\|(U \cap \{z : f(z) \ne 0\}) \ge \boldsymbol{\alpha}(m) r^m$$

which would be incompatible with the last inequality of 3.6 for small r even if Γ would be allowed to depend additionally on V and δ.

3.8. If $a \ge 0$, $b \ge 0$, $c > 0$ and $d > 0$ then

$$\inf\{at^c + bt^{-d} : 0 < t < \infty\} = \left((d/c)^{c/(c+d)} + (d/c)^{-d/(c+d)} \right) a^{d/(c+d)} b^{c/(c+d)}.$$

3.9 Lemma. *Suppose m, n, p, U, V, and ψ are as in 3.3, $p < m$, $V \in \mathbf{RV}_m(U)$, $\boldsymbol{\Theta}^m(\|V\|, z) \ge 1$ for $\|V\|$ almost all z, K is a compact subset of U, H is the set of all $z \in \operatorname{spt} \|V\|$ such that*

$$\|V\|\, \mathbf{B}(z, r) \ge (40)^{-m} (\boldsymbol{\gamma}(m) m)^{-m} r^m \quad \textit{whenever } 0 < r < \infty,\ \mathbf{B}(z, r) \subset K,$$

$\phi \in \mathscr{D}^0(U)$, $0 \leq \phi \leq 1$, $\operatorname{spt} \phi \subset K$, $1 \leq q \leq \infty$, $1/p + 1/q \geq 1$, $a \in \mathbf{R}^n$, $T \in \mathbf{G}(n,m)$, $h : U \to \mathbf{R}$ *with* $h(z) = \operatorname{dist}(z-a, T)$ *for* $z \in U$, *and*

$$\alpha = \psi(K)^{1/p}, \quad \beta = \big(\textstyle\int \phi(z)^2 |S_\natural - T_\natural|^2 \, \mathrm{d}V(z,S) \big)^{1/2},$$
$$\gamma = (\phi^2 \|V\| \llcorner H)_{(q)}(h) \qquad \textit{if } q < \infty,$$
$$\gamma = \sup\{ h(z) : z \in \operatorname{spt} \|V\|, \phi(z) > 0 \} \qquad \textit{if } q = \infty,$$
$$\xi = (\|V\| \llcorner H)_{(2)}(|D\phi| h).$$

Then

$$\beta^2 \leq \Gamma \big(\alpha^{mp/(m-p)} + (\alpha\gamma)^{1/(1/p+1/q)} \big) + (16 + 4m)\xi^2$$

where Γ *is a positive, finite number depending only on* m, p, *and* q.

Proof. Assume $a = 0$, hence $h(z) = |T_\natural^\perp(z)|$ for $z \in U$. Use 3.6 with $\delta = \frac{1}{40}$ to obtain f and define $V_1, V_2 \in \mathbf{RV}_m(U)$ by

$$V_1(A) = \textstyle\int_A^* f(z) \, \mathrm{d}V(z,S) \quad \text{for } A \subset U \times \mathbf{G}(n,m)$$

and $V_2 = V - V_1$. Using [Fed69, 2.10.19 (4)], one remarks

$$f(z) = 1 \text{ and } \operatorname{ap} Df(z) = 0 \text{ for } \|V\| \text{ almost all } z \in U \sim H,$$
$$\textstyle\int \phi(z)^2 |S_\natural - T_\natural|^2 \, \mathrm{d}V_1(z,S) \leq 4m \Gamma_{3.6}(m,p) \, \alpha^{mp/(m-p)},$$
$$\|\delta V_2\| \leq (1-f)\|\delta V\| + |\operatorname{ap} Df| \|V\|, \quad \|V\|_{(p)}(|\operatorname{ap} Df|) \leq (400)^m \alpha.$$

Defining $g = \phi^2 (T_\natural^\perp | U)$, one obtains

$$\textstyle\int \phi(z)^2 |S_\natural - T_\natural|^2 \, \mathrm{d}V_2(z,S) \leq 4 |(\delta V_2)(g)| + 16\xi^2$$

as in [Bra78, 5.5]. If $1/p + 1/q = 1$ then the conclusion is a consequence of the preceding remarks and Hölder's inequality. Therefore suppose $1/p + 1/q > 1$, hence $p < \infty$ and $q < \infty$.

Letting $0 < t < \infty$, $r = 1 - q(1 - 1/p)$, and defining $\eta : \{ s : 0 \leq s < \infty \} \to \{ s : 0 \leq s \leq 1 \}$ by $\eta(s) = \inf\{1, ts^{-r}\}$ for $0 \leq s < \infty$, one observes $0 < r \leq 1$ and

$$0 \leq s\eta(s) \leq ts^{1-r} \quad \text{whenever } 0 < s < \infty,$$
$$|s\eta'(s)| + |1 - \eta(s)| \leq 1 \quad \text{whenever } t^{1/r} < s < \infty.$$

Moreover, defining $\eta_1 : U \to \mathbf{R}^n$, $\eta_2 : U \to \mathbf{R}^n$ by

$$\eta_1(z) = \eta(|T_\natural^\perp(z)|) T_\natural^\perp(z), \quad \eta_2(z) = (1 - \eta(|T_\natural^\perp(z)|)) T_\natural^\perp(z)$$

whenever $z \in U$,

$$Z_1 = U \cap \big\{ z : 0 < h(z) < t^{1/r} \big\}, \quad Z_2 = U \cap \big\{ z : t^{1/r} < h(z) \big\},$$

one notes $\eta_1 + \eta_2 = T_\natural^\perp | U$ and computes

$$\langle v, D\eta_2(z) \rangle = -\eta'(|T_\natural^\perp(z)|) \frac{T_\natural^\perp(z) \bullet v}{|T_\natural^\perp(z)|} T_\natural^\perp(z) + (1 - \eta(|T_\natural^\perp(z)|)) T_\natural^\perp(v)$$

for $z \in Z_2$, $v \in \mathbf{R}^n$, hence

$$\|D\eta_2(z)\| \leq 1 \quad \text{for } z \in Z_2$$

and for $z \in U$

$$|\eta_1(z)| \leq th(z)^{1-r} \quad \text{if } r < 1, \qquad |\eta_1(z)| \leq t \quad \text{if } r = 1.$$

Letting $g_1 = \phi^2\eta_1$, $g_2 = \phi^2\eta_2$, one notes $g_1 + g_2 = g$ and infers $|g_1| = \phi^2|\eta_1|$,

$$\begin{aligned}\|Dg_2(z)\| &\leq 2\phi(z)|D\phi(z)|h(z) + \phi^2(z)\|D\eta_2(z)\| \\ &\leq 2\phi^2(z) + |D\phi(z)|^2 h(z)^2 \leq 2\phi^2(z)t^{-q/r}h(z)^q + |D\phi(z)|^2 h(z)^2\end{aligned}$$

for $z \in Z_2$. Since $Dg_2(z) = 0$ for $z \in Z_1$ and ϕ, $D\phi$, and h are continuous, approximating g_1 and g_2 by smooth functions yields that $|(\delta V_2)(g)|$ does not exceed

$$t\|\delta V_2\|(\phi^2 h^{1-r}) + m\|V_2\|\big(2t^{-q/r}\phi^2 h^q + |D\phi|^2 h^2\big) \quad \text{if } r < 1,$$
$$t\|\delta V_2\|(\phi^2) + m\|V_2\|\left(2t^{-q}\phi^2 h^q + |D\phi|^2 h^2\right) \quad \text{if } r = 1,$$

hence, using Hölder's inequality and recalling the remarks of the first paragraph, one obtains

$$|(\delta V_2)(g)| \leq t(800)^m \alpha\gamma^{1-r} + 2mt^{-q/r}\gamma^q + m\xi^2 \quad \text{if } r < 1,$$
$$|(\delta V_2)(g)| \leq t(800)^m \alpha + 2mt^{-q}\gamma^q + m\xi^2 \quad \text{if } r = 1.$$

The conclusion is now a consequence of 3.8. □

3.10 Remark. Using the inequality relating arithmetic and geometric means (cf. [Fed69, 2.4.13]), one obtains for $0 < \lambda < \infty$

$$(\alpha\gamma)^{1/(1/p+1/q)} \leq \tfrac{2(1/p+1/q)-1}{2(1/p+1/q)}(\alpha/\lambda)^{\frac{2}{2(1/p+1/q)-1}} + \tfrac{1}{2(1/p+1/q)}(\lambda\gamma)^2.$$

Note, concerning the exponent of α, if $1/q = 1/2 - 1/m$, then $\frac{2}{2(1/p+1/q)-1} = \frac{mp}{m-p}$.

3.11 Remark. The estimate for $|(\delta V_2)(g)|$ is adapted from Brakke [Bra78, 5.5] where $p \in \{1, 2\}$ and $q = 2$.

3.12 Remark. One cannot replace h by the distance from two planes parallel to T, as may be seen from the estimates for the catenoid in 3.2 considering $R \to \infty$. This behaviour is in contrast to the Sobolev Poincaré type inequality in [Men09b, 3.4].

3.13 Lemma. *Suppose m, n, p, U, and V are as in 3.3, $\phi \in \mathscr{D}^0(U)$, $\phi \geq 0$, $1 \leq q \leq \infty$, $1/p + 1/q \geq 1$, $a \in \mathbf{R}^n$, $T \in \mathbf{G}(n, m)$, $h : U \to \mathbf{R}$ with $h(z) = \operatorname{dist}(z - a, T)$ for $z \in U$, and*

$$\alpha = \|\delta V\|(\phi^2) \quad \text{if } p = 1, \qquad \alpha = (\phi^2\|V\|)_{(p)}(\mathbf{h}(V;\cdot)) \quad \text{if } p > 1,$$
$$\beta = \left(\textstyle\int \phi(z)^2 |S_\natural - T_\natural|^2 \, dV(z, S)\right)^{1/2}, \quad \xi = (\|V\|)_{(2)}(|D\phi|h),$$
$$\gamma = (\phi^2\|V\|)_{(q)}(h) \quad \text{if } q < \infty, \qquad \gamma = (\phi^2\|\delta V\|)_{(\infty)}(h) \quad \text{if } q = \infty.$$

Then

$$\beta^2 \leq \Gamma(\alpha\gamma)^{1/(1/p+1/q)} + (16 + 4m)\xi^2$$

where Γ is a positive, finite number depending only on m, p, and q.

Proof. The proof of 3.9 has been designed such that a proof of the present assertion results when the arguments involving the function f are omitted. $\square$

4 Approximation by $\mathbf{Q}_Q(\mathbf{R}^{n-m})$ valued functions

The purpose of this section is to establish the necessary adaptions and extensions of the approximation by $\mathbf{Q}_Q(\mathbf{R}^{n-m})$ valued functions carried out in [Men09b, 2.18]. This is done in 4.8 (1)–(8) and supplemented by a basic estimate concerning the partial differential equation satisfied by the "average" of the approximating function in 4.8 (9) leaving the estimates more directly related to the purposes of the present paper to Section 7. The results are based on those in [Men09b, §2]. However, as their statements are sometimes rather long, only the statement of the "multilayer monotonicity with variable offset", has been duplicated here in 4.6.

4.1 Definition. A subset of a topological space is called *universally measurable* if and only if it is measurable with respect to every Borel measure on that space.

A function between topological spaces is *universally measurable* if and only if every preimage of an open set is universally measurable.

4.2 Remark. The corresponding definition for measures defined on Borel families can found for example in [CV77, III.21].

4.3 Remark. If $f : X \to Y$ is a Borel function and A is a universally measurable subset of Y, then $f^{-1}[A]$ is universally measurable as may be verified with the help of [Fed69, 2.1.2].

4.4 Remark. The universally measurable sets form a Borel family.

4.5 Lemma. *Suppose X is a complete, separable metric space, Y is a Hausdorff topological space, $f : X \to Y$ is continuous, B is a Borel subset of X, and $g : B \to \{t : 0 \leq t \leq \infty\}$ is a Borel function.*

Then $h : Y \to \{t : 0 \leq t \leq \infty\}$ defined by

$$h(y) = \sum_{B \cap f^{-1}[\{y\}]} g \quad \textit{whenever } y \in Y$$

is universally measurable.

Proof. [Fed69, 2.10.10, 2.3.1 (6)] may be adapted by use of [Fed69, 2.2.13, 2.3.3] to obtain the conclusion. $\square$

4.6 Lemma (Multilayer monotonicity with variable offset, cf. [Men09b, 2.12])**.** *Suppose $n, Q \in \mathscr{P}$, $0 \leq M < \infty$, $\delta > 0$, and $0 \leq s < 1$.*

Then there exists a positive, finite number ε with the following property.

If $m \in \mathscr{P}$, $m < n$, $Z \subset \mathbf{R}^n$, $T \in \mathbf{G}(n, m)$, $0 \leq d < \infty$, $0 < r < \infty$, $0 < t < \infty$, $f : Z \to \mathbf{R}^n$,

$$|T_\natural(z_1 - z_2)| \leq s|z_1 - z_2|, \quad |T_\natural(f(z_1) - f(z_2))| \leq s|f(z_1) - f(z_2)|,$$
$$f(z) - z \in T \cap \mathbf{B}(0, d), \quad d \leq Mt, \quad d + t \leq r$$

for $z, z_1, z_2 \in Z$, $V \in \mathbf{IV}_m\big(\bigcup\{\mathbf{U}(z, r) : z \in Z\}\big)$, $\|\delta V\|$ is a Radon measure,

$$\textstyle\sum_{z \in Z} \mathbf{\Theta}^m_*(\|V\|, z) \geq Q - 1 + \delta, \quad \|V\| \, \mathbf{U}(z, r) \leq M \boldsymbol{\alpha}(m) r^m$$

whenever $z \in Z \cap \operatorname{spt} \|V\|$, *and*

$$\|\delta V\| \mathbf{B}(z,\varrho) \le \varepsilon \|V\|(\mathbf{B}(z,\varrho))^{1-1/m},$$
$$\textstyle\int_{\mathbf{B}(z,\varrho)\times\mathbf{G}(n,m)} |S_\natural - T_\natural| \,\mathrm{d}V(\xi,S) \le \varepsilon \|V\| \mathbf{B}(z,\varrho),$$

whenever $0 < \varrho < r$, $z \in Z \cap \operatorname{spt} \|V\|$, *then*

$$\|V\|\big(\textstyle\bigcup\big\{\mathbf{U}(f(z),t) \cap \{\xi : |T_\natural(\xi - z)| > s|\xi - z|\} : z \in Z\big\}\big) \ge (Q-\delta)\boldsymbol{\alpha}(m)t^m.$$

4.7 Lemma. *Suppose* X, Y *are normed vector spaces,* $f : X \to Y$ *is of class* 1, $a \in X$, $0 < r < \infty$, $Q \in \mathscr{P}$, $x_i \in \mathbf{B}(a,r)$ *for* $i = 1,\ldots,Q$, *and* $\gamma = \operatorname{Lip}(Df|\mathbf{B}(a,r))$.

Then

$$\left| \frac{1}{Q}\sum_{i=1}^{Q} f(x_i) - f\left(\frac{1}{Q}\sum_{i=1}^{Q} x_i\right)\right| \le \gamma r^2.$$

Proof. Let $P : X \to Y$ by defined by $P(x) = f(a) + \langle x - a, Df(a)\rangle$ for $x \in X$. Then for $x \in \mathbf{B}(a,r)$

$$|f(x) - P(x)| = \big|\big\langle x-a, \textstyle\int_0^1 Df(a + t(x-a)) - Df(a) \,\mathrm{d}\mathscr{L}^1 t\big\rangle\big| \le (\gamma/2)r^2.$$

Since $\frac{1}{Q}\sum_{i=1}^{Q} P(x_i) = P(Q^{-1}\sum_{i=1}^{Q} x_i)$, this implies the conclusion. □

4.8 Lemma. *Suppose* $n, Q \in \mathscr{P}$, $0 < L < \infty$, $1 \le M < \infty$, *and* $0 < \delta_i \le 1$ *for* $i \in \{1,2,3,4,5\}$.

Then there exists a positive, finite number ε *with the following property.*

If $m \in \mathscr{P}$, $m < n$, $0 < r < \infty$, $0 < h \le \infty$, $h > 2\delta_4 r$, $T = \operatorname{im} \mathbf{p}^*$,

$$U = (\mathbf{R}^m \times \mathbf{R}^{n-m}) \cap \{(x,y) : \operatorname{dist}((x,y), \mathbf{C}(T,0,r,h)) < 2r\},$$

$V \in \mathbf{IV}_m(U)$, $\|\delta V\|$ *is a Radon measure,*

$$(Q - 1 + \delta_1)\boldsymbol{\alpha}(m)r^m \le \|V\|(\mathbf{C}(T,0,r,h)) \le (Q + 1 - \delta_2)\boldsymbol{\alpha}(m)r^m,$$
$$\|V\|(\mathbf{C}(T,0,r,h+\delta_4 r) \sim \mathbf{C}(T,0,r,h-2\delta_4 r)) \le (1-\delta_3)\boldsymbol{\alpha}(m)r^m,$$
$$\|V\|(U) \le M\boldsymbol{\alpha}(m)r^m,$$

$0 < \delta \le \varepsilon$, B *denotes the set of all* $z \in \mathbf{C}(T,0,r,h)$ *with* $\boldsymbol{\Theta}^{*m}(\|V\|,z) > 0$ *such that*

$$\text{either} \quad \|\delta V\| \mathbf{B}(z,\varrho) > \delta \|V\|(\mathbf{B}(z,\varrho))^{1-1/m} \quad \text{for some } 0 < \varrho < 2r,$$
$$\text{or} \quad \textstyle\int_{\mathbf{B}(z,\varrho)\times\mathbf{G}(n,m)} |S_\natural - T_\natural| \,\mathrm{d}V(\xi,S) > \delta \|V\| \mathbf{B}(z,\varrho) \quad \text{for some } 0 < \varrho < 2r,$$

$A = \mathbf{C}(T,0,r,h) \sim B$, $A(x) = A \cap \{z : \mathbf{p}(z) = x\}$ *for* $x \in \mathbf{R}^m$, X_1 *is the set of all* $x \in \mathbf{R}^m \cap \mathbf{B}(0,r)$ *such that*

$$\textstyle\sum_{z\in A(x)} \boldsymbol{\Theta}^m(\|V\|,z) = Q \quad \text{and} \quad \boldsymbol{\Theta}^m(\|V\|,z) \in \mathscr{P} \cup \{0\} \text{ for } z \in A(x),$$

X_2 *is the set of all* $x \in \mathbf{R}^m \cap \mathbf{B}(0,r)$ *such that*

$$\textstyle\sum_{z\in A(x)} \boldsymbol{\Theta}^m(\|V\|,z) \le Q - 1 \quad \text{and} \quad \boldsymbol{\Theta}^m(\|V\|,z) \in \mathscr{P} \cup \{0\} \text{ for } z \in A(x),$$

$N = \mathbf{R}^m \cap \mathbf{B}(0,r) \sim (X_1 \cup X_2)$, *and* $f : X_1 \to \mathbf{Q}_Q(\mathbf{R}^{n-m})$ *is characterised by the requirement*

$$\boldsymbol{\Theta}^m(\|V\|,z) = \boldsymbol{\Theta}^0(\|f(x)\|, \mathbf{q}(z)) \quad \text{whenever } x \in X_1 \text{ and } z \in A(x),$$

then the following nine statements hold:

(1) X_1 *and* X_2 *are universally measurable, and* $\mathscr{L}^m(N) = 0$.

(2) A *and* B *are Borel sets and*

$$\mathbf{q}[A \cap \operatorname{spt} \|V\|] \subset \mathbf{B}(0, h - \delta_4 r).$$

(3) $\mathbf{p}[A \cap \{z : \boldsymbol{\Theta}^m(\|V\|, z) = Q\}] \subset X_1$.

(4) *The function* f *is Lipschitzian with* $\operatorname{Lip} f \leq L$.

(5) *For* $\mathscr{L}^m$ *almost all* $x \in X_1$ *the following is true:*

(a) *The function* f *is approximately strongly affinely approximable at* x.

(b) *If* $(x, y) \in \operatorname{graph}_Q f$ *then*

$$\operatorname{Tan}^m(\|V\|, (x, y)) = \operatorname{Tan}\big(\operatorname{graph}_Q \operatorname{ap} Af(x), (x, y)\big) \in \mathbf{G}(n, m).$$

(6) *If* $a \in \mathbf{C}(T, 0, r, h)$, $0 < \varrho \leq r - |\mathbf{p}(a)|$, $|\mathbf{q}(a)| + \delta_4 \varrho \leq h$, *and*

$$\begin{aligned} B_{a,\varrho} &= \mathbf{C}(T, a, \varrho, \delta_4 \varrho) \cap B, \\ C_{a,\varrho} &= \mathbf{B}(\mathbf{p}(a), \varrho) \sim (X_1 \sim \mathbf{p}[B_{a,\varrho}]), \\ D_{a,\varrho} &= \mathbf{C}(T, a, \varrho, \delta_4 \varrho) \cap \mathbf{p}^{-1}[C_{a,\varrho}], \end{aligned}$$

then $B_{a,\varrho}$ *is a Borel set and* $C_{a,\varrho}$ *and* $D_{a,\varrho}$ *are universally measurable.*

(7) *If* a, ϱ, $B_{a,\varrho}$, $C_{a,\varrho}$, *and* $D_{a,\varrho}$ *are as in* (6) *and*

$$\begin{aligned} \operatorname{graph}_Q f | \mathbf{B}(\mathbf{p}(a), \varrho) &\subset \mathbf{C}(T, a, \varrho, \delta_4 \varrho / 2), \\ \|V\|(\mathbf{C}(T, a, \varrho, \delta_4 \varrho)) &\geq (Q - 1/4) \boldsymbol{\alpha}(m) \varrho^m, \end{aligned}$$

then

$$\mathscr{L}^m(C_{a,\varrho}) + \|V\|(D_{a,\varrho}) \leq \Gamma_{(7)} \, \|V\|(B_{a,\varrho})$$

with $\Gamma_{(7)} = 3 + 2Q + (12Q + 6)5^m$.

(8) *Suppose* H *denotes the set of all* $z \in \mathbf{C}(T, 0, r, h)$ *such that*

$$\begin{aligned} \|\delta V\| \, \mathbf{U}(z, 2r) &\leq \varepsilon \, \|V\|(\mathbf{U}(z, 2r))^{1-1/m}, \\ \textstyle\int_{\mathbf{U}(z,2r) \times \mathbf{G}(n,m)} |S_\natural - T_\natural| \, \mathrm{d}V(z, S) &\leq \varepsilon \, \|V\| \, \mathbf{U}(z, 2r), \\ \|V\| \, \mathbf{B}(z, \varrho) &\geq \delta_5 \boldsymbol{\alpha}(m) \varrho^m \quad \text{for } 0 < \varrho < 2r. \end{aligned}$$

Then there exists a positive, finite number $\varepsilon_{(8)}$ *depending only on* m, δ_2, *and* δ_4 *with the following property:*

If $c \in \mathbf{R}^m \cap \mathbf{U}(0, r)$, $0 < \varrho \leq r - |c|$, $\mathscr{L}^m(\mathbf{B}(c, \varrho) \sim X_1) \leq \varepsilon_{(8)} \boldsymbol{\alpha}(m) \varrho^m$, $\emptyset \neq P \subset \mathbf{C}(T, \mathbf{p}^*(c), \varrho)$, *for every* $z \in P$ *and* $x \in \mathbf{B}(c, \varrho)$ *there exists* y *with* $(x, y) \in P$ *and* $|y - \mathbf{q}(z)| \leq |x - \mathbf{p}(z)|$, *and* $d : \mathbf{C}(T, \mathbf{p}^*(c), \varrho, h) \to \mathbf{R}$ *and* $g : X_1 \cap \mathbf{B}(c, \varrho) \to \mathbf{R}$ *are defined by*

$$\begin{aligned} d(z) &= \inf\{|\mathbf{q}(\xi - z)| : \xi \in P, \mathbf{p}(\xi) = \mathbf{p}(z)\} && \text{for } z \in \mathbf{C}(T, \mathbf{p}^*(c), \varrho, h), \\ g(x) &= \sup\{d(x, y) : y \in \operatorname{spt} f(x)\} && \text{for } x \in X_1 \cap \mathbf{B}(c, \varrho), \end{aligned}$$

then $\operatorname{Lip} d \leq 2^{1/2}$, $\operatorname{Lip} g \leq 2^{1/2}(1+L)$, *and*

$$\begin{aligned}&(\|V\| \llcorner H \cap \mathbf{C}(T, \mathbf{p}^*(c), \varrho, h))_{(q)}(d)\\ &\leq \Gamma_{(8)} Q\big((\mathscr{L}^m \llcorner \mathbf{B}(c,\varrho) \cap X_1)_{(q)}(g) + \mathscr{L}^m(\mathbf{B}(c,\varrho) \sim X_1)^{1/q+1/m}\big)\end{aligned}$$

whenever $1 \leq q \leq \infty$ *where* $\Gamma_{(8)}$ *is a positive, finite number depending only on* m.

(9) *If* a, ϱ, $C_{a,\varrho}$, $D_{a,\varrho}$ *are as in* (6),

$$\operatorname{graph}_Q f|\mathbf{B}(\mathbf{p}(a),\varrho) \subset \mathbf{C}(T, a, \varrho, \delta_4 \varrho/2),$$

$g : \mathbf{R}^m \to \mathbf{R}^{n-m}$, $\operatorname{Lip} g < \infty$, $g|X_1 = \boldsymbol{\eta}_Q \circ f$, $\tau \in \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m})$, $\theta \in \mathscr{D}(\mathbf{R}^m, \mathbf{R}^{n-m})$, $\eta \in \mathscr{D}^0(\mathbf{R}^{n-m})$,

$$\begin{aligned}\operatorname{spt}\theta \subset \mathbf{U}(\mathbf{p}(a),\varrho), \qquad 0 \leq \eta(y) \leq 1 \quad \text{for } y \in \mathbf{R}^{n-m},\\ \operatorname{spt}\eta \subset \mathbf{U}(\mathbf{q}(a), \delta_4\varrho), \quad \mathbf{B}(\mathbf{q}(a), \delta_4\varrho/2) \subset \operatorname{Int}(\mathbf{R}^{n-m} \cap \{y : \eta(y) = 1\}),\end{aligned}$$

and $\Psi^\S$ *denotes the nonparametric integrand associated to the area integrand* Ψ, *then*

$$\begin{aligned}&\big|Q{\textstyle\int}\big\langle D\theta(x), D\Psi_0^\S(Dg(x))\big\rangle \,\mathrm{d}\mathscr{L}^m x - (\delta V)((\eta\circ\mathbf{q})\cdot(\mathbf{q}^*\circ\theta\circ\mathbf{p}))\big|\\ &\quad\leq \gamma_1 Q m^{1/2} \operatorname{Lip} g {\textstyle\int}_{C_{a,\varrho}} |D\theta| \,\mathrm{d}\mathscr{L}^m\\ &\qquad + \gamma_2 {\textstyle\int}_{E_{a,\varrho}\sim C_{a,\varrho}} |D\theta(x)| |\operatorname{ap} Af(x) \,(+)(-\tau)|^2 \,\mathrm{d}\mathscr{L}^m x\\ &\qquad + m^{1/2} {\textstyle\int}_{D_{a,\varrho}} |D((\eta\circ\mathbf{q})\cdot(\mathbf{q}^*\circ\theta\circ\mathbf{p}))| \,\mathrm{d}\|V\|\end{aligned}$$

where

$$\begin{aligned}\gamma_1 &= \sup \|D^2\Psi_0^\S\|[\mathbf{B}(0, m^{1/2}\operatorname{Lip} g)],\\ \gamma_2 &= \operatorname{Lip}\big(D^2\Psi_0^\S|\mathbf{B}(0, m^{1/2}(L+2\|\tau\|))\big),\\ E_{a,\varrho} &= \mathbf{B}(\mathbf{p}(a),\varrho) \cap X_1 \cap \{x : \boldsymbol{\Theta}^0(\|f(x)\|, g(x)) \neq Q\}.\end{aligned}$$

Choice of constants. One can assume $2L \leq \delta_4$ and $\delta_5 \leq (2\boldsymbol{\gamma}(m)m)^{-m}/\boldsymbol{\alpha}(m)$ whenever $m \in \mathscr{P}$ with $m < n$.

Choose $0 < s_0 < 1$, $0 < s < 1$ close to 1 satisfying

$$(s_0^{-2} - 1)^{1/2} \leq \delta_4/2, \quad (s^{-2}-1)^{1/2} \leq \inf\{\delta_4/4, L\}$$

and define $\varepsilon > 0$ so small that

$$1 - n\varepsilon^2 \geq 1/2, \quad (1-n\varepsilon^2)(Q - 1/4) \geq Q - 1/2$$

and not larger than the infimum of the following numbers corresponding to $m \in \mathscr{P}$ with $m < n$

$$\begin{gathered}\varepsilon_{[\text{Men09b, 2.18}]}(n-m, m, Q, L, M, \delta_1, \delta_2, \delta_3, \delta_4, \delta_5), \quad (2\boldsymbol{\gamma}(m))^{-1},\\ \varepsilon_{4.6}(n, Q+1, M, \inf\{\delta_2/2, (2\boldsymbol{\gamma}(m)m)^{-m}/\boldsymbol{\alpha}(m)\}, s) \quad \varepsilon_{4.6}(n, Q, M, 1/4, s),\\ \varepsilon_{[\text{Men09b, 2.13}]}(n-m, m, 1, \delta_2, 0, s_0, M).\end{gathered}$$

Clearly, δ satisfies the same inequalities as ε and one can assume $r = 1$. □

Proof of (1) (2) (4) (5). The sets X_1 and X_2 are universally measurable by 4.4 and 4.5. Noting the sets Y and Z defined in the proof of [Men09b, 2.18 (1) (2)] equal X_1 and X_2 and satisfy $\mathscr{L}^m(\mathbf{B}(0,1)\sim(Y\cup Z))=0$, the assertion follows from [Men09b, 2.18 (1) (2) (4) (7)]. □

Proof of (3). Let $\eta = \inf\{\delta_2/2, (2\boldsymbol{\gamma}(m)m)^{-m}/\boldsymbol{\alpha}(m)\}$, consider $z\in A$ with $\boldsymbol{\Theta}^m(\|V\|,z)=Q$, $Z=A(\mathbf{p}(z))$, note, using (2), that

$$\mathbf{U}(\xi-\mathbf{p}^*(\mathbf{p}(z)),1)\cap\{\kappa : |T_\natural(\kappa-\xi)>s|\kappa-\xi|\}\subset\mathbf{C}(T,0,1,h)$$

for $\xi\in A(\mathbf{p}(z))$ and apply 4.6 with

$$Q,\ \delta,\ d,\ r,\ t,\text{ and } f$$
$$\text{replaced by } Q+1,\ \eta,\ 1,\ 2,\ 1,\text{ and } \boldsymbol{\tau}_{-\mathbf{p}^*(\mathbf{p}(z))}|Z$$

to obtain $\sum_{\xi\in A(\mathbf{p}(z))}\boldsymbol{\Theta}^m_*(\|V\|,\xi)<Q+\eta$, hence [Men09a, 2.5] implies (3). □

Proof of (6). The set $\mathbf{p}[B_{a,\varrho}]$ is universally measurable by [Fed69, 2.2.13], hence $C_{a,\varrho}$, $D_{a,\varrho}$ are universally measurable sets by 4.3, 4.4. □

Proof of (7). Let ν denote the Radon measure characterised by

$$\nu(Z)=\textstyle\int_Z\|\Lambda_m(\mathbf{p}|S)\|\,\mathrm{d}V(z,S)$$

whenever Z is a Borel subset of U, and note

$$|S_\natural-T_\natural|\leq\varepsilon\quad\text{for } V \text{ almost all } (z,S)\in A\times\mathbf{G}(n,m),$$

hence $1-\|\Lambda_m(\mathbf{p}|S)\|\leq 1-\|\Lambda_m(T_\natural|S)\|^2\leq m\varepsilon^2$ for those (z,S) by [Men09b, 2.16]. Therefore

$$(1-m\varepsilon^2)\,\|V\|\llcorner A\leq\nu\llcorner A.$$

This implies the *coarea estimate*

$$\begin{aligned}&(1-m\varepsilon^2)\,\|V\|\big(\mathbf{C}(T,a,\varrho,\delta_4\varrho)\cap\mathbf{p}^{-1}[W]\big)\\ &\leq\|V\|\big(B_{a,\varrho}\cap\mathbf{p}^{-1}[W]\big)+Q\mathscr{L}^m(X_1\cap W)+(Q-1)\mathscr{L}^m(X_2\cap W)\end{aligned}$$

for every subset W of $\mathbf{R}^m$; in fact the estimate holds for every Borel set by [Fed69, 3.2.22 (3)] and $\mathbf{p}_\#(\|V\|\llcorner B_{a,\varrho})$ is a Radon measure by [Fed69, 2.2.17]. In particular, taking $W=\mathbf{B}(\mathbf{p}(a),\varrho)$ yields

$$(1-m\varepsilon^2)\|V\|(\mathbf{C}(T,a,\varrho,\delta_4\varrho))\leq\|V\|(B_{a,\varrho})+Q\boldsymbol{\alpha}(m)\varrho^m,$$

thus one can assume, since $8Q+6\leq\Gamma_{(7)}$, that

$$\|V\|(B_{a,\varrho})\leq\tfrac14\boldsymbol{\alpha}(m)\varrho^m.$$

Next, it will be shown that this assumption implies

$$\mathscr{L}^m(X_1\cap\mathbf{B}(\mathbf{p}(a),\varrho))>0;$$

in fact, using the coarea estimate with $W = \mathbf{B}(\mathbf{p}(a), \varrho)$, one obtains

$$\begin{aligned}
&(Q - 1/2)\boldsymbol{\alpha}(m)\varrho^m \\
&\quad \le (1 - m\varepsilon^2)\|V\|(\mathbf{C}(T, a, \varrho, \delta_4\varrho)) \\
&\quad \le \|V\|(B_{a,\varrho}) + Q\mathscr{L}^m(X_1 \cap \mathbf{B}(\mathbf{p}(a), \varrho)) + (Q-1)\mathscr{L}^m(X_2 \cap \mathbf{B}(\mathbf{p}(a), \varrho)) \\
&\quad \le (Q - 1/2)\boldsymbol{\alpha}(m)\varrho^m + \mathscr{L}^m(X_1 \cap \mathbf{B}(\mathbf{p}(a), \varrho)) - \tfrac{1}{4}\mathscr{L}^m(X_2 \cap \mathbf{B}(\mathbf{p}(a), \varrho)), \\
&\mathscr{L}^m(X_2 \cap \mathbf{B}(\mathbf{p}(a), \varrho)) \le 4\,\mathscr{L}^m(X_1 \cap \mathbf{B}(\mathbf{p}(a), \varrho)), \quad \mathscr{L}^m(X_1 \cap \mathbf{B}(\mathbf{p}(a), \varrho)) > 0.
\end{aligned}$$

In order to estimate $\mathscr{L}^m(X_2 \cap \mathbf{B}(\mathbf{p}(a), \varrho))$, the following assertion will be proven. *If $x \in X_2 \cap \mathbf{B}(\mathbf{p}(a), \varrho)$ and $\boldsymbol{\Theta}^m(\mathscr{L}^m \llcorner \mathbf{R}^m \sim X_2, x) = 0$, then there exist $\zeta \in \mathbf{R}^m$ and $0 < t < \infty$ with*

$$x \in \mathbf{B}(\zeta, t) \subset \mathbf{B}(\mathbf{p}(a), \varrho), \quad \mathscr{L}^m\,\mathbf{B}(\zeta, 5t) \le 6 \cdot 5^m\,\|V\|\big(B_{a,\varrho} \cap \mathbf{p}^{-1}[\mathbf{B}(\zeta, t)]\big).$$

Since $\mathscr{L}^m(X_1 \cap \mathbf{B}(\mathbf{p}(a), \varrho)) > 0$, some element $\mathbf{B}(\zeta, t)$ of the family of balls

$$\{\mathbf{B}((1-\theta)x + \theta\mathbf{p}(a), \theta\varrho) : 0 < \theta \le 1\}$$

will satisfy

$$x \in \mathbf{B}(\zeta, t) \subset \mathbf{B}(\mathbf{p}(a), \varrho), \quad 0 < \mathscr{L}^m(X_1 \cap \mathbf{B}(\zeta, t)) \le \tfrac{1}{2}\mathscr{L}^m(X_2 \cap \mathbf{B}(\zeta, t)).$$

Hence there exists $\eta \in X_1 \cap \mathbf{U}(\zeta, t)$. Noting for $\xi \in A(\eta)$ with $\boldsymbol{\Theta}^m(\|V\|, \xi) > 0$

$$\begin{gathered}
\mathbf{U}(\boldsymbol{\tau}_{\mathbf{p}^*(\zeta-\eta)}(\xi), t) \subset \mathbf{p}^{-1}[\mathbf{B}(\zeta, t)], \quad \xi \in \operatorname{spt} f(\eta) \subset \mathbf{B}(\mathbf{q}(a), \delta_4\varrho/2), \\
(s^{-2} - 1)^{1/2}|\mathbf{p}(\kappa - \xi)| \le \delta_4 t/2 \le \delta_4\varrho/2 \quad \text{for } \kappa \in \mathbf{p}^{-1}[\mathbf{B}(\zeta, t)],
\end{gathered}$$

the inclusion

$$\mathbf{U}(\boldsymbol{\tau}_{\mathbf{p}^*(\zeta-\eta)}(\xi), t) \cap \{\kappa : |\mathbf{p}(\kappa - \xi)| > s|\kappa - \xi|\} \subset \mathbf{C}(T, a, \varrho, \delta_4\varrho) \cap \mathbf{p}^{-1}[\mathbf{B}(\zeta, t)]$$

is valid for such ξ and 4.6 can be applied with

$$\begin{gathered}
\delta,\ Z,\ d,\ r, \text{ and } f \text{ replaced by} \\
1/4,\ A(\eta) \cap \{\xi : \boldsymbol{\Theta}^m(\|V\|, \xi) > 0\},\ t,\ 2, \\
\text{and } \boldsymbol{\tau}_{\mathbf{p}^*(\zeta-\eta)} | A(\eta) \cap \{\xi : \boldsymbol{\Theta}^m(\|V\|, \xi) > 0\}
\end{gathered}$$

to obtain

$$(Q - 1/4)\boldsymbol{\alpha}(m)t^m \le \|V\|\big(\mathbf{C}(T, a, \varrho, \delta_4\varrho) \cap \mathbf{p}^{-1}[\mathbf{B}(\zeta, t)]\big).$$

The coarea estimate with $W = \mathbf{B}(\zeta, t)$ now implies

$$\begin{aligned}
&(Q - 1/2)\boldsymbol{\alpha}(m)t^m - \|V\|(B_{a,\varrho} \cap \mathbf{p}^{-1}[\mathbf{B}(\zeta, t)]) \\
&\quad \le Q\mathscr{L}^m(X_1 \cap \mathbf{B}(\zeta, t)) + (Q-1)\mathscr{L}^m(X_2 \cap \mathbf{B}(\zeta, t)) \\
&\quad = (Q - 1/2)\boldsymbol{\alpha}(m)t^m + \tfrac{1}{2}\mathscr{L}^m(X_1 \cap \mathbf{B}(\zeta, t)) - \tfrac{1}{2}\mathscr{L}^m(X_2 \cap \mathbf{B}(\zeta, t)),
\end{aligned}$$

hence, recalling $\mathscr{L}^m(X_1 \cap \mathbf{B}(\zeta, t)) \le \frac{1}{2}\mathscr{L}^m(X_2 \cap \mathbf{B}(\zeta, t))$,

$$\tfrac{2}{3}\mathscr{L}^m(\mathbf{B}(\zeta, t)) \le \mathscr{L}^m(X_2 \cap \mathbf{B}(\zeta, t)) \le 4\,\|V\|\big(B_{a,\varrho} \cap \mathbf{p}^{-1}[\mathbf{B}(\zeta, t)]\big)$$

and the assertion follows.

The assumption of the last assertion is satisfied for $\mathscr{L}^m$ almost all $x \in X_2 \cap \mathbf{B}(\mathbf{p}(a), \varrho)$ by [Fed69, 2.9.11] and [Fed69, 2.8.5] implies

$$\mathscr{L}^m(X_2 \cap \mathbf{B}(\mathbf{p}(a), \varrho)) \leq 6 \cdot 5^m \|V\|(B_{a,\varrho}).$$

Clearly,

$$\mathscr{L}^m(\mathbf{p}[B_{a,\varrho}]) \leq \mathscr{H}^m(B_{a,\varrho}) \leq \|V\|(B_{a,\varrho}).$$

Since $C_{a,\varrho} \sim N \subset (X_2 \cap \mathbf{B}(\mathbf{p}(a), \varrho)) \cup \mathbf{p}[B_{a,\varrho}]$, it follows

$$\mathscr{L}^m(C_{a,\varrho}) \leq (1 + 6 \cdot 5^m)\|V\|(B_{a,\varrho}).$$

Finally, applying the coarea estimate with $W = C_{a,\varrho}$ yields

$$\begin{aligned}(1 - m\varepsilon^2)\|V\|(D_{a,\varrho}) &\leq \|V\|(B_{a,\varrho}) + Q\mathscr{L}^m(C_{a,\varrho}) \\ &\leq (1 + Q + 6Q \cdot 5^m)\|V\|(B_{a,\varrho})\end{aligned}$$

and the conclusion follows. □

Proof of (8). Choose $0 < \lambda \leq 1$ such that

$$\lambda \leq \inf\{\lambda_{[\text{Men09b, 2.18 (4)}]}(m, \delta_2, \delta_4), \lambda_{[\text{Men09b, 2.13}]}(m, \delta_2, s_0)/2\}$$

and define $\varepsilon_{(8)} = (1/2)(\lambda/6)^m \leq 1$.

Suppose $z_1, z_2 \in \mathbf{C}(T, \mathbf{p}^*(c), \varrho, h)$ and $\xi_1 \in P$ with $\mathbf{p}(\xi_1) = \mathbf{p}(z_1)$. Then there exists $\xi_2 \in P$ such that $\mathbf{p}(\xi_2) = z_2$ and $|\mathbf{q}(\xi_1 - \xi_2)| \leq |\mathbf{p}(\xi_1 - \xi_2)|$, hence

$$\begin{aligned}|\mathbf{q}(\xi_2 - z_2)| &\leq |\mathbf{q}(\xi_2 - \xi_1)| + |\mathbf{q}(\xi_1 - z_1)| + |\mathbf{q}(z_1 - z_2)| \\ &\leq 2^{1/2}|z_1 - z_2| + |\mathbf{q}(\xi_1 - z_1)|\end{aligned}$$

and $\operatorname{Lip} d \leq 2^{1/2}$.

Suppose $x_1, x_2 \in X_1 \cap \mathbf{B}(c, \varrho)$, $y_1 \in \operatorname{spt} f(x_1)$. Then there exists $y_2 \in \operatorname{spt} f(x_2)$ with $|y_1 - y_2| \leq L|x_1 - x_2|$, hence

$$d(x_1, y_1) \leq 2^{1/2}|(x_1, y_1) - (x_2, y_2)| + d(x_2, y_2) \leq 2^{1/2}(1 + L)|x_1 - x_2| + g(x_2)$$

and $\operatorname{Lip} g \leq 2^{1/2}(1 + L)$.

First, *the case* $q < \infty$ will be treated. Note $A \cap \operatorname{spt}\|V\| \subset H$ and $H \cap \mathbf{p}^{-1}[X_1] = \operatorname{graph}_Q f$ by [Men09b, 2.18 (4)], let $\psi = \|V\| \llcorner H \cap \mathbf{C}(T, \mathbf{p}^*(c), \varrho, h)$ and recall

$$(\mathbf{p}_\# \psi) \llcorner X_1 \leq 2(\mathbf{p}_\#(\nu \llcorner H)) \llcorner X_1 \leq 2Q\mathscr{L}^n \llcorner X_1$$

with ν as in the proof of (7). Using

$$\begin{aligned}&H \cap \mathbf{C}(T, \mathbf{p}^*(c), \varrho, h) \cap \mathbf{p}^{-1}[X_1] \cap \{z : d(z) > \gamma\} \\ &\subset H \cap \mathbf{p}^{-1}[X_1 \cap \mathbf{B}(c, \varrho) \cap \{x : g(x) > \gamma\}]\end{aligned}$$

for $0 < \gamma < \infty$, one infers

$$(\psi \llcorner \mathbf{p}^{-1}[X_1])_{(q)}(d) \leq 2Q(\mathscr{L}^m \llcorner X_1 \cap \mathbf{B}(c, \varrho))_{(q)}(g).$$

Therefore it remains to estimate $(\psi \llcorner U \sim \mathbf{p}^{-1}[X_1])_{(q)}(d)$.

Whenever $x \in \mathbf{B}(c, \varrho) \sim \operatorname{Clos} X_1$ there exist $\zeta \in \mathbf{R}^m$, $0 < t \leq (2\varepsilon_{(8)})^{1/m} \varrho = \lambda\varrho/6$ such that

$$x \in \mathbf{B}(\zeta, t) \subset \mathbf{B}(c, \varrho), \quad \mathscr{L}^m(\mathbf{B}(\zeta, t) \cap X_1) = \mathscr{L}^m(\mathbf{B}(\zeta, t) \sim X_1)$$

as may be verified by consideration of the family of closed balls

$$\{\mathbf{B}(\theta c + (1-\theta)x, \theta\varrho) : 0 < \theta \leq (2\varepsilon_{(8)})^{1/m}\}.$$

Therefore [Fed69, 2.8.5] yields a countable set I and $\zeta_i \in \mathbf{R}^m$, $0 < t_i \leq \lambda\varrho/6$ and $x_i \in X_1 \cap \mathbf{B}(\zeta_i, t_i)$ for each $i \in I$ such that

$$\begin{gathered}\mathbf{B}(\zeta_i, t_i) \subset \mathbf{B}(c, \varrho), \quad \mathscr{L}^m(\mathbf{B}(\zeta_i, t_i) \cap X_1) = \mathscr{L}^m(\mathbf{B}(\zeta_i, t_i) \sim X_1), \\ \mathbf{B}(\zeta_i, t_i) \cap \mathbf{B}(\zeta_j, t_j) = \emptyset \quad \text{whenever } i, j \in I \text{ with } i \neq j, \\ \mathbf{B}(c, \varrho) \sim \operatorname{Clos} X_1 \subset \bigcup\{E_i : i \in I\} \subset \mathbf{B}(c, \varrho)\end{gathered}$$

where $E_i = \mathbf{B}(\zeta_i, 5t_i) \cap \mathbf{B}(c, \varrho)$ for $i \in I$. Let

$$h_i = g(x_i), \quad Z_i = A(x_i) \cap \{\xi : \boldsymbol{\Theta}^m(\|V\|, \xi) \in \mathscr{P}\}$$

for $i \in I$, $J = I \cap \{i : h_i \geq 24t_i\}$, and $K = I \sim J$.

In view of [Men09b, 2.18 (5)] there holds

$$\begin{aligned}&(\psi \llcorner U \sim \mathbf{p}^{-1}[X_1])_{(q)}(d) \\ &\quad \leq (\psi \llcorner \mathbf{p}^{-1}[\textstyle\bigcup\{E_j : j \in J\}])_{(q)}(d) + (\psi \llcorner \mathbf{p}^{-1}[\textstyle\bigcup\{E_k : k \in K\}])_{(q)}(d).\end{aligned}$$

In order to estimate the terms on the right hand side, two observations will be useful. Firstly, *if* $i \in I$, $z \in H \cap \mathbf{C}(T, \mathbf{p}^*(c), \varrho, h) \cap \mathbf{p}^{-1}[E_i]$, *then*

$$d(z) \leq 24t_i + h_i;$$

in fact $|\mathbf{p}(z) - x_i| \leq 6t_i \leq \lambda\varrho \leq \lambda$ and [Men09b, 2.18 (4)] yields a point $\xi \in Z_i$ with $|\mathbf{q}(z - \xi)| \leq L|\mathbf{p}(z - \xi)|$, hence

$$\begin{gathered}|z - \xi| \leq (1 + L)|\mathbf{p}(z - \xi)| = (1 + L)|\mathbf{p}(z) - x_i| \leq 12t_i, \\ d(z) \leq 2^{1/2}|z - \xi| + d(\xi) \leq 24t_i + h_i.\end{gathered}$$

Moreover, since

$$H \cap \mathbf{C}(T, \mathbf{p}^*(c), \varrho, h) \cap \mathbf{p}^{-1}[E_i] \subset \bigcup\{\mathbf{B}(\xi, 12t_i) : \xi \in Z_i\},$$

one may apply [Men09b, 2.13 (1)], verifying

$$\mathbf{U}(z - \mathbf{p}^*(x_i), 1) \cap \{\xi : |\mathbf{p}(\xi - z)| > s_0|\xi - z|\} \subset \mathbf{C}(T, 0, 1, h)$$

whenever $z \in A(x_i)$ with the help of (2), with

m, n, δ_1, s, λ, X, d, r, t, ζ, μ, and τ replaced by
$n - m$, m, 1, 0, $\lambda_{[\text{Men09b, 2.13 (1)}]}(m, \delta_2, s_0)$, Z_i, 1, 2, 1, $-\mathbf{p}^*(x_i)$, $\|V\|$, and $12t_i$

to obtain the second observation, *namely*

$$\psi\big(\mathbf{p}^{-1}[E_i]\big) \le (Q+1)\boldsymbol{\alpha}(m)(12t_i)^m \quad \textit{whenever } i \in I.$$

Now, the first term will be estimated. Note, if $j \in J$, then

$$\begin{aligned} d(z) \le 2h_j &\quad \text{whenever } z \in H \cap \mathbf{C}(T, \mathbf{p}^*(c), \varrho, h) \cap \mathbf{p}^{-1}[E_j], \\ 2h_j \le 3g(x) &\quad \text{whenever } x \in X_1 \cap \mathbf{B}(\zeta_j, t_j), \end{aligned}$$

because

$$g(x) \ge g(x_j) - 4|x_j - x| \ge h_j - 8t_j \ge 2h_j/3.$$

Using this fact and the preceding observations, one estimates with $J(\gamma) = J \cap \{j : 2h_j > \gamma\}$ for $0 < \gamma < \infty$

$$\begin{aligned} &\psi\big(\mathbf{p}^{-1}[\bigcup\{E_j : j \in J\}] \cap \{z : d(z) > \gamma\}\big) \le \textstyle\sum_{j \in J(\gamma)} \psi\big(\mathbf{p}^{-1}[E_j]\big) \\ &\quad \le \textstyle\sum_{j \in J(\gamma)} (Q+1)\boldsymbol{\alpha}(m)(12t_j)^m \\ &\quad \le (Q+1)(12)^m \mathscr{L}^m \big(\bigcup\{\mathbf{B}(\zeta_j, t_j) : j \in J(\gamma)\}\big) \\ &\quad \le 2(Q+1)(12)^m \mathscr{L}^m \big(\bigcup\{X_1 \cap \mathbf{B}(\zeta_j, t_j) : j \in J(\gamma)\}\big) \\ &\quad \le 2(Q+1)(12)^m \mathscr{L}^m (X_1 \cap \mathbf{B}(c, \varrho) \cap \{x : g(x) > \gamma/3\}, \end{aligned}$$

hence

$$(\psi \llcorner \mathbf{p}^{-1}[\bigcup\{E_j : j \in J\}])_{(q)}(d) \le Q(12)^{m+1} (\mathscr{L}^m \llcorner X_1 \cap \mathbf{B}(c, \varrho))_{(q)}(g).$$

To estimate the second term, one notes

$$d(z) < 48t_k \quad \text{whenever } k \in K,\ z \in H \cap \mathbf{C}(T, \mathbf{p}^*(c), \varrho, h) \cap \mathbf{p}^{-1}[E_k].$$

Therefore one estimates with $K(\gamma) = K \cap \{k : 48t_k > \gamma\}$ for $0 < \gamma < \infty$ and $u : \mathbf{R}^m \to \mathbf{R}$ defined by $u = \sum_{i \in I} t_i b_i$ where b_i is the characteristic function of $\mathbf{B}(\zeta_i, t_i)$

$$\begin{aligned} &\psi\big(\mathbf{p}^{-1}[\bigcup\{E_k : k \in K\}] \cap \{z : d(z) > \gamma\}\big) \le \textstyle\sum_{k \in K(\gamma)} \psi\big(\mathbf{p}^{-1}[E_k]\big) \\ &\quad \le \textstyle\sum_{k \in K(\gamma)} (Q+1)\boldsymbol{\alpha}(m)(12t_k)^m \\ &\quad \le (Q+1)(12)^m \mathscr{L}^m (\bigcup\{\mathbf{B}(\zeta_k, t_k) : k \in K(\gamma)\}) \\ &\quad \le (Q+1)(12)^m \mathscr{L}^m (\mathbf{R}^m \cap \{x : u(x) > \gamma/(48)\}), \end{aligned}$$

hence

$$(\psi \llcorner \mathbf{p}^{-1}[\bigcup\{E_k : k \in K\}])_{(q)}(d) \le Q(12)^{m+2} \mathscr{L}^m{}_{(q)}(u).$$

Combining these two estimates and

$$\begin{aligned} \mathscr{L}^m(\bigcup\{\mathbf{B}(\zeta_i, t_i) : i \in I\}) &\le 2\mathscr{L}^m(\mathbf{B}(c, \varrho) \sim X_1), \\ \textstyle\int |u|^q \, \mathrm{d}\mathscr{L}^m &= \boldsymbol{\alpha}(m)^{-q/m} \textstyle\sum_{i \in I} \mathscr{L}^m(\mathbf{B}(\zeta_i, t_i))^{1+q/m} \\ &\le \boldsymbol{\alpha}(m)^{-q/m} \big(\textstyle\sum_{i \in I} \mathscr{L}^m(\mathbf{B}(\zeta_i, t_i))\big)^{1+q/m}, \\ (\mathscr{L}^m)_{(q)}(u) &\le 4\boldsymbol{\alpha}(m)^{-1/m} \mathscr{L}^m(\mathbf{B}(c, \varrho) \sim X_1)^{1/q+1/m}, \end{aligned}$$

one obtains the conclusion for $q < \infty$.

The case $q = \infty$ follows by taking the limit $q \to \infty$ with the help of [Fed69, 2.4.17]. □

Proof of (9). Let I, f_i be associated to f as in 2.3, and define $C_i = \operatorname{dmn} f_i$ for $i \in I$ and $G = \operatorname{graph}_Q f$. Note

$$G \cap \mathbf{p}^{-1}[\mathbf{B}(\mathbf{p}(a),\varrho) \sim C_{a,\varrho}] = G \cap \mathbf{C}(T,a,\varrho,\delta_4\varrho/2) \sim \mathbf{p}^{-1}[C_{a,\varrho}],$$
$$\mathbf{p}[B_{a,\varrho}] \subset C_{a,\varrho}, \quad \|V\|\big(\mathbf{C}(T,a,\varrho,\delta_4\varrho) \sim (G \cup \mathbf{p}^{-1}[C_{a,\varrho}])\big) = 0.$$

Therefore one computes using 2.6 and recalling that $C_{a,\varrho}$, $D_{a,\varrho}$, and, by 4.3, also $\mathbf{p}^{-1}[C_{a,\varrho}]$ are universally measurable

$$\begin{aligned}
&\sum_{i \in I} \textstyle\int_{C_i \cap \mathbf{B}(\mathbf{p}(a),\varrho) \sim C_{a,\varrho}} \big\langle D\theta(x), D\Psi_0^\S(\operatorname{ap} Df_i(x)) \big\rangle \,\mathrm{d}\mathscr{L}^m x \\
&= \delta\big(V \llcorner (G \cap \mathbf{p}^{-1}[\mathbf{B}(\mathbf{p}(a),\varrho) \sim C_{a,\varrho}]) \times \mathbf{G}(n,m)\big)(\mathbf{q}^* \circ \theta \circ \mathbf{p}) \\
&= \delta\big(V \llcorner (G \cap \mathbf{C}(T,a,\varrho,\delta_4\varrho/2) \sim \mathbf{p}^{-1}[C_{a,\varrho}]) \times \mathbf{G}(n,m)\big)((\eta \circ \mathbf{q}) \cdot (\mathbf{q}^* \circ \theta \circ \mathbf{p})) \\
&= \delta\big(V \llcorner (\mathbf{C}(T,a,\varrho,\delta_4\varrho) \sim \mathbf{p}^{-1}[C_{a,\varrho}]) \times \mathbf{G}(n,m)\big)((\eta \circ \mathbf{q}) \cdot (\mathbf{q}^* \circ \theta \circ \mathbf{p})) \\
&= (\delta V)((\eta \circ \mathbf{q}) \cdot (\mathbf{q}^* \circ \theta \circ \mathbf{p})) - \delta(V \llcorner (D_{a,\varrho} \times \mathbf{G}(n,m)))((\eta \circ \mathbf{q}) \cdot (\mathbf{q}^* \circ \theta \circ \mathbf{p})),
\end{aligned}$$

hence

$$\begin{aligned}
&Q \textstyle\int \big\langle D\theta(x), D\Psi_0^\S(Dg(x)) \big\rangle \,\mathrm{d}\mathscr{L}^m x - (\delta V)((\eta \circ \mathbf{q}) \cdot (\mathbf{q}^* \circ \theta \circ \mathbf{p})) \\
&= Q \textstyle\int_{C_{a,\varrho}} \big\langle D\theta(x), D\Psi_0^\S(Dg(x)) \big\rangle \,\mathrm{d}\mathscr{L}^m x \\
&\quad + Q\Big(\textstyle\int_{\mathbf{B}(\mathbf{p}(a),\varrho) \sim C_{a,\varrho}} \big\langle D\theta(x), D\Psi_0^\S(Dg(x)) \big\rangle \,\mathrm{d}\mathscr{L}^m x \\
&\qquad - \frac{1}{Q} \sum_{i \in I} \textstyle\int_{C_i \cap \mathbf{B}(\mathbf{p}(a),\varrho) \sim C_{a,\varrho}} \big\langle D\theta(x), D\Psi_0^\S(\operatorname{ap} Df_i(x)) \big\rangle \,\mathrm{d}\mathscr{L}^m x \Big) \\
&\quad - \delta(V \llcorner (D_{a,\varrho} \times \mathbf{G}(n,m)))((\eta \circ \mathbf{q}) \cdot (\mathbf{q}^* \circ \theta \circ \mathbf{p})).
\end{aligned}$$

The first summand may be estimated using

$$D\Psi_0^\S(0) = 0, \quad \|D\Psi_0^\S(\alpha)\| \le \gamma_1 |\alpha| \le \gamma_1 m^{1/2} \operatorname{Lip} g$$

for $\alpha \in \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m})$ with $\|\alpha\| \le \operatorname{Lip} g$. The second summand can be treated noting

$$Dg(x) = \frac{1}{Q} \sum_{i \in I(x)} \operatorname{ap} Df_i(x) \quad \text{where } I(x) = I \cap \{ i : x \in \operatorname{dmn} \operatorname{ap} Df_i \}$$

for $\mathscr{L}^m$ almost all $x \in \mathbf{B}(\mathbf{p}(a),\varrho) \sim C_{a,\varrho}$ and applying 4.7 with

$$\begin{gathered}
X,\ Y,\ f,\ a,\ r, \text{ and } \{x_1, \ldots, x_Q\} \\
\text{replaced by } \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m}),\ \operatorname{Hom}(\operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m}), \mathbf{R}),\ D\Psi_0^\S,\ \tau, \\
Q^{-1/2} |\operatorname{ap} Af(x) \,(+)(-\tau)|, \text{ and } \{\operatorname{ap} Df_i(x) : i \in I(x)\}
\end{gathered}$$

for $\mathscr{L}^m$ almost all $x \in E_{a,\varrho} \sim C_{a,\varrho}$. Finally, the third summand is estimated by use of

$$|S_\natural \bullet \beta| \le m^{1/2} |\beta| \quad \text{for } S \in \mathbf{G}(n,m),\ \beta \in \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n). \qquad \square$$

4.9 Remark. Concerning measurability, note that $\mathscr{L}^m$ measurability of W does not imply $\|V\|$ measurability of $\mathbf{p}^{-1}[W]$ but only ν measurability. An example is provided by taking $n-m=1$, $m>1$, W to be a $\mathscr{H}^{m-1}$ nonmeasurable subset of $\mathbf{S}^{m-1}$ and $V \in \mathbf{IV}_m(\mathbf{R}^m \times \mathbf{R}^{n-m})$ such that $\|V\| = \mathscr{H}^m \llcorner \mathbf{p}^{-1}[\mathbf{S}^{m-1}]$ as may be verified by use of [Fed69, 2.2.4, 2.6.2, 3.2.23]. In the case $W = C_{a,\varrho}$ this difficulty could also have been resolved by making use of $\mathbf{p}^{-1}[X_1 \sim \mathbf{p}[B_{a,\varrho}]] \cap B_{a,\varrho} = \emptyset$.

4.10 Remark. If a and ϱ are as in (6), $a \in A$, $\boldsymbol{\Theta}^m(\|V\|, a) = Q$, $0 < s < 1$, $(s^{-2}-1)^{1/2} \leq \delta_4$, $\delta \leq \varepsilon_{4.6}(n, Q, M, 1/4, s)$, then

$$\mathbf{U}(a,\varrho) \cap \{\xi : |\mathbf{p}(\xi - a)| > s|\xi - a|\} \subset \mathbf{C}(T, a, \varrho, \delta_4 \varrho)$$

and 4.6 applied with

$$\delta,\ Z,\ d,\ r,\ t, \text{ and } f \text{ replaced by}$$
$$1/4,\ \{a\},\ 0,\ 2,\ \varrho, \text{ and } \mathbf{1}_{\{a\}}$$

yields

$$\|V\|(\mathbf{C}(T, a, \varrho, \delta_4 \varrho)) \geq (Q - 1/4)\boldsymbol{\alpha}(m)\varrho^m.$$

Moreover, if additionally $L \leq \delta_4/2$ then (3) implies $a \in \operatorname{graph}_Q f$ and

$$\operatorname{graph}_Q f | \mathbf{B}(\mathbf{p}(a), \varrho) \subset \mathbf{C}(T, a, \varrho, \delta_4 \varrho/2).$$

5 An interpolation inequality

In this section an interpolation inequality for weakly differentiable functions defined in a ball $\mathbf{U}(a,r)$ with $a \in \mathbf{R}^m$, $0 < r < \infty$ with values in $\mathbf{R}^{n-m}$ is proven (see 5.3) which states that the Lebesgue seminorm of a function can be controlled by a small multiple of a suitable Lebesgue seminorm of its weak derivative and a large multiple of the $\mathbf{L}_1(\mathscr{L}^m \llcorner A, \mathbf{R}^{n-m})$ seminorm of the function where A is subset of $\mathbf{U}(a,r)$ which is large in $\mathscr{L}^m$ measure. The possibility to neglect a set of small $\mathscr{L}^m$ measure will be important in Section 7. The proof is accomplished following essentially the usual lines (see e.g. [GT01, Theorem 7.27]). The case of Lipschitzian functions with values in $\mathbf{Q}_Q(\mathbf{R}^{n-m})$ then is a simple consequence of Almgren's bi-Lipschitzian embedding of $\mathbf{Q}_Q(\mathbf{R}^{n-m})$ into $\mathbf{R}^{PQ}$ for some P, see 5.4. Finally, two auxiliary statements are included in 5.5 and 5.6 for later reference.

5.1 Lemma. *Suppose $m, n \in \mathscr{P}$, $1 \leq \zeta \leq m < n$, either $\zeta = m = 1$ or $\zeta < m$, $q = \infty$ if $m = 1$, $q = m\zeta/(m-\zeta)$ if $m > 1$, U is an open, bounded, convex subset of $\mathbf{R}^m$, A is an $\mathscr{L}^m$ measurable subset of U with $\mathscr{L}^m(A) > 0$, $u \in \mathbf{W}^{1,1}(U, \mathbf{R}^{n-m})$ and $h = \fint_A u \,\mathrm{d}\mathscr{L}^m$.*

Then

$$|u - h|_{q;U} \leq \Gamma \frac{(\operatorname{diam} U)^m}{\mathscr{L}^m(A)} |\mathbf{D}u|_{\zeta;U}$$

where Γ is a positive, finite number depending only on m and ζ.

Proof. If $\zeta = m = 1$ then u is $\mathscr{L}^1 \llcorner \mathbf{U}(a,r)$ almost equal to an absolutely continuous function by [Fed69, 4.5.9 (30), 4.5.16] and the assertion follows from [Fed69, 2.9.20].

If $\zeta < m$ this fact can be obtained by combining the method of [GT01, Lemma 7.16] with estimates for convolutions, see e.g. O'Neil [O'N63]. □

5.2. Suppose $a, x \in \mathbf{R}^m$, $0 < \varrho \le 2r < \infty$, $x \in \mathbf{U}(a,r)$ and $b = a$ if $|x-a| < \varrho/2$ and $b = x + (\varrho/2)(a-x)/|a-x|$ else. Then one readily verifies $\mathbf{U}(b,\varrho/2) \subset \mathbf{U}(a,r) \cap \mathbf{U}(x,\varrho)$.

5.3 Lemma. *Suppose $m, n \in \mathscr{P}$, $1 \le \zeta \le m < n$, either $\zeta = m = 1$ or $\zeta < m$, $q = \infty$ if $m = 1$, $q = m\zeta/(m-\zeta)$ if $m > 1$, $1 \le \xi \le q$, $\zeta \le s \le q$, $0 < \lambda < \infty$, $a \in \mathbf{R}^m$, $0 < r < \infty$, $u \in \mathbf{W}^{1,1}(\mathbf{U}(a,r), \mathbf{R}^{n-m})$, A is an $\mathscr{L}^m$ measurable subset of $\mathbf{U}(a,r)$, and $\mathscr{L}^m(\mathbf{U}(a,r) \sim A) \le \lambda \le (1/2)\boldsymbol{\alpha}(m) r^m$.*

Then

$$|u|_{q;a,r} \le \Gamma \lambda^{1/\zeta - 1/s} |\mathbf{D}u|_{s;a,r} + 2^{5m+2} \lambda^{1/q - 1/\xi} |u|_{\xi;A}$$

where Γ is a positive, finite number depending only on m and ζ.

Proof. Define $\Delta_1 = \Gamma_{5.1}(m,\zeta)\boldsymbol{\alpha}(m)^{-1} 2^{3m+2}$, $\Delta_2 = 2^{m+1}$ and $\Gamma = 2^{4m+1}\Delta_1$. Let $\varrho = \lambda^{1/m} \boldsymbol{\alpha}(m)^{-1/m} 2^{1+1/m}$, note $\varrho \le 2r$ and define

$$E(b,t) = \mathbf{U}(a,r) \cap \mathbf{U}(b,t) \quad \text{whenever } b \in \mathbf{R}^m,\ 0 < t < \infty.$$

One estimates, using 5.2,

$$\begin{gathered}\mathscr{L}^m(E(b,\varrho) \sim A) \le \lambda = 2^{-1-m} \boldsymbol{\alpha}(m) \varrho^m \le \mathscr{L}^m(E(b,\varrho))/2 \le \mathscr{L}^m(A \cap E(b,\varrho)), \\ \mathscr{L}^m(E(b,\varrho)) \le \boldsymbol{\alpha}(m)\varrho^m = 2^{m+1}\lambda,\end{gathered}$$

whenever $b \in \mathbf{U}(a,r)$. Therefore one applies 5.1 with $h_b = \fint_{A \cap E(b,\varrho)} u \, \mathrm{d}\mathscr{L}^m$ to obtain

$$|u|_{q;E(b,\varrho)} \le \Gamma_{5.1}(m,\zeta) 2^{2m+1} \boldsymbol{\alpha}(m)^{-1} |\mathbf{D}u|_{\zeta;E(b,\varrho)} + 2^{(m+1)/q} \lambda^{1/q} |h_b|$$

for $b \in \mathbf{U}(a,r)$. Using Hölder's inequality, this yields

$$|u|_{q;E(b,\varrho)} \le \Delta_1 \lambda^{1/\zeta - 1/s} |\mathbf{D}u|_{s;E(b,\varrho)} + \Delta_2 \lambda^{1/q - 1/\xi} |u|_{\xi;A \cap E(b,\varrho)}$$

for $b \in \mathbf{U}(a,r)$. If $q = \infty$, the conclusion is now evident.

If $q < \infty$, choosing a maximal set B (with respect to inclusion) such that

$$B \subset \mathbf{U}(a,r), \quad \{E(b,\varrho/2) : b \in B\} \text{ is disjointed,}$$

one notes for $x \in B$ and $S_x = B \cap \{b : E(b,\varrho) \cap E(x,\varrho) \ne \emptyset\}$

$$\mathbf{U}(a,r) \subset \bigcup \{E(b,\varrho) : b \in B\}, \quad \operatorname{card} S_x \le 2^{4m};$$

in fact for the estimate one uses 5.2 to infer

$$\begin{aligned} E(b,\varrho) &\subset E(x,3\varrho) \quad \text{whenever } b \in S_x, \\ (\operatorname{card} S_x)\boldsymbol{\alpha}(m) 2^{-2m} \varrho^m &\le \textstyle\sum_{b \in S_x} \mathscr{L}^m(E(b,\varrho/2)) \\ &\le \mathscr{L}^m(E(x,3\varrho)) \le \boldsymbol{\alpha}(m) 3^m \varrho^m. \end{aligned}$$

Therefore, as $q \geq \sup\{s, \xi\}$,

$$\textstyle\sum_{b\in B} |\mathbf{D}u|^q_{s;E(b,\varrho)} \leq \big(\sum_{b\in B} |\mathbf{D}u|^s_{s;E(b,\varrho)}\big)^{q/s} \leq \big(2^{4m} |\mathbf{D}u|_{s;a,r}\big)^q,$$

$$\textstyle\sum_{b\in B} |u|^q_{\xi;A\cap E(b,\varrho)} \leq \big(\sum_{b\in B} |u|^\xi_{\xi;A\cap E(b,\varrho)}\big)^{q/\xi} \leq \big(2^{4m} |u|_{\xi;A}\big)^q,$$

hence one obtains form the estimate of the preceding paragraph

$$\begin{aligned} |u|^q_{q;a,r} &\leq 2^{q-1} \textstyle\sum_{b\in B} \Big(\big(\Delta_1 \lambda^{1/\zeta-1/s} |\mathbf{D}u|_{s;E(b,\varrho)}\big)^q + \big(\Delta_2 \lambda^{1/q-1/\xi} |u|_{\xi;A\cap E(b,\varrho)}\big)^q \Big) \\ &\leq \big(2^{4m+1} \Delta_1 \lambda^{1/\zeta-1/s} |\mathbf{D}u|_{s;a,r}\big)^q + \big(2^{4m+1} \Delta_2 \lambda^{1/q-1/\xi} |u|_{\xi;A}\big)^q. \end{aligned}$$

and the conclusion follows. □

5.4 Lemma. *Suppose $m, n, Q \in \mathscr{P}$, $m < n$, $q = \infty$ if $m = 1$, $2 \leq q < \infty$ if $m = 2$, $2 \leq q \leq 2m/(m-2)$ if $m > 2$, $a \in \mathbf{R}^m$, $0 < r < \infty$, $f : \mathbf{U}(a, r) \to \mathbf{Q}_Q(\mathbf{R}^{n-m})$ is Lipschitzian, $0 < \eta \leq 1/2$, and A is an $\mathscr{L}^m$ measurable subset of $\mathbf{U}(a, r)$ with $\mathscr{L}^m(\mathbf{U}(a, r) \sim A) \leq \eta \boldsymbol{\alpha}(m) r^m$, then*

$$r^{-m/q} |f|_{q;a,r} \leq \Gamma\big(\eta^{1/q+1/m-1/2} r^{1-m/2} |Af|_{2;a,r} + \eta^{1/q-1} r^{-m} |f|_{1;A}\big)$$

where Γ is a positive, finite number depending only on n, Q, and q.

Proof. Let P and $\boldsymbol{\xi} : \mathbf{Q}_Q(\mathbf{R}^{n-m}) \to \mathbf{R}^{PQ}$ with $\operatorname{Lip} \boldsymbol{\xi} < \infty$ be as in Almgren [Alm00, 1.2 (3)]. Define $u = \boldsymbol{\xi} \circ f$, $\mu = 1/q + 1/m - 1/2 \geq 0$, $\nu = 1 - 1/q \geq 1/2$, $\zeta = 1$ if $m = 1$ and $\zeta = qm/(m + q)$ if $m > 1$, hence $1 \leq \zeta < m$ and $\zeta m/(m - \zeta) = q$ if $m > 1$. From 5.3 applied with λ, s and ξ replaced by $\eta \boldsymbol{\alpha}(m) r^m$, 2, and 1 one obtains

$$r^{-m/q} |u|_{q;a,r} \leq \Delta\big(\eta^\mu r^{1-m/2} |Du|_{2;a,r} + \eta^{-\nu} r^{-m} |u|_{1;A}\big)$$

where $\Delta = \sup\big\{\Gamma_{5.3}(m, \zeta) \boldsymbol{\alpha}(m)^{1/\zeta-1/2}, 2^{5m+2} \boldsymbol{\alpha}(m)^{1/q-1}\big\}$. Since

$$\begin{gathered} \boldsymbol{\xi}(Q[\![0]\!]) = 0, \quad 0 < \operatorname{Lip} \boldsymbol{\xi} < \infty, \quad \boldsymbol{\xi} \text{ is univalent}, \quad \operatorname{Lip} \boldsymbol{\xi}^{-1} < \infty, \\ (\operatorname{Lip} \boldsymbol{\xi})^{-1} |u(x)| \leq \mathscr{G}(f(x), Q[\![0]\!]) \leq \operatorname{Lip} \boldsymbol{\xi}^{-1} \, |u(x)| \quad \text{for } x \in \mathbf{U}(a, r), \\ |Du(x)| \leq \operatorname{Lip} \boldsymbol{\xi} \, |Af(x)| \quad \text{for } x \in \operatorname{dmn} Du \end{gathered}$$

by Almgren [Alm00, 1.1 (6), 1.2 (3), 1.4 (3)], the conclusion follows. □

5.5 Lemma. *Suppose $k, m, n \in \mathscr{P}$, $m < n$, $a \in \mathbf{R}^m$, $0 < r < \infty$, and $u : \mathbf{U}(a, r) \to \mathbf{R}^{n-m}$ is of class k.*

Then

$$\textstyle\sum_{i=0}^k r^i |D^i u|_{\infty;a,r} \leq \Gamma\big(r^k |D^k u|_{\infty;a,r} + r^{-m} |u|_{1;a,r}\big)$$

where Γ is a positive, finite number depending only on k and n.

Proof. Assuming $r = 1$, this is a consequence of Ehring's lemma, see e.g. [Wlo87, Theorem I.7.3], and Arzelà's and Ascoli's theorem. □

5.6 Lemma. *Suppose $m, n \in \mathscr{P}$, $m < n$, $a \in \mathbf{R}^m$, $0 < r < \infty$, and $u \in \mathbf{W}^{1,2}(\mathbf{U}(a, r), \mathbf{R}^{n-m})$.*

Then there exists $h \in \mathbf{R}^{n-m}$ with

$$|u - h|_{2;a,r} \leq \Gamma r |\mathbf{D}u|_{2;a,r}$$

where Γ is a positive, finite number depending only on n.

Proof. This is Poincaré's inequality, see e.g. [GT01, (7.45)]. □

6 Some estimates concerning linear second order elliptic systems

The purpose of the present section is to gather some standard estimates precisely in the form needed in Section 7. Proofs are included for the convenience of the reader.

6.1. The following situation will occur repeatedly: $m, n \in \mathscr{P}$, $m < n$, $0 < c \leq M < \infty$, and $\Upsilon \in \bigodot^2 \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m})$ with $\|\Upsilon\| \leq M$ is strongly elliptic with ellipticity bound c, i.e. Υ is an $\mathbf{R}$ valued bilinear form on $\operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m})$ with $\Upsilon(\sigma, \tau) \leq M|\sigma||\tau|$ whenever $\sigma, \tau \in \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m})$ and

$$\textstyle\int \Upsilon(D\theta(x), D\theta(x)) - c|D\theta(x)|^2 \, \mathrm{d}\mathscr{L}^m x \geq 0 \quad \text{whenever } \theta \in \mathscr{D}(\mathbf{R}^m, \mathbf{R}^{n-m}).$$

Following [Fed69, 5.2.11], one associates to any $\Upsilon \in \bigodot^2 \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m})$ a linear function $S : \bigodot^2(\mathbf{R}^m, \mathbf{R}^{n-m}) \cong (\bigodot^2 \mathbf{R}^m) \otimes \mathbf{R}^{n-m} \to \mathbf{R}^{n-m}$ characterised by

$$\langle (\xi \odot \psi) y, S \rangle \bullet \upsilon = \langle (\xi\, y, \psi\, \upsilon), \Upsilon \rangle + \langle (\psi\, y, \xi\, \upsilon), \Upsilon \rangle$$

whenever $\xi, \psi \in \bigodot^1 \mathbf{R}^m$, $y, \upsilon \in \mathbf{R}^{n-m}$; here $\xi\, y \in \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m})$ is given by $(\xi\, y)(x) = \xi(x) y$ for $x \in \mathbf{R}^m$. Applying this construction with the area integrand Ψ to $D^2 \Psi_0^\S(\sigma)$ for each $\sigma \in \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m})$, one obtains a function $C : \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m}) \to \operatorname{Hom}\big(\bigodot^2(\mathbf{R}^m, \mathbf{R}^{n-m}), \mathbf{R}^{n-m}\big)$ which satisfies

$$\langle \phi, C(\sigma) \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n-m} \sum_{k=1}^{m} \sum_{l=1}^{n-m} \big\langle (X_i \upsilon_j, X_k \upsilon_l), D^2 \Psi_0^\S(\sigma) \big\rangle (\phi(e_i, e_k) \bullet \upsilon_j) \upsilon_l$$

for $\phi \in \bigodot^2(\mathbf{R}^m, \mathbf{R}^{n-m})$ where $e_1, \ldots, e_m$ and $X_1, \ldots, X_m$ are dual orthonormal bases of $\mathbf{R}^m$ and $\bigodot^1 \mathbf{R}^m$, and $\upsilon_1, \ldots, \upsilon_{n-m}$ form an orthonormal base of $\mathbf{R}^{n-m}$. Hence whenever U is an open subset of $\mathbf{R}^m$, $u \in \mathbf{W}^{2,1}(U, \mathbf{R}^{n-m})$ is Lipschitzian, $v \in \mathbf{W}^{2,1}(U, \mathbf{R}^{n-m})$, $\sigma \in \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m})$, and $\theta \in \mathscr{D}(U, \mathbf{R}^{n-m})$ one obtains by partial integration the formulae

$$-\textstyle\int_U \big\langle D\theta(x), D\Psi_0^\S(Du(x)) \big\rangle \, \mathrm{d}\mathscr{L}^m x = \int_U \theta(x) \bullet \big\langle \mathbf{D}^2 u(x), C(Du(x)) \big\rangle \, \mathrm{d}\mathscr{L}^m x,$$
$$-\textstyle\int_U \big\langle D\theta(x) \odot \mathbf{D} v(x), D^2 \Psi_0^\S(\sigma) \big\rangle \, \mathrm{d}\mathscr{L}^m x = \int_U \theta(x) \bullet \big\langle \mathbf{D}^2 v(x), C(\sigma) \big\rangle \, \mathrm{d}\mathscr{L}^m x,$$

here $\odot$ denotes multiplication in $\bigodot_* \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m})$, see [Fed69, 1.9.1].

6.2 Lemma. *Suppose m, n, c, M, and Υ are as in 6.1, $a \in \mathbf{R}^m$, $0 < r < \infty$, $v \in \mathbf{W}^{1,2}(\mathbf{U}(a, r), \mathbf{R}^{n-m})$, $T \in \mathscr{D}'(\mathbf{U}(a, r), \mathbf{R}^{n-m})$ with $|T|_{-1,2;a,r} < \infty$.*

Then there exists an $\mathscr{L}^m \llcorner \mathbf{U}(a, r)$ almost unique $u \in \mathbf{W}^{1,2}(\mathbf{U}(a, r), \mathbf{R}^{n-m})$ such that

$$-\textstyle\int_{\mathbf{U}(a,r)} \langle D\theta(x) \odot \mathbf{D} u(x), \Upsilon \rangle \, \mathrm{d}\mathscr{L}^m x = T(\theta) \quad \text{for } \theta \in \mathscr{D}(\mathbf{U}(a, r), \mathbf{R}^{n-m}),$$
$$u - v \in \mathbf{W}_0^{1,2}(\mathbf{U}(a, r), \mathbf{R}^{n-m}).$$

Moreover, for every affine function $P : \mathbf{R}^m \to \mathbf{R}^{n-m}$

$$|\mathbf{D}(u - v)|_{2;a,r} \leq c^{-1} \big(M |\mathbf{D}(v - P)|_{2;a,r} + |T|_{-1,2;a,r} \big).$$

Proof. To prove existence, assume $v = 0$, let R denote the extension of T to $\mathbf{W}_0^{1,2}(\mathbf{U}(a,r), \mathbf{R}^{n-m})$ by continuity and observe that one can take u to be a minimiser of

$$\tfrac{1}{2} \textstyle\int_{\mathbf{U}(a,r)} \langle \mathbf{D}u(x) \odot \mathbf{D}u(x), \Upsilon \rangle \, \mathrm{d}\mathscr{L}^m x + R(u)$$

in $\mathbf{W}_0^{1,2}(\mathbf{U}(a,r), \mathbf{R}^{n-m})$

To prove the estimate, assuming $P = 0$ by possibly replacing u, v, P by $u - P$, $v - P$, 0, one lets θ approximate $u - v$ in $\mathbf{W}_0^{1,2}(\mathbf{U}(a,r), \mathbf{R}^{n-m})$ to obtain

$$c|\mathbf{D}(u-v)|_{2;a,r}^2 \leq \big(M|\mathbf{D}(v-P)|_{2;a,r} + |T|_{-1,2;a,r} \big) |\mathbf{D}(u-v)|_{2;a,r}.$$

The uniqueness follows from the estimate. □

6.3 Remark. If $T = 0$ then u is $\mathscr{L}^m \llcorner \mathbf{U}(a,r)$ almost equal to an analytic Υ harmonic function by [Fed69, 5.2.5, 6].

6.4 Lemma. *Suppose m, n, c, M, Υ, and S are as in 6.1, $0 < \alpha < 1$, $a \in \mathbf{R}^m$, $0 < r < \infty$, $u : \mathbf{U}(a,r) \to \mathbf{R}^{n-m}$ is of class 2, $D^2 u$ locally satisfies a Hölder condition with exponent α, $f : \mathbf{U}(a,r) \to \mathbf{R}^{n-m}$, and $S \circ D^2 u = f$.*

Then

$$r^{-\alpha} |D^2 u|_{\infty;a,r/2} + \mathbf{h}_\alpha(D^2 u | \mathbf{B}(a,r/2)) \leq \Gamma \big(r^{-2-\alpha-m} |u|_{1;a,r} + \mathbf{h}_\alpha(f) \big)$$

where Γ is a positive, finite number depending only on n, c, M, and α.

Proof. Interpolating by use of Ehring's lemma, see e.g. [Wlo87, Theorem I.7.3], and Arzelà's and Ascoli's theorem, it is enough to prove the assertion remaining when the term $r^{-\alpha}|D^2 u|_{\infty;a,r/2}$ is omitted.

Considering slightly smaller r, one may assume $\mathbf{h}_\alpha(D^2 u) < \infty$.

Applying [Fed69, 5.2.14] to the partial derivatives of u and using Ehring's lemma as above, one infers the existence of a positive, finite number Δ depending only on n, c, M, and α such that

$$\begin{aligned} \mathbf{h}_\alpha(D^2 u | \mathbf{B}(b,s)) \leq {} & 2^{-6-m} \mathbf{h}_\alpha(D^2 u | \mathbf{B}(b,2s)) \\ & + \Delta \big(s^{-2-\alpha-m} |u|_{1;b,2s} + \mathbf{h}_\alpha(f | \mathbf{B}(b,2s)) \big) \end{aligned}$$

whenever $b \in \mathbf{R}^m$, $0 < s < \infty$ and $\mathbf{B}(b,2s) \subset \mathbf{U}(a,r)$.

Defining $h : \mathbf{U}(a,r) \to \mathbf{R}$ by $h(x) = \frac{1}{4} \operatorname{dist}(x, \mathbf{R}^m \sim \mathbf{U}(a,r))$ for $x \in \mathbf{U}(a,r)$,

$$\mu = \sup \big\{ h(b)^{2+\alpha+m} \mathbf{h}_\alpha(D^2 u | \mathbf{B}(b, h(b))) : b \in \mathbf{U}(a,r) \big\}$$

and noting $\mu \leq r^{2+\alpha+m} \mathbf{h}_\alpha(D^2 u) < \infty$, one estimates for $b \in \mathbf{U}(a,r)$

$$\begin{gathered} \begin{aligned} \mathbf{h}_\alpha(D^2 u | \mathbf{B}(b, h(b))) \leq {} & 2^{-6-m} \mathbf{h}_\alpha(D^2 u | \mathbf{B}(b, 2h(b))) \\ & + \Delta \big(h(b)^{-2-\alpha-m} |u|_{1;a,r} + \mathbf{h}_\alpha(f) \big), \end{aligned} \\ |h(b) - h(c)| \leq (\operatorname{Lip} h)|b - c| \leq h(b)/2, \; h(b) \leq 2h(c) \qquad \text{for } c \in \mathbf{B}(b, 2h(b)), \\ h(b)^{2+\alpha+m} \mathbf{h}_\alpha(D^2 u | \mathbf{B}(b, 2h(b))) \leq 2^{4+\alpha+m} \mu, \\ h(b)^{2+\alpha+m} \mathbf{h}_\alpha(D^2 u | \mathbf{B}(b, h(b))) \leq \mu/2 + \Delta \big(|u|_{1;a,r} + r^{2+\alpha+m} \mathbf{h}_\alpha(f) \big), \end{gathered}$$

hence

$$(r/4)^{2+\alpha+m} \mathbf{h}_\alpha(D^2 u | \mathbf{B}(a, r/2)) \leq 2^{5+m} \mu \leq 2^{6+m} \Delta \big(|u|_{1;a,r} + r^{2+\alpha+m} \mathbf{h}_\alpha(f) \big)$$

and the remaining assertion is evident. □

6.5 Remark. Similar absorption procedures can be found for example in [Fed69, 5.2.14] or [GT01, Theorem 9.11].

6.6 Lemma. *Suppose m, n, c, M, and Υ are as in 6.1, $2 \leq p < \infty$, $a \in \mathbf{R}^m$, and $0 < r < \infty$.*

Then for every $f \in \mathbf{L}_p(\mathscr{L}^m \llcorner \mathbf{U}(a,r), \mathbf{R}^{n-m})$ there exists an $\mathscr{L}^m \llcorner \mathbf{U}(a,r)$ almost unique $u \in \mathbf{W}_0^{1,p}(\mathbf{U}(a,r), \mathbf{R}^{n-m})$ such that

$$-\int_{\mathbf{U}(a,r)} \langle D\theta(x) \odot \mathbf{D}u(x), \Upsilon \rangle \, \mathrm{d}\mathscr{L}^m x = (\theta, f)_{a,r} \quad \text{for } \theta \in \mathscr{D}(\mathbf{U}(a,r), \mathbf{R}^{n-m}).$$

Moreover, $u \in \mathbf{W}^{2,p}(\mathbf{U}(a,r), \mathbf{R}^{n-m})$ and

$$\textstyle\sum_{i=0}^{2} r^{i-2} |\mathbf{D}^i u|_{p;a,r} \leq \Gamma |f|_{p;a,r}$$

where Γ is a positive, finite number depending only on n, c, M, and p.

Proof. See [Giu03, p. 368-370]. □

6.7 Remark. The condition $p \geq 2$ can, of course, be replaced by $p > 1$. For example [Giu03, Theorem 10.15] extends to this case via duality and the estimate of the second order derivatives can be carried out by using the method of difference quotients starting from a suitably localised version of the theorem cited.

6.8 Lemma. *Suppose m, n, c, M, and Υ are as in 6.1, $a \in \mathbf{R}^m$, $0 < r < \infty$, $u \in \mathbf{W}_0^{1,1}(\mathbf{U}(a,r), \mathbf{R}^{n-m})$, $T \in \mathscr{D}'(\mathbf{U}(a,r), \mathbf{R}^{n-m})$, and*

$$-\int_{\mathbf{U}(a,r)} \langle D\theta(x) \odot \mathbf{D}u(x), \Upsilon \rangle \, \mathrm{d}\mathscr{L}^m x = T(\theta) \quad \text{for } \theta \in \mathscr{D}(\mathbf{U}(a,r), \mathbf{R}^{n-m}).$$

Then

$$|u|_{1;a,r} \leq \Gamma r |T|_{-1,1;a,r}$$

where Γ is a positive, finite number depending only on n, c, and M.

Proof. Let $p = 2m$ and $q = p/(p-1)$ and assume $r = 1$.

Whenever $\theta \in \mathscr{D}(\mathbf{U}(a,r), \mathbf{R}^{n-m})$ one obtains $\eta \in \mathbf{W}_0^{1,p}(\mathbf{U}(a,r), \mathbf{R}^{n-m})$ from 6.6 such that with $\Delta_1 = \Gamma_{6.6}(n,c,M,p)$

$$-\int_{\mathbf{U}(a,1)} \langle D\zeta(x) \odot \mathbf{D}\eta(x), \Upsilon \rangle \, \mathrm{d}\mathscr{L}^m x = (\zeta, \theta)_{a,1} \quad \text{for } \zeta \in \mathscr{D}(\mathbf{U}(a,1), \mathbf{R}^{n-m}),$$
$$\textstyle\sum_{i=0}^{2} |\mathbf{D}^i \eta|_{p;a,1} \leq \Delta_1 |\theta|_{p;a,1},$$

hence by [GT01, Theorem 7.26 (ii)]

$$|\mathbf{D}\eta|_{\infty;a,1} \leq \Delta_2 \big(|\mathbf{D}\eta|_{p;a,1} + |\mathbf{D}^2\eta|_{p;a,1} \big) \leq \Delta_1 \Delta_2 |\theta|_{p;a,1}$$

where Δ_2 is a positive, finite number depending only on n and p. Approximating and u by $\zeta_i \in \mathscr{D}(\mathbf{U}(a,1), \mathbf{R}^{n-m})$ in $\mathbf{W}_0^{1,1}(\mathbf{U}(a,1), \mathbf{R}^{n-m})$ and η by a sequence $\eta_i \in \mathscr{D}(\mathbf{U}(a,r), \mathbf{R}^{n-m})$ such that

$$\eta_i \to \eta \quad \text{in } \mathbf{W}^{1,p}(\mathbf{U}(a,1), \mathbf{R}^{n-m}) \text{ as } i \to \infty, \quad \lim_{i \to \infty} |D\eta_i|_{\infty;a,1} = |\mathbf{D}\eta|_{\infty;a,1},$$

one obtains

$$(\theta, u)_{a,1} = -\textstyle\fint_{\mathbf{U}(a,1)} \langle \mathbf{D}\eta(x) \odot \mathbf{D}u(x), \Upsilon \rangle \, \mathrm{d}\mathscr{L}^m x \leq |T|_{-1,1;a,1} |\mathbf{D}\eta|_{\infty;a,1}.$$

Therefore (cp. [Fed69, 2.4.16])

$$|u|_{1;a,1} \leq \boldsymbol{\alpha}(m)^{1/p} |u|_{q;a,1} \leq \boldsymbol{\alpha}(m)^{1/p} \Delta_1 \Delta_2 |T|_{-1,1;a,1}$$

and one may take $\Gamma = \sup\{\boldsymbol{\alpha}(i)^{1/p}\Delta_1\Delta_2 : n > i \in \mathscr{P}\}$. □

6.9 Remark. If $m > 1$ the estimate may be sharpened to

$$\sup\left\{t\mathscr{L}^m(\mathbf{U}(a,r) \cap \{x : |u(x)| > t\})^{1-1/m} : 0 < t < \infty\right\} \leq \Gamma |T|_{-1,1;a,r};$$

in fact one may follow the same line of arguments with the Lorentz space $\mathbf{L}_{m,1}$ replacing $\mathbf{L}_p$.

7 Estimates concerning the quadratic tilt-excess

The estimates of the present section constitute the core of the proof of the pointwise regularity theorem, Theorem 8.3, in Section 8. All constructions are based on the approximation by a $\mathbf{Q}_Q(\mathbf{R}^{n-m})$ valued function of Section 4. First, in 7.1 and 7.2 some lower mass bounds are derived by a simple adaption of [Sim83, Theorem 17.7] and a straightforward use of Allard [All72, 6.4]. Then, in 7.3 several auxiliary estimates concerning the approximation by a $\mathbf{Q}_Q(\mathbf{R}^{n-m})$ valued function in 4.8 are carried out. In 7.4 the main elliptic estimates are established, see below for a more detailed description. Finally, a reformulation of a special case of 7.4 (9) replacing any reference to the specific approximating functions used there by quantities more tightly connected to the varifold is provided in 7.5 for use in [Men09c].

Next, an overview of the constructions in 7.4 is given. One considers cylinders centred at a fixed point $a \in \mathbf{R}^n$ with projection $c \in \mathbf{R}^m$. For any radius ϱ functions u_ϱ solving a Dirichlet problem in $\mathbf{U}(c,\varrho)$ for a suitable linear elliptic system with constant coefficients with the "average" g of the approximating $\mathbf{Q}_Q(\mathbf{R}^{n-m})$ valued function f as boundary values are defined. It is readily seen in 7.4 (6) that $\phi_1(\varrho) = |D^2 u|_{\infty;c,\varrho/2}$, the leading quantity in the iteration, is controlled by the tilt-excess of the varifold and mean curvature. More importantly, an estimate of $|u - g|_{1;c,\varrho}$ mainly in terms of mean curvature is established in 7.4 (7) by use of 6.8. Using this estimate, the iteration inequality for ϕ_1 follows in 7.4 (8). In order to derive an iteration inequality for the tilt-excess of the varifold, i.e. controlling the tilt-excess basically by ϕ_1 and mean curvature, the estimate 7.4 (9) is established. It asserts that $|f\,(+)(-P)|_{1;X}$ with $P : \mathbf{R}^m \to \mathbf{R}^{n-m}$ an affine function and X a large (with respect to $\mathscr{L}^m$) subset of $\mathbf{U}(c,\varrho/2)$ together with mean curvature essentially controls the tilt-excess. Here the coercive estimates of Section 3, the interpolation procedure of Section 5 and the adaptions of the Sobolev Poincaré type estimates of [Men09b] in 4.8 (8) are used. Assuming that f agrees with its "average" g on a large set, for example because the density of the varifold is at least Q on a large set, the iteration inequality for the tilt-excess is then primarily a consequence of Taylor's expansion, see 7.4 (10). Finally, both iteration inequalities are iterated in 7.4 (11) as long as the afore-mentioned density condition is satisfied on the scales involved. As all the preceding estimates only hold under various side conditions which have to be checked at each iteration step and the interdependence of the various constants occurring is not entirely straightforward, the iteration procedure is presented in some detail to ease verification.

Finally, it should be mentioned that the current iteration procedure has to be carried out within a fixed coordinate systems as differences of functions corresponding to different iteration steps have to be computed, see the Introduction and 7.4 (8). Though this fact does not pose a serious difficulty it nevertheless contributes significantly to the level of technicality, see for example the definition of J_4 and 7.3 (8). However, regarding a possible application of the techniques of the present paper in partial regularity problems for systems of elliptic equations, this difficulty as well as several other technicalities would not be present.

7.1 Lemma. *Suppose $m, n \in \mathscr{P}$, $m \leq n$, $a \in \mathbf{R}^n$, $0 < r < \infty$, $V \in \mathbf{V}_m(\mathbf{U}(a,r))$, $a \in \operatorname{spt}\|V\|$, $1 \leq p < \infty$, $0 < \alpha \leq 1$, $0 \leq M < \infty$, and*

$$\|\delta V\|\,\mathbf{B}(a,\varrho) \leq M\|V\|(\mathbf{B}(a,\varrho))^{1-1/p}\varrho^{m/p+\alpha-1}r^{-\alpha} \quad \textit{for } 0 < \varrho < r.$$

Then

$$\big(\varrho^{-m}\|V\|\,\mathbf{U}(a,\varrho)\big)^{1/p} + Mp^{-1}\alpha^{-1}\varrho^{\alpha}r^{-\alpha}$$

is monotone increasing in ϱ for $0 < \varrho < r$. In particular, $0 \leq \mathbf{\Theta}^m(\|V\|, a) < \infty$.

Proof. Suppose $0 < \lambda < 1$ and $\phi \in \mathscr{E}^0(\mathbf{R})$ with $\phi' \leq 0$ and $\phi(t) = 1$ for $-\infty < t \leq \lambda$ and $\phi(t) = 0$ for $1 \leq t < \infty$ and $f : \mathbf{R} \cap \{\varrho : 0 < \varrho < r\} \to \mathbf{R}$ is defined by $f(\varrho) = \varrho^{-m}\int \phi(\varrho^{-1}|z-a|)\,\mathrm{d}\|V\|z$ for $0 < \varrho < r$. Then one obtains as in [Sim83, Theorem 17.7] that

$$\begin{aligned} f'(\varrho) &\geq \varrho^{-m-1}(\delta V)_z\big(\phi(\varrho^{-1}|z-a|)(z-a)\big) \\ &\geq -M(\varrho^{-m}\|V\|\,\mathbf{U}(z,\varrho))^{1-1/p}\varrho^{\alpha-1}r^{-\alpha} \geq -M\big(\lambda^{-m}f(\lambda^{-1}\varrho)\big)^{1-1/p}\varrho^{\alpha-1}r^{-\alpha} \end{aligned}$$

for $0 < \varrho < \lambda r$, hence multiplying by $p^{-1}f(\varrho)^{1/p-1}$ and integrating yields

$$f(t)^{1/p} - f(s)^{1/p} \geq -Mp^{-1}r^{-\alpha}\textstyle\int_s^t (\lambda^{-m}f(\varrho/\lambda)/f(\varrho))^{1-1/p}\varrho^{\alpha-1}\,\mathrm{d}\mathscr{L}^1\varrho$$

for $0 < s < t < \lambda r$. Thus, approximating the characteristic function of $\mathbf{R} \cap \{t : t < 1\}$ by such ϕ and letting λ tend to 1 implies the conclusion. ☐

7.2 Lemma. *Suppose $n, Q \in \mathscr{P}$, $0 < \alpha \leq 1$, $1 \leq p < \infty$, and $0 < \delta \leq 1$.*

Then there exists a positive, finite number ε with the following property.

If $n > m \in \mathscr{P}$, $a \in \mathbf{R}^n$, $0 < r < \infty$, $U = \mathbf{U}(a,r) \cap \{z : |T_\natural^\perp(z-a)| < \delta r\}$, $V \in \mathbf{IV}_m(U)$, ψ is related to V and p as in 3.3, $T \in \mathbf{G}(n,m)$,

$$\mathbf{\Theta}^{*m}(\|V\|, a) \geq Q - 1 + \delta, \quad \textstyle\int |S_\natural - T_\natural|\,\mathrm{d}V(z,S) \leq \varepsilon r^m,$$
$$\varrho^{1-m/p}\psi(U \cap \mathbf{B}(a,\varrho))^{1/p} \leq \varepsilon(\varrho/r)^{\alpha} \quad \textit{whenever } 0 < \varrho < r,$$

then

$$\|V\|(U) \geq (Q-\delta)\boldsymbol{\alpha}(m)r^m.$$

Proof. If the lemma were false for some n, Q, α, p, and δ, there would exist a sequence ε_i with $\varepsilon_i \downarrow 0$ as $i \to \infty$ and sequences m_i, a_i, r_i, U_i, V_i, ψ_i, and T_i showing that $\varepsilon = \varepsilon_i$ does not have the asserted property.

One could assume for some $m \in \mathscr{P}$, $a \in \mathbf{R}^n$, $T \in \mathbf{G}(n,m)$

$$m_i = m, \quad a_i = a, \quad r_i = 1, \quad T_i = T$$

whenever $i \in \mathscr{P}$. Abbreviating $U = \mathbf{U}(a,1) \cap \{z : |T_\natural^\perp(z-a)| < \delta\}$ one would deduce for large i

$$\|V_i\|(U \cap \mathbf{U}(a,\varrho)) \geq (Q-1+\delta/2)\boldsymbol{\alpha}(m)\varrho^m \quad \text{whenever } 0 < \varrho < \delta$$

from 7.1 in conjunction with Hölder's inequality. Clearly, also

$$\|V_i\|(U) \leq (Q-\delta)\boldsymbol{\alpha}(m) \quad \text{for } i \in \mathscr{P}.$$

By Allard [All72, 6.4], possibly passing to a subsequence, there would exist $V \in \mathbf{IV}_m(U)$ such that $\delta V = 0$ and

$$\begin{aligned} V_i(f) \to V(f) \quad &\text{as } i \to \infty \text{ for } f \in \mathscr{K}(U \times \mathbf{G}(n,m)), \\ S = T \quad &\text{for } V \text{ almost all } (z,S) \in U \times \mathbf{G}(n,m), \end{aligned}$$

hence, noting [Men09b, 2.1],

$$\boldsymbol{\Theta}^m(\|V\|, a) \geq Q, \quad \boldsymbol{\alpha}(m)Q \leq \|V\|(U) \leq \boldsymbol{\alpha}(m)(Q-\delta),$$

a contradiction. □

7.3 Lemma. *Suppose the hypotheses of 4.8 are satisfied with $h = 3r$, i.e. suppose $m, n, Q \in \mathscr{P}$, $m < n$, $0 < L < \infty$, $1 \leq M < \infty$, and $0 < \delta_i \leq 1$ for $i \in \{1,2,3,4,5\}$, $\varepsilon = \varepsilon_{4.8}(n,Q,L,M,\delta_1,\delta_2,\delta_3,\delta_4,\delta_5)$, $0 < r < \infty$, $T = \operatorname{im} \mathbf{p}^*$,*

$$U = (\mathbf{R}^m \times \mathbf{R}^{n-m}) \cap \{(x,y) : \operatorname{dist}((x,y), \mathbf{C}(T,0,r,3r)) < 2r\},$$

$V \in \mathbf{IV}_m(U)$, $\|\delta V\|$ is a Radon measure,

$$\begin{aligned} (Q-1+\delta_1)\boldsymbol{\alpha}(m)r^m \leq \|V\|(\mathbf{C}(T,0,r,3r)) &\leq (Q+1-\delta_2)\boldsymbol{\alpha}(m)r^m, \\ \|V\|(\mathbf{C}(T,0,r,3r+\delta_4 r) \sim \mathbf{C}(T,0,r,3r-2\delta_4 r)) &\leq (1-\delta_3)\boldsymbol{\alpha}(m)r^m, \\ \|V\|(U) &\leq M\boldsymbol{\alpha}(m)r^m, \end{aligned}$$

$0 < \delta \leq \varepsilon$, B denotes the set of all $z \in \mathbf{C}(T,0,r,3r)$ with $\boldsymbol{\Theta}^{*m}(\|V\|, z) > 0$ such that

$$\begin{aligned} &\textit{either} \quad \|\delta V\| \, \mathbf{B}(z,\varrho) > \delta \, \|V\|(\mathbf{B}(z,\varrho))^{1-1/m} \quad \textit{for some } 0 < \varrho < 2r, \\ &\textit{or} \quad \textstyle\int_{\mathbf{B}(z,\varrho)\times\mathbf{G}(n,m)} |S_\natural - T_\natural| \, \mathrm{d}V(\xi,S) > \delta \, \|V\| \, \mathbf{B}(z,\varrho) \quad \textit{for some } 0 < \varrho < 2r, \end{aligned}$$

$A = \mathbf{C}(T,0,r,3r) \sim B$, $A(x) = A \cap \{z : \mathbf{p}(z) = x\}$ for $x \in \mathbf{R}^m$, X_1 is the set of all $x \in \mathbf{R}^m \cap \mathbf{B}(0,r)$ such that

$$\textstyle\sum_{z \in A(x)} \boldsymbol{\Theta}^m(\|V\|, z) = Q \quad \textit{and} \quad \boldsymbol{\Theta}^m(\|V\|, z) \in \mathscr{P} \cup \{0\} \textit{ for } z \in A(x),$$

and $f : X_1 \to \mathbf{Q}_Q(\mathbf{R}^{n-m})$ is characterised by the requirement

$$\boldsymbol{\Theta}^m(\|V\|, z) = \boldsymbol{\Theta}^0(\|f(x)\|, \mathbf{q}(z)) \quad \textit{whenever } x \in X_1 \textit{ and } z \in A(x).$$

Suppose additionally:

(1) *Suppose* $L \le \delta_4/8$, $\delta \le \inf\{1, (2\boldsymbol{\gamma}(m))^{-1}\}$, $a \in \operatorname{Int} \mathbf{C}(T, 0, r, 3r)$, $c = \mathbf{p}(a)$, *and* $0 < \kappa < \infty$.

(2) *Suppose* $F : \mathbf{R}^m \to \mathbf{Q}_Q(\mathbf{R}^{n-m})$ *with* $F|X_1 = f$ *and* $\operatorname{Lip} F \le \Gamma_{(2)} \operatorname{Lip} f$ *where* $\Gamma_{(2)}$ *is a positive, finite number depending only on* $n - m$ *and* Q, *see [Alm00, Theorem 1.3 (2)]. Moreover, let* $g = \boldsymbol{\eta}_Q \circ F$.

(3) *Suppose either* $p = m = 1$ *or* $1 \le p < m$ *and* p, ψ *are related to* V *as in 3.3.*

(4) *Define* $J = \{\varrho : 0 < \varrho < \infty\}$ *and* $\phi_2 : J \times \mathbf{G}(n, m) \to \mathbf{R}$ *and* $\phi_3 : J \to \mathbf{R}$, $\phi_4 : J \to \mathbf{R}$ *by*

$$\begin{aligned} \phi_2(\varrho, R) &= \big(\varrho^{-m} \textstyle\int_{(U \cap \mathbf{C}(T,a,\varrho,\delta_4\varrho)) \times \mathbf{G}(n,m)} |S_\natural - R_\natural|^2 \, \mathrm{d}V(z, S)\big)^{1/2} \\ \phi_3(\varrho) &= \varrho^{1-m/p} \psi(U \cap \mathbf{C}(T, a, \varrho, \delta_4\varrho))^{1/p} \\ \phi_4(\varrho) &= \delta^{-mp/(m-p)} \phi_3(\varrho)^{mp/(m-p)} \qquad \text{if } m > 1, \\ \phi_4(\varrho) &= 0 \qquad \text{if } m = 1, \end{aligned}$$

whenever $\varrho \in J$, $R \in \mathbf{G}(n, m)$.[3]

(5) *For* $0 < \varrho < \infty$ *suppose* $T_\varrho \in \mathbf{G}(n, m)$ *is defined such that*

$$\phi_2(\varrho, T_\varrho) \le \phi_2(\varrho, R) \quad \text{whenever } R \in \mathbf{G}(n, m).$$

(6) *Define*

$$\begin{aligned} J_0 &= J \cap \{\varrho : 0 < \varrho \le r - |\mathbf{p}(a)|, |\mathbf{q}(a)| + \delta_4\varrho \le 3r\}, \\ J_1 &= J \cap \{\varrho : \mathbf{p}[T_\varrho] = \mathbf{R}^m\} \\ J_2 &= J \cap \{\varrho : \|\delta V\|(U \cap \mathbf{C}(T, a, \varrho, \delta_4\varrho)) \le \kappa\varrho^{m-1}\}, \\ J_3 &= J \cap \{\varrho : \textstyle\int_{(U \cap \mathbf{C}(T,a,\varrho,\delta_4\varrho)) \times \mathbf{G}(n,m)} |S_\natural - T_\natural| \, \mathrm{d}V(z, S) \le \kappa\varrho^m\}, \\ J_4 &= J \cap \{\varrho : \varrho + t/\delta_4 \in J_2 \cap J_3 \text{ for } 0 \le t < 2r\}, \\ J_5 &= J_0 \cap \{\varrho : \|V\|(\mathbf{C}(T, a, \varrho, \delta_4\varrho/4)) \ge \boldsymbol{\alpha}(m)(Q - 1/4)\varrho^m\}. \end{aligned}$$

and $T_\varrho = \sigma_\varrho \in \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m})$ *for* $\varrho \in J_1$.

(7) *Define* $B_{a,\varrho}$, *and* $C_{a,\varrho}$ *for* $\varrho \in J_0$ *as in 4.8 (6), i.e.*

$$B_{a,\varrho} = \mathbf{C}(T, a, \varrho, \delta_4\varrho) \cap B, \quad C_{a,\varrho} = \mathbf{B}(\mathbf{p}(a), \varrho) \sim (X_1 \sim \mathbf{p}[B_{a,\varrho}]),$$

and H *as in 4.8 (8), i.e.* H *denotes the set of all* $z \in \mathbf{C}(T, 0, r, 3r)$ *such that*

$$\begin{gathered} \|\delta V\| \, \mathbf{U}(z, 2r) \le \varepsilon \, \|V\|(\mathbf{U}(z, 2r))^{1-1/m}, \\ \textstyle\int_{\mathbf{U}(z,2r) \times \mathbf{G}(n,m)} |S_\natural - T_\natural| \, \mathrm{d}V(z, S) \le \varepsilon \, \|V\| \, \mathbf{U}(z, 2r), \\ \|V\| \, \mathbf{B}(z, \varrho) \ge \delta_5 \boldsymbol{\alpha}(m)\varrho^m \quad \text{for } 0 < \varrho < 2r. \end{gathered}$$

Then the following six conclusions hold:

[3] The symbol ϕ_1 will denote the leading iteration quantity introduced in 7.4 (3).

(8) *There exists a positive finite number* $\varepsilon_{(8)}$ *depending only on* m, δ_4, *and* δ *with the following property.*

If $R \in \mathbf{G}(n,m)$, $|R_\natural - T_\natural| \le \delta/2$, $\varrho \in J_0 \cap J_4$, $\kappa \le \varepsilon_{(8)}$, *then*

$$\varrho^{-m}\|V\|(B_{a,\varrho}) \le 2^m \boldsymbol{\beta}(n)\left(4\delta^{-2}\phi_2(2\varrho,R)^2 + \phi_4(2\varrho)\right).$$

Moreover, $4\delta^{-2}\phi_2(2\varrho,R)^2$ *may be replaced by* $\delta^{-1}\kappa$.

(9) *There exists a positive, finite number* $\varepsilon_{(9)}$ *depending only on* m, δ_4, δ_5, *and* ε *with the following property.*

If $8r/\delta_4 \in J_2 \cap J_3$ *and* $\kappa \le \varepsilon_{(9)}$, *then* H *is the set of all* $z \in \mathbf{C}(T,0,r,3r)$ *such that*

$$\|V\|\,\mathbf{B}(z,t) \ge \delta_5 \boldsymbol{\alpha}(m) t^m \quad \text{whenever } 0 < t < 2r.$$

(10) *If* $0 < \alpha \le 1$ *and* $0 < \delta_6 \le 1$ *then there exists a positive, finite number* $\varepsilon_{(10)}$ *depending only on* n, Q, δ_4, p, α, *and* δ_6 *with the following property.*

If $\boldsymbol{\Theta}^{*m}(\|V\|,a) \ge Q - 1 + \delta_6$, $\varrho \in J_0 \cap J_3$, $\kappa \le \varepsilon_{(10)}$, *and*

$$\phi_3(t) \le \varepsilon_{(10)}(t/\varrho)^\alpha \quad \text{for } 0 < t < \varrho,$$

then $\varrho \in J_5$.

(11) *There exists a positive, finite number* $\varepsilon_{(11)}$ *depending only on* n, δ_4, *and* δ *with the following three properties.*

(a) *If* $\varrho \in J_0 \cap J_4$, $\kappa \le \varepsilon_{(11)}$, *and* $\phi_4(2\varrho) \le 2^{-m}\boldsymbol{\beta}(n)^{-1}\boldsymbol{\alpha}(m)(1/8)$, *then*

$$\|V\|(\mathbf{C}(T,a,\varrho,\delta_4\varrho)) \le (Q+1/2)\boldsymbol{\alpha}(m)\varrho^m.$$

(b) *If, additionally to the conditions of* (11a), $\varrho \in J_5$, *then*

$$\operatorname{graph}_Q f|\mathbf{B}(c,\varrho) \subset \mathbf{C}(T,a,\varrho,\delta_4\varrho/2).$$

(c) *If, additionally to the conditions of* (11a) *and* (11b), $0 < \lambda < \infty$,

$$\kappa \le 2^{-m}\boldsymbol{\beta}(n)^{-1}\boldsymbol{\alpha}(m)\lambda(2\Gamma_{4.8(7)}(Q,m))^{-1}\delta,$$
$$\phi_4(2\varrho) \le 2^{-m}\boldsymbol{\beta}(n)^{-1}\boldsymbol{\alpha}(m)\lambda(2\Gamma_{4.8(7)}(Q,m))^{-1},$$

then

$$\mathscr{L}^m(C_{a,\varrho}) \le \lambda \boldsymbol{\alpha}(m)\varrho^m.$$

(12) *If* $\varrho \in J_4 \cap J_5$, $\kappa \le \inf\{\varepsilon_{(8)}(m,\delta_4,\delta), \varepsilon_{(11)}(n,\delta_4,\delta)\}$, *and*

$$\sigma \in \operatorname{Hom}(\mathbf{R}^m,\mathbf{R}^{n-m}), \quad \|\sigma\| \le n^{-1/2}\delta/2, \quad \sigma = R \in \mathbf{G}(n,m),$$

then

$$\varrho^{-m}{\textstyle\int_{\mathbf{U}(c,\varrho)}}|AF(x)\,(+)(-\sigma)|^2\,\mathrm{d}\mathscr{L}^m x \le \Gamma_{(12)}\left(\phi_2(2\varrho,R)^2 + \phi_4(2\varrho)\right)$$

where $\Gamma_{(12)}$ *is a positive, finite number depending only on* n, Q, *and* δ.

(13) If $\varrho \in J_0 \cap J_1$, $\varrho/8 \leq s \leq t \leq \varrho$, $0 < \lambda \leq 1$, and

$$\|\sigma_\varrho\| \leq n^{-1/2}/4, \quad \phi_2(\varrho, T_\varrho) \leq \lambda^{1/2} 2^{-2m-3} \boldsymbol{\alpha}(m)^{1/2},$$
$$\|V\|(\mathbf{C}(T, a, s, \delta_4 s)) \geq \lambda \boldsymbol{\alpha}(m) s^m,$$

then $t \in J_1$ and

$$\|\sigma_\varrho - \sigma_t\| \leq \lambda^{-1/2} 2^{2m+2} \boldsymbol{\alpha}(m)^{-1/2} \phi_2(\varrho, T_\varrho).$$

Proof of (8). Let

$$\varepsilon_{(8)} = \inf\big\{(1/2)(4\boldsymbol{\gamma}(m)m)^{1-m}(\delta_4)^{m-1}\delta, (4\boldsymbol{\gamma}(m)m)^{-m}(\delta_4)^m\delta\big\}.$$

Define the sets $B'_{a,\varrho}$ and $B''_{a,\varrho}$ by

$$B'_{a,\varrho} = B_{a,\varrho} \cap \{z : \|\delta V\| \, \mathbf{B}(z,t) > \delta \, \|V\|(\mathbf{B}(z,t))^{1-1/m} \text{ for some } 0 < t < 2r\},$$
$$B''_{a,\varrho} = B_{a,\varrho} \sim B'_{a,\varrho}$$

and D to be the set of all $z \in \operatorname{spt} \|V\|$ such that

$$\limsup_{t \to 0+} \frac{\|\delta V\| \, \mathbf{B}(z,t)}{\|V\|(\mathbf{B}(z,t))^{1-1/m}} > 0.$$

Note $\|V\|(D) = 0$ by [Fed69, 2.9.5].

First, the following assertion will be shown. *If $m = 1$ then $B'_{a,\varrho} \sim D = \emptyset$ and if $m > 1$ then for $z \in B'_{a,\varrho} \sim D$ there exists $0 < t < \delta_4 \varrho$ such that*

$$\|V\| \, \mathbf{B}(z,t) \leq \delta^{-mp/(m-p)} \psi(\mathbf{B}(z,t))^{m/(m-p)}.$$

For this purpose assume $z \in B'_{a,\varrho} \sim D$ and define

$$t = \inf\big\{s : \|\delta V\| \, \mathbf{B}(z,s) > \delta \, \|V\|(\mathbf{B}(z,s))^{1-1/m}\big\}.$$

One infers $0 < t < 2r$ and

$$\|\delta V\| \, \mathbf{B}(z,t) \geq \delta \, \|V\|(\mathbf{B}(z,t))^{1-1/m} \geq (\delta/\Delta_1) t^{m-1}$$

by [Men09a, 2.5] where $\Delta_1 = (2\boldsymbol{\gamma}(m)m)^{m-1}$ since $\delta \leq (2\boldsymbol{\gamma}(m))^{-1}$. Noting

$$\varrho + t/\delta_4 \in J_2, \quad \mathbf{B}(z,t) \subset U \cap \mathbf{C}(T, a, \varrho + t/\delta_4, \delta_4(\varrho + t/\delta_4)),$$

one obtains

$$(\delta/\Delta_1) t^{m-1} \leq \kappa(\varrho + t/\delta_4)^{m-1}, \quad m > 1,$$
$$t \leq (\varrho + t/\delta_4)(\kappa\Delta_1/\delta)^{1/(m-1)} < (\varrho + t/\delta_4)\delta_4/2, \quad t < \delta_4 \varrho.$$

The assertion now follows from the definition of t in conjunction with Hölder's inequality.

The preceding assertion yields

$$\|V\|(B'_{a,\varrho}) = 0 \quad \text{if } m = 1,$$
$$\|V\|(B'_{a,\varrho}) \leq \delta^{-mp/(m-p)} \boldsymbol{\beta}(n) \psi(U \cap \mathbf{C}(T, a, 2\varrho, 2\delta_4\varrho))^{m/(m-p)} \quad \text{if } m > 1;$$

in fact if $m > 1$ there exist countable disjointed families $F_1, \ldots, F_{\boldsymbol{\beta}(n)}$ of closed balls such that

$$B'_{a,\varrho} \sim D \subset \bigcup \bigcup \{F_i : i = 1, \ldots, \boldsymbol{\beta}(n)\},$$
$$\|V\|(S) \le \Delta_2 \psi(S)^{m/(m-p)}, \quad S \subset U \cap \mathbf{C}(T, a, 2\varrho, 2\delta_4 \varrho)$$

whenever $S \in \bigcup \{F_i : i = 1, \ldots, \boldsymbol{\beta}(n)\}$ where $\Delta_2 = \delta^{-mp/(m-p)}$, hence

$$\|V\|(B'_{a,\varrho}) = \|V\|(B'_{a,\varrho} \sim D) \le \Delta_2 \textstyle\sum_{i=1}^{\boldsymbol{\beta}(n)} \sum_{S \in F_i} \psi(S)^{m/(m-p)}$$
$$\le \Delta_2 \textstyle\sum_{i=1}^{\boldsymbol{\beta}(n)} \big(\sum_{S \in F_i} \psi(S)\big)^{m/(m-p)} \le \Delta_2 \boldsymbol{\beta}(n) \psi(U \cap \mathbf{C}(T, a, 2\varrho, 2\delta_4\varrho))^{m/(m-p)}.$$

Next, it will be shown that *for $z \in B''_{a,\varrho}$ there exists $0 < t \le \delta_4 \varrho$ such that*

$$\|V\| \, \mathbf{B}(z,t) \le 4\delta^{-2} \textstyle\int_{\mathbf{B}(z,t) \times \mathbf{G}(n,m)} |S_\natural - R_\natural|^2 \, \mathrm{d}V(z,S),$$
$$\|V\| \, \mathbf{B}(z,t) < \delta^{-1} \textstyle\int_{\mathbf{B}(z,t) \times \mathbf{G}(n,m)} |S_\natural - T_\natural| \, \mathrm{d}V(z,S).$$

In fact, one can take any $0 < t < 2r$ satisfying the last inequality since this firstly implies, using [Men09a, 2.5], $\delta \le (2\boldsymbol{\gamma}(m))^{-1}$ and $\varrho + t/\delta_4 \in J_3$,

$$(2\boldsymbol{\gamma}(m)m)^{-m} t^m \le \|V\| \, \mathbf{B}(z,t) < \delta^{-1} \textstyle\int_{\mathbf{B}(z,t) \times \mathbf{G}(n,m)} |S_\natural - T_\natural| \, \mathrm{d}V(z,S)$$
$$\le \delta^{-1} \textstyle\int_{(U \cap \mathbf{C}(T, a, \varrho + t/\delta_4, \delta_4(\varrho + t/\delta_4))) \times \mathbf{G}(n,m)} |S_\natural - T_\natural| \, \mathrm{d}V(z,S) \le (\kappa/\delta)(\varrho + t/\delta_4)^m,$$
$$t \le (2\boldsymbol{\gamma}(m)m)(\kappa/\delta)^{1/m}(\varrho + t/\delta_4) \le (\varrho + t/\delta_4)\delta_4/2, \quad t \le \delta_4 \varrho,$$

and secondly, using $|R_\natural - T_\natural| \le \delta/2$ and Hölder's inequality,

$$\|V\| \, \mathbf{B}(z,t) \le 2\delta^{-1} \textstyle\int_{\mathbf{B}(z,t) \times \mathbf{G}(n,m)} |S_\natural - R_\natural| \, \mathrm{d}V(z,S),$$
$$\|V\| \, \mathbf{B}(z,t) \le 4\delta^{-2} \textstyle\int_{\mathbf{B}(z,t) \times \mathbf{G}(n,m)} |S_\natural - R_\natural|^2 \, \mathrm{d}V(z,S).$$

Since $2\varrho \in J_3$ and

$$\mathbf{B}(z,t) \subset U \cap \mathbf{C}(T, a, 2\varrho, 2\delta_4\varrho) \quad \text{whenever } z \in B''_{a,\varrho},\ 0 < t \le \delta_4 \varrho,$$

the assertion implies

$$\|V\|(B''_{a,\varrho}) \le 4\delta^{-2} \boldsymbol{\beta}(n) \textstyle\int_{(U \cap \mathbf{C}(T, a, 2\varrho, 2\delta_4\varrho)) \times \mathbf{G}(n,m)} |S_\natural - R_\natural|^2 \, \mathrm{d}V(z,S),$$
$$\|V\|(B''_{a,\varrho}) \le \boldsymbol{\beta}(n) \delta^{-1} \kappa (2\varrho)^m.$$

and the conclusion follows. □

Proof of (9). Defining

$$\varepsilon_{(9)} = \varepsilon \inf\{4^{1-m}(\delta_4)^{m-1}(\delta_5 \boldsymbol{\alpha}(m))^{1-1/m}, 4^{-m}(\delta_4)^m \delta_5 \boldsymbol{\alpha}(m)\},$$

one estimates for $z \in \mathbf{C}(T, 0, r, 3r)$

$$\|\delta V\| \, \mathbf{U}(z, 2r) \le \|\delta V\|(U \cap \mathbf{C}(T, a, 4r, 8r))$$
$$\le \kappa (8r/\delta_4)^{m-1} \le \varepsilon \big(\delta_5 \boldsymbol{\alpha}(m)(2r)^m\big)^{1-1/m},$$
$$\textstyle\int_{\mathbf{U}(z,2r) \times \mathbf{G}(n,m)} |S_\natural - T_\natural| \, \mathrm{d}V(z,S) \le \int_{(U \cap \mathbf{C}(T, a, 4r, 8r)) \times \mathbf{G}(n,m)} |S_\natural - T_\natural| \, \mathrm{d}V(z,S)$$
$$\le \kappa (8r/\delta_4)^m \le \varepsilon \delta_5 \boldsymbol{\alpha}(m)(2r)^m$$

and the conclusion follows. □

Proof of (10). Defining $\varepsilon_{(10)} = (\delta_4)^n \varepsilon_{7.2}(n, Q, \alpha, p, \inf\{\delta_6, \delta_4/4\})$ and noting

$$\begin{aligned} &\psi(\mathbf{B}(a,t) \cap \{z : \operatorname{dist}(z-a,T) < \delta_4\varrho/4\})^{1/p} \leq \psi(\mathbf{C}(T,a,t,\delta_4 \inf\{t/\delta_4, \varrho/4\}))^{1/p} \\ &\qquad \leq \varepsilon_{(10)}(t/\delta_4)^{m/p+\alpha-1}\varrho^{-\alpha} \leq \varepsilon_{(10)}(\delta_4)^{-m/p} t^{m/p+\alpha-1}\varrho^{-\alpha} \end{aligned}$$

for $0 < t < \varrho$, the assertion follows from 7.2 with δ, r replaced by $\inf\{\delta_6, \delta_4/4\}$, ϱ. ☐

Proof of (11). Define $\varepsilon_{(11)}$ to be the infimum of all numbers

$$\inf\left\{2^{-n}\boldsymbol{\beta}(n)^{-1}\boldsymbol{\alpha}(i)(1/8)\delta, 2^{-3}n^{-1}\boldsymbol{\alpha}(i), \varepsilon_{(8)}(i,\delta_4,\delta)\right\}$$

corresponding to $n > i \in \mathscr{P}$.

If the conclusion of (11b) were not true, one would infer

$$\begin{gathered} \operatorname{spt} f(x) \sim \mathbf{B}(\mathbf{q}(a), \delta_4\varrho/4) \neq \emptyset, \\ \textstyle\sum_{y \in \mathbf{B}(\mathbf{q}(a),\delta_4\varrho/4) \cap \operatorname{spt} f(x)} \boldsymbol{\Theta}^0(\|f(x)\|, y) \leq Q-1 \end{gathered}$$

whenever $x \in \operatorname{dmn} f | \mathbf{B}(c,\varrho)$ by (1) and 4.8 (4) and therefore by 4.8 (1) (2) and [Fed69, 3.2.22 (3)] one would obtain

$$\textstyle\int_{\mathbf{C}(T,a,\varrho,\delta_4\varrho/4)\cap A} \|\Lambda_m(\mathbf{p}|S)\| \, \mathrm{d}V(z,S) \leq (Q-1)\boldsymbol{\alpha}(m)\varrho^m,$$

hence by [Men09b, 2.16] and (8) with R replaced by T, noting $\varrho \in J_4 \subset J_3$,

$$\begin{aligned} &\|V\|(\mathbf{C}(T,a,\varrho,\delta_4\varrho/4)) - (Q-1)\boldsymbol{\alpha}(m)\varrho^m \\ &\qquad \leq \|V\|(B_{a,\varrho}) + 2m \textstyle\int_{\mathbf{C}(T,a,\varrho,\delta_4\varrho/4)} |S_\natural - T_\natural| \, \mathrm{d}V(z,S) \leq (1/2)\boldsymbol{\alpha}(m)\varrho^m \end{aligned}$$

in contradiction to $\varrho \in J_5$.

Using similarly

$$\textstyle\sum_{y \in A(x)} \boldsymbol{\Theta}^0(\|V\|, (x,y)) \leq Q \quad \text{for } x \in X_1 \cup X_2,$$

one obtains (11a).

To prove (11c), one estimates with 4.8 (7) and (8) with R replaced by T

$$\mathscr{L}^m(C_{a,\varrho}) \leq \Gamma_{4.8(7)}(Q,m)\|V\|(B_{a,\varrho}) \leq \lambda\boldsymbol{\alpha}(m)\varrho^m. \qquad ☐$$

Proof of (12). Denote by X_1' the set of all $x \in X_1$ such that 4.8 (5) is true for x and note $\mathscr{L}^m(X_1 \sim X_1') = 0$. Since

$$|\operatorname{ap} AF(x)\,(+)(-\sigma)| \leq (1 + \operatorname{Lip} F)(Qm)^{1/2} \leq (1 + \Gamma_{(2)}(n-m,Q))(Qm)^{1/2}$$

for $x \in \operatorname{dmn} \operatorname{ap} AF$, one may assume

$$\phi_4(2\varrho) \leq 2^{-m}\boldsymbol{\beta}(n)^{-1}\boldsymbol{\alpha}(m)(1/8).$$

Next, it will shown with $G = \operatorname{graph}_Q f$

$$\begin{gathered} \mathbf{B}(c,\varrho) \cap X_1' \cap \{x : |\operatorname{ap} Af(x)\,(+)(-\sigma)| > \gamma\} \\ \subset \mathbf{p}\big[\mathbf{C}(T,a,\varrho,\delta_4\varrho) \cap G \cap \{z : |\operatorname{Tan}^m(\|V\|,z)_\natural - R_\natural| > 2^{-1}(Qm)^{-1/2}\gamma\}\big] \end{gathered}$$

whenever $0 < \gamma < \infty$. In fact, if x is a member of the first set there exist $y \in \operatorname{spt} f(x)$ and $\tau \in \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^{n-m})$ such that

$$\tau = \operatorname{Tan}^m(\|V\|, (x,y)), \quad |\tau - \sigma| > Q^{-1/2}\gamma,$$

hence, noting $\|\sigma\| \leq 1$ and $\big\| \operatorname{Tan}^m(\|V\|,(x,y))_\natural - T_\natural \big\| \leq \|\tau\| \leq L \leq 1/2$ by 3.1,

$$\|\sigma - \tau\| \leq 2 \big\| \operatorname{Tan}^m(\|V\|,(x,y))_\natural - R_\natural \big\|$$

by 3.1, and the inclusion follows, since $(x,y) \in \mathbf{C}(T,a,\varrho,\delta_4\varrho)$ by (11b). Therefore, since $\boldsymbol{\Theta}^m(\|V\|,z) \geq 1$ for $z \in G$,

$$\begin{aligned}
&\|V\|(\mathbf{C}(T,a,\varrho,\delta_4\varrho) \cap \{z : |\operatorname{Tan}^m(\|V\|,z)_\natural - R_\natural| > 2^{-1}(Qm)^{-1/2}\gamma\}) \\
&\quad \geq \mathscr{H}^m(\mathbf{C}(T,a,\varrho,\delta_4\varrho) \cap G \cap \{z : |\operatorname{Tan}^m(\|V\|,z)_\natural - R_\natural| > 2^{-1}(Qm)^{-1/2}\gamma\}) \\
&\quad \geq \mathscr{L}^m(\mathbf{B}(c,\varrho) \cap X_1 \cap \{x : |\operatorname{ap} Af(x)\,(+)(-\sigma)| > \gamma\})
\end{aligned}$$

and one obtains

$$\varrho^{-m} \textstyle\int_{\mathbf{U}(c,\varrho)\cap X_1} |\operatorname{ap} Af(x)\,(+)(-\sigma)|^2 \,\mathrm{d}\mathscr{L}^m \leq 2^{m+2} Qm\, \phi_2(2\varrho, R)^2.$$

Recalling the first paragraph of the proof, and noting

$$|R_\natural - T_\natural| \leq n^{1/2}\|R_\natural - T_\natural\| \leq n^{1/2}\|\sigma\| \leq \delta/2$$

by 3.1 and $\mathbf{U}(c,\varrho) \sim X_1 \subset C_{a,\varrho}$, the conclusion follows combining (11b), (8) and 4.8 (7). □

Proof of (13). Using Hölder's inequality, one obtains

$$\begin{aligned}
|(T_t)_\natural - (T_\varrho)_\natural| &\leq \|V\|(\mathbf{C}(T,a,s,\delta_4 s))^{-1/2}\big(t^{m/2}\phi_2(t,T_t) + \varrho^{m/2}\phi_2(\varrho,T_\varrho)\big) \\
&\leq \lambda^{-1/2} 2^{2m+1} \boldsymbol{\alpha}(m)^{-1/2} \phi_2(\varrho, T_\varrho),
\end{aligned}$$

since $t^{m/2}\phi_2(t,T_t) \leq \varrho^{m/2}\phi_2(\varrho,T_\varrho)$. Noting by 3.1

$$\begin{gathered}
|(T_t)_\natural - T_\natural| \leq |(T_t)_\natural - (T_\varrho)_\natural| + |(T_\varrho)_\natural - T_\natural| \\
\leq \lambda^{-1/2} 2^{2m+1} \boldsymbol{\alpha}(m)^{-1/2} \phi_2(\varrho,T_\varrho) + n^{1/2}\|\sigma_\varrho\| \leq 1/2, \\
\|(T_t)_\natural - T_\natural\| \leq 1/2, \quad T_t \cap \ker \mathbf{p} = \{0\}, \quad t \in J_1,
\end{gathered}$$

one applies 3.1 with S, S_1, S_2 replaced by T, T, T_t to infer

$$\begin{gathered}
\|\sigma_t\|^2 \leq (1 + \|\sigma_t\|^2)\|(T_t)_\natural - T_\natural\|^2, \\
\|\sigma_t\|^2 \leq \|(T_t)_\natural - T_\natural\|^2/(1 - \|(T_t)_\natural - T_\natural\|^2) \leq 2\|(T_t)_\natural - T_\natural\|^2 \leq 1/2,
\end{gathered}$$

Now, 3.1 with S, S_1, S_2 replaced by T, T_t, T_ϱ implies

$$\|\sigma_t - \sigma_\varrho\| \leq 2|(T_t)_\natural - (T_\varrho)_\natural|. \qquad □$$

7.4 Lemma. *Suppose m, n, Q, L, M, δ_1, δ_2, δ_3, δ_4, δ_5, ε, r, T, U, V, δ, X_1, f, a, c, κ, F, p, ψ, J, ϕ_2, ϕ_3, ϕ_4, T_ϱ, J_0, J_1, J_2, J_3, J_4, J_5, and σ_ϱ are as in 7.3. Suppose additionally:*

(1) Suppose Ψ and C are as in 6.1.

(2) *Whenever* $\varrho \in J_1$ *suppose* u_ϱ *denotes the unique analytic function in* $\mathbf{W}^{1,2}(\mathbf{U}(c,\varrho),\mathbf{R}^{n-m})$ *such that*

$$\langle D^2 u_\varrho(x), C(\sigma_\varrho)\rangle = 0 \quad \textit{for } x \in \mathbf{U}(c,\varrho),$$
$$u_\varrho - g \in \mathbf{W}_0^{1,2}(\mathbf{U}(c,\varrho),\mathbf{R}^{n-m}),$$

see 6.1–6.3 and [Fed69, 5.1.2, 10].

(3) *Define the function* $\phi_1 : J_1 \to \mathbf{R}$ *by* $\phi_1(\varrho) = |D^2 u_\varrho|_{\infty;c,\varrho/2}$ *for* $\varrho \in J_1$.

(4) *Suppose* $0 < \tau \le 1$ *and* $\tau = 1$ *if* $m = 1$, $p/2 \le \tau < \frac{mp}{2(m-p)}$ *if* $m = 2$ *and* $\tau = \frac{mp}{2(m-p)}$ *if* $m > 2$.

Then the following seven conclusions hold:

(5) *There exists a positive, finite number* $\Gamma_{(5)}$ *depending only on* n *such that*

$$D^2\Psi_0^\S(\sigma) \textit{ is strongly elliptic with ellipticity bound } (\Gamma_{(5)})^{-1},$$
$$\|D^2\Psi_0^\S(\sigma)\| \le \Gamma_{(5)}$$

whenever $\sigma \in \operatorname{Hom}(\mathbf{R}^m,\mathbf{R}^{n-m})$ *with* $\|\sigma\| \le 1$.

(6) *If* $\varrho \in J_4 \cap J_5$, $2\varrho \in J_0 \cap J_1$, $\|\sigma_{2\varrho}\| \le n^{-1/2}\inf\{\delta/2, 1/4\}$, *and*

$$\phi_2(2\varrho, T_{2\varrho}) \le 2^{-2m-4}\boldsymbol{\alpha}(m)^{1/2},$$
$$\kappa \le \inf\{\varepsilon_{\mathit{7.3(8)}}(m,\delta_4,\delta), \varepsilon_{\mathit{7.3(11)}}(n,\delta_4,\delta)\},$$

then

$$\phi_1(\varrho) \le \Gamma_{(6)}\varrho^{-1}\big(\phi_2(2\varrho,T_{2\varrho}) + \phi_4(2\varrho)^{1/2}\big)$$

where $\Gamma_{(6)}$ *is a positive, finite number depending only on* n, Q, *and* δ.

(7) *If* $\varrho \in J_1 \cap J_4 \cap J_5$, $\|\sigma_\varrho\| \le 1$, $2\varrho \in J_1$, $\|\sigma_{2\varrho}\| \le n^{-1/2}\delta/2$,

$$\kappa \le \inf\{\varepsilon_{\mathit{7.3(8)}}(m,\delta_4,\delta), \varepsilon_{\mathit{7.3(11)}}(n,\delta_4,\delta)\},$$
$$\phi_4(2\varrho) \le 2^{-m}\boldsymbol{\beta}(n)^{-1}\boldsymbol{\alpha}(m)(1/8),$$

then

$$\varrho^{-m-1}|u_\varrho - g|_{1;c,\varrho} \le \Gamma_{(7)}\big(\phi_2(2\varrho,T_{2\varrho})^2 + \phi_3(2\varrho)\big)$$

where $\Gamma_{(7)}$ *is a positive, finite number depending only on* m, n, Q, δ_4, δ, *and* p.

(8) *There exists a positive, finite number* $\varepsilon_{(8)}$ *depending only on* n, δ_4, *and* δ *with the following property.*
If $\varrho \in J$, $2\varrho \in J_0 \cap J_1$, $\|\sigma_{2\varrho}\| \le n^{-1/2}\delta/4$, $\kappa \le \varepsilon_{(8)}$, *and for* $s \in \{\varrho/4, \varrho\}$

$$s \in J_4 \cap J_5, \quad \phi_4(2s) \le 2^{-m}\boldsymbol{\beta}(n)^{-1}\boldsymbol{\alpha}(m)(1/8),$$

then

$$\phi_1(\varrho/4) \le \phi_1(\varrho) + \Gamma_{(8)}\big(\phi_1(\varrho)\phi_2(\varrho,T_\varrho) + \varrho^{-1}(\phi_2(2\varrho,T_{2\varrho})^2 + \phi_3(2\varrho))\big)$$

where $\Gamma_{(8)}$ *is a positive, finite number depending only on* m, n, Q, δ_4, δ *and* p.

(9) *There exists a positive, finite number $\varepsilon_{(9)}$ depending only on m, n, Q, δ_2, ε, δ, and p with the following property.*

If $\delta_4 = 1$, $\delta_5 = (40)^{-m}(\boldsymbol{\gamma}(m)m)^{-m}/\boldsymbol{\alpha}(m)$, $0 < \eta < 2^{-m}$, $P : \mathbf{R}^m \to \mathbf{R}^{n-m}$ is affine, $\operatorname{Lip} P \leq n^{-1/2}\delta/2$, $R = \operatorname{im} D(\mathbf{p}^* + \mathbf{q}^* \circ P)(0)$, $\varrho \in J$, X is an $\mathscr{L}^m$ measurable subset of $\mathbf{U}(c, \varrho/2) \cap X_1$,

$$\mu = 1/2 \quad \text{if } m = 1, \quad \mu = 1/m \quad \text{if } m > 1,$$
$$\varrho/2 \in J_4 \cap J_5, \quad 8r \in J_2 \cap J_3, \quad \varrho \in J_1, \quad \|\sigma_\varrho\| \leq n^{-1/2}\delta/2,$$
$$\kappa \leq \varepsilon_{(9)}, \quad \phi_3(\varrho) \leq \varepsilon_{(9)}, \quad \mathscr{L}^m(\mathbf{U}(c, \varrho/2) \sim X) \leq \eta\boldsymbol{\alpha}(m)(\varrho/2)^m,$$

then for $0 < \lambda \leq 1$

$$\begin{aligned}\phi_2(\varrho/4, R) \leq \Gamma_{(9)}\Big(&\big(\lambda + \phi_2(\varrho, T_\varrho)^{2/m}\big)\phi_2(\varrho, T_\varrho) + (\lambda + \eta^\mu)\phi_2(\varrho, R) \\ &+ \eta^{-1}\varrho^{-m-1}|f\,(+)(-P)|_{1;X} + \lambda^{-\tau}\phi_3(\varrho)^\tau\Big)\end{aligned}$$

where $\Gamma_{(9)}$ is a positive, finite number depending only on m, n, Q, δ, p, and τ.

(10) *There exists a positive, finite number $\varepsilon_{(10)}$ depending only on m, n, Q, δ_2, ε, δ, and p with the following property.*

If $\delta_4 = 1$, $\delta_5 = (40)^{-m}(\boldsymbol{\gamma}(m)m)^{-m}/\boldsymbol{\alpha}(m)$, $0 < \eta < 2^{-m}$, $\varrho \in J$,

$$\mu = 1/2 \quad \text{if } m = 1, \quad \mu = 1/m \quad \text{if } m > 1,$$
$$\{\varrho/2, \varrho\} \subset J_4 \cap J_5, \quad 2\varrho \in J_0 \cap J_1, \quad \|\sigma_{2\varrho}\| \leq n^{-1/2}\delta/4,$$
$$8r \in J_2 \cap J_3, \quad \kappa \leq \varepsilon_{(10)}, \quad \phi_3(2\varrho) \leq \varepsilon_{(10)},$$
$$\mathscr{L}^m(\mathbf{U}(c, \varrho/2) \sim \{x : \boldsymbol{\Theta}^0(\|f(x)\|, g(x)) = Q\}) \leq \eta\boldsymbol{\alpha}(m)(\varrho/2)^m,$$

then for $0 < \lambda \leq 1$

$$\begin{aligned}\phi_2(\varrho/4, T_{\varrho/4}) \leq \Gamma_{(10)}\Big(&\big(\lambda + \eta^\mu + \eta^{-1}\phi_2(2\varrho, T_{2\varrho})^{\inf\{1, 2/m\}}\big)\phi_2(2\varrho, T_{2\varrho}) \\ &+ \eta^{-1}\varrho\phi_1(\varrho) + (\eta^{-1} + \lambda^{-\tau})\phi_3(2\varrho)^\tau\Big)\end{aligned}$$

where $\Gamma_{(10)}$ is a positive, finite number depending only on m, n, Q, δ, p, and τ.

(11) *Let $\delta_4 = 1$, $\delta_5 = (40)^{-m}(\boldsymbol{\gamma}(m)m)^{-m}/\boldsymbol{\alpha}(m)$, $\delta = \inf\{1, \varepsilon, (2\boldsymbol{\gamma}(m))^{-1}\}$, $0 < \alpha \leq 1$, and $0 < \delta_6 \leq 1$.*

Then there positive, finite numbers γ_i for $i \in \{1, 2, 3\}$ and a positive, finite number $\varepsilon_{(11)}$ both depending only on m, n, Q, L, M, δ_1, δ_2, δ_3, p, τ, α, and δ_6 with the following property.

If $a \in \mathbf{C}(T, 0, r/2, 2r)$, $\boldsymbol{\Theta}^{*m}(\|V\|, a) \geq Q - 1 + \delta_6$, $0 < t \leq \frac{r}{64}$, $0 < \gamma \leq 1$,

$$\phi_2(8r, T) \leq \varepsilon_{(11)}, \quad \phi_2(8r, T_{8r}) \leq \varepsilon_{(11)}\gamma,$$
$$\|V\|(\mathbf{C}(T, a, \varrho, \varrho) \cap \{z : \boldsymbol{\Theta}^m(\|V\|, z) \leq Q - 1\}) \leq \varepsilon_{(11)}\boldsymbol{\alpha}(m)\varrho^m$$

whenever $t \leq \varrho \leq r/8$, and

$$\phi_3(\varrho)^\tau \leq \gamma\gamma_3(\varrho/r)^{\alpha\tau} \quad \text{whenever } 0 < \varrho \leq 8r,$$

then, in case $\alpha\tau < 1$,

$$\varrho \in J_1 \quad \textit{and} \quad \varrho\phi_1(\varrho) \le \gamma\gamma_1(\varrho/r)^{\alpha\tau} \quad \textit{for } t \le \varrho \le r/4,$$
$$\phi_2(\varrho, T_\varrho) \le \gamma\gamma_2(\varrho/r)^{\alpha\tau} \quad \textit{for } t \le \varrho \le r$$

and, in case $\alpha\tau = 1$,

$$\varrho \in J_1 \quad \textit{and} \quad \varrho\phi_1(\varrho) \le \gamma\gamma_1(\varrho/r)(1 + \log(r/\varrho)) \quad \textit{for } t \le \varrho \le r/4,$$
$$\phi_2(\varrho, T_\varrho) \le \gamma\gamma_2(\varrho/r)(1 + \log(r/\varrho)) \quad \textit{for } t \le \varrho \le r.$$

Proof of (5). This follows from [Fed69, 5.1.2, 10]. □

Proof of (6). Note by 7.3 (13) applied with ϱ, s, t, λ replaced by 2ϱ, ϱ, ϱ, $1/2$

$$\varrho \in J_1, \quad \|\sigma_\varrho\| \le \|\sigma_{2\varrho}\| + 2^{2m+3}\boldsymbol{\alpha}(m)^{-1/2}\phi_2(2\varrho, T_{2\varrho}) \le 1.$$

Since $u_\varrho - \sigma_{2\varrho}$ is $D^2\Psi_0^\S(\sigma_\varrho)$ harmonic, applying [Fed69, 5.2.5] yields, noting (5),

$$|D^2 u_\varrho|_{\infty;c,\varrho/2} \le \Delta_1 \varrho^{-1-m/2} |D(u_\varrho - \sigma_{2\varrho})|_{2;c,\varrho}$$

where $\Delta_1 = 2^{n+5} n^{n+1} \Gamma_{(5)}(n)^n \sup\{\boldsymbol{\alpha}(i)^{-1/2} : n > i \in \mathscr{P}\}$. Using 6.2, one obtains

$$|D(u_\varrho - \sigma_{2\varrho})|_{2;c,\varrho} \le |D(u_\varrho - g)|_{2;c,\varrho} + |D(g - \sigma_{2\varrho})|_{2;c,\varrho} \le \Delta_2 |D(g - \sigma_{2\varrho})|_{2;c,\varrho}$$

where $\Delta_2 = 1 + \Gamma_{(5)}(n)^2$. Taking $\Gamma_{(6)} = \Delta_1\Delta_2\Gamma_{7.3(12)}(n, Q, \delta)^{1/2}$, the conclusion now follows from 7.3 (12) with σ replaced by $\sigma_{2\varrho}$. □

Proof of (7). Suppose B, and $B_{a,t}$, $C_{a,t}$ for $t \in J_0$ are as in 7.3. Define $S, R \in \mathscr{D}'(\mathbf{U}(c,\varrho), \mathbf{R}^{n-m})$ by

$$S(\theta) = -\textstyle\int_{\mathbf{U}(c,\varrho)} \big\langle D\theta(x), D\Psi_0^\S(Dg(x)) \big\rangle \,\mathrm{d}\mathscr{L}^m x,$$
$$R(\theta) = -\textstyle\int_{\mathbf{U}(c,\varrho)} \big\langle D\theta(x) \odot Dg(x), D^2\Psi_0^\S(\sigma_\varrho) \big\rangle \,\mathrm{d}\mathscr{L}^m x$$

whenever $\theta \in \mathscr{D}(\mathbf{U}(c,\varrho), \mathbf{R}^{n-m})$. Since u_ϱ is $D^2\Psi_0^\S(\sigma_\varrho)$ harmonic,

$$|u_\varrho - g|_{1;c,\varrho} \le \Delta_1 \varrho |R|_{-1,1;c,\varrho} \tag{I}$$

by 6.8 and (5) where $\Delta_1 = \Gamma_{6.8}(n, \Gamma_{(5)}(n)^{-1}, \Gamma_{(5)}(n))$. One computes for $x \in \operatorname{dmn} Dg$

$$D\Psi_0^\S(Dg(x)) - D\Psi_0^\S(\sigma_\varrho) - (Dg(x) - \sigma_\varrho) \lrcorner D^2\Psi_0^\S(\sigma_\varrho)$$
$$= (Dg(x) - \sigma_\varrho) \lrcorner \textstyle\int_0^1 D^2\Psi_0^\S(tDg(x) + (1-t)\sigma_\varrho) - D^2\Psi_0^\S(\sigma_\varrho)\,\mathrm{d}\mathscr{L}^1 t,$$
$$\|D^2\Psi_0^\S(tDg(x) + (1-t)\sigma_\varrho) - D^2\Psi_0^\S(\sigma_\varrho)\|$$
$$\le \operatorname{Lip}(D^2\Psi_0^\S|\mathbf{B}(0,\gamma))\, t |Dg(x) - \sigma_\varrho| \qquad \text{for } 0 \le t \le 1$$

where $\gamma = m^{1/2}\sup\{1, \Gamma_{(2)}(n-m, Q)\}$, hence, since

$$\textstyle\int_{\mathbf{U}(c,\varrho)} \langle D\theta(x), \beta\rangle \,\mathrm{d}\mathscr{L}^m x = 0$$

for $\theta \in \mathscr{D}(\mathbf{U}(c,\varrho), \mathbf{R}^{n-m})$ and $\beta \in \{D\Psi_0^\S(\sigma_\varrho), \sigma_\varrho \lrcorner D^2\Psi_0^\S(\sigma_\varrho)\}$,

$$\varrho^{-m}|S-R|_{-1,1;c,\varrho} \leq \Delta_2 \varrho^{-m} \textstyle\int_{\mathbf{U}(c,\varrho)} |Dg(x) - \sigma_\varrho|^2 \,\mathrm{d}\mathscr{L}^m x$$

where Δ_2 is a positive, finite number depending only on n and Q. Therefore by 7.3 (12) with σ replaced by $\sigma_{2\varrho}$

$$\varrho^{-m}|S-R|_{-1,1;c,\varrho} \leq \Delta_3 \big(\phi_2(2\varrho, T_{2\varrho})^2 + \phi_4(2\varrho)\big) \tag{II}$$

where $\Delta_3 = \Delta_2 \Gamma_{7.3(12)}(n,Q,\delta)$.

Let $\theta \in \mathscr{D}(\mathbf{U}(c,\varrho), \mathbf{R}^{n-m})$ with $|D\theta|_{\infty;c,\varrho} \leq 1$ and $\eta \in \mathscr{D}^0(\mathbf{R}^{n-m})$ with

$$\begin{gathered}\operatorname{spt}\eta \subset \mathbf{U}(\mathbf{q}(a), \delta_4\varrho), \quad \mathbf{B}(\mathbf{q}(a), \delta_4\varrho/2) \subset \operatorname{Int}(\mathbf{R}^{n-m} \cap \{y : \eta(y) = 1\}), \\ 0 \leq \eta(y) \leq 1, \quad |D\eta(y)| \leq 4(\delta_4)^{-1}\varrho^{-1} \qquad \text{for } y \in \mathbf{R}^{n-m}.\end{gathered}$$

From 4.8 (9) with τ replaced by $\sigma_{2\varrho}$ one infers with $D_{a,\varrho} = \mathbf{C}(T,a,\varrho,\delta_4\varrho) \cap \mathbf{p}^{-1}[C_{a,\varrho}]$, noting 7.3 (11b) and $|\theta|_{\infty;c,\varrho} \leq \varrho$,

$$\begin{aligned}&|QS(\theta) + (\delta V)((\eta \circ \mathbf{q}) \cdot (\mathbf{q}^* \circ \theta \circ \mathbf{p}))| \\ &\quad \leq \Delta_4 \big(\mathscr{L}^m(C_{a,\varrho}) + \textstyle\int_{\mathbf{U}(c,\varrho)} |AF(x)\,(+)(-\sigma_{2\varrho})|^2 \,\mathrm{d}\mathscr{L}^m x + \|V\|(D_{a,\varrho})\big)\end{aligned}$$

where Δ_4 is a positive, finite number depending only on n, Q, and δ_4. By 4.8 (7), noting 7.3 (11b), and 7.3 (12) with σ replaced by $\sigma_{2\varrho}$

$$\begin{aligned}&\varrho^{-m}|QS(\theta) + (\delta V)((\eta \circ \mathbf{q}) \cdot (\mathbf{q}^* \circ \theta \circ \mathbf{p}))| \\ &\leq \Delta_4 \Gamma_{4.8(7)}(Q,m) \varrho^{-m} \|V\|(B_{a,\varrho}) + \Delta_4 \Gamma_{7.3(12)}(n,Q,\delta)\big(\phi_2(2\varrho, T_{2\varrho})^2 + \phi_4(2\varrho)\big)\end{aligned}$$

Therefore one obtains in view of 7.3 (8), since $|(T_\varrho)_\natural - T_\natural| \leq n^{1/2}\|(T_{2\varrho})_\natural - T_\natural\| \leq n^{1/2}\|\sigma_{2\varrho}\| \leq \delta/2$ by 3.1,

$$\varrho^{-m}|QS(\theta) + (\delta V)((\eta \circ \mathbf{q}) \cdot (\mathbf{q}^* \circ \theta \circ \mathbf{p}))| \leq \Delta_5 \big(\phi_2(2\varrho, T_{2\varrho})^2 + \phi_4(2\varrho)\big) \tag{III}$$

where Δ_5 is a positive, finite number depending only on n, Q, δ_4, and δ. Also, using 7.3 (11a) and Hölder's inequality, recalling $|\theta|_{\infty;c,\varrho} \leq \varrho$,

$$\varrho^{-m}|(\delta V)((\eta \circ \mathbf{q}) \cdot (\mathbf{q}^* \circ \theta \circ \mathbf{p}))| \leq (\boldsymbol{\alpha}(m)(Q+1/2))^{1-1/p} \phi_3(\varrho). \tag{IV}$$

Finally, noting

$$\begin{gathered}\phi_3(2\varrho) = \delta\phi_4(2\varrho)^{\frac{m-p}{mp}} \leq \delta\big(2^{-m}\boldsymbol{\beta}(n)^{-1}\boldsymbol{\alpha}(m)(1/8)\big)^{\frac{m-p}{mp}} \quad \text{if } m > 1, \\ \phi_4(2\varrho) \leq \Delta_6 \phi_3(2\varrho)\end{gathered}$$

where $\Delta_6 = \delta^{-1}\big(2^{-m}\boldsymbol{\beta}(n)^{-1}\boldsymbol{\alpha}(m)(1/8)\big)^{1-\frac{m-p}{mp}}$, the conclusion may be obtained by combining (I), (II), (III) and (IV). □

Proof of (8). Define $\varepsilon_{(8)}$ to be the infimum of all numbers

$$\inf\big\{\varepsilon_{7.3(8)}(i,\delta_4,\delta), \varepsilon_{7.3(11)}(n,\delta_4,\delta), 2^{-4n-5}n^{-2}\boldsymbol{\alpha}(i)\delta^2\big\}$$

corresponding to $n > i \in \mathscr{P}$.

Noting

$$\phi_1(\varrho/4) \le \phi_1(\varrho) + |D^2(u_{\varrho/4} - u_\varrho)|_{\infty;c,\varrho/8},$$

only $|D^2(u_{\varrho/4} - u_\varrho)|_{\infty;c,\varrho/8}$ needs to be estimated. Since $\varrho < 2r$ as $2\varrho \in J_0$ and $\varrho \in J_4$, one notes

$$2\varrho \in J_3, \quad \phi_2(2\varrho, T_{2\varrho}) \le \phi_2(2\varrho, T) \le (2m^{1/2}\kappa)^{1/2}.$$

Therefore one may apply 7.3 (13) for each $t \in \{\varrho/4, \varrho/2, \varrho\}$ with ϱ, s, λ replaced by 2ϱ, $\varrho/4$, $1/2$ to obtain $\{\varrho/4, \varrho/2, \varrho\} \subset J_1$ and

$$\sup\{\|\sigma_{\varrho/4}\|, \|\sigma_{\varrho/2}\|, \|\sigma_\varrho\|\} \le \|\sigma_{2\varrho}\| + 2^{2m+3}\boldsymbol{\alpha}(m)^{-1/2}\phi_2(2\varrho, T_{2\varrho}) \le n^{-1/2}\delta/2.$$

Computing for $x \in \mathbf{U}(c, \varrho/4)$

$$\left\langle D^2(u_\varrho - u_{\varrho/4})(x), C(\sigma_{\varrho/4})\right\rangle = \left\langle D^2 u_\varrho(x), C(\sigma_{\varrho/4}) - C(\sigma_\varrho)\right\rangle,$$

one infers from 6.4 with c, M, Υ, α, a, r, and u replaced by $\Gamma_{(5)}(n)^{-1}$, $\Gamma_{(5)}(n)$ $D^2\Psi_0^\S(\sigma_{\varrho/4})$, $1/2$, c, $\varrho/4$, and $u_\varrho - u_{\varrho/4}$ that

$$\begin{aligned} &|D^2(u_\varrho - u_{\varrho/4})|_{\infty;c,\varrho/8} \\ &\quad \le \Delta_1\big(\varrho^{-2-m}|u_\varrho - u_{\varrho/4}|_{1;c,\varrho/4} + \varrho^{1/2}\mathbf{h}_{1/2}(D^2u_\varrho|\mathbf{B}(c,\varrho/4))\|\sigma_{\varrho/4} - \sigma_\varrho\|\big) \end{aligned}$$

where Δ_1 is a positive, finite number depending only on n. Since

$$\varrho^{1/2}\mathbf{h}_{1/2}(D^2u_\varrho|\mathbf{B}(c,\varrho/4)) \le \Delta_2\phi_1(\varrho)$$

by [Fed69, 5.2.5] and (5) for some positive, finite number Δ_2 depending only on n, the conclusion now follows, noting 7.3 (13), by applying (7) twice, once with ϱ as given and once with ϱ replaced by $\varrho/4$. □

Proof of (9). Define $q = \infty$ if $m = 1$ and $q = (\frac{1}{2\tau} + \frac{1}{2} - \frac{1}{p})^{-1}$ if $m > 1$ and note $2 \le q < \infty$ if $m = 2$ and $q = 2m/(m-2)$ if $m > 2$ and

$$1/p + 1/q \ge 1, \quad \tau = (2(1/p + 1/q) - 1)^{-1}.$$

With $\delta_4 = 1$ and $\delta_5 = (40)^{-m}(\boldsymbol{\gamma}(m)m)^{-m}/\boldsymbol{\alpha}(m)$ define

$$\begin{gathered}
\Delta_1 = \inf\big\{\varepsilon_{7.3(8)}(m,\delta_4,\delta), \varepsilon_{7.3(9)}(m,\delta_4,\delta_5,\varepsilon), \varepsilon_{7.3(11)}(n,\delta_4,\delta), \\
2^{-m}\boldsymbol{\beta}(n)^{-1}\boldsymbol{\alpha}(m)\varepsilon_{4.8(8)}(m,\delta_2,\delta_4)(2\Gamma_{4.8(7)}(Q,m))^{-1}\delta\big\}, \\
\Delta_2 = \inf\big\{1, (2\boldsymbol{\gamma}(1))^{-1}\big\}, \\
\Delta_3 = \inf\big\{1, 2^{-m}\boldsymbol{\beta}(n)^{-1}\boldsymbol{\alpha}(m)\inf\{\varepsilon_{4.8(8)}(m,\delta_2,\delta_4)(2\Gamma_{4.8(7)}(Q,m))^{-1}, 1/8\}\big\}, \\
\varepsilon_{(9)} = \inf\big\{\Delta_1, 2^{-1}m^{-1/2}, \Delta_2, \delta(\Delta_3)^{1/p-1/m}\big\}, \quad \Delta_4 = \sup\{2^m\Gamma_{5.4}(n,Q,q), 1\}, \\
\Delta_5 = \sup\big\{2\Gamma_{5.4}(n,Q,\infty), 2^m\Gamma_{5.4}(n,Q,2), 1\big\}, \quad \Delta_6 = \Gamma_{7.3(12)}(n,Q,\delta)^{1/2}\delta^{-\tau}, \\
\Delta_7 = \sup\{Q\Gamma_{4.8(8)}(m), 1\}, \quad \Delta_8 = 2^{m+2}\delta^{-2}\boldsymbol{\beta}(n), \\
\Delta_9 = 19/(2^{1/2}\cdot 40 + 19), \quad \Delta_{10} = \Gamma_{3.13}(m,p,q) \quad \text{if } m = 1, \\
\Delta_{10} = \Gamma_{3.9}(m,p,q) \quad \text{if } m > 1, \\
\Delta_{11} = 2^m\sup\big\{2(\Delta_{10})^{1/2}, 2(16+4m)^{1/2}|\Delta_9 - 1/4|^{-1}\big\}, \\
\Delta_{12} = \big(4(\Delta_4 + \Delta_5)\Delta_7(\Delta_8)^2\delta^{-\tau} + 1\big)\Delta_{11}, \quad \Gamma_{(9)} = \Delta_{12}(2 + \Delta_6).
\end{gathered}$$

It will be shown that $\varepsilon_{(9)}$ and $\Gamma_{(9)}$ have the asserted property.

Suppose B, A, $B_{a,t}$, $C_{a,t}$, and H for $t \in J_0$ are as in 7.3.

Since $\varrho/2 \in J_0 \cap J_4$, it follows

$$\varrho/2 < 2r, \quad \varrho \in J_3, \quad \phi_2(\varrho, T_\varrho) \leq \phi_2(\varrho, T) \leq (2m^{1/2}\kappa)^{1/2}.$$

One therefore obtains

$$\kappa \leq \Delta_1, \quad \phi_2(\varrho, T_\varrho) \leq 1, \quad \phi_3(\varrho) \leq \Delta_2, \quad \phi_4(\varrho) \leq \Delta_3. \tag{I}$$

Applying 5.4 with a, r, f, and A replaced by c, $\varrho/2$, $F\,(+)(-P)|\mathbf{U}(c,\varrho/2)$, and X, noting 4.8 (4), one obtains

$$\begin{aligned} &\varrho^{-1-m/q}|F\,(+)(-P)|_{q;c,\varrho/2} \\ &\quad \leq \Delta_4\big(\varrho^{-m/2}|A(F\,(+)(-P))|_{2;c,\varrho/2} + \eta^{1/q-1}\varrho^{-m-1}|f\,(+)(-P)|_{1;X}\big). \end{aligned} \tag{II}$$

Similarly, one obtains

$$\begin{aligned} &\varrho^{-1-m/2}|F\,(+)(-P)|_{2;c,\varrho/2} \\ &\quad \leq \Delta_5\big(\eta^\mu\varrho^{-m/2}|A(F\,(+)(-P))|_{2;c,\varrho/2} + \eta^{-1}\varrho^{-m-1}|f\,(+)(-P)|_{1;X}\big). \end{aligned} \tag{III}$$

Applying 7.3 (12) applied with ϱ, σ replaced by $\varrho/2$, $DP(0)$ yields, noting $\phi_4(\varrho) \leq 1$ by (I) and $1/2 \geq \tau(1/p - 1/m)$,

$$\varrho^{-m/2}|A(F\,(+)(-P))|_{2;c,\varrho/2} \leq \Delta_6\big(\phi_2(\varrho, R) + \phi_3(\varrho)^\tau\big). \tag{IV}$$

Define $d : \mathbf{R}^n \to \mathbf{R}$ by

$$d(z) = \inf\{(|\mathbf{p}(z-\xi)|^2 + |\mathbf{q}(z-\xi)|^2)^{1/2} : \xi \in \mathbf{R}^n,\ P(\mathbf{p}(\xi)) = \mathbf{q}(\xi)\}$$

whenever $z \in \mathbf{R}^n$ and note, taking $\xi = (\mathbf{p}^* + \mathbf{q}^* \circ P)(\mathbf{p}(z))$,

$$d(z) \leq |P(\mathbf{p}(z)) - \mathbf{q}(z)| \quad \text{for } z \in \mathbf{R}^n.$$

Hence, defining $d_{4.8(8)}$ and $g_{4.8(8)}$ to be the functions defined in 4.8 (8) under the names "d" and "g" with

$$\varrho,\ P \text{ replaced by } \varrho/2,\ \mathbf{C}(T, \mathbf{p}^*(c), \varrho/2) \cap \{z : P(\mathbf{p}(z)) = \mathbf{q}(z)\},$$

one infers

$$\begin{gathered} d|\mathbf{C}(T, \mathbf{p}^*(c), \varrho/2, 3r) \leq d_{4.8(8)}, \\ g_{4.8(8)}(x) \leq \mathscr{G}(f(x), Q[\![P(x)]\!]) = \mathscr{G}((f\,(+)(-P))(x), Q[\![0]\!]) \end{gathered}$$

for $x \in X_1 \cap \mathbf{B}(c, \varrho/2)$. Therefore 4.8 (8) with ϱ, P replaced as in the definition of $d_{4.8(8)}$ and $g_{4.8(8)}$ yields, noting

$$\mathscr{L}^m(\mathbf{B}(c,\varrho/2) \sim X_1) \leq \mathscr{L}^m(C_{a,\varrho/2}) \leq \varepsilon_{4.8(8)}(m, \delta_2, \delta_4)\boldsymbol{\alpha}(m)(\varrho/2)^m$$

by 7.3 (11c) with ϱ replaced by $\varrho/2$ and (I),

$$\begin{aligned} &(\|V\| \llcorner H \cap \mathbf{C}(T, \mathbf{p}^*(c), \varrho/2, 3r))_{(s)}(d) \\ &\quad \leq \Delta_7\big(|F\,(+)(-P)|_{s;c,\varrho/2} + \mathscr{L}^m(\mathbf{B}(c,\varrho/2) \sim X_1)^{1/s+1/m}\big) \end{aligned} \tag{V}$$

whenever $1 \le s \le \infty$. Using 4.8 (7) with ϱ replaced by $\varrho/2$ in conjunction with 7.3 (11b) with ϱ replaced by $\varrho/2$, one estimates

$$\mathscr{L}^m(\mathbf{B}(c,\varrho/2) \sim X_1) \le \mathscr{L}^m(C_{a,\varrho/2}) \le \Gamma_{4.8(7)}(Q,m)\|V\|(B_{a,\varrho/2}),$$

hence by 7.3 (8) with ϱ and R replaced by $\varrho/2$ and T_ϱ, noting (I) and $|(T_\varrho)_\natural - T_\natural| \le n^{1/2}\|(T_\varrho)_\natural - T_\natural\| \le n^{1/2}\|\sigma_\varrho\| \le \delta/2$ by 3.1,

$$\varrho^{-m}\mathscr{L}^m(\mathbf{B}(c,\varrho/2) \sim X_1) \le \Delta_8\big(\phi_2(\varrho,T_\varrho)^2 + \phi_4(\varrho)\big). \qquad \text{(VI)}$$

In order to apply 3.9, first define $K = \mathbf{C}(T,\mathbf{p}^*(c),\varrho,\varrho)$ and $H_{3.9}$ to be the set defined in 3.9 under the name "H", i.e. the set of all $z \in \operatorname{spt}\|V\|$ such that

$$\|V\|\,\mathbf{B}(z,t) \ge (40)^{-m}(\boldsymbol{\gamma}(m)m)^{-m}t^m \quad \text{whenever } 0 < t < \infty,\ \mathbf{B}(z,t) \subset K.$$

One infers that

$$\mathbf{C}(T,a,\varrho,\varrho) \cap \operatorname{spt}\|V\| \subset H_{3.9} \quad \text{if } m = 1,$$
$$H_{3.9} \cap \mathbf{C}(T,a,\Delta_9\varrho,\Delta_9\varrho) \subset H;$$

in fact the first inclusion follows by [Men09a, 2.5] and (I) whereas concerning the second inclusion $\eta < 2^{-m}$ implies by 7.3 (11b) with ϱ replaced by $\varrho/2$ the existence of $\xi \in A \cap \mathbf{C}(T,a,\varrho/4,\varrho/4)$ hence, verifying $1/4 < \Delta_9 < 1/2$ and $2^{3/2}\Delta_9/(1-\Delta_9) \le \frac{19}{20}$, one obtains for $z \in \mathbf{C}(T,a,\Delta_9\varrho,\Delta_9\varrho)$, $(1-\Delta_9)\varrho < t < 2r$

$$|\xi - z| \le 2^{3/2}\Delta_9\varrho \le 2^{3/2}\Delta_9 t/(1-\Delta_9) \le \tfrac{19}{20}t, \quad \mathbf{B}(z,t) \supset \mathbf{B}(\xi,t/(20)),$$
$$\|V\|\,\mathbf{B}(z,t) \ge \|V\|\,\mathbf{B}(\xi,t/(20)) \ge (40)^{-m}(\boldsymbol{\gamma}(m)m)^{-m}t^m = \delta_5\boldsymbol{\alpha}(m)t^m$$

by [Men09a, 2.5] since $\delta \le (2\boldsymbol{\gamma}(m))^{-1}$ and, noting (I), the inclusion follows from 7.3 (9) as $\mathbf{B}(z,(1-\Delta_9)\varrho) \subset K$. Choose $\phi \in \mathscr{D}^0(U)$ such that

$$0 \le \phi(x) \le 1 \quad \text{and} \quad |D\phi(x)| \le 2\cdot(\Delta_9 - 1/4)^{-1}\varrho^{-1} \qquad \text{for } x \in U,$$
$$\phi(x) = 1 \quad \text{for } x \in \mathbf{C}(T,a,\varrho/4,\varrho/4),$$
$$\operatorname{spt}\phi \subset \mathbf{C}(T,a,\Delta_9\varrho,\Delta_9\varrho) \subset K \cap \operatorname{Int}\mathbf{C}(T,a,\varrho/2,\varrho/2).$$

One now applies 3.13 if $m = 1$ and 3.9 if $m > 1$ both with a and T replaced by $(\mathbf{p}^* + \mathbf{q}^* \circ P)(0)$ and $\operatorname{im} D(\mathbf{p}^* + \mathbf{q}^* \circ P)(0)$ to obtain with $\alpha_m = 0$ if $m = 1$ and $\alpha_m = (\varrho^{1-m/p}\alpha)^{\frac{mp}{m-p}}$ if $m > 1$

$$\varrho^{-m}\beta^2 \le \Delta_{10}\big(\alpha_m + (\varrho^{1-m/p}\alpha\varrho^{-1-m/q}\gamma)^{1/(1/p+1/q)}\big) + (16+4m)\varrho^{-m}\xi^2;$$

here α, β, γ, and ξ are as in 3.9 and 3.13 respectively. Noting $(\alpha_m)^{1/2} \le \phi_3(\varrho)^\tau$, since $\phi_3(\varrho) \le 1$ by (I), and using the inequality relating arithmetic and geometric means as in 3.10, one infers

$$\begin{aligned}\phi_2(\varrho/4,R) \le \Delta_{11}\big(&\lambda\varrho^{-1-m/q}(\|V\| \llcorner H \cap \mathbf{C}(T,\mathbf{p}^*(c),\varrho/2,3r))_{(q)}(d) \\ &+ \lambda^{-\tau}\phi_3(\varrho)^\tau + \varrho^{-1-m/2}(\|V\| \llcorner H \cap \mathbf{C}(T,\mathbf{p}^*(c),\varrho/2,3r))_{(2)}(d)\big).\end{aligned} \qquad \text{(VII)}$$

Finally, the estimates (II)–(VII) are combined as follows: Firstly,

$$\begin{aligned}\phi_2(\varrho/4,R) &\le \Delta_{11}\lambda^{-\tau}\phi_3(\varrho)^\tau \\ &+ \Delta_7\Delta_{11}\lambda\varrho^{-1-m/q}\big(|F(+)(-P)|_{q;c,\varrho/2} + \mathscr{L}^m(\mathbf{B}(c,\varrho/2) \sim X_1)^{1/q+1/m}\big) \\ &+ \Delta_7\Delta_{11}\varrho^{-1-m/2}\big(|F(+)(-P)|_{2;c,\varrho/2} + \mathscr{L}^m(\mathbf{B}(c,\varrho/2) \sim X_1)^{1/2+1/m}\big)\end{aligned}$$

by (VII) and (V). Then, by (II), (III), and (VI)

$$\begin{aligned}\phi_2(\varrho/4,R) \le{}& \Delta_{11}\lambda^{-\tau}\phi_3(\varrho)^\tau \\ &+ \Delta_7\Delta_{11}(\Delta_4+\Delta_5)(\lambda+\eta^\mu)\varrho^{-m/2}|A(F\,(+)(-P))|_{2;c,\varrho/2} \\ &+ \Delta_7\Delta_{11}(\Delta_4+\Delta_5)(\eta^{1/q-1}+\eta^{-1})\varrho^{-1-m}|f\,(+)(-P)|_{1;X} \\ &+ 2\Delta_7(\Delta_8)^{1/q+1/m}\Delta_{11}\lambda\big(\phi_2(\varrho,T_\varrho)^{2/q+2/m}+\phi_4(\varrho)^{1/q+1/m}\big) \\ &+ 2\Delta_7(\Delta_8)^{1/2+1/m}\Delta_{11}\big(\phi_2(\varrho,T_\varrho)^{1+2/m}+\phi_4(\varrho)^{1/2+1/m}\big).\end{aligned}$$

Finally, using $\phi_2(\varrho,T_\varrho)\le 1$ and $\phi_4(\varrho)\le 1$ by (I), $q\ge 2$, and $\tau\le\frac{mp}{2(m-p)}\le\big(\frac{1}{q}+\frac{1}{m}\big)\frac{mp}{m-p}$ if $m>1$ this simplifies to

$$\begin{aligned}\phi_2(\varrho/4,R) \le \Delta_{12}\Big(&\lambda^{-\tau}\phi_3(\varrho)^\tau + \big(\lambda+\phi_2(\varrho,T_\varrho)^{2/m}\big)\phi_2(\varrho,T_\varrho) \\ &+ (\lambda+\eta^\mu)\varrho^{-m/2}|A(F\,(+)(-P))|_{2;c,\varrho/2} + \eta^{-1}\varrho^{-m-1}|f\,(+)(-P)|_{1;X}\Big)\end{aligned}$$

and the conclusion is a consequence of (IV). □

Proof of (10). With $\delta_4=1$ and $\delta_5=(40)^{-m}(\boldsymbol{\gamma}(m)m)^{-m}/\boldsymbol{\alpha}(m)$ define

$$\begin{gathered}\Delta_1=\inf\{\varepsilon_{7.3(8)}(m,\delta_4,\delta),\varepsilon_{7.3(11)}(n,\delta_4,\delta),\varepsilon_{(9)}(m,n,Q,\delta_2,\varepsilon,\delta,p)\}, \\ \Delta_2=6(2m\Gamma_{(5)}(n))^{m+1}\boldsymbol{\alpha}(m)^{-1/2},\quad \Delta_3=\Delta_2\big(\Gamma_{(5)}(n)^2+1\big), \\ \Delta_4=\Delta_3\Gamma_{7.3(12)}(n,Q,\delta)^{1/2}, \\ \Delta_5=\inf\big\{2^{-2m-5}\boldsymbol{\alpha}(m)n^{-1/2}\delta,(\Delta_4)^{-1}n^{-1/2}\delta/4,1\big\}, \\ \Delta_6=\inf\big\{1,2^{-m}\varepsilon_{(9)}(m,n,Q,\delta_2,\varepsilon,\delta,p)\big\}, \\ \Delta_7=\inf\big\{(\Delta_4)^{-2}n^{-1}\delta^2 2^{-4},2^{-m}\boldsymbol{\beta}(n)^{-1}\boldsymbol{\alpha}(m)(1/8),2^{-m}\big\}, \\ \varepsilon_{(10)}=\inf\big\{\Delta_1,2^{-1}m^{-1/2}(\Delta_5)^2,\Delta_6,\delta(\Delta_7)^{1/p-1/m}\big\}.\end{gathered}$$

Moreover, define

$$\begin{gathered}\Delta_8=\Gamma_{(7)}(m,n,Q,\delta_4,\delta,p),\quad \Delta_9=\Gamma_{5.6}(n)\boldsymbol{\alpha}(m)^{1/2}, \\ \Delta_{10}=\Delta_9\Gamma_{7.3(12)}(n,Q,\delta)^{1/2},\quad \Delta_{11}=2^{m+1}\Gamma_{5.5}(2,n), \\ \Delta_{12}=\Delta_{11}\sup\{\boldsymbol{\alpha}(m),\Delta_8+2^m\Delta_{10}\delta^{-\tau}\}, \\ \Delta_{13}=(Q+1)^{1/2}\boldsymbol{\alpha}(m)^{1/2}\Delta_{12}n^{1/2}+2^m,\quad \Delta_{14}=Q^{1/2}\sup\{\boldsymbol{\alpha}(m),\Delta_8\}, \\ \Gamma_{(10)}=\Gamma_{(9)}(m,n,Q,\delta,p,\tau)(2^{m+1}+2\Delta_{13}+\Delta_{14}).\end{gathered}$$

It will be shown that $\varepsilon_{(10)}$ and $\Gamma_{(10)}$ have the asserted property.

Since $\varrho\in J_4$ and $2\varrho\in J_0$, it follows

$$\varrho<2r,\quad 2\varrho\in J_3,\quad \phi_2(2\varrho,T)\le(2m^{1/2}\kappa)^{1/2}.$$

One therefore obtains

$$\begin{gathered}\kappa\le\Delta_1,\quad \phi_2(2\varrho,T)\le\Delta_5,\quad \phi_3(2\varrho)\le\Delta_6,\quad \phi_4(2\varrho)\le\Delta_7, \\ \varrho\in J_1,\quad \|\sigma_\varrho\|\le n^{-1/2}\delta/2;\end{gathered}\tag{I}$$

in fact the first four inequalities are directly implied by the definition of $\varepsilon_{(10)}$ and the last two statements follow from 7.3 (13) applied with ϱ, s, t, λ replaced by 2ϱ, ϱ, ϱ, $1/2$ since $\phi_2(2\varrho, T_{2\varrho}) \leq 2^{-2m-5}\boldsymbol{\alpha}(m)n^{-1/2}\delta$ by the second inequality.

Define $P : \mathbf{R}^m \to \mathbf{R}^{n-m}$ by $P(x) = u_\varrho(c) + \langle x - c, Du_\varrho(c)\rangle$ for $x \in \mathbf{R}^m$. One verifies

$$\operatorname{Lip} P = \|DP(0)\| \leq n^{-1/2}\delta/2; \tag{II}$$

in fact using [Fed69, 5.2.5], 6.2, 7.3 (12) with σ replaced by 0, and (I)

$$\begin{aligned}\|DP(0)\| = \|Du_\varrho(c)\| &\leq \Delta_2 \varrho^{-m/2} |Du_\varrho|_{2;c,\varrho} \\ &\leq \Delta_2 \varrho^{-m/2}\big(|D(u_\varrho - g)|_{2;c,\varrho} + |Dg|_{2;c,\varrho}\big) \\ &\leq \Delta_3 \varrho^{-m/2} |Dg|_{2;c,\varrho} \leq \Delta_4\big(\phi_2(2\varrho, T) + \phi_4(2\varrho)^{1/2}\big) \leq n^{-1/2}\delta/2.\end{aligned}$$

Taylor's expansion yields

$$\varrho^{-m-1}|u_\varrho - P|_{1;c,\varrho/2} \leq \boldsymbol{\alpha}(m)\varrho|D^2 u_\varrho|_{\infty;c,\varrho/2}. \tag{III}$$

Noting (I), one obtains from (7) that

$$\varrho^{-m-1}|u_\varrho - g|_{1;c,\varrho} \leq \Delta_8\big(\phi_2(2\varrho, T_{2\varrho})^2 + \phi_3(2\varrho)^\tau\big). \tag{IV}$$

By 5.6 with a, r, u replaced by c, $\varrho/2$, $(g - \sigma_\varrho)|\mathbf{U}(c, \varrho/2)$ there exists an affine function $R : \mathbf{R}^m \to \mathbf{R}^{n-m}$ with $DR(0) = \sigma_\varrho$ such that

$$\varrho^{-m-1}|g - R|_{1;c,\varrho/2} \leq \Delta_9 \varrho^{-m/2}|D(g - R)|_{2;c,\varrho/2},$$

hence by 7.3 (12) with ϱ, σ replaced by $\varrho/2$, σ_ϱ, noting (I),

$$\varrho^{-m-1}|g - R|_{1;c,\varrho/2} \leq \Delta_{10}\big(\phi_2(\varrho, T_\varrho) + \phi_4(\varrho)^{1/2}\big). \tag{V}$$

Since by 5.5 with k, a, r, u replaced by 2, c, $\varrho/2$, $P - R$

$$\begin{aligned}|DP(0) - \sigma_\varrho| = |D(P-R)(0)| &\leq \Delta_{11}\varrho^{-1-m}|P - R|_{1;c,\varrho/2} \\ &\leq \Delta_{11}\varrho^{-1-m}\big(|P - u_\varrho|_{1;c,\varrho/2} + |u_\varrho - g|_{1;c,\varrho/2} + |g - R|_{1;c,\varrho/2}\big),\end{aligned}$$

one obtains from (III)–(V), noting $\sup\{\phi_2(2\varrho, T_{2\varrho}), \phi_3(2\varrho), \phi_4(\varrho)\} \leq 1$ by (I) and $1/2 \geq \tau(1/p - 1/m)$,

$$|DP(0) - \sigma_\varrho| \leq \Delta_{12}\big(\varrho\phi_1(\varrho) + \phi_2(2\varrho, T_{2\varrho}) + \phi_3(2\varrho)^\tau\big),$$

hence using 7.3 (11a) and 3.1

$$\phi_2(\varrho, S) \leq \Delta_{13}\big(\varrho\phi_1(\varrho) + \phi_2(2\varrho, T_{2\varrho}) + \phi_3(2\varrho)^\tau\big) \tag{VI}$$

where $S = \operatorname{im} D(\mathbf{p}^* + \mathbf{q}^* \circ P)(0)$.

Define $X = \mathbf{U}(c, \varrho/2) \cap X_1 \cap \{x : \boldsymbol{\Theta}^0(\|f(x)\|, g(x)) = Q\}$ and note

$$|f\,(+)(-P)|_{1;X} \leq Q^{1/2}(|g - u_\varrho|_{1;c,\varrho} + |u_\varrho - P|_{1;c,\varrho/2}).$$

Combining this with (III) and (IV) yields

$$\varrho^{-1-m}|f\,(+)(-P)|_{1;X} \leq \Delta_{14}\big(\varrho\phi_1(\varrho) + \phi_2(2\varrho, T_{2\varrho})^2 + \phi_3(2\varrho)^\tau\big).$$

Therefore noting (I), (II) and 4.8 (1) and applying (9) with R replaced by S, one obtains in conjunction with (VI) the conclusion. ☐

Proof of (11). As the assertion does not involve κ it may be restricted to a specific value. One defines

$$
\begin{aligned}
\Delta_1 &= \sup\{\Gamma_{(8)}(m,n,Q,\delta_4,\delta,p),\Gamma_{(10)}(m,n,Q,\delta,p,\tau),1\},\\
\eta &= \inf\big\{(48\Delta_1)^{-n},2^{-n}\big\},\\
\kappa &= \inf\big\{\varepsilon_{7.3(8)}(m,\delta_4,\delta),\varepsilon_{7.3(10)}(n,Q,\delta_4,p,\alpha,\delta_6),\varepsilon_{7.3(11)}(n,\delta_4,\delta),\\
&\qquad \varepsilon_{(8)}(n,\delta_4,\delta),2^{-m-2}\boldsymbol{\beta}(n)^{-1}\boldsymbol{\alpha}(m)\eta\Gamma_{4.8(7)}(Q,m)^{-1},\\
&\qquad \varepsilon_{(10)}(m,n,Q,\delta_2,\varepsilon,\delta,p)\big\},\\
\Delta_2 &= \inf\big\{1,2^{-m}\boldsymbol{\beta}(n)^{-1}\boldsymbol{\alpha}(m)\inf\{\eta(4\Gamma_{4.8(7)}(Q,m))^{-1},1/8\}\big\},\\
\Delta_3 &= \inf\big\{2^{-2m}\sup\{(Q+1)\boldsymbol{\alpha}(m),1\}^{-1}\kappa,1,\varepsilon_{(10)}(m,n,Q,\delta_2,\varepsilon,\delta,p),\\
&\qquad (\Delta_2)^{1/p-1/m}\delta,2^{-9m}\sup\{M\boldsymbol{\alpha}(m),1\}^{-1}\kappa\big\},\\
\Delta_4 &= \inf\big\{(\Delta_3/8)^{\tau},\varepsilon_{7.3(10)}(n,Q,\delta_4,p,\alpha,\delta_6)^{\tau},\\
&\qquad \big(\alpha p\boldsymbol{\alpha}(m)^{1/p}((Q-1+\delta_6)^{1/p}-(Q-1+\delta_6/2)^{1/p})\big)^{\tau}\big\},\\
\Delta_5 &= \inf\big\{2^{-2m}(Q+1)^{-1/2}\boldsymbol{\alpha}(m)^{-1/2}\kappa,2^{-m-2}\boldsymbol{\alpha}(m)^{1/2}\big\},\\
\Delta_6 &= n^{-1/2}\inf\big\{\delta/4,2^{-m-1}\sup\{(Q+1)\boldsymbol{\alpha}(m),1\}^{-1}\Delta_5\big\},\\
\Delta_7 &= \inf\big\{n^{-1/2}\inf\{\delta/2,1/4\},\Delta_6/2\big\},\\
\Delta_8 &= 1-4^{\alpha\tau-1} \quad \text{if } \alpha\tau<1,\\
\Delta_8 &= \log 4 \quad \text{if } \alpha\tau=1,\\
\Delta_9 &= \inf\big\{2^{-2m-4}\boldsymbol{\alpha}(m)^{1/2},2^{-2m-4}\boldsymbol{\alpha}(m)^{1/2}(1-2^{-\alpha\tau})\Delta_6,2^{-m-1}\Delta_5,1,\\
&\qquad (3\Delta_1)^{-1}\Delta_8,\tfrac{1}{576}(\Delta_1)^{-2}\eta\Delta_8,(48\Delta_1)^{-n}\eta^n\big\},\\
\Delta_{10} &= \Gamma_{(6)}(n,Q,\delta),\\
\Delta_{11} &= \inf\big\{\delta^{\tau}2^{-7}(\Delta_{10})^{-1},\tfrac{1}{24}\Delta_8(\Delta_1)^{-1}\big\},\\
\lambda &= (48\Delta_1)^{-1},\\
\Delta_{12} &= \big(24\Delta_1(\eta^{-1}+\lambda^{-\tau})\big)^{-1},\\
\gamma_2 &= (e/4)\Delta_9,\\
\gamma_1 &= \eta(24\Delta_1)^{-1}\gamma_2,\\
\gamma_3 &= \inf\{\Delta_4,\Delta_{11}\gamma_1,\Delta_{12}\gamma_2\},\\
\varepsilon_{(11)} &= \inf\big\{2^{-8m}\sup\{M\boldsymbol{\alpha}(m),1\}^{-1}\kappa,2^{-6m-4}\boldsymbol{\alpha}(m)^{1/2},\\
&\qquad 2^{-5m-3}\boldsymbol{\alpha}(m)^{1/2}\Delta_7,2^{-5m}\Delta_5,2^{-4m-7}(\Delta_{10})^{-1}\gamma_1,2^{-5m-6}\gamma_2,\eta/2\big\};
\end{aligned}
$$

here e denotes Euler's number. It will be shown that γ_i and $\varepsilon_{(11)}$ have the asserted property.

Suppose $C_{a,t}$ for $t \in J_0$ is as in 7.3.

First, note that

$$\phi_3(\varrho)^{\tau} \leq \gamma\gamma_3(\varrho/r)^{\alpha\tau} \quad \text{for } 0 < t \leq 8r \tag{I}$$

implies, noting $\gamma_3 \leq \Delta_4$,

$$\phi_3(\varrho) \leq \Delta_3 \quad \text{and} \quad \phi_4(\varrho) \leq \Delta_2 \quad \text{whenever } 0 < \varrho \leq 8r. \tag{I'}$$

Next, some auxiliary assertions will be shown:

$$\mathbf{R} \cap \{\varrho : 0 < \varrho \le r/2\} \subset J_0, \tag{II}$$
$$\mathbf{R} \cap \{\varrho : \tfrac{r}{64} \le \varrho \le r\} \subset J_1, \tag{III}$$
$$\mathbf{R} \cap \{\varrho : \tfrac{r}{64} \le \varrho \le 8r\} \subset J_2 \cap J_3, \tag{IV}$$
$$\mathbf{R} \cap \{\varrho : \tfrac{r}{64} \le \varrho \le 4r\} \subset J_4, \tag{V}$$
$$\mathbf{R} \cap \{\varrho : \tfrac{r}{64} \le \varrho \le r/2\} \subset J_5, \tag{VI}$$
$$\|V\|(\mathbf{C}(T,a,\varrho,\varrho)) \ge (Q-1+\delta_4/2)\boldsymbol{\alpha}(m)\varrho^m \quad \text{whenever } 0 < \varrho \le r/2, \tag{VII}$$
$$\|\sigma_\varrho\| \le \Delta_7 \quad \text{whenever } \tfrac{r}{64} \le \varrho \le r, \tag{VIII}$$

Proof of (II)**.** This follows from $a \in \mathbf{C}(T,0,r/2,2r)$.

Proof of (IV)**.** For $\frac{r}{64} \le \varrho \le 8r$ one computes, using Hölder's inequality and (I'),

$$\begin{aligned}
&\|\delta V\|(U \cap \mathbf{C}(T,a,\varrho,\varrho)) \le \|V\|(U)^{1-1/p}\psi(U \cap \mathbf{C}(T,a,8r,8r))^{1/p} \\
&\qquad \le \sup\{M\boldsymbol{\alpha}(m),1\} r^{m-m/p}(8r)^{m/p-1}\phi_3(8r) \\
&\qquad \le \Delta_3 \sup\{M\boldsymbol{\alpha}(m),1\} 2^{9m}(\tfrac{r}{64})^{m-1} \le \kappa\varrho^{m-1}, \\
&\textstyle\int_{(U\cap\mathbf{C}(T,a,\varrho,\varrho))\times\mathbf{G}(n,m)} |S_\natural - T_\natural| \,\mathrm{d}V(z,S) \le \|V\|(U)^{1/2}(8r)^{m/2}\phi_2(8r,T) \\
&\qquad \le \sup\{M\boldsymbol{\alpha}(m),1\} 2^{8m}\varepsilon_{(11)}(\tfrac{r}{64})^m \le \kappa\varrho^m.
\end{aligned}$$

Proof of (V)**.** This follows from (IV).

Proof of (VI)**.** Let $\frac{r}{64} \le \varrho \le r/2$. One computes for $0 < t < \varrho$, (I) and $\gamma_3 \le \Delta_4$,

$$\phi_3(t) \le (\Delta_4)^{1/\tau}(t/r)^\alpha \le \varepsilon_{7.3(10)}(n,Q,\delta_4,p,\alpha,\delta_6)(t/\varrho)^\alpha.$$

Therefore, noting (II) and (IV), (VI) is implied by 7.3 (10).

Proof of (VII)**.** Applying 7.1 with r, M, ϱ replaced by ϱ, $(\Delta_4)^{1/\tau}$, ϱ in conjunction with Hölder's inequality, noting (I) and $\gamma_3 \le \Delta_4$, yields

$$\begin{aligned}
\big(\varrho^{-m}\|V\|(\mathbf{C}(T,a,\varrho,\varrho))\big)^{1/p} &\ge ((Q-1+\delta_6)\boldsymbol{\alpha}(m))^{1/p} - (\Delta_4)^{1/\tau}\alpha^{-1}p^{-1} \\
&\ge ((Q-1+\delta_6/2)\boldsymbol{\alpha}(m))^{1/p}.
\end{aligned}$$

Proof of (III) **and** (VIII)**.** Let $\frac{r}{64} \le \varrho \le r$. Using Hölder's inequality and $\varrho/2 \le \inf\{\varrho, r/2\} \in J_5$ by (VI), one estimates

$$\begin{aligned}
&\|(T_\varrho)_\natural - T_\natural\| \le \|V\|(U \cap \mathbf{C}(T,a,\varrho,\varrho))^{-1/2}\varrho^{m/2}\big(\phi_2(\varrho,T_\varrho) + \phi_2(\varrho,T)\big) \\
&\qquad \le \boldsymbol{\alpha}(m)^{-1/2}2^{m/2+3/2}\phi_2(\varrho,T) \le \boldsymbol{\alpha}(m)^{-1/2}2^{5m+2}\phi_2(8r,T) \\
&\qquad \le \boldsymbol{\alpha}(m)^{-1/2}2^{5m+2}\varepsilon_{(11)} \le 1/2,
\end{aligned}$$

hence $T_\varrho \cap \ker \mathbf{p} = \{0\}$ and $\varrho \in J_1$, i.e. (III). Now, 3.1 applied with S, S_1, S_2 replaced by T, T, T_ϱ yields

$$\|\sigma_\varrho\|^2 \le (1 + \|\sigma_\varrho\|^2)\|(T_\varrho)_\natural - T_\natural\|^2,$$
$$\|\sigma_\varrho\|^2 \le \|(T_\varrho)_\natural - T_\natural\|^2/(1 - \|(T_\varrho)_\natural - T_\natural\|^2) \le 2\|(T_\varrho)_\natural - T_\natural\|^2,$$
$$\|\sigma_\varrho\| \le 2\|(T_\varrho)_\natural - T_\natural\| \le \boldsymbol{\alpha}(m)^{-1/2}2^{5m+3}\varepsilon_{(11)} \le \Delta_7.$$

Having shown the auxiliary assertions (II)–(VIII), one chooses $j \in \mathscr{P}$ such that $\frac{r}{64} < 4^j t \le \frac{r}{16}$ and defines $t_i = 4^{j+1-i}t$ whenever $i \in \mathscr{P}$, $i \le j+1$ in order to show inductively the following assertions whenever $i \in \mathscr{P}$, $i \le j+1$:

$$\mathbf{R} \cap \{\varrho : t_i \le \varrho \le r\} \subset J_4 \tag{IX}$$
$$\mathbf{R} \cap \{\varrho : t_i \le \varrho \le r/2\} \subset J_5, \tag{X}$$
$$\mathbf{R} \cap \{\varrho : t_i \le \varrho \le r\} \subset J_1, \tag{XI}$$
$$\|\sigma_\varrho\| \le \Delta_6 \quad \text{for } t_i \le \varrho \le r, \tag{XII}$$
$$\phi_2(\varrho, T) \le \Delta_5 \quad \text{for } t_i \le \varrho \le r, \tag{XIII}$$
$$\begin{aligned} &\phi_1(\varrho) \le \gamma\gamma_1\varrho^{-1+\alpha\tau}r^{-\alpha\tau} \quad \text{whenever } t_i \le \varrho \le r/4,\ \alpha\tau < 1, \\ &\phi_1(\varrho) \le \gamma\gamma_1 r^{-1}(1 + \log(r/\varrho)) \quad \text{whenever } t_i \le \varrho \le r/4,\ \alpha\tau = 1, \end{aligned} \tag{XIV}$$
$$\begin{aligned} &\phi_2(\varrho, T_\varrho) \le \gamma\gamma_2(\varrho/r)^{\alpha\tau} \quad \text{whenever } t_i \le \varrho \le r,\ \alpha\tau < 1, \\ &\phi_2(\varrho, T_\varrho) \le \gamma\gamma_2(\varrho/r)(1 + \log(r/\varrho)) \quad \text{whenever } t_i \le \varrho \le r,\ \alpha\tau = 1. \end{aligned} \tag{XV}$$

One verifies that $(\text{XV})_i$ implies

$$\phi_2(\varrho, T_\varrho) \le \Delta_9(\varrho/r)^{\alpha\tau/2} \quad \text{whenever } t_i \le \varrho \le r, \tag{XV'}$$
$$\phi_2(\varrho, T_\varrho) \le \Delta_9(1 + \log(r/\varrho))^{-1} \quad \text{whenever } t_i \le \varrho \le r,\ \alpha\tau = 1; \tag{XV''}$$

here and in the remaining proof references to equations involving the inductive parameter will be supplemented by the value of this parameter as index.

Proof of $(\text{IX})_1$**,** $(\text{X})_1$ **and** $(\text{XI})_1$**.** Since $t_1 = 4^j t \ge \frac{r}{64}$ the assertions follow from (V), (III) and (VI).

Proof of $(\text{XII})_1$**.** Since $t_1 \ge \frac{r}{64}$ and $\Delta_7 \le \Delta_6$, this follows from (VIII).

Proof of $(\text{XIII})_1$**.** For $t_1 \le \varrho \le r$

$$\phi_2(\varrho, T) \le 2^{5m}\phi_2(8r, T) \le 2^{5m}\varepsilon_{(11)} \le \Delta_5.$$

Proof of $(\text{XIV})_1$**.** Let $\frac{r}{64} \le \varrho \le r/4$ and note

$$\varrho \in J_4 \cap J_5 \quad \text{by (V) and (VI)}, \quad 2\varrho \in J_0 \cap J_1 \quad \text{by (II) and (III)},$$
$$\|\sigma_{2\varrho}\| \le n^{-1/2}\inf\{\delta/2, 1/4\} \quad \text{by (VIII)},$$
$$\phi_2(2\varrho, T_{2\varrho}) \le 2^{4m}\phi_2(8r, T) \le 2^{4m}\varepsilon_{(11)} \le 2^{-2m-4}\boldsymbol{\alpha}(m)^{1/2}.$$

Therefore by (6), using $\phi_4(2\varrho) \le 1$ by (I'), $1/2 \ge \tau(1/p - 1/m)$, (I) and $\gamma_3 \le \Delta_{11}\gamma_1$,

$$\begin{aligned} \varrho\phi_1(\varrho) &\le \Delta_{10}\big(\phi_2(2\varrho, T_{2\varrho}) + \phi_4(2\varrho)^{1/2}\big) \le \Delta_{10}\big(2^{4m}\phi_2(8r, T_{8r}) + \delta^{-\tau}\phi_3(2\varrho)^\tau\big) \\ &\le \gamma\Delta_{10}\big(2^{4m}\varepsilon_{(11)} + \delta^{-\tau}\Delta_{11}\gamma_1\big) \le \gamma\gamma_1\tfrac{1}{64} \le \gamma\gamma_1(\varrho/r)^{\alpha\tau}. \end{aligned}$$

Proof of $(\mathrm{XV})_1$**.** For $\frac{r}{64} \leq \varrho \leq r$ one estimates

$$\phi_2(\varrho, T_\varrho) \leq 2^{5m}\phi_2(8r, T_{8r}) \leq 2^{5m}\varepsilon_{(11)}\gamma \leq \gamma\gamma_2 \tfrac{1}{64} \leq \gamma\gamma_2(\varrho/r)^{\alpha\tau}.$$

Therefore the assertions $(\mathrm{IX})_1$–$(\mathrm{XV})_1$ are proven in the case $i = 1$. Suppose now that the assertions $(\mathrm{IX})_i$–$(\mathrm{XV})_i$ hold for some $i \in \mathscr{P}$ with $i \leq j$. Note $t_i \leq t_1 = 4^j t \leq \frac{r}{16}$. Since $t_i \in J_0 \cap J_4$ by (II) and $(\mathrm{IX})_i$ and

$$\phi_4(2t_i) \leq \Delta_2 \leq 2^{-m}\boldsymbol{\beta}(n)^{-1}\boldsymbol{\alpha}(m)(1/8)$$

by (I'), 7.3 (11a) with ϱ replaced by t_i implies

$$\|V\|(\mathbf{C}(T, a, \varrho, \varrho)) \leq (Q+1)\boldsymbol{\alpha}(m)4^m\varrho^m \quad \text{for } t_{i+1} \leq \varrho \leq t_i. \tag{XVI}$$

Proof of $(\mathrm{IX})_{i+1}$**,** $(\mathrm{X})_{i+1}$ **and** $(\mathrm{XI})_{i+1}$**.** Let $t_{i+1} \leq \varrho \leq t_i$. Note $\varrho \in J_0$ by (II). One estimates, using Hölder's inequality, (XVI) and (I'),

$$\begin{aligned}\|\delta V\|(\mathbf{C}(T, a, \varrho, \varrho)) &\leq \|V\|(\mathbf{C}(T, a, \varrho, \varrho))^{1-1/p}\psi(\mathbf{C}(T, a, t_i, t_i))^{1/p} \\ &\leq \sup\{(Q+1)\boldsymbol{\alpha}(m), 1\}4^m\varrho^{m-1}\Delta_3 \leq \kappa\varrho^{m-1},\end{aligned}$$

hence $\varrho \in J_2$. Similarly, using $(\mathrm{XIII})_i$,

$$\begin{aligned}&\textstyle\int_{\mathbf{C}(T,a,\varrho,\varrho)\times\mathbf{G}(n,m)} |S_\natural - T_\natural| \,\mathrm{d}V(z, S) \\ &\quad\leq \|V\|(\mathbf{C}(T, a, \varrho, \varrho))^{1/2}\big(\textstyle\int_{\mathbf{C}(T,a,t_i,t_i)\times\mathbf{G}(n,m)} |S_\natural - T_\natural|^2 \,\mathrm{d}V(z, S)\big)^{1/2} \\ &\quad\leq (Q+1)^{1/2}\boldsymbol{\alpha}(m)^{1/2}4^m\varrho^m\Delta_5 \leq \kappa\varrho^m\end{aligned}$$

and $\varrho \in J_3$. Together with $(\mathrm{IX})_i$ this implies

$$\mathbf{R} \cap \{s : t_{i+1} \leq s < 2r\} \subset J_2 \cap J_3, \quad \mathbf{R} \cap \{s : t_{i+1} \leq s \leq r\} \subset J_4,$$

hence $(\mathrm{IX})_{i+1}$. One computes for $0 < t < \varrho$, using (II), (I) and $\gamma_3 \leq \Delta_4$,

$$\phi_3(t) \leq (\Delta_4)^{1/\tau}(t/r)^\alpha \leq \varepsilon_{7.3(10)}(n, Q, \delta_4, p, \alpha, \delta_6)(t/\varrho)^\alpha.$$

Therefore, noting (II) and $(\mathrm{IX})_{i+1}$, 7.3 (10) implies $(\mathrm{X})_{i+1}$. To prove $\varrho \in J_1$, one estimates

$$\begin{aligned}\|(T_\varrho)_\natural - T_\natural\| &\leq \|V\|(\mathbf{C}(T, a, \varrho, \varrho))^{-1/2}\varrho^{m/2}\big(\phi_2(\varrho, T_\varrho) + \phi_2(\varrho, T)\big) \\ &\leq \|V\|(\mathbf{C}(T, a, t_{i+1}, t_{i+1}))^{-1/2}(t_i)^{m/2}\big(\phi_2(t_i, T_{t_i}) + \phi_2(t_i, T)\big) \\ &\leq \boldsymbol{\alpha}(m)^{-1/2}2^m(\Delta_9 + \Delta_5) \leq 1/2\end{aligned}$$

by $(\mathrm{X})_{i+1}$ and $(\mathrm{XV'})_i$, $(\mathrm{XIII})_i$, hence

$$T_\varrho \cap \ker \mathbf{p} = \{0\}, \quad \varrho \in J_1.$$

Proof of $(\mathrm{XII})_{i+1}$**.** Let $t_{i+1} \leq \varrho \leq t_i$ and define $\varrho_k = 4^{k-1}\varrho$ for $k \in \mathscr{P}$. Since $\varrho \leq t_i \leq r/4$, there exists $l \in \mathscr{P}$ such that $\frac{r}{16} < \varrho_l \leq r/4$. Note

$$\varrho_k \in J_1 \cap J_5 \quad \text{for } k = 1, \ldots, l$$

by $(\mathrm{XI})_{i+1}$ and $(\mathrm{X})_{i+1}$. Also, by $(\mathrm{XII})_i$,

$$\|\sigma_{\varrho_k}\| \le n^{-1/2}/4 \quad \text{whenever } k \in \mathscr{P},\ 2 \le k \le l$$

and, by $(\mathrm{XV'})_i$,

$$\phi_2(\varrho_k, T_{\varrho_k}) \le \Delta_9 \le 2^{-2m-4}\boldsymbol{\alpha}(m)^{1/2} \quad \text{whenever } k \in \mathscr{P},\ 2 \le k \le l.$$

Now, applying 7.3 (13) with ϱ, s, t, λ replaced by ϱ_k, ϱ_{k-1}, ϱ_{k-1}, $1/2$ and using $(\mathrm{XV'})_i$, one obtains

$$\|\sigma_{\varrho_{k-1}} - \sigma_{\varrho_k}\| \le 2^{2m+3}\boldsymbol{\alpha}(m)^{-1/2}\phi_2(\varrho_k, T_{\varrho_k}) \le 2^{2m+3}\boldsymbol{\alpha}(m)^{-1/2}\Delta_9(\varrho_k/r)^{\alpha\tau/2}$$

whenever $k \in \mathscr{P}$, $2 \le k \le l$. Therefore by (VIII)

$$\begin{aligned}
\|\sigma_\varrho\| &\le \|\sigma_{\varrho_l}\| + \textstyle\sum_{k=2}^{l}\|\sigma_{\varrho_{k-1}} - \sigma_{\varrho_k}\| \\
&\le \Delta_7 + 2^{2m+3}\boldsymbol{\alpha}(m)^{-1/2}\Delta_9 r^{-\alpha\tau/2}\textstyle\sum_{k=2}^{l}(4^{k-1}\varrho)^{\alpha\tau/2} \\
&\le \Delta_7 + 2^{2m+3}\boldsymbol{\alpha}(m)^{-1/2}\Delta_9(4^{l-1}\varrho/r)^{\alpha\tau/2}\textstyle\sum_{k=0}^{\infty}2^{-k\alpha\tau} \\
&\le \Delta_7 + 2^{2m+3}\boldsymbol{\alpha}(m)^{-1/2}(1-2^{-\alpha\tau})^{-1}\Delta_9 \le \Delta_6.
\end{aligned}$$

Proof of $(\mathrm{XIII})_{i+1}$**.** For $t_{i+1} \le \varrho \le t_i$, $\varrho \in J_0$ by (II) and

$$\phi_2(\varrho, T) \le \phi_2(\varrho, T_\varrho) + \varrho^{-m/2}\|V\|(\mathbf{C}(T, a, \varrho, \varrho))^{1/2}|T_\natural - (T_\varrho)_\natural|$$

by Hölder's inequality. By $(\mathrm{XV'})_i$ and (XVI)

$$\phi_2(\varrho, T) \le 2^m\Delta_9 + 2^m \sup\{(Q+1)\boldsymbol{\alpha}(m), 1\}|T_\natural - (T_\varrho)_\natural|.$$

Also by 3.1, noting $\varrho \in J_1$ by $(\mathrm{XI})_{i+1}$ and $(\mathrm{XII})_{i+1}$,

$$|T_\natural - (T_\varrho)_\natural| \le n^{1/2}\|T_\natural - (T_\varrho)_\natural\| \le n^{1/2}\|\sigma_\varrho\| \le n^{1/2}\Delta_6,$$

hence

$$\phi_2(\varrho, T) \le 2^m\Delta_9 + 2^m \sup\{(Q+1)\boldsymbol{\alpha}(m), 1\}n^{1/2}\Delta_6 \le \Delta_5.$$

Proof of $(\mathrm{XIV})_{i+1}$**.** Let $t_{i+1} \le \varrho \le t_i$. It will be shown that the hypotheses of (8) are satisfied with ϱ replaced by 4ϱ; in fact $\varrho \le t_1 \le \frac{r}{16}$,

$$8\varrho \in J_0 \cap J_1 \quad \text{by (II) and } (\mathrm{XI})_i, \qquad \|\sigma_{8\varrho}\| \le n^{-1/2}\delta/4 \quad \text{by } (\mathrm{XII})_i,$$

and for $s \in \{\varrho, 4\varrho\}$

$$\begin{gathered}
s \in J_4 \cap J_5 \quad \text{by } (\mathrm{IX})_{i+1} \text{ and } (\mathrm{X})_{i+1}, \\
\phi_4(2s) \le 2^{-m}\boldsymbol{\beta}(n)^{-1}\boldsymbol{\alpha}(m)(1/8) \quad \text{by (I')}.
\end{gathered}$$

Therefore, in case $\alpha\tau < 1$, (8) implies, using $(\mathrm{XIV})_i$, $(\mathrm{XV'})_i$, $(\mathrm{XV})_i$, $\phi_3(8\varrho) \le 1$ by (I'), (I) and $\gamma_2 = (24\Delta_1)\eta^{-1}\gamma_1$, $\gamma_3 \le \Delta_{11}\gamma_1$,

$$\begin{aligned}
\phi_1(\varrho) &\le \phi_1(4\varrho) + \Delta_1\big(\phi_1(4\varrho)\phi_2(4\varrho, T_{4\varrho}) + \varrho^{-1}(\phi_2(8\varrho, T_{8\varrho})^2 + \phi_3(8\varrho))\big) \\
&\le \gamma\varrho^{-1+\alpha\tau}r^{-\alpha\tau}\big(4^{\alpha\tau-1}\gamma_1 + \Delta_1\Delta_9\gamma_1 + 8\Delta_1\Delta_9\gamma_2 + 8\Delta_1\gamma_3\big) \\
&\le \gamma\gamma_1\varrho^{-1+\alpha\tau}r^{-\alpha\tau}\big(\Delta_8 + \Delta_1\Delta_9 + 192(\Delta_1)^2\eta^{-1}\Delta_9 + 8\Delta_1\Delta_{11}\big) \\
&\le \gamma\gamma_1\varrho^{-1+\alpha\tau}r^{-\alpha\tau}.
\end{aligned}$$

Similarly, in case $\alpha\tau = 1$, (8) implies, using $(\mathrm{XIV})_i$, $(\mathrm{XV"})_i$, $(\mathrm{XV})_i$, (I) and $\gamma_2 = (24\Delta_1)\eta^{-1}\gamma_1$, $\gamma_3 \leq \Delta_{11}\gamma_1$,

$$\begin{aligned}\phi_1(\varrho) &\leq \phi_1(4\varrho) + \Delta_1\big(\phi_1(4\varrho)\phi_2(4\varrho, T_{4\varrho}) + \varrho^{-1}(\phi_2(8\varrho, T_{8\varrho})^2 + \phi_3(8\varrho))\big)\\ &\leq \gamma r^{-1}\big((1 + \log(r/\varrho) - \log 4)\gamma_1 + \Delta_1\Delta_9\gamma_1 + 8\Delta_1\Delta_9\gamma_2 + 8\Delta_1\gamma_3\big)\\ &\leq \gamma\gamma_1 r^{-1}\big((1 + \log(r/\varrho) - \Delta_8) + \Delta_1\Delta_9 + 192(\Delta_1)^2\eta^{-1}\Delta_9 + 8\Delta_1\Delta_{11}\big)\\ &\leq \gamma\gamma_1 r^{-1}(1 + \log(r/\varrho)).\end{aligned}$$

Proof of $(\mathrm{XV})_{i+1}$**.** Let $t_{i+1} \leq \varrho \leq t_i$. First, it will be shown that the hypotheses of 7.3 (11b) and 7.3 (11c) are satisfied with ϱ, λ replaced by 2ϱ, $\eta/2$; in fact

$$\begin{gathered}2\varrho \in J_4 \cap J_5 \quad \text{by } (\mathrm{IX})_{i+1} \text{ and } (\mathrm{X})_{i+1},\\ \phi_4(4\varrho) \leq 2^{-m}\boldsymbol{\beta}(n)^{-1}\boldsymbol{\alpha}(m)\inf\big\{\eta(4\Gamma_{4.8(7)}(Q,m))^{-1}, 1/8\big\} \quad \text{by (I')}.\end{gathered}$$

Next, it will be shown that the hypotheses of (10) are satisfied with ϱ replaced by 4ϱ; in fact, noting $t \leq \varrho \leq \frac{r}{16}$,

$$\begin{gathered}\{2\varrho, 4\varrho\} \subset J_4 \cap J_5 \quad \text{by } (\mathrm{IX})_{i+1}, \text{ and } (\mathrm{X})_{i+1},\\ 8\varrho \in J_0 \cap J_1 \quad \text{by (II) and } (\mathrm{XI})_i, \qquad \|\sigma_{8\varrho}\| \leq n^{-1/2}\delta/4 \quad \text{by } (\mathrm{XII})_i,\\ 8r \in J_2 \cap J_3 \quad \text{by (IV)}, \qquad \phi_3(8\varrho) \leq \varepsilon_{(10)}(m,n,Q,\delta_2,\varepsilon,\delta,p) \quad \text{by (I')},\end{gathered}$$

$$\begin{aligned}\mathbf{U}(c,2\varrho) \sim &\{x : \boldsymbol{\Theta}^0(\|f(x)\|, g(x)) = Q\}\\ &\subset C_{a,2\varrho} \cup \mathbf{p}\big[\mathbf{C}(T,a,2\varrho,2\varrho) \cap \{z : Q > \boldsymbol{\Theta}^m(\|V\|, z) \in \mathscr{P}\}\big],\end{aligned}$$

by 7.3 (11b) with ϱ replaced by 2ϱ, hence

$$\begin{aligned}&\mathscr{L}^m(\mathbf{U}(c,2\varrho) \sim \{x : \boldsymbol{\Theta}^0(\|f(x)\|, g(x)) = Q\})\\ &\qquad \leq (\eta/2)\boldsymbol{\alpha}(m)(2\varrho)^m + \varepsilon_{(11)}\boldsymbol{\alpha}(m)(2\varrho)^m \leq \eta\boldsymbol{\alpha}(m)(2\varrho)^m\end{aligned}$$

by 7.3 (11c) with ϱ, λ replaced by 2ϱ, $\eta/2$. Therefore, in case $\alpha\tau < 1$, (10) implies, using $(\mathrm{XV'})_i$, $(\mathrm{XV})_i$, $(\mathrm{XIV})_i$, (I), and $\gamma_1 = \eta(24\Delta_1)^{-1}\gamma_2$, $\gamma_3 \leq \Delta_{12}\gamma_2$,

$$\begin{aligned}\phi_2(\varrho, T_\varrho) \leq\ &\Delta_1\Big(\big(\lambda + \eta^{1/n} + \eta^{-1}\phi_2(8\varrho, T_{8\varrho})^{\inf\{1,2/m\}}\big)\phi_2(8\varrho, T_{8\varrho})\\ &\qquad + \eta^{-1}4\varrho\phi_1(4\varrho) + (\eta^{-1} + \lambda^{-\tau})\phi_3(8\varrho)^\tau\Big)\\ \leq\ &\gamma(\varrho/r)^{\alpha\tau}\Big(8\Delta_1\big(\lambda + \eta^{1/n} + \eta^{-1}(\Delta_9)^{1/n}\big)\gamma_2\\ &\qquad + 4\Delta_1\eta^{-1}\gamma_1 + 8\Delta_1(\eta^{-1} + \lambda^{-\tau})\gamma_3\Big)\\ \leq\ &\gamma(\varrho/r)^{\alpha\tau}\big(\tfrac{1}{6}\gamma_2 + \tfrac{1}{6}\gamma_2 + \tfrac{1}{6}\gamma_2 + \tfrac{1}{6}\gamma_2 + \tfrac{1}{3}\gamma_2\big) = \gamma\gamma_2(\varrho/r)^{\alpha\tau}.\end{aligned}$$

Similarly, in case $\alpha\tau = 1$, (10) implies, using $(\mathrm{XV'})_i$, $(\mathrm{XV})_i$, $(\mathrm{XIV})_i$, (I), and $\gamma_1 = \eta(24\Delta_1)^{-1}\gamma_2$, $\gamma_3 \leq \Delta_{12}\gamma_2$,

$$\begin{aligned}\phi_2(\varrho, T_\varrho) \leq\ &\gamma(\varrho/r)(1 + \log(r/\varrho))\Big(8\Delta_1\big(\lambda + \eta^{1/n} + \eta^{-1}(\Delta_9)^{1/n}\big)\gamma_2\\ &\qquad + 4\Delta_1\eta^{-1}\gamma_1 + 8\Delta_1(\eta^{-1} + \lambda^{-\tau})\gamma_3\Big)\\ \leq\ &\gamma\gamma_2(\varrho/r)(1 + \log(r/\varrho)).\end{aligned}$$

Therefore the assertions $(\mathrm{IX})_i$–$(\mathrm{XV})_i$ are verified whenever $i \in \mathscr{P}$, $i \leq j+1$. The conclusion now follows from $(\mathrm{XI})_{j+1}$, $(\mathrm{XIV})_{j+1}$ and $(\mathrm{XV})_{j+1}$. □

7.5 Lemma. *Suppose $m, n, Q \in \mathscr{P}$, $m < n$, either $p = m = 1$ or $1 < p < m = 2$ or $1 \leq p < m > 2$ and $\frac{mp}{m-p} = 2$, $0 < \delta \leq 1$, and $1 \leq M < \infty$.*

Then there exist positive, finite numbers ε and Γ with the following property.

If $a \in \mathbf{R}^n$, $0 < r < \infty$, $V \in \mathbf{IV}_m(\mathbf{U}(a, 6r))$, ψ and p are related to V as in 3.3, $T \in \mathbf{G}(n, m)$, Z is a $\|V\|$ measurable subset of $\mathbf{C}(T, a, r, 3r)$,

$$
\begin{gathered}
(Q - 1/2)\boldsymbol{\alpha}(m) r^m \leq \|V\|(\mathbf{C}(T, a, r, 3r)) \leq (Q + 1/2)\boldsymbol{\alpha}(m) r^m, \\
\|V\|(\mathbf{C}(T, a, r, 4r) \sim \mathbf{C}(T, a, r, r)) \leq (1/2)\boldsymbol{\alpha}(m) r^m, \\
\|V\| \, \mathbf{U}(a, 6r) \leq M \boldsymbol{\alpha}(m) r^m, \quad \|V\|(\mathbf{C}(T, a, r/2, r/2)) \geq (Q - 1/4)\boldsymbol{\alpha}(m)(r/2)^m, \\
\|V\|(\mathbf{C}(T, a, r, 3r) \sim Z) \leq \varepsilon \boldsymbol{\alpha}(m) r^m, \quad \big({\textstyle\int} |S_\natural - T_\natural|^2 \, \mathrm{d} V(z, S) \big)^{1/2} \leq \varepsilon r^{m/2},
\end{gathered}
$$

then

$$
\begin{aligned}
& \big(r^{-m} {\textstyle\int_{\mathbf{C}(T, a, r/4, r/4) \times \mathbf{G}(n, m)}} |S_\natural - T_\natural|^2 \, \mathrm{d} V(z, S) \big)^{1/2} \\
& \quad \leq \delta \big(r^{-m} {\textstyle\int_{\mathbf{C}(T, a, r, r) \times \mathbf{G}(n, m)}} |S_\natural - T_\natural|^2 \, \mathrm{d} V(z, S) \big)^{1/2} \\
& \qquad + \Gamma \big(r^{-m-1} {\textstyle\int_Z} \operatorname{dist}(z - a, T) \, \mathrm{d}\|V\| z + r^{1 - m/p} \psi(\mathbf{U}(a, 6r))^{1/p} \big).
\end{aligned}
$$

Proof. Define

$$
\begin{gathered}
L = 1/8, \quad \delta_1 = \delta_2 = \delta_3 = 1/2, \quad \delta_4 = 1, \quad \delta_5 = (40)^{-m} (\boldsymbol{\gamma}(m) m)^{-m} / \boldsymbol{\alpha}(m), \\
\Delta_1 = \varepsilon_{4.8}(n, Q, L, M, \delta_1, \delta_2, \delta_3, \delta_4, \delta_5), \quad \Delta_2 = \inf \big\{ 1, (2\boldsymbol{\gamma}(m))^{-1}, \Delta_1 \big\}, \\
\mu = 1/2 \quad \text{if } m = 1, \quad \mu = 1/m \quad \text{if } m > 1, \quad \Delta_3 = \Gamma_{7.4(9)}(m, n, Q, \Delta_2, p, 1), \\
\eta = \inf \big\{ \delta^{1/\mu} (4\Delta_3)^{-1/\mu}, 2^{-m-1} \big\}, \quad \lambda = \inf \big\{ \delta (4\Delta_3)^{-1}, 1 \big\}, \\
\kappa = \inf \big\{ \varepsilon_{7.4(9)}(m, n, Q, \delta_2, \Delta_1, \Delta_2, p), \varepsilon_{7.3(11)}(n, \delta_4, \Delta_2), \\
2^{-m-2} \boldsymbol{\beta}(n)^{-1} \boldsymbol{\alpha}(m) \eta \Gamma_{4.8(7)}(Q, m)^{-1} \Delta_2 \big\}, \\
\Delta_4 = \inf \big\{ (M \boldsymbol{\alpha}(m))^{-1/2} 2^{-m} \kappa, \boldsymbol{\alpha}(m)^{1/2} 2^{-m-4} n^{-1/2} \Delta_2, \\
(M \boldsymbol{\alpha}(m))^{-1/2} \delta^{m/2} (4\Delta_3)^{-m/2} \big\}, \\
\varepsilon = \inf \big\{ \Delta_4, 2^{-m-1} \eta \big\}, \\
\Delta_5 = 2^{-m} \boldsymbol{\beta}(n)^{-1} \boldsymbol{\alpha}(m) \inf \big\{ \eta \Gamma_{4.8(7)}(Q, m)^{-1}/4, 1/8 \big\}, \\
\Delta_6 = \inf \big\{ (M \boldsymbol{\alpha}(m))^{1/p - 1} 2^{1-m} \kappa, \varepsilon_{7.4(9)}(m, n, Q, \delta_2, \Delta_1, \Delta_2, p), \\
\Delta_2 (\Delta_5)^{1/p - 1/m} \big\}, \\
\Gamma = \sup \big\{ \Delta_3 Q^{1/2} \eta^{-1}, \Delta_3 \lambda^{-1}, (4(Q+1)\boldsymbol{\alpha}(m) m)^{1/2} (\Delta_6)^{-1} \big\}.
\end{gathered}
$$

It will be shown that ε and Γ have the asserted property.

Suppose a, r, V, ψ, p, T, and Z satisfy the hypotheses in the body of the lemma.

By the definition of Γ and

$$
r^{-m} {\textstyle\int_{\mathbf{C}(T, a, r/4, r/4) \times \mathbf{G}(n, m)}} |S_\natural - T_\natural|^2 \, \mathrm{d} V(z, S) \leq 4(Q+1)\boldsymbol{\alpha}(m) m
$$

one may assume that

$$
r^{1 - m/p} \psi(\mathbf{U}(a, 6r))^{1/p} \leq \Delta_6.
$$

Additionally, one may assume that Z is a Borel set and that $a = 0$, $T = \operatorname{im} \mathbf{p}^*$ using isometries and identifying $\mathbf{R}^n \simeq \mathbf{R}^m \times \mathbf{R}^{n-m}$.

Defining A, X_1, f, c, ϕ_2, ϕ_3, ϕ_4, T_ϱ, J_1, J_2, J_3, J_4, J_5, σ_ϱ, and $C_{a,\varrho}$ as in 7.3 and $X = \mathbf{U}(c, r/2) \cap X_1 \sim \mathbf{p}[A \sim Z]$, next, the hypotheses of 7.4 (9) with δ, P, ϱ replaced by Δ_2, 0, r will be verified. The $\mathscr{L}^m$ measurability of X is a consequence of 4.8 (2) and [Fed69, 2.2.13]. One estimates

$$\textstyle\int |S_\natural - T_\natural| \,\mathrm{d}V(z,S) \le (M\boldsymbol{\alpha}(m))^{1/2} r^m \Delta_4 \le \kappa (r/2)^m,$$
$$\|\delta V\| \,\mathbf{U}(a, 6r) \le (M\boldsymbol{\alpha}(m))^{1-1/p} r^{m-1} \Delta_6 \le \kappa (r/2)^{m-1},$$

hence $r/2 \in J_4 \cap J_5$ and $8r \in J_2 \cap J_3$. Also

$$\begin{aligned} \|(T_r)_\natural - T_\natural\| &\le \|V\|(\mathbf{C}(T,a,r/2,r/2))^{-1/2} 2\phi_2(6r,T)(6r)^{m/2} \\ &\le 2^{m+2}\boldsymbol{\alpha}(m)^{-1/2}\Delta_4 \le 1/2, \\ & T_r \cap \ker \mathbf{p} = \{0\}, \quad r \in J_1 \end{aligned}$$

and, using 3.1 with S, S_1, S_2 replaced by T, T, T_r,

$$\|\sigma_r\|^2 \le (1 + \|\sigma_r\|^2)\|(T_r)_\natural - T_\natural\|^2,$$
$$\|\sigma_r\|^2 \le \|(T_r)_\natural - T_\natural\|^2/(1 - \|(T_r)_\natural - T_\natural\|^2) \le 2\|(T_r)_\natural - T_\natural\|^2,$$
$$\|\sigma_r\| \le 2\|(T_r)_\natural - T_\natural\| \le 2^{m+3}\boldsymbol{\alpha}(m)^{-1/2}\Delta_4 \le n^{-1/2}\Delta_2/2.$$

Noting $\phi_4(r) \le \Delta_5$, one infers from 7.3 (11c) with ϱ, λ replaced by $r/2$, $\eta/2$ that

$$\mathscr{L}^m(C_{a,r/2}) \le (\eta/2)\boldsymbol{\alpha}(m)(r/2)^m.$$

Combining this with

$$\mathscr{L}^m(\mathbf{p}[A \sim Z]) \le \mathscr{H}^m(A \sim Z) \le \|V\|(\mathbf{C}(T,a,r,3r) \sim Z) \le (\eta/2)\boldsymbol{\alpha}(m)(r/2)^m,$$
$$\mathbf{U}(c, r/2) \sim X \subset C_{a,r/2} \cup \mathbf{p}[A \sim Z],$$

one obtains

$$\mathscr{L}^m(\mathbf{U}(c,r/2) \sim X) \le \eta\boldsymbol{\alpha}(m)(r/2)^m.$$

Now, applying 7.4 (9) with δ, P, ϱ, and τ replaced by Δ_2, 0, r, and 1 yields

$$\begin{aligned} \phi_2(r/4,T) \le{} & \Delta_3\Big(\big(\lambda + ((M\boldsymbol{\alpha}(m))^{1/2}\Delta_4)^{2/m} + (\lambda + \eta^\mu)\big)\phi_2(r,T) \\ & + \eta^{-1} r^{-m-1} |f|_{1;X} + \lambda^{-1}\phi_3(r)\Big) \\ \le{} & \delta\phi_2(r,T) + \Gamma\big(Q^{-1/2} r^{-m-1}|f|_{1;X} + \phi_3(r)\big). \end{aligned}$$

Finally, noting

$$X \cap \big\{x : \mathscr{G}(f(x), Q[\![0]\!]) > Q^{1/2}\gamma\big\} \subset \mathbf{p}\big[A \cap Z \cap \{z : \operatorname{dist}(z-a,T) > \gamma\}\big]$$

for $0 < \gamma < \infty$, one obtains

$$Q^{-1/2}|f|_{1;X} \le \textstyle\int_Z \operatorname{dist}(z-a,T)\,\mathrm{d}\|V\|z$$

and the conclusion follows. □

8 The pointwise regularity theorem

Here, after verifying the hypotheses of the approximation by a $\mathbf{Q}_Q(\mathbf{R}^{n-m})$ valued function in 8.2, the pointwise regularity theorem is deduced from 7.4 (11) in 8.3. An example demonstrating the sharpness of the modulus of continuity obtained in case $\alpha\tau = 1$ and $m > 1$ is provided in 8.5. Finally, a corollary concerning almost everywhere decay rates is included in 8.6.

8.1 Lemma. *Suppose $1 < n \in \mathscr{P}$, $0 < \delta \leq 1$, $0 \leq \lambda < 1$, and $0 \leq M < \infty$.*

Then there exists a positive, finite number ε with the following property.

If $n > m \in \mathscr{P}$, $a \in \mathbf{R}^n$, $0 < r < \infty$, $T \in \mathbf{G}(n,m)$, $V \in \mathbf{IV}_m(\mathbf{U}(a,r))$ and

$$\begin{gathered} \|V\|\,\mathbf{U}(a,r) \leq M\boldsymbol{\alpha}(m)r^m, \quad \|\delta V\|\,\mathbf{U}(a,r) \leq \varepsilon \|V\|(\mathbf{U}(a,r))^{1-1/m}, \\ \textstyle\int |S_\natural - T_\natural| \,\mathrm{d}V(z,S) \leq \varepsilon \|V\|\,\mathbf{U}(a,r), \\ \|V\|\,\mathbf{B}(a,\varrho) \geq \delta\boldsymbol{\alpha}(m)\varrho^m \quad \text{for } 0 < \varrho < r, \end{gathered}$$

then

$$\|V\|(\mathbf{U}(a,r) \cap \{z : |T_\natural(z-a)| > \lambda|z-a|\}) \geq (1-\delta)\boldsymbol{\alpha}(m)r^m.$$

Proof. This is a special case of [Men09b, 2.2]. □

8.2 Lemma. *Suppose $m, n, Q \in \mathscr{P}$, $m < n$, either $p = m = 1$ or $1 \leq p < m$, $0 < \alpha \leq 1$, $1 \leq M < \infty$, $0 < \mu \leq 1$, and $0 < \delta_i \leq 1$ for $i \in \{1,2\}$.*

Then there exists a positive, finite number ε with the following property.

If $a \in \mathbf{R}^n$, $0 < r < \infty$, $V \in \mathbf{IV}_m(\mathbf{U}(a,r))$, ψ is related to p and V as in 3.3, $T \in \mathbf{G}(n,m)$,

$$\begin{gathered} \Delta = \inf\big\{\mu, (1+M^2)^{-1/2}\big(1-(1-\delta_1/2)^{1/m}(1-\delta_1/4)^{-1/m}\big)\big\}, \\ \boldsymbol{\Theta}^{*m}(\|V\|,a) \geq Q-1+\delta_2, \quad \|V\|\,\mathbf{U}(a,r) \leq (Q+1-\delta_1)\boldsymbol{\alpha}(m)r^m, \\ \textstyle\int |S_\natural - T_\natural| \,\mathrm{d}V(z,S) \leq \varepsilon r^m, \\ \varrho^{1-m/p}\psi(\mathbf{B}(a,\varrho))^{1/p} \leq \varepsilon(\varrho/r)^\alpha \quad \text{whenever } 0 < \varrho < r, \end{gathered}$$

then with $s = \Delta r$

$$\|V\|(\mathbf{C}(T,a,s,Ms) \sim \mathbf{C}(T,a,s,\delta_2 s)) \leq \delta_2\boldsymbol{\alpha}(m)s^m.$$

Proof. Define Δ as in the hypotheses of the body of the lemma, $\lambda = \big(1 - (\Delta\delta_2/4)^2\big)^{1/2}$,

$$\Delta_1 = \varepsilon_{8.1}(n, \inf\{(2\boldsymbol{\gamma}(m)m)^{-m}/\boldsymbol{\alpha}(m), \delta_1/4\}, \lambda, 2(Q+1)),$$

let ε be the infimum of the following five numbers

$$\begin{gathered} \varepsilon_{7.2}(n,Q,\alpha,p,\inf\{\delta_1/3, \Delta\delta_2/2\}), \quad ((Q+1)\boldsymbol{\alpha}(m))^{1/p-1}(4\boldsymbol{\gamma}(m)m)^{1-m}\Delta_1, \\ (4\boldsymbol{\gamma}(m)m)^{-m}\Delta_1, \quad (2\boldsymbol{\gamma}(m))^{-1}, \quad (\delta_2\Delta^m\boldsymbol{\alpha}(m)\boldsymbol{\beta}(n)^{-1})^{1/p-1/m}(2\boldsymbol{\gamma}(m))^{-1} \end{gathered}$$

and suppose that m, a, r, V, ψ, T and s satisfy the hypotheses in the body of the lemma.

First, note by 7.2 with δ replaced by $\inf\{\delta_1/3, \Delta\delta_2/2\}$

$$\|V\|(\mathbf{U}(a,r) \cap \{z : |T_\natural^\perp(z-a)| < \delta_2 s/2\}) \geq \boldsymbol{\alpha}(m)(Q-\delta_1/3)r^m.$$

Define A to be set of all $z \in \operatorname{spt} \|V\|$ such that

$$\|\delta V\| \, \mathbf{B}(z,t) \leq (2\boldsymbol{\gamma}(m))^{-1} \|V\|(\mathbf{B}(z,t))^{1-1/m}$$

whenever $0 < t < \infty$ and $\mathbf{B}(z,t) \subset \mathbf{U}(a,r)$. Next, the following assertion will be proven:

$$A \cap \mathbf{C}(T,a,s,Ms) \subset \mathbf{C}(T,a,s,\delta_2 s).$$

For this purpose suppose $z \in A \cap \operatorname{spt} \|V\| \cap \mathbf{C}(T,a,s,Ms)$ and abbreviate $t = \operatorname{dist}(z, \mathbf{R}^n \sim \mathbf{U}(a,r))$. Since $\Delta < (1+M^2)^{-1/2}$, one notes $\mathbf{C}(T,a,s,Ms) \subset \mathbf{U}(a,r)$ and $t > 0$. From [Men09a, 2.5] one obtains

$$\|V\| \, \mathbf{B}(z,\varrho) \geq (2\boldsymbol{\gamma}(m)m)^{-m} \varrho^m \quad \text{for } 0 < \varrho < t.$$

Therefore, noting

$$\begin{gathered}
t \geq r - (1+M^2)^{1/2}\Delta r, \quad (t/r)^m \geq (1-\delta_1/2)(1-\delta_1/4)^{-1} \geq 2/3, \\
\|V\| \, \mathbf{U}(z,t) \leq \|V\| \, \mathbf{U}(a,r) \leq (Q+1)\boldsymbol{\alpha}(m) r^m \leq 2(Q+1)\boldsymbol{\alpha}(m) t^m, \\
\begin{aligned}
\|\delta V\| \, \mathbf{U}(z,t) \leq \|\delta V\| \, \mathbf{U}(a,r) &\leq \big((Q+1)\boldsymbol{\alpha}(m)\big)^{1-1/p} \varepsilon r^{m-1} \\
&\leq \big((Q+1)\boldsymbol{\alpha}(m)\big)^{1-1/p} (4\boldsymbol{\gamma}(m)m)^{m-1} \varepsilon \|V\|(\mathbf{U}(z,t))^{1-1/m} \\
&\leq \Delta_1 \|V\|(\mathbf{U}(z,t))^{1-1/m},
\end{aligned} \\
\begin{aligned}
\textstyle\int_{\mathbf{U}(z,t)\times \mathbf{G}(n,m)} |S_\natural - T_\natural| \, \mathrm{d}V(\xi,S) &\leq \textstyle\int |S_\natural - T_\natural| \, \mathrm{d}V(\xi,S) \\
&\leq \varepsilon r^m \leq \varepsilon (4\boldsymbol{\gamma}(m)m)^m \|V\| \, \mathbf{U}(z,t) \leq \Delta_1 \|V\| \, \mathbf{U}(z,t),
\end{aligned}
\end{gathered}$$

one uses 8.1 with δ, M, a, and r replaced by $\inf\{(2\boldsymbol{\gamma}(m)m)^{-m}/\boldsymbol{\alpha}(m), \delta_1/4\}$, $2(Q+1)$, z, and t to infer

$$\begin{aligned}
\|V\|(\mathbf{U}(z,t) \cap \{\xi : |T_\natural(\xi - z)| > \lambda |\xi - z|\}) &\geq (1-\delta_1/4)\boldsymbol{\alpha}(m) t^m \\
&\geq (1-\delta_1/2)\boldsymbol{\alpha}(m) r^m.
\end{aligned}$$

Since $\|V\| \, \mathbf{U}(a,r) \leq (Q+1-\delta_1)\boldsymbol{\alpha}(m) r^m$, this implies together with the second paragraph that the intersection of

$$T_\natural^\perp[\mathbf{U}(z,t) \cap \{\xi : |T_\natural(\xi - z)| > \lambda |\xi - z|\}] \quad \text{and} \quad \mathbf{R}^n \cap \{\xi : |T_\natural^\perp(\xi - a)| < \delta_2 s/2\}$$

cannot be empty. Now, estimating for $\xi \in \mathbf{U}(z,t)$ with $|T_\natural(\xi - z)| > \lambda |\xi - z|$

$$|T_\natural^\perp(\xi - z)| \leq (1-\lambda^2)^{1/2} |\xi - z| \leq 2(1-\lambda^2)^{1/2} r = \delta_2 s/2,$$

one obtains $|T_\natural^\perp(z-a)| \leq \delta_2 s$ and the inclusion follows.

If $m = 1$ then $A = \operatorname{spt} \|V\|$ and the conclusion is evident. Hence suppose $m > 1$. The assertion of the preceding paragraph implies with the help of Besicovitch's covering theorem and Hölder's inequality the existence of countable disjointed families of closed balls $F_1, \ldots, F_{\boldsymbol{\beta}(n)}$ such that

$$\begin{gathered}
\operatorname{spt} \|V\| \cap \mathbf{C}(T,a,s,Ms) \sim \mathbf{C}(T,a,s,\delta_2 s) \subset \textstyle\bigcup \bigcup \{F_i : i = 1, \ldots, \boldsymbol{\beta}(n)\}, \\
S \subset \mathbf{U}(a,r), \qquad \|V\|(S) \leq \Delta_2 \psi(S)^{m/(m-p)}
\end{gathered}$$

whenever $S \in \bigcup\{F_i : i = 1, \ldots, \boldsymbol{\beta}(n)\}$ where $\Delta_2 = (2\boldsymbol{\gamma}(m))^{mp/(m-p)}$, hence

$$\begin{aligned} \|V\|(\mathbf{C}(T,a,s,Ms) \sim \mathbf{C}(T,a,s,\delta_2 s)) &\le \Delta_2 \textstyle\sum_{i=1}^{\boldsymbol{\beta}(n)} \sum_{S\in F_i} \psi(S)^{m/(m-p)} \\ &\le \Delta_2 \textstyle\sum_{i=1}^{\boldsymbol{\beta}(n)} \big(\sum_{S\in F_i} \psi(S)\big)^{m/(m-p)} \le \Delta_2 \boldsymbol{\beta}(n) \psi(\mathbf{U}(a,r))^{m/(m-p)} \\ &\le (2\boldsymbol{\gamma}(m)\varepsilon)^{mp/(m-p)} \boldsymbol{\beta}(n) r^m \le \delta_2 \boldsymbol{\alpha}(m) s^m. \end{aligned}$$

□

8.3 Theorem. *Suppose $m, n, Q \in \mathscr{P}$, $m < n$, either $p = m = 1$ or $1 \le p < m$, $0 < \delta \le 1$, $0 < \alpha \le 1$, $0 < \tau \le 1$, and $\tau = 1$ if $m = 1$, $p/2 \le \tau < \frac{mp}{2(m-p)}$ if $m = 2$ and $\tau = \frac{mp}{2(m-p)}$ if $m > 2$.*

Then there exist positive, finite numbers ε and Γ with the following property.

If $a \in \mathbf{R}^n$, $0 < r < \infty$, $V \in \mathbf{IV}_m(\mathbf{U}(a,r))$, p and ψ are related to V as in 3.3, $T \in \mathbf{G}(n,m)$, $\omega : \mathbf{R} \cap \{t : 0 < t \le 1\} \to \mathbf{R}$ with $\omega(t) = t^{\alpha\tau}$ if $\alpha\tau < 1$ and $\omega(t) = t(1 + \log(1/t))$ if $\alpha\tau = 1$ whenever $0 < t \le 1$, and $0 < \gamma \le \varepsilon$,

$$\begin{gathered} \boldsymbol{\Theta}^{*m}(\|V\|, a) \ge Q - 1 + \delta, \quad \|V\|\,\mathbf{U}(a,r) \le (Q + 1 - \delta)\boldsymbol{\alpha}(m) r^m, \\ \big(r^{-m} \textstyle\int |S_\natural - T_\natural|^2 \,\mathrm{d}V(z,S)\big)^{1/2} \le \gamma, \\ \|V\|(\mathbf{B}(a,\varrho) \cap \{z : \boldsymbol{\Theta}^m(\|V\|, z) \le Q - 1\}) \le \varepsilon \boldsymbol{\alpha}(m) \varrho^m \quad \text{for } 0 < \varrho < r, \\ \varrho^{1-m/p} \psi(\mathbf{B}(a,\varrho))^{1/p} \le \gamma^{1/\tau} (\varrho/r)^\alpha \quad \text{for } 0 < \varrho < r, \end{gathered}$$

then $\boldsymbol{\Theta}^m(\|V\|, a) = Q$, $R = \operatorname{Tan}^m(\|V\|, a) \in \mathbf{G}(n,m)$ and

$$\big(\varrho^{-m} \textstyle\int_{\mathbf{U}(a,\varrho)\times\mathbf{G}(n,m)} |S_\natural - R_\natural|^2 \,\mathrm{d}V(z,S)\big)^{1/2} \le \Gamma\gamma\omega(\varrho/r) \quad \text{whenever } 0 < \varrho \le r.$$

Proof. Define, noting $(\boldsymbol{\gamma}(m)m)^{-m} \le \boldsymbol{\alpha}(m)$ by, e.g., [Men09a, 2.4],

$$\begin{gathered} \Delta_1 = \inf\big\{1/6, (17)^{-1/2}\big(1 - (1 - \delta/2)^{1/m}(1 - \delta/4)^{-1/m}\big)\big\}, \\ \delta_1 = \delta/2, \quad \delta_2 = \delta/4, \quad \delta_3 = 1 - \delta/4, \quad \delta_4 = 1, \\ \delta_5 = (40)^{-m}(\boldsymbol{\gamma}(m)m)^{-m}/\boldsymbol{\alpha}(m), \quad \delta_6 = \delta, \quad L = \delta_4/8, \quad M = (\Delta_1)^{-m}(Q+1), \\ \delta' = \inf\big\{1, \varepsilon_{4.8}(n, Q, L, M, \delta_1, \delta_2, \delta_3, \delta_4, \delta_5), (2\boldsymbol{\gamma}(m))^{-1}\big\}, \\ \eta = \inf\{1, (Q + 1 - \delta/2)^{1/m}(Q + 1 - 3\delta/4)^{-1/m} - 1\} \end{gathered}$$

and apply 7.4 (11) with δ replaced by δ' to obtain γ_i for $i \in \{2,3\}$. Define

$$\begin{aligned} \Delta_2 = {} & \inf\big\{(Q + 1 - 3\delta/4)^{1/p} - (Q + 1 - \delta)^{1/p}, \\ & \qquad (Q - 1 + \delta)^{1/p} - (Q - 1 + \delta/2)^{1/p}\big\}, \\ \Delta_3 = {} & \inf\big\{(\Delta_1)^{m/2}\varepsilon_{7.4(11)}(m, n, Q, L, M, \delta_1, \delta_2, \delta_3, p, \tau, \alpha, \delta_6), \gamma_3\big\}, \\ \varepsilon = {} & \inf\{(\alpha p \boldsymbol{\alpha}(m)^{1/p}\Delta_2)^\tau, \\ & (Q+1)^{-1/2}\boldsymbol{\alpha}(m)^{-1/2}\varepsilon_{8.2}(m, n, Q, p, \alpha, 4, 1/6, \delta, \inf\{\eta, \delta/4\}), \\ & \varepsilon_{8.2}(m, n, Q, p, \alpha, 4, 1/6, \delta, \inf\{\eta, \delta/4\})^\tau, \Delta_3, 1\} \end{aligned}$$

and also

$$\begin{gathered} \Delta_4 = \sup\big\{\gamma_2(\Delta_1\Delta_3)^{-1}, (\Delta_1)^{-m/2-1}\big\}, \quad \Delta_5 = (1 - 2^{-\alpha\tau})^{-1} \quad \text{if } \alpha\tau < 1, \\ \Delta_5 = 2 + 2\log 2 \quad \text{if } \alpha\tau = 1, \quad \Delta_6 = 2^{m+2}\delta^{-1}\boldsymbol{\alpha}(m)^{-1/2}\Delta_4\Delta_5, \\ \Gamma = \Delta_4 + (Q+1)^{1/2}\boldsymbol{\alpha}(m)^{1/2}\Delta_6. \end{gathered}$$

Suppose a, r, V, ψ, T, and ω satisfy the hypotheses of the body of the theorem.

Let $s = \Delta_1 r$. Applying 7.1 twice with M replaced by ε^τ in conjunction with Hölder's inequality, one deduces the *mass bounds*:

$$(Q - 1 + \delta/2)\boldsymbol{\alpha}(m)\varrho^m \leq \|V\|\,\mathbf{U}(a, \varrho) \leq (Q + 1 - 3\delta/4)\boldsymbol{\alpha}(m)\varrho^m$$

for $0 < \varrho \leq r$. From 8.2 applied with M, μ, δ_1, δ_2 replaced by 4, 1/6, δ, $\inf\{\eta, \delta/4\}$ one obtains, noting $\int |S_\natural - T_\natural| \,\mathrm{d}V(z, S) \leq (Q+1)^{1/2}\boldsymbol{\alpha}(m)^{1/2}\varepsilon r^m$ by Hölder's inequality,

$$\|V\|(\mathbf{C}(T, a, s, 4s) \sim \mathbf{C}(T, a, s, \eta s)) \leq (\delta/4)\boldsymbol{\alpha}(m)s^m.$$

Together this implies, noting $(1 + \eta)s \leq r$,

$$\begin{gathered}
\begin{aligned}
\|V\|\,\mathbf{U}(a, (1 + \eta)s) &\leq (Q + 1 - 3\delta/4)\boldsymbol{\alpha}(m)(1 + \eta)^m s^m \\
&\leq (Q + 1 - \delta/2)\boldsymbol{\alpha}(m)s^m,
\end{aligned} \\
\mathbf{C}(T, a, s, 3s) \subset (\mathbf{C}(T, a, s, 4s) \sim \mathbf{C}(T, a, s, \eta s)) \cup \mathbf{U}(a, (1 + \eta)s) \\
\|V\|(\mathbf{C}(T, a, s, 3s)) \leq (Q + 1 - \delta/4)\boldsymbol{\alpha}(m)s^m, \\
\|V\|(\mathbf{C}(T, a, s, 3s)) \geq \|V\|\,\mathbf{U}(a, s) \geq (Q - 1 + \delta/2)\boldsymbol{\alpha}(m)s^m,
\end{gathered}$$

hence, using isometries and identifying $\mathbf{R}^n \simeq \mathbf{R}^m \times \mathbf{R}^{n-m}$, one may assume that $a = 0$, and the hypotheses of 7.3 and 7.4 are satisfied with r, δ replaced by s, δ'.

Defining $\phi : (\mathbf{R} \cap \{\varrho : 0 < \varrho \leq r\}) \times \mathbf{G}(n, m) \to \mathbf{R}$ by

$$\phi(\varrho, R) = \big(\varrho^{-m} {\textstyle\int_{\mathbf{U}(a,\varrho)\times\mathbf{G}(n,m)}} |S_\natural - R_\natural|^2 \,\mathrm{d}V(z, S)\big)^{1/2}$$

for $0 < \varrho \leq r$, $R \in \mathbf{G}(n, m)$ and choosing $T_\varrho \in \mathbf{G}(n, m)$ such that

$$\phi(\varrho, T_\varrho) \leq \phi(\varrho, R) \quad \text{whenever } 0 < \varrho \leq r \text{ and } R \in \mathbf{G}(n, m)$$

and noting $\varepsilon \leq \Delta_3$ and $\Delta_1 \leq 1/4$, one obtains from 7.4 (11) with r, δ and γ, replaced by s, δ' and γ/Δ_3 that

$$\phi(\varrho, T_\varrho) \leq (\gamma/\Delta_3)\gamma_2\omega(\varrho/s) \quad \text{for } 0 < \varrho \leq s.$$

One infers the *tilt estimate*

$$\phi(\varrho, T_\varrho) \leq \Delta_4\gamma\omega(\varrho/r) \quad \text{for } 0 < \varrho \leq r.$$

Next, it will be shown that a similar estimate holds with T_ϱ replaced by a suitable $R \in \mathbf{G}(n, m)$. Using the lower mass bound, one notes for $0 < \varrho/2 \leq t \leq \varrho \leq r$

$$\begin{aligned}
|(T_\varrho)_\natural - (T_t)_\natural| &\leq 2^{m+1}\delta^{-1}\boldsymbol{\alpha}(m)^{-1/2}\varrho^{-m/2}\big(\varrho^{m/2}\phi(\varrho, T_\varrho) + t^{m/2}\phi(t, T_t)\big) \\
&\leq 2^{m+2}\delta^{-1}\boldsymbol{\alpha}(m)^{-1/2}\phi(\varrho, T_\varrho).
\end{aligned}$$

This implies inductively for $0 < t \leq \varrho \leq r$

$$|(T_t)_\natural - (T_\varrho)_\natural| \leq 2^{m+2}\delta^{-1}\boldsymbol{\alpha}(m)^{-1/2}{\textstyle\sum_{\nu=0}^{\infty}}\phi(2^{-\nu}\varrho, T_{2^{-\nu}\varrho}),$$

hence, noting that the tilt estimate yields

$$\textstyle\sum_{\nu=0}^{\infty}\phi(2^{-\nu}\varrho, T_{2^{-\nu}\varrho}) \leq \Delta_4\gamma\sum_{\nu=0}^{\infty}(2^{-\nu}\varrho/r)^{\alpha\tau} = \Delta_4\Delta_5\gamma\omega(\varrho/r) \quad \text{if } \alpha\tau < 1,$$

$$\begin{aligned}\textstyle\sum_{\nu=0}^{\infty}\phi(2^{-\nu}\varrho, T_{2^{-\nu}\varrho}) &\leq \Delta_4\gamma\textstyle\sum_{\nu=0}^{\infty}(2^{-\nu}\varrho/r)(1+\log(r/\varrho)+\nu\log 2)\\ &\leq \Delta_4\gamma(\varrho/r)(1+\log(r/\varrho))\big(2+\log 2\textstyle\sum_{\nu=0}^{\infty}2^{-\nu}\nu\big) = \Delta_4\Delta_5\gamma\omega(\varrho/r)\end{aligned}$$

if $\alpha\tau = 1$, there exists $R \in \mathbf{G}(n,m)$ with

$$|R_\natural - (T_\varrho)_\natural| \leq \Delta_6\gamma\omega(\varrho/r) \quad \text{whenever } 0 < \varrho \leq r.$$

Combining this with the tilt estimate, one obtains, using the upper mass bound,

$$\phi(\varrho, R) \leq \phi(\varrho, T_\varrho) + (Q+1)^{1/2}\boldsymbol{\alpha}(m)^{1/2}\Delta_6\gamma\omega(\varrho/r) \leq \Gamma\gamma\omega(\varrho/r) \quad \text{for } 0 < \varrho \leq r.$$

Since $0 \leq \boldsymbol{\Theta}^m(\|V\|, a) < \infty$ by 7.1, one now infers from Allard [All72, 6.4] in conjunction with, e.g., [Men09b, 2.1] that

$$\varrho^{-m}\textstyle\int f((z-a)/\varrho, S)\,\mathrm{d}V(z,S) \to Q\int_R f(z,R)\,\mathrm{d}\mathscr{H}^m z \quad \text{as } \varrho \to 0+$$

for $f \in \mathscr{K}(\mathbf{R}^n \times \mathbf{G}(n,m))$, hence $\boldsymbol{\Theta}^m(\|V\|, a) = Q$ and $R = \operatorname{Tan}^m(\|V\|, a)$. □

8.4 Remark. If $\alpha\tau < 1$ and $m > 2$, then τ cannot be replaced by any larger number.

An example is provided as follows. Defining $\eta = \frac{\alpha p}{m-p}$, choosing for each $i \in \mathscr{P}$ an m dimensional sphere M_i of radius $\varrho_i = 2^{-i-\eta i-2}$ with $M_i \subset \mathbf{U}(a, 2^{-i}) \sim \mathbf{B}(a, 2^{-i-1})$, one readily verifies that one may take $V \in \mathbf{IV}_m(\mathbf{R}^n)$ such that $\|V\| = Q\mathscr{H}^m \llcorner T + \mathscr{H}^m \llcorner M$ where $M = \bigcup_{i=1}^{\infty} M_i$ and r sufficiently small.

8.5 Remark. In case $\alpha\tau = 1$, $m > 1$, it can happen that

$$\liminf_{\varrho\to 0+}\big(\varrho^{-m}\textstyle\int_{\mathbf{U}(a,\varrho)\times\mathbf{G}(n,m)}|S_\natural - R_\natural|^2\,\mathrm{d}V(z,S)\big)^{1/2}\omega(\varrho/r)^{-1} > 0.$$

To construct an example, assume $n - m = 1$, with $\mathbf{C} = \mathbf{R}^2$ take $u : \mathbf{C} \to \mathbf{R}$ of class 1 such that

$$u\big(re^{\mathbf{i}\theta}\big) = r^2(\log r)\cos(2\theta) \quad \text{for } 0 < r < \infty,\ \theta \in \mathbf{R},$$

and verify, using the homogeneity of u,

$$\begin{gathered}\operatorname{Lap} u\big(re^{\mathbf{i}\theta}\big) = 4\cos(2\theta) \quad \text{for } 0 < r < \infty,\ \theta \in \mathbf{R},\\ |D^i u(x)| \leq \Gamma|x|^{2-i}(1+\log(1/|x|)) \quad \text{for } x \in \mathbf{U}(0,1) \sim \{0\},\ i \in \{1,2\}\end{gathered}$$

where Γ is a positive, finite number, hence computing with C as in 6.1, noting [Fed69, 5.1.9],

$$\big\langle D^2u(x), C(Du(x))\big\rangle = \operatorname{Lap} u(x) + \big\langle D^2u(x), C(Du(x)) - C(0)\big\rangle$$

for $x \in \mathbf{R}^2 \sim \{0\}$, one obtains, since $Du(0) = 0$,

$$\begin{gathered}\big\langle D^2u, C \circ Du\big\rangle \in \mathbf{L}_\infty(\mathscr{L}^2 \llcorner \mathbf{U}(0,1)),\\ u|\mathbf{U}(0,1) \in \mathbf{W}^{2,q}(\mathbf{U}(0,1)) \quad \text{for } 1 \leq q < \infty.\end{gathered}$$

Choosing $g \in \mathbf{O}^*(m,2)$ and defining $f = u \circ g$, one may now take V associated to f as in 2.6 with $Q = 1$.

8.6 Corollary. *Suppose m, n, p, U, and V are as in 3.3, either $m \in \{1,2\}$ and $0 < \tau < 1$ or $\sup\{2,p\} < m$ and $\tau = \frac{mp}{2(m-p)} < 1$, and $V \in \mathbf{IV}_m(U)$.*

Then there holds for V almost all $(a,T) \in U \times \mathbf{G}(n,m)$ that

$$\limsup_{r \to 0+} r^{-\tau - m/2} \big(\textstyle\int_{\mathbf{U}(a,r) \times \mathbf{G}(n,m)} |S_\natural - T_\natural|^2 \, \mathrm{d}V(z,S) \big)^{1/2} < \infty.$$

Proof. From [Fed69, 2.9.13, 5] one infers that for $\|V\|$ almost all $a \in U$ there exists $Q \in \mathscr{P}$ and $T \in \mathbf{G}(n,m)$ such that for $f \in \mathscr{K}(\mathbf{R}^n \times \mathbf{G}(n,m))$

$$\lim_{r \to 0+} r^{-m} \textstyle\int f(r^{-1}(z-a), S) \, \mathrm{d}V(z,S) = Q \int_T f(z,T) \, \mathrm{d}\mathscr{H}^m z,$$
$$\mathbf{\Theta}^m(\|V\| \llcorner \{z : \mathbf{\Theta}^m(\|V\|, z) \leq Q - 1\}, a) = 0, \quad \mathbf{\Theta}^{*m}(\psi, a) < \infty,$$

hence for such a one may apply 8.3 with r sufficiently small and $\alpha = 1$ to infer the conclusion. □

8.7 Remark. The examples in [Men09a, 1.2] with $q_1 = q_2 = 2$ and $\alpha_1 = \alpha_2$ slightly larger than $\frac{mp}{m-p}$ show that τ cannot be replaced by any larger number provided $m > 2$. However, it will be shown in [Men09c] that "$< \infty$" can be replaced by "$= 0$".

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