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Background-independent charges in Topologically Massive Gravity

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ABSTRACT: We construct background-independent Noether charges in Topologically Massive Gravity with negative cosmological constant using its first-order formulation.

The procedure is carried out by keeping track of the surface terms in the variation of the action, regardless the value of the gravitational Chern-Simons coupling μ . In particular, this method provides a definition of conserved quantities for solutions at the chiral point $\mu\ell = 1$ (ℓ is the AdS radius) that contain logarithmic terms (Log Gravity).

It is also shown that the charge formula gives a finite result for warped AdS black holes without the need for any background-subtraction procedure.

KEYWORDS: Chern-Simons Theories, Classical Theories of Gravity, Black Holes, Space-Time Symmetries

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1 Introduction

Standard Einstein-Hilbert gravity in three spacetime dimensions is topological in the sense that it does not possess local degrees of freedom. Inclusion of a gravitational Chern-Simons term in the action gives rise to propagating gravitons in an inequivalent theory known as Topologically Massive Gravity (*TMG*) [1].

Nevertheless, it is commonly expected that the addition of higher-derivative terms to the standard gravity action would render the theory unstable. Indeed, in an anti-de Sitter context, the theory exhibits gravitons of negative mass around AdS₃ background. In case of flat space, one has the possibility of changing the sign of the Newton constant G , what turns the theory unitary. However, for gravity with negative cosmological constant, this trick would also make negative the energy of BTZ black holes and, thus, it cannot be performed [2, 3].

The renewed interest in TMG with cosmological constant has been mainly sparked by the claim that the theory becomes stable and chiral for a special value of the topological mass parameter μ , such that $\mu\ell = 1$, where ℓ is the AdS radius [4]. It has been argued that, at this *chiral point*, the left-moving central charge c_L of the boundary conformal algebra vanishes and the local degree of freedom that corresponds to a negative mass graviton becomes a pure-gauge state. This idea presents a new perspective on the problem of achieving a unitary theory in AdS₃, especially in the light of the recent Witten's proposal [5, 6].

However, several authors provided evidence against this conjecture, manifested as the existence of left-moving excitations at the chiral point [2, 7–13], but that obey a relaxed version of Brown-Henneaux boundary conditions [7, 14, 15]. In addition, an exact solution of TMG compatible with Grumiller-Johansson fall-off conditions, but not Brown-Henneaux ones was explicitly shown in ref. [16].

Therefore, given the above results, the only possibility of chiral gravity conjecture to be true is by considering from the beginning Brown-Henneaux boundary conditions as part of the setup [17].

Nevertheless, the extensive discussion on the topic was extremely enlightening to learn about the subtleties of the boundary conditions in the context of AdS₃/CFT₂ correspondence.

In the gauge/gravity duality framework, in order to extract the holographic data of the theory, usually a regularized Brown-York stress tensor is required [18]. The same procedure is used to define background-independent charges in the gravity theory. However, it is well-known that, due to the presence of higher-derivative terms in TMG there is no analog of Gibbons-Hawking term to define the Dirichlet variational problem for the boundary metric h_{ij} . As a consequence, a quasilocal stress-tensor cannot be read off directly from the variation of the action as $T^{ij} = (2/\sqrt{-h}) (\delta I/\delta h_{ij})$. Only for asymptotically AdS spacetimes, a holographic stress tensor can be obtained considering a Fefferman-Graham-type expansion of the metric [19, 20] (see [21] for the inclusion of leading and subleading logarithms in the metric).

On the other hand, the construction of charges is important to achieve the proof of positivity of energy and stability of the solutions, that remains as an open question. Background-subtraction approaches to conserved quantities [13, 22, 23] may be useful to deal with this issue, but are not particularly insightful about the problem of holography. Taking this into consideration, we derive here background-independent charge formulas for TMG in view of a possible holographic description in the non asymptotically AdS case.

2 Obtaining TMG from first-order formalism

In first-order formalism, Topologically Massive Gravity with negative cosmological constant $\Lambda = -1/\ell^2$ is described by the action

$$I = -\frac{1}{16\pi G} \int_M \epsilon_{ABC} \left(R^{AB} + \frac{1}{3\ell^2} e^A e^B \right) e^C + \frac{1}{32\pi G\mu} \int_M (L_{CS}(\omega) + 2\lambda_A T^A) + \int_{\partial M} B, \tag{2.1}$$

where M is a three-dimensional manifold with local coordinates x^μ , G is the (positive) gravitational constant, μ is a constant parameter with dimension of mass and L_{CS} is the gravitational Chern-Simons 3-form

$$L_{CS}(\omega) = \omega^{AB} d\omega_{BA} + \frac{2}{3} \omega^A_B \omega^B_C \omega^C_A. \tag{2.2}$$

The fundamental fields of the theory are the dreibein $e^A = e^A_\mu dx^\mu$ and the spin connection $\omega^{AB} = \omega^{AB}_\mu dx^\mu$, which define the curvature 2-form $R^{AB} = \frac{1}{2} R^{AB}_{\mu\nu} dx^\mu dx^\nu = d\omega^{AB} +$

$\omega^A{}_C \omega^{CB}$ and the torsion 2-form $T^A = \frac{1}{2} T^A{}_{\mu\nu} dx^\mu dx^\nu = De^A$. The covariant derivative acts on a Lorentz vector V^A as $DV^A = dV^A + \omega^A{}_B V^B$. For simplicity, we have omitted wedge products \wedge between the differential forms.

The boundary term B will be discussed in the next section.

The introduction of a Lagrange multiplier 1-form $\lambda_A(x)$ implements a torsionless condition on the manifold and consistently recovers the dynamics of TMG [11, 13, 24, 25].

Indeed, an arbitrary variation of the fields in the above action produces the equations of motion

$$\begin{aligned} \delta I = & -\frac{1}{16\pi G} \int_M \delta\omega^{AB} \left[\epsilon_{ABC} T^C + \frac{1}{\mu} (R_{AB} + \lambda_A e_B) \right] \\ & -\frac{1}{16\pi G} \int_M \delta e^A \left[\epsilon_{ABC} \left(R^{BC} + \frac{1}{\ell^2} e^B e^C \right) - \frac{1}{\mu} D\lambda_A \right] \\ & -\frac{1}{16\pi G\mu} \int_M \delta\lambda_A T^A + \int_{\partial M} \Theta(\delta e, \delta\omega), \end{aligned} \quad (2.3)$$

where the surface term is

$$\Theta = -\frac{1}{16\pi G} \epsilon_{ABC} \delta\omega^{AB} e^C + \frac{1}{32\pi G\mu} (\delta\omega^{AB} \omega_{BA} + 2\delta e^A \lambda_A) + \delta B. \quad (2.4)$$

The field equations then read

$$T^A = 0, \quad (2.5)$$

$$\frac{1}{\mu} \left[R^{AB} + \frac{1}{2} (\lambda^A e^B - \lambda^B e^A) \right] + \epsilon^{ABC} T_C = 0, \quad (2.6)$$

$$\epsilon_{ABC} \left(R^{BC} + \frac{1}{\ell^2} e^B e^C \right) - \frac{1}{\mu} D\lambda_A = 0. \quad (2.7)$$

From eqs. (2.5) and (2.6), the Lagrange multiplier can be solved in terms of the Schouten tensor of the manifold

$$S_{\mu\nu} = (Ric)_{\mu\nu} - \frac{1}{4} \mathcal{G}_{\mu\nu} R \quad (2.8)$$

as

$$\lambda_\mu^A = -2 e^{A\nu} S_{\mu\nu}. \quad (2.9)$$

The manifold is endowed with a spacetime metric $\mathcal{G}_{\mu\nu} = \eta_{AB} e_\mu^A e_\nu^B$.

The relation (2.7) together with eq. (2.9) leads to the usual field equation of TMG in second-order formalism,

$$R_{\mu\nu} - \frac{1}{2} \mathcal{G}_{\mu\nu} R - \frac{1}{\ell^2} \mathcal{G}_{\mu\nu} + \frac{1}{\mu} C_{\mu\nu} = 0. \quad (2.10)$$

Here $C^\mu{}_\nu = \frac{1}{\sqrt{-\mathcal{G}}} \epsilon^{\mu\alpha\beta} \nabla_\alpha S_{\beta\nu}$ denotes the Cotton tensor obtained from varying the gravitational Chern-Simons term with respect to the metric. In our conventions, the Levi-Civita tensor density $\epsilon^{\mu\alpha\beta}$ is defined as $\epsilon^{tr\phi} = +1$.

3 Noether charges

The bulk theory described by eq. (2.1) is invariant under diffeomorphisms $\delta x^\mu = \xi^\mu(x)$ in the sense that the equations of motion remain covariant, even though the full action is not invariant.

For an action $I = \int_M L$ in terms of a Lagrangian $L = \frac{1}{3!} L_{\mu\nu\lambda} dx^\mu dx^\nu dx^\lambda$, the corresponding Noether current J^μ has the form

$$*J = \Theta(\mathcal{L}_\xi \omega^{AB}, \mathcal{L}_\xi e^A) - I_\xi L, \quad (3.1)$$

where the star $*$ denotes the Hodge dual and $I_\xi L = \frac{1}{2} \xi^\mu L_{\mu\nu\lambda} dx^\nu dx^\lambda$ is the contraction operator acting on L . The symbol \mathcal{L}_ξ stands for the Lie derivative, which acts on the spin connection and vielbein as

$$\begin{aligned} \mathcal{L}_\xi \omega_\mu^{AB} &= \partial_\mu \xi^\nu \omega_\nu^{AB} + \xi^\nu \partial_\nu \omega_\mu^{AB} \\ &= D_\mu (\xi^\nu \omega_\nu^{AB}) + \xi^\nu R_{\nu\mu}^{AB}, \end{aligned} \quad (3.2)$$

and

$$\begin{aligned} \mathcal{L}_\xi e_\mu^A &= \partial_\mu \xi^\nu e_\nu^A + \xi^\nu \partial_\nu e_\mu^A \\ &= D_\mu (\xi^\nu e_\nu^A) + \xi^\nu T_{\nu\mu}^A - (\xi^\nu \omega_\nu^{AB}) e_{B\mu}, \end{aligned} \quad (3.3)$$

respectively.

Generally speaking, the conservation equation $d*J = 0$ implies, by virtue of the Poincaré lemma, that the current can always be written locally as the exterior derivative of certain quantity. It is the procedure developed in this section which permits to write down the current as $*J = dQ(\xi)$ globally and, therefore, to identify the Noether charge for a Killing vector $\xi = \xi^\mu \partial_\mu$.

For the purpose of construction of conserved quantities in TMG, we shall consider a boundary term B given by

$$B = \frac{1}{32\pi G} \epsilon_{ABC} \omega^{AB} e^C, \quad (3.4)$$

i.e., a half of the Gibbons-Hawking (*GH*) term. Supplementing the Einstein-Hilbert action with B sets a well-posed variational principle when the extrinsic curvature is held fixed at the boundary, and makes the action finite for asymptotically AdS spacetimes [26]. This point can be seen by performing an explicit comparison to Balasubramanian-Kraus regularization [18] (see appendix A). The unusual factor in eq. (3.4) respect to the one carried by the GH term arises naturally in the gauge formulation of standard gravity Lagrangian as a single Chern-Simons density for $SO(2, 2)$ group, and it is the simplest case of a regularization scheme known as Kounterterm method [27].

It is expected that any modification of the *effective* cosmological constant due to the Chern-Simons coupling μ would leave the form of B unchanged, what should not be the case of the counterterm coupling in ref. [18].

Plugging in eq. (3.4) into the general form of the surface term (2.4), one finds

$$\Theta = -\frac{1}{32\pi G} \epsilon_{ABC} (\delta\omega^{AB} e^C - \omega^{AB} \delta e^C) + \frac{1}{32\pi G\mu} (\delta\omega^{AB} \omega_{BA} + 2\delta e^A \lambda_A). \quad (3.5)$$

Splitting the Lagrangian as

$$L = L_1 + \frac{1}{32\pi G\mu} (L_{\text{CS}}(\omega) + 2\lambda_A T^A), \quad (3.6)$$

where L_1 is the Einstein-Hilbert part Lagrangian with the boundary term in eq. (3.4), makes clear that the total charge $Q(\xi)$ is the sum of two contributions

$$Q(\xi) = Q_1(\xi) + Q_2(\xi). \quad (3.7)$$

The first term, $Q_1(\xi)$, is associated to the standard gravity part [26]

$$Q_1(\xi) = -\frac{1}{32\pi G} \int_{\Sigma_\infty} \epsilon_{ABC} (\xi^\mu \omega_\mu^{AB} e_\nu^C + \xi^\mu e_\mu^A \omega_\nu^{BC}) dx^\nu, \quad (3.8)$$

where Σ_∞ is the boundary of the spatial section at constant time. The construction of the second term, proportional to μ^{-1} , $Q_2(\xi)$, is carried out below.

Since the torsion vanishes, the spin-connection ω_μ^{AB} is a function of the dreibein e_μ^A

$$\omega_\mu^{AB} = -e^{B\alpha} \partial_\mu e_\alpha^A + \Gamma_{\nu\mu}^\lambda e_\lambda^A e^{B\nu}, \quad (3.9)$$

where $\Gamma_{\nu\mu}^\lambda$ is the Christoffel symbol. In Gauss-normal form of the metric,

$$ds^2 = \mathcal{G}_{\mu\nu} dx^\mu dx^\nu = N^2(r) dr^2 + h_{ij}(r, x) dx^i dx^j, \quad (3.10)$$

where r is the Schwarzschild-like radial coordinate (the boundary is at $r \rightarrow \infty$), some of the components of the Christoffel symbol are related to the extrinsic curvature $K_{ij} = \frac{1}{2N} h'_{ij}$ as

$$\Gamma_{ij}^r = -\frac{1}{N} K_{ij}, \quad \Gamma_{rj}^i = N K_j^i. \quad (3.11)$$

Here, the prime denotes radial derivative $\partial/\partial r$.

The line element at the boundary of the spacetime in the frame (3.10) can be, in turn, written in a time-like ADM form

$$h_{ij} dx^i dx^j = -N_\Sigma^2(t) dt^2 + \sigma_{\underline{mn}} (N^{\underline{m}} dt + dy^{\underline{m}}) (N^{\underline{n}} dt + dy^{\underline{n}}), \quad (3.12)$$

generated by a unit normal $u_i = (-N_\Sigma, \vec{0})$. It is clear that for a constant-time slice, $\sigma_{\underline{mn}}$ is the metric on Σ_∞ .

For a given set of asymptotic Killing vectors $\{\xi^i\}$, the charge (3.8) expressed in terms of boundary tensors is

$$Q_1(\xi) = \int_{\Sigma_\infty} \sqrt{\sigma} dy \xi^j q_{(1)j}^i u_i, \quad (3.13)$$

where the integrand reads

$$q_{(1)j}^i = \frac{1}{16\pi G} \epsilon_{mk} (\delta_l^k K_j^m + \delta_j^k K_l^m) \epsilon^{il}, \quad (3.14)$$

and we have used the convention $\epsilon^{rij} = -\epsilon^{ij}$.

The Noether current for the second part in the Lagrangian (3.6) uses the corresponding surface term

$$\Theta_{(2)} = \frac{1}{32\pi G\mu} (\mathcal{L}_\xi \omega^{AB} \omega_{BA} + 2\mathcal{L}_\xi e^A \lambda_A), \quad (3.15)$$

and the explicit expressions of Lie derivative on the fields, eqs. (3.2) and (3.3), such that

$$\begin{aligned} *J_{(2)} &= \frac{1}{32\pi G\mu} (\partial_\alpha \xi^\mu \omega_\mu^{AB} + \xi^\mu \partial_\mu \omega_\alpha^{AB}) \omega_{BA\beta} dx^\alpha \wedge dx^\beta \\ &\quad + \frac{1}{16\pi G\mu} (\partial_\alpha \xi^\mu e_\mu^A + \xi^\mu \partial_\mu e_\alpha^A) \lambda_{A\beta} dx^\alpha \wedge dx^\beta \\ &\quad - \frac{1}{32\pi G\mu} \left[I_\xi L_{\text{CS}}(\omega) + \xi^\mu (\lambda_{A\mu} T_{\alpha\beta}^A - 2\lambda_{A\alpha} T_{\mu\beta}^A) dx^\alpha \wedge dx^\beta \right], \end{aligned} \quad (3.16)$$

where the contraction operator acting on the gravitational Chern-Simons term produces

$$I_\xi L_{\text{CS}}(\omega) = \xi^\mu (\omega_\mu^{AB} \partial_\alpha \omega_{BA\beta} - \omega_\alpha^{AB} \partial_\mu \omega_{BA\beta} + \omega_\alpha^{AB} \partial_\beta \omega_{BA\mu} + 2\omega_{\mu B}^A \omega_{\alpha C}^B \omega_{\beta A}^C) dx^\alpha \wedge dx^\beta. \quad (3.17)$$

We show now that the different contributions in the current can be rearranged as a total derivative plus the equations of motion in the form

$$*J = d\mathcal{Q} + (\xi \cdot \omega^{AB})(\text{e.o.m.})_{AB} + (\xi \cdot e^A)(\text{e.o.m.})_A + (\xi \cdot \lambda_A)(\text{e.o.m.})^A.$$

Indeed, the part proportional to $1/\mu$ is

$$\begin{aligned} J_{(2)}^\lambda &= \frac{1}{64\pi G\mu} \frac{\epsilon^{\lambda\alpha\beta}}{\sqrt{-\mathcal{G}}} \left[\partial_\alpha (\xi^\mu \omega_\mu^{AB} \omega_{BA\beta} + 2\xi^\mu e_\mu^A \lambda_{A\beta}) \right. \\ &\quad \left. - (\xi^\mu \omega_\mu^{AB}) (R_{BA\alpha\beta} + 2\lambda_{A\beta} e_{B\alpha}) - (\xi^\mu \lambda_{A\mu}) T_{\alpha\beta}^A - 2(\xi^\mu e_\mu^A) D_\alpha \lambda_{A\beta} \right]. \end{aligned} \quad (3.18)$$

On the other hand, in the derivation of the first part of the Noether charge, the corresponding current is

$$\begin{aligned} J_{(1)}^\lambda &= \frac{1}{64\pi G\mu} \frac{\epsilon^{\lambda\alpha\beta}}{\sqrt{-\mathcal{G}}} \left[\partial_\alpha (-\epsilon_{ABC} (\xi^\mu \omega_\mu^{AB} e_\beta^C + \xi^\mu e_\mu^A \omega_\beta^{BC})) \right. \\ &\quad \left. - (\xi^\mu \omega_\mu^{AB}) \epsilon_{ABC} T_{\alpha\beta}^C + 2(\xi^\mu e_\mu^A) \epsilon_{ABC} \left(\frac{1}{2} R_{\alpha\beta}^{BC} + \frac{1}{\ell^2} e_\alpha^B e_\beta^C \right) \right], \end{aligned} \quad (3.19)$$

what makes easy reading off the second piece of the Noether charge from the first line of eq. (3.16),

$$Q_2(\xi) = \frac{1}{32\pi G\mu} \int_{\Sigma_\infty} (\xi^\mu \omega_\mu^{AB} \omega_{BA\nu} + 2\xi^\mu e_\mu^A \lambda_{A\nu}) dx^\nu. \quad (3.20)$$

It can be shown that the Noether charge $Q(\xi)$ in eq. (3.7) is equivalent (up to finite terms that do not modify the value of the mass and angular momentum) to a Hamiltonian charge defined in ref. [28] associated to the same isometry ξ (see appendix D).

In what follows, we first evaluate the charges derived in this section for the Bañados-Teitelboim-Zanelli (BTZ) black hole [29, 30], obtained from a global AdS spacetime through identifications that preserve the constant-curvature property. After that, we consider TMG solutions with more general asymptotics.

3.1 Solutions with constant curvature

For spacetimes which are locally AdS, the Riemann tensor satisfies

$$R_{\mu\nu}^{\alpha\beta} = -\frac{1}{\ell^2} \delta_{[\mu\nu]}^{[\alpha\beta]}, \quad (3.21)$$

what implies a particular value of the Lagrange multiplier (2.9) in terms of the dreibein as

$$\lambda_\mu^A = \frac{1}{\ell^2} e_\mu^A. \quad (3.22)$$

Constant-curvature condition solves trivially the equations of motion of TMG.

In that case, the corresponding Noether charge is

$$Q_2(\xi) = \frac{1}{32\pi G\mu} \int_{\Sigma_\infty} \left(\xi^\mu \omega_\mu^{AB} \omega_{BA\nu} + \frac{2}{\ell^2} \xi^\mu e_\mu^A e_{A\nu} \right) dx^\nu. \quad (3.23)$$

By virtue of the torsionless condition for the spin connection (3.9) and the Gaussian form of the Christoffel symbol (3.11), one can rewrite the charge (3.23) for asymptotic Killing vectors in terms of quantities defined at the boundary, in a similar form to eq. (3.13)

$$Q_{(2)}(\xi) = \int_{\partial\Sigma} \sqrt{\sigma} dy \xi^j q_{(2)j}^i u_i, \quad (3.24)$$

where

$$\begin{aligned} q_{(2)j}^i &= \frac{1}{32\pi G\mu} \left(\Gamma_{\nu j}^\lambda \Gamma_{\lambda k}^\nu + \frac{2}{\ell^2} h_{jk} \right) \frac{\epsilon^{ik}}{\sqrt{-h}} \\ &= \frac{1}{32\pi G\mu} \left(-2K_{jl} K_k^l + \Gamma_{mj}^l(h) \Gamma_{lk}^m(h) + \frac{2}{\ell^2} h_{jk} \right) \frac{\epsilon^{ik}}{\sqrt{-h}}. \end{aligned} \quad (3.25)$$

We have dropped a few terms containing derivatives of the boundary zweibein of the type $\partial_i e_j^A$. These are scheme-dependent contributions, because they arise from the difference between $L_{\text{CS}}(\omega)$ in (2.2) and the gravitational Chern-Simons in terms of the Christoffel symbol $L_{\text{CS}}(\Gamma)$

$$L_{\text{CS}}(\Gamma) = \epsilon^{\mu\nu\lambda} \left(\Gamma_{\mu\alpha}^\beta \partial_\nu \Gamma_{\lambda\beta}^\alpha + \frac{2}{3} \Gamma_{\mu\alpha}^\gamma \Gamma_{\nu\beta}^\alpha \Gamma_{\lambda\gamma}^\beta \right) \quad (3.26)$$

which can be expressed as a topological invariant plus a boundary term,

$$L_{\text{CS}}(\Gamma) - L_{\text{CS}}(\omega) = \epsilon^{\mu\nu\lambda} \left[-\frac{1}{3} (e_A^\alpha \partial_\mu e_\beta^A) (e_B^\beta \partial_\nu e_\sigma^B) (e_C^\sigma \partial_\lambda e_\alpha^C) + \partial_\mu (\omega_\nu^{AB} \partial_\lambda e_{B\alpha} e_A^\alpha) \right]. \quad (3.27)$$

The rotating solution of Einstein-Hilbert AdS gravity in three dimensions is the BTZ black hole given by the metric

$$ds^2 = \mathcal{G}_{\mu\nu} dx^\mu dx^\nu = -f^2(r) dt^2 + \frac{dr^2}{f^2(r)} + r^2 (N_\phi(r) dt + d\phi)^2, \quad (3.28)$$

with the azimuthal angle $0 \leq \phi \leq 2\pi$ and where the metric function and angular shift read

$$f^2(r) = -8GM + \frac{r^2}{\ell^2} + \frac{16G^2 J^2}{r^2}, \quad N_\phi(r) = -\frac{4GJ}{r^2}. \quad (3.29)$$

The horizons of the BTZ black hole are the roots of the equation $f^2(r) = 0$, that is,

$$r_{\pm}^2 = 4GM\ell^2 \left(1 \pm \sqrt{1 - \frac{J^2}{\ell^2 M^2}} \right). \quad (3.30)$$

Using eqs. (3.14) and (3.25) for the metric (3.28) give a formula for the total mass

$$\begin{aligned} \mathcal{M} &\equiv Q(\partial_t) = \lim_{r \rightarrow \infty} \frac{1}{8G} \left[\left(-f^2 + \frac{r^2}{\ell^2} + r^2 N_\phi^2 \right) + \frac{2}{\mu} \frac{r^2 N_\phi}{\ell^2} \right], \\ &= M - \frac{J}{\mu \ell^2}, \end{aligned} \quad (3.31)$$

whereas for the total angular momentum we obtain

$$\begin{aligned} \mathcal{J} &\equiv Q(\partial_\phi) = \lim_{r \rightarrow \infty} \frac{1}{8G} \left[-2r^2 N_\phi + \frac{1}{\mu} \left(f^2 - \frac{r^2}{\ell} - r^2 N_\phi^2 \right) \right], \\ &= J - \frac{M}{\mu}. \end{aligned} \quad (3.32)$$

The above Noether charges agree with the standard results computed by canonical methods [31–35] and holographic procedures [16, 19, 20], and they satisfy the first law of black hole thermodynamics

$$T \delta S = \delta \mathcal{M} + \Omega \delta \mathcal{J}, \quad (3.33)$$

where T is the Hawking temperature

$$T = \frac{1}{4\pi} f'(r_+) = \frac{1}{2\pi r_+} \left(\frac{r_+^2}{\ell^2} - \frac{16G^2 J^2}{r_+^2} \right), \quad (3.34)$$

and Ω is the angular velocity of the horizon

$$\Omega = N_\phi(r_+) = -\frac{r_-}{\ell r_+}. \quad (3.35)$$

The relation (3.33) is valid only if one considers a contribution to the entropy proportional to the inner horizon due to the gravitational Chern-Simons term, that is,

$$S = \frac{2\pi r_+}{4G} \left(1 - \frac{1}{\ell \mu} \frac{r_-}{r_+} \right). \quad (3.36)$$

The general contribution to the entropy from the gravitational Chern-Simons term was first computed by Solodukhin [20] using the conical singularity method, which does not rely on the equations of motion. For BTZ black hole, the correction to the macroscopic entropy due to the inner horizon was derived from the first law of black hole thermodynamics in ref. [36], where the Noether charges in first-order formalism were also found. Solodukhin’s general formula has been reobtained by means of Tachikawa’s procedure [37], which incorporates corrections to the Wald’s formula due to the fact that $L_{CS}(\Gamma)$ is not invariant under diffeomorphisms.

The form of the charges for the constant-curvature case is identical to the one derived from a Lagrangian that is the linear combination of two Chern-Simons densities for the $SO(2,2)$ group, constructed up with invariant tensors of different parity (see appendix A)

$$\tilde{L} = -\frac{\ell}{16\pi G} \left\langle AdA + \frac{2}{3} A^3 \right\rangle_1 + \frac{1}{16\pi G\mu} \left\langle AdA + \frac{2}{3} A^3 \right\rangle_2. \quad (3.37)$$

Indeed, following the Noether procedure (see ref. [26]), one can obtain the Noether charge associated to a Killing vector ξ^μ for an arbitrary Chern-Simons action. For the above case, the conserved quantity takes the form

$$\tilde{Q}(\xi) = -\frac{\ell}{16\pi G} \int_{\Sigma_\infty} \xi^\mu \langle A_\mu A_\nu \rangle_1 dx^\nu + \frac{1}{16\pi G\mu} \int_{\Sigma_\infty} \xi^\mu \langle A_\mu A_\nu \rangle_2 dx^\nu. \quad (3.38)$$

Substituting the traces for the generators of the AdS group defined in appendix A, eqs. (A.8) and (A.9), it is straightforward to reproduce the charges (3.8) and (3.23) from the corresponding pieces in the last expression.

The Lagrangian density (3.37) induces a topological theory in four dimensions which, in the conventions of the appendix A, reads

$$d\tilde{L} = -\frac{\ell}{16\pi G} \langle F^2 \rangle_1 + \frac{1}{16\pi G\mu} \langle F^2 \rangle_2. \quad (3.39)$$

Defining the dual of the field strength in the indices of the universal covering of AdS as

$$* \mathcal{F}^{AB} = \frac{1}{2} \epsilon^{ABCD} \mathcal{F}_{CD}, \quad (3.40)$$

the equation (3.39) can be cast into the equivalent form

$$d\tilde{L} = -\frac{\ell}{64\pi G} \epsilon_{ABCD} \left(\mathcal{F}^{AB} \mathcal{F}^{CD} + \frac{1}{\mu\ell} * \mathcal{F}^{AB} \mathcal{F}^{CD} \right), \quad (3.41)$$

$$= -\frac{\ell}{32\pi G} \left\langle (F \pm *F)^2 \right\rangle_1 - \frac{\ell}{16\pi G} \left(\frac{1}{\mu\ell} \mp 1 \right) \langle F * F \rangle_1, \quad (3.42)$$

where, in the last line, one employs the identity

$$\langle F^2 \rangle_1 = \frac{1}{2} (\langle F^2 \rangle_1 + \langle *F^2 \rangle_1), \quad (3.43)$$

consistent with $\eta_{AB} = \text{diag}(-1, 1, 1, -1)$. The form of eq. (3.42) makes clear that when the Chern-Simons coupling is $\mu\ell = \mp 1$, $d\tilde{L}$ vanishes identically for a globally (anti) self-dual AdS curvature ($*F = \pm F$). This fact resembles on the case of inclusion of topological invariants in the four-dimensional AdS gravity action, where globally (anti) self-dual solutions in the Weyl tensor are interpreted as the vacuum state of the theory [38].

3.2 Perturbed Extreme BTZ black hole

An axisymmetric solution of Topologically Massive Gravity at the chiral point $\mu\ell = 1$ has been found recently in ref. [16]. The Riemann tensor for this space is no longer constant,

because the metric contains a logarithmic branch. It is, however, an asymptotically AdS spacetime defined in terms of the *relaxed* boundary conditions given by Grumiller and Johansson [7, 14]. Therefore, this is the first exact stationary solution obtained for Log Gravity, as only pp-wave solutions existed previously in the literature [39].

For this solution, the line element in the ADM form is

$$ds^2 = -\mathcal{N}_\perp^2(r) dt^2 + \frac{dr^2}{N^2(r)} + R^2(r) (d\phi - \mathcal{N}_\phi(r) dt)^2, \quad (3.44)$$

with

$$\mathcal{N}_\perp^2(r) = N^2(r) - r^2 N_\phi^2(r) - N_k^2(r) + R^2(r) \mathcal{N}_\phi^2(r), \quad (3.45)$$

$$R^2(r) = r^2 + \ell^2 N_k^2(r), \quad (3.46)$$

$$\mathcal{N}_\phi(r) = \frac{r^2 N_\phi(r) + \ell N_k^2(r)}{R^2(r)}. \quad (3.47)$$

The functions N , N_ϕ and N_k in the above expressions are defined as

$$N^2(r) = \frac{r^2}{\ell^2} - 8\pi GM + \frac{16\pi^2 G^2 M^2 \ell^2}{r^2}, \quad (3.48)$$

$$N_\phi(r) = \frac{4\pi GM\ell}{r^2}, \quad (3.49)$$

$$N_k^2(r) = k \log \left(\frac{r^2 - 4\pi GM\ell^2}{r_0^2} \right), \quad (3.50)$$

where the constants k and r_0 are two arbitrary parameters.

Summing the contributions of the charge formulas (3.8) and (3.20), the total mass and angular momentum are

$$\mathcal{M} = \frac{k}{2G}, \quad \mathcal{J} = -\frac{k\ell}{2G}. \quad (3.51)$$

The dependence on the parameter k makes evident the fact that, when the logarithmic branch is switched off, the charges vanish identically, that corresponds to the extreme BTZ black hole ($J = -M\ell$ in the conventions of refs. [4, 16]). The results in eq. (3.51) have been verified in different frameworks, which are the Clement's Super Angular Momentum (SAM) method [40] and the Barnich-Brandt-Compère approach to conserved quantities [41, 42].

There is, however, a discrepancy in a global factor in the charges (3.51) respect to the ones computed in the original reference. As we mentioned in the Introduction, TMG does not lend itself to a clear definition of quasilocal stress tensor. This means that any attempt to integrate by parts the variations of the extrinsic curvature $\delta K_{ij} = \frac{1}{2N} \partial_r (\delta h_{ij}) + \dots$ and to cast the variation of the action in the form $\delta I = \frac{1}{2} \int_{\partial M} \sqrt{-h} \tau^{ij} \delta h_{ij}$ would necessarily be ambiguous in the definition of the new Brown-York stress tensor τ^{ij} . Thus, the mismatch of (3.51) with the conserved quantities calculated in ref. [16] must be due to this problem inherent to TMG.

3.3 Warped AdS₃ black holes

It has been argued in ref. [43] that TMG could present stable backgrounds which are spacelike, timelike or lightlike warped anti-de Sitter spaces. In this work, the authors also

conjecture the form of the corresponding left and right moving central charges, simply based on an entropy argument. The proposal for the value of c_L was explicitly verified from the asymptotic symmetry algebra in [44].

It is well-known that BTZ black hole is obtained as the quotient of global AdS₃ space by a discrete subgroup of $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ isometries. In a similar way, since warped AdS black holes are locally warped AdS₃ with isometries $SL(2, \mathbb{R}) \times U(1)$, they are also constructed as identifications of warped AdS.

Spacelike (timelike) warped AdS black hole solutions of TMG are produced by an identification along a Killing vector of a $SL(2, \mathbb{R}) \times U(1)$ -invariant vacuum, where the $U(1)$ isometry is spacelike (timelike) [43].

For convenience, the Chern-Simons coupling is redefined in terms of a dimensionless parameter as $\nu = \mu\ell/3$.

For illustrative purposes, we compute here the conserved quantities associated to the spacelike stretched black holes ($\nu^2 > 1$). The ADM form of the metric reads

$$ds^2 = -N^2(r) dt^2 + \frac{\ell^2 dr^2}{4R^2(r)N^2(r)} + R^2(r) (d\phi + N_\phi(r)dt)^2, \quad (3.52)$$

where

$$R^2(r) = \frac{r}{4} \left[3r(\nu^2 - 1) + (\nu^2 + 3)(r_+ + r_-) - 4\nu\sqrt{r_+r_-(\nu^2 + 3)} \right], \quad (3.53)$$

$$N^2(r) = \frac{(\nu^2 + 3)(r - r_+)(r - r_-)}{4R^2(r)}, \quad (3.54)$$

$$N_\phi(r) = \frac{2\nu r - \sqrt{r_+r_-(\nu^2 + 3)}}{2R^2(r)}, \quad (3.55)$$

and ϕ is the angle with a period 2π .

Spacelike stretched black holes of this type were first obtained –in a slightly different coordinate system– in ref. [23].

In the general case, the integrand of the second part of the Noether charge (3.20) can be expressed as

$$q_{(2)j}^i = \frac{\ell}{96\pi G\nu} \left(-2K_{jl}K_k^l + \Gamma_{mj}^l(h)\Gamma_{lk}^m(h) - 4S_{jk} \right) \frac{\epsilon^{ik}}{\sqrt{-h}}, \quad (3.56)$$

where S_{jk} are the boundary components of the Schouten tensor (2.8),

$$\begin{aligned} S_{ij} = & \mathcal{R}_{ij}(h) - \frac{1}{4} h_{ij} \mathcal{R}(h) - \frac{1}{N} \left(\partial_r K_{ij} - \frac{1}{2} h_{ij} \partial_r K \right) \\ & + 2K_{ik}K_j^k - K K_{ij} + \frac{1}{4} h_{ij} \left(K^2 + K^{kl} K_{kl} \right), \end{aligned} \quad (3.57)$$

as a consequence of Gauss-Codazzi relations listed in appendix B. Here, $\mathcal{R}_{ij}(h)$ and $\mathcal{R}(h)$ are the Ricci tensor and Ricci scalar of the boundary, respectively.

For a stationary metric described by eq. (3.52), the Schouten tensor S_μ^ν takes the general form found in appendix C. In particular, for warped AdS black holes with functions

eqs. (3.53)–(3.55), it is given by

$$S_\nu^\mu = \begin{pmatrix} -\frac{4\nu^2-3}{2\ell^2} & 0 & 0 \\ 0 & \frac{2\nu^2-3}{2\ell^2} & 0 \\ -\frac{3(\nu^2-1)(2\nu r - \sqrt{r_+ r_- (\nu^2+3)})}{2\ell^2} & 0 & \frac{2\nu^2-3}{2\ell^2} \end{pmatrix}. \quad (3.58)$$

In terms of $R(r)$, $N(r)$ and $N_\phi(r)$, the conserved quantities for the metric (3.52) are

$$\begin{aligned} \mathcal{M} = & -\frac{R}{8\ell G} (2N^2 R' - R(N^2)' + R^3(N_\phi^2)') - \\ & -\frac{R^3}{12\ell G\nu} \left[N^2 (2(RN_\phi)'' + 3R'N_\phi') + 3R^3(N_\phi')^2 N_\phi \right. \\ & \left. + (N^2)' \left(N_\phi R' - \frac{1}{2} R N_\phi' \right) - R N_\phi (N^2)'' \right] \end{aligned} \quad (3.59)$$

and

$$\mathcal{J} = \frac{R^3}{4\ell G} \left[\ell^2 R N_\phi' + \frac{1}{3\nu} \left(3\ell^2 R^3 (N_\phi')^2 + 2N (R'N)' - R (N^2)'' \right) \right], \quad (3.60)$$

that, substituting eqs. (3.53)–(3.55), produces

$$\mathcal{M} = \frac{(\nu^2+3)}{48G\ell} \left(r_+ + r_- - \frac{1}{\nu} \sqrt{r_+ r_- (\nu^2+3)} \right) \quad (3.61)$$

and

$$\mathcal{J} = \frac{\nu(\nu^2+3)}{96G\ell} \left[\left(r_+ + r_- - \frac{1}{\nu} \sqrt{r_+ r_- (\nu^2+3)} \right)^2 - \frac{(5\nu^2+3)}{4\nu^2} (r_+ - r_-)^2 \right]. \quad (3.62)$$

Up to a factor 2 in eq. (3.61), the above quantities correspond to the conserved quantities computed in ref. [43] using the Abbott-Deser-Tekin approach [22] and also the ones obtained by the Hamiltonian procedure in first-order formalism in ref. [35]. The mismatch in the mass respect to the value obtained by ADT formula ($\mathcal{M} = (1/2) \mathcal{M}_{\text{ADT}}$) is similar to the one found for ACL black holes [32], when the charges are computed by SAM method [23].

4 Summary and outlook

In this work, we have used the surface terms that come from the first-order formulation of TMG with negative cosmological constant to construct background-independent charges from the Noether theorem. It has been shown that this definition of conserved quantities gives finite results for locally AdS black holes, solutions of Log Gravity and warped AdS black holes without the need for any background-subtraction procedure.

In ref. [45], the derivation of TMG from an action that depends explicitly on the torsion through a constraint imposed by a Lagrange multiplier has also been employed to obtain the Witten-Nester charges in the supersymmetric extension of this gravity theory. The generality of the procedure makes possible a comparison between Witten-Nester and ADT charges and a further generalization of the latter to an arbitrary background. This also

shows the equivalence with the conserved quantities discussed in ref. [23]. Background-independence is encoded in the charges obtained by the SAM method, but it is not obvious to us that one can reconstruct the covariant formulas found here from the expressions given in a particular frame in ref. [23]. In any case, it would be interesting to explore the relation of the existing methods to the charges presented in this paper.

In gravity with standard AdS asymptotics, the existence of background-independent conserved quantities is a consequence of holographic renormalization, in the sense that this procedure identifies the divergences in the action and provides a systematic construction of the counterterms needed to get rid of them.

In TMG, the situation is kind of the opposite as the obtention of background-independent charges above may be regarded as a step toward a holographic formulation of the theory.

In that respect, it is clear that the method carried out here should reduce to the discussion on holographic stress tensor by Kraus-Larsen [19] and Solodukhin [20] for asymptotically locally AdS spacetimes.

For a modified asymptotic behavior of the metric that includes logarithmic terms, the finiteness of the Noether charges shown in subsection 3.2 suggests that holographic renormalization might be performed in Log Gravity using a metric frame which considers relaxed Brown-Henneaux (or Fefferman-Graham) fall-off conditions [7, 14, 17].

The case of asymptotically warped AdS₃ is expected to be both conceptual and technically more challenging. However, new insight on boundary conditions for warped AdS black holes (see, e.g., [46]) and recent study of the asymptotic structure of the field equations for TMG discussed in ref. [21] could make possible a better understanding of the theory in the context of AdS/CFT correspondence.

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A AdS group

The AdS group in three dimensions is isomorphic to the four-dimensional rotation group SO(2,2) that leaves invariant the quadratic form

$$\eta_{AB} x^A x^B = -x_0^2 + x_1^2 + x_2^2 - x_3^2 = -\ell^2. \tag{A.1}$$

The group has six generators $J_{AB} = -J_{BA}$, which can be decomposed as $J_{AB} = \{J_{AB}, P_A\}$, with $A, B, \dots = 0, 1, 2$, where J_{AB} are Lorentz rotations and $P_A = J_{A3}$ are AdS translations.

SO(2, 2) generators satisfy the algebra

$$[J_{\underline{AB}}, J_{\underline{CD}}] = \eta_{\underline{BC}} J_{\underline{AD}} - \eta_{\underline{AC}} J_{\underline{BD}} + \eta_{\underline{AD}} J_{\underline{BC}} - \eta_{\underline{BD}} J_{\underline{AC}}, \quad (\text{A.2})$$

or, in terms of the generators J_{AB} and P_A ,

$$\begin{aligned} [J_{AB}, J_{CD}] &= \eta_{BC} J_{AD} - \eta_{AC} J_{BD} + \eta_{AD} J_{BC} - \eta_{BD} J_{AC}, \\ [J_{AB}, P_C] &= \eta_{BC} P_A - \eta_{AC} P_B, \\ [P_A, P_C] &= J_{AC}. \end{aligned} \quad (\text{A.3})$$

The Lie-algebra valued gauge connection $A = A_\mu dx^\mu$ takes the form

$$\begin{aligned} A &= \frac{1}{2} W^{\underline{AB}} J_{\underline{AB}} \\ &= \frac{1}{2} \omega^{AB} J_{AB} + \frac{1}{\ell} e^A P_A, \end{aligned} \quad (\text{A.4})$$

and the corresponding AdS curvature $F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$ is

$$\begin{aligned} F &= dA + A \wedge A = \frac{1}{2} \mathcal{F}^{\underline{AB}} J_{\underline{AB}} \\ &= \frac{1}{2} \left(R^{AB} + \frac{1}{\ell^2} e^A \wedge e^B \right) J_{AB} + \frac{1}{\ell} T^A P_A. \end{aligned} \quad (\text{A.5})$$

Bianchi identity for AdS group $\nabla_{\text{AdS}} F = 0$ implies the standard differential and algebraic Bianchi identities

$$D(\omega) R^{AB} = 0, \quad D(\omega) T^A = R^{AB} \wedge e_B. \quad (\text{A.6})$$

In a similar fashion as for four-dimensional Lorentz group, there exist two inequivalent invariant tensors for SO(2, 2), that is,

$$\begin{aligned} \langle J_{\underline{AB}} J_{\underline{CD}} \rangle_1 &= \epsilon_{\underline{ABCD}}, \\ \langle J_{\underline{AB}} J_{\underline{CD}} \rangle_2 &= \eta_{[\underline{AB}], [\underline{CD}]} \equiv \eta_{\underline{AD}} \eta_{\underline{BC}} - \eta_{\underline{BD}} \eta_{\underline{AC}}, \end{aligned} \quad (\text{A.7})$$

whose only non-vanishing components are

$$\langle J_{AB} P_C \rangle_1 = \epsilon_{ABC}, \quad (\text{A.8})$$

with the convention $\epsilon_{ABC3} \equiv \epsilon_{ABC}$, and

$$\begin{aligned} \langle J_{AB} J_{CD} \rangle_2 &= \eta_{AD} \eta_{BC} - \eta_{BD} \eta_{AC}, \\ \langle P_A P_B \rangle_2 &= \eta_{AB}. \end{aligned} \quad (\text{A.9})$$

A different choice of the trace of the AdS generators in a Chern-Simons form for SO(2, 2) group

$$L_{\text{CS}}(A) = \left\langle AdA + \frac{2}{3} A^3 \right\rangle_{\text{AdS}} \quad (\text{A.10})$$

produces different gravity actions. Indeed, taking (A.8) as the corresponding invariant tensor, the Chern-Simons density is proportional to the Lagrangian of standard gravity

$$\left\langle AdA + \frac{2}{3}A^3 \right\rangle_1 = \frac{1}{\ell} \epsilon_{ABC} \left(R^{AB} + \frac{1}{3\ell^2} e^A e^B \right) e^C - \frac{1}{2\ell} d(\epsilon_{ABC} \omega^{AB} e^C). \quad (\text{A.11})$$

In the Gauss-normal frame (3.10), it is easy to see that the boundary term in eq. (3.4) is half of the Gibbons-Hawking term, because

$$\frac{1}{32\pi G} \epsilon_{ABC} \omega^{AB} e^C = \frac{1}{16\pi G} \sqrt{-h} K. \quad (\text{A.12})$$

Any asymptotically AdS spacetime metric can be put in a Fefferman-Graham form [47]

$$ds^2 = \frac{\ell^2}{4\rho^2} d\rho^2 + \frac{1}{\rho} g_{ij} dx^i dx^j, \quad (\text{A.13})$$

where $g_{ij}(x, \rho)$ accepts a regular expansion near the conformal boundary $\rho = 0$

$$g_{ij}(x, \rho) = g_{(0)ij} + \rho g_{(1)ij} + \dots \quad (\text{A.14})$$

The above expansion results in an asymptotic behavior for the extrinsic curvature

$$K_j^i = \frac{1}{\ell} (\delta_j^i - \rho k_j^i), \quad (\text{A.15})$$

with the tensor k_j^i given by

$$k_j^i = g_{(1)ij} + \mathcal{O}(\rho). \quad (\text{A.16})$$

In three dimensions, Einstein's equation does not fully determine $g_{(1)ij}$ from the initial data $g_{(0)ij}$, but only implies

$$\nabla_{(0)i} g_{(1)j}^i = 0 \quad (\text{A.17})$$

and that the trace is

$$g_{(1)} = -\frac{\ell^2}{2} \mathcal{R}_{(0)}. \quad (\text{A.18})$$

It was shown in ref. [26] that adding and subtracting the Gibbons-Hawking term from the action

$$I = I_{EH} + \frac{1}{16\pi G} \int_{\partial M} \sqrt{-h} K, \quad (\text{A.19})$$

one is able to recover the Balasubramanian-Kraus counterterm plus a topological invariant of the metric $g_{(0)}$. Indeed, with the help of relations (A.13)–(A.18) we have

$$-\frac{1}{16\pi G} \sqrt{-h} K = -\frac{1}{8\pi G \ell} \left(\sqrt{-h} + \frac{\ell^2}{4} \sqrt{-g_{(0)}} \mathcal{R}_{(0)} \right). \quad (\text{A.20})$$

In turn, the *exotic* copy of CS for AdS group is the sum of the gravitational Chern-Simons term plus the translational Chern-Simons term $e_A T^A$

$$\left\langle AdA + \frac{2}{3}A^3 \right\rangle_2 = \frac{1}{2} \left(L_{CS}(\omega) + \frac{2}{\ell^2} e_A T^A \right). \quad (\text{A.21})$$

For a given AdS radius, the above relation singles out (3.22) as the value of the Lagrange multiplier λ_A that produces a symmetry-enhancement in the Mielke-Baekler theory [48] from local Lorentz to AdS group in the gravity action (2.1).

B Gauss-Codazzi relations

We consider a spacetime described by the radial foliation

$$ds^2 = \mathcal{G}_{\mu\nu} dx^\mu dx^\nu = N^2(r) dr^2 + h_{ij}(r, x) dx^i dx^j, \quad (\text{B.1})$$

which defines the extrinsic curvature as

$$K_{ij} = \frac{1}{2N} h'_{ij}. \quad (\text{B.2})$$

In this frame, the non-vanishing components of the Christoffel symbol $\Gamma_{\mu\nu}^\lambda(\mathcal{G})$ are

$$\begin{aligned} \Gamma_{rr}^r &= \frac{N'}{N}, & \Gamma_{ij}^r &= -\frac{1}{N} K_{ij}, \\ \Gamma_{jr}^i &= N K_j^i, & \Gamma_{jk}^i &= \Gamma_{jk}^i(h). \end{aligned} \quad (\text{B.3})$$

(The metric h_{ij} lowers and raises indices of tensors constructed from h_{ij} and K_{ij} .) Then, the curvature takes the Gauss-Codazzi form

$$R_{jr}^{ir} = -\frac{1}{N} (K_j^i)' - (K^2)^i_j, \quad (\text{B.4})$$

$$R_{jk}^{ir} = -\frac{1}{N} (\nabla_j K_k^i - \nabla_k K_j^i), \quad (\text{B.5})$$

$$R_{kr}^{ij} = N (\nabla^i K_k^j - \nabla^j K_k^i), \quad (\text{B.6})$$

$$R_{kl}^{ij} = \mathcal{R}_{kl}^{ij}(h) - K_k^i K_l^j + K_l^i K_k^j, \quad (\text{B.7})$$

where ∇ is the covariant derivative defined in terms of the Christoffel symbol of the boundary $\Gamma_{jk}^i(h)$ and $\mathcal{R}_{kl}^{ij}(h)$ is the Riemann tensor of the boundary.

As a consequence, Ricci tensor and Ricci scalar can be written as

$$R_j^i = \mathcal{R}_j^i(h) - K_j^i K - \frac{1}{N} (K_j^i)', \quad (\text{B.8})$$

$$R_j^r = \frac{1}{N} (\nabla_j K - \nabla_k K_j^k), \quad (\text{B.9})$$

$$R_r^r = -\frac{1}{N} K' - K_j^i K_i^j, \quad (\text{B.10})$$

and

$$R = \mathcal{R}(h) - K^2 - K_j^i K_i^j - \frac{2}{N} K'. \quad (\text{B.11})$$

C Schouten tensor for stationary metric

For a metric of the form

$$ds^2 = -N^2(r) dt^2 + \frac{\ell^2 dr^2}{4R^2(r)N^2(r)} + R^2(r) (d\phi + N_\phi(r) dt)^2, \quad (\text{C.1})$$

the components of the Schouten tensor read

$$\begin{aligned}
 S_{tt} = & -\frac{1}{2\ell^2} \left(4N^4(R')^2 + 4N^2(R^4)'(N_\phi^2)' + \frac{1}{2}(R^4)'N_\phi^2(N^2)' - 4R^4N_\phi^2(N')^2 + 5R^6N_\phi^2(N_\phi')^2 \right. \\
 & - 4R^4N_\phi^2NN'' + 4R^2N_\phi^2(R')^2N^2 + 8N^2R^4N_\phi N_\phi'' - \frac{1}{2}(N^4)'(R^2)' + 4R^3N_\phi^2N^2R'' \\
 & \left. - 4N^2R^2(N')^2 - 4N^3R^2N'' + 3N^2R^4(N_\phi')^2 + 4N^4RR'' \right), \quad (\text{C.2})
 \end{aligned}$$

$$\begin{aligned}
 S_{t\phi} = & -\frac{R^2}{2\ell^2} (8(R^2)'N^2N_\phi' + 4R^2N^2N_\phi'' + (R^2)'(N^2)'N_\phi + 5R^4(N_\phi')^2N_\phi + 4(R')^2N^2N_\phi \\
 & + 4RN^2R''N_\phi - 4N_\phi R^2(N')^2 - 4N_\phi R^2NN''), \quad (\text{C.3})
 \end{aligned}$$

$$S_{rr} = -\frac{1}{8R^2N^2} (4R^2(N')^2 + 4R^2NN'' - 3R^4(N_\phi')^2 + 4N^2RR'' + 4(R')^2N^2 + (R^2)'(N^2)'), \quad (\text{C.4})$$

$$S_{\phi\phi} = -\frac{R^2}{2\ell^2} ((R^2)'(N^2)' + 5R^4(N_\phi')^2 + 4(R')^2N^2 + 4N^2RR'' - 4R^2(N')^2 - 4R^2NN''). \quad (\text{C.5})$$

D Noether charges and Wald Hamiltonians

A Hamiltonian $H(\xi)$ describes the dynamics generated by the vector field ξ^μ , which is related to Noether current and surface term in eq. (3.1) by

$$\begin{aligned}
 \delta H(\xi) &= \delta \int_{\partial M} *J - \int_{\partial M} d(\xi \cdot \Theta) \\
 &= \delta \int_{\Sigma_\infty} \mathcal{Q}(\xi) - \int_{\Sigma_\infty} \xi \cdot \Theta, \quad (\text{D.1})
 \end{aligned}$$

where we have used the Stokes' theorem for $*J = d\mathcal{Q}(\xi)$ to integrate the Noether charge $Q(\xi) = \int_{\Sigma_\infty} \mathcal{Q}(\xi)$. The Hamiltonian exists if there is a $(D-1)$ -form \mathcal{B} (D is the dimension of spacetime) such that the second term is a total variation,

$$\int_{\Sigma_\infty} \xi \cdot \Theta(\phi, \delta\phi) = \delta \int_{\Sigma_\infty} \xi \cdot \mathcal{B}(\phi). \quad (\text{D.2})$$

Only if this is possible, one can write down the Wald Hamiltonian as [28]

$$H(\xi) = \int_{\Sigma_\infty} (\mathcal{Q}(\xi) - \xi \cdot \mathcal{B}). \quad (\text{D.3})$$

It is clear that the integrability criterion (D.2) requires to identify precise boundary conditions, what is not always possible. In general, the procedure of integrating (D.2) breaks general covariance, such that \mathcal{B} is a non-covariant correction to the charge.

There are, however, other instances where \mathcal{B} can be covariantly integrated and seen as a contribution coming from a boundary term in the action, what we show below for standard gravity.

Let us consider Einstein-Hilbert AdS action in three dimensions without boundary terms,

$$I_0 = -\frac{1}{16\pi G} \int_M \epsilon_{ABC} \left(R^{AB} + \frac{1}{3\ell^2} e^A e^B \right) e^C, \quad (\text{D.4})$$

whose on-shell variation gives a boundary term

$$\Theta_0(\delta e, \delta \omega) = \delta \omega^{AB} \frac{\delta I_0}{\delta R^{AB}} = -\frac{1}{16\pi G} \epsilon_{ABC} \delta \omega^{AB} e^C. \quad (\text{D.5})$$

The Noether charge derived from action (D.4), associated to a Killing vector is the three-dimensional equivalence of Komar integral, that is,

$$Q_0(\xi) = \int_{\Sigma_\infty} \xi^\mu \omega_\mu^{AB} \frac{\delta I_0}{\delta R^{AB}} = -\frac{1}{16\pi G} \int_{\Sigma_\infty} \epsilon_{ABC} \xi^\mu \omega_\mu^{AB} e^C. \quad (\text{D.6})$$

The Komar formula is obtained in an arbitrary dimension from the dimensional continuation of (D.4) and gives $Q_0(\partial_t) = \frac{D-3}{D-2} M + \lim_{r \rightarrow \infty} \frac{\text{Vol}(S^{D-2})}{8\pi G \ell^2} r^{D-1}$ for a spherical Schwarzschild-AdS black hole. This means that the correct mass for the BTZ black hole comes necessarily from the addition of boundary terms.

We can implement the integrability condition (D.2) for the surface term (D.5) by demanding a boundary condition, whose fully-covariant version is written as

$$\epsilon_{ABC} \delta \omega^{AB} e^C = \epsilon_{ABC} \omega^{AB} \delta e^C, \quad \text{on } \partial M. \quad (\text{D.7})$$

This boundary condition can be explicitly realized in Fefferman-Graham frame (A.13)–(A.15) by the conditions $K_j^i = \frac{1}{\ell} \delta_j^i + \mathcal{O}(\rho)$ and $\delta K_j^i = \mathcal{O}(\rho)$, which are compatible with asymptotically AdS spacetimes [26].

Using (D.7), we can integrate \mathcal{B} in eq. (D.2) in the following way,

$$\begin{aligned} \xi \cdot \Theta_0(\delta e, \delta \omega) &= -\frac{1}{16\pi G} \frac{1}{2} \xi \cdot (\epsilon_{ABC} \delta \omega^{AB} e^C + \epsilon_{ABC} \omega^{AB} \delta e^C) + \mathcal{O}(1) \\ &= -\delta \left[\xi \cdot \left(\frac{1}{32\pi G} \epsilon_{ABC} \omega^{AB} e^C \right) \right] + \mathcal{O}(1), \end{aligned} \quad (\text{D.8})$$

and therefore, \mathcal{B} is equal to (up to a finite contribution) minus the boundary term (3.4) that regularizes the Einstein-Hilbert part of the action,

$$\mathcal{B} = -B + \mathcal{O}(1) = -\frac{1}{32\pi G} \epsilon_{ABC} \omega^{AB} e^C + \mathcal{O}(1). \quad (\text{D.9})$$

A similar procedure can be carried out in four-dimensional AdS gravity in order to integrate out the boundary term necessary for the regularization of the conserved quantities, which also provides a fully-covariant correction to the Komar charge [49].

The additional term in (D.8) is responsible for a finite difference between the Noether charge $Q_1(\xi)$ in eq. (3.8) and the corresponding Hamiltonian, because

$$\delta H_1(\xi) = \delta Q_1(\xi) - \int_{\Sigma_\infty} \xi \cdot \Theta_1 \quad (\text{D.10})$$

where Θ_1 is the first part of the surface term (3.5) or, equivalently, $\Theta_1 = \Theta_0 + \delta B$.

To this end, let us calculate the finite contribution of $\Theta_1(\delta e, \delta \omega)$

$$\begin{aligned}\Theta_1(\delta e, \delta \omega) &= -\frac{1}{32\pi G} \epsilon_{ABC} (\delta \omega^{AB} e^C - \omega^{AB} \delta e^C) \\ &= -d^2x \frac{\sqrt{-h}}{16\pi G} \left[\left(K_j^i - \frac{1}{2} K \delta_j^i \right) (h^{-1} \delta h)_i^j + \delta K \right],\end{aligned}\quad (\text{D.11})$$

and, using expansion of the fields in FG frame (A.13)–(A.18), we find

$$\begin{aligned}\Theta_1 &= d^2x \frac{\sqrt{-g}}{16\pi G \ell} (k_j^i - k \delta_j^i) (g^{-1} \delta g)_i^j + d^2x \frac{1}{16\pi G \ell} \delta(\sqrt{-g} k) \\ &= \frac{1}{16\pi G \ell} d^2x \left[H_j^i (g_{(0)}^{-1} \delta g_{(0)})_i^j - \frac{\ell^2}{2} \delta(\sqrt{-g_{(0)}} \mathcal{R}_{(0)}) \right],\end{aligned}\quad (\text{D.12})$$

where we have defined the tensor $H_j^i = g_{(1)j}^i - g_{(1)} \delta_j^i$ which has conformal weight 2 and, by virtue of eq. (A.17), is covariantly conserved. The second term is a two-dimensional topological invariant. In this respect, H_j^i gives rise to an ambiguity in the definition of the Wald Hamiltonian and thus, to a difference between the Noether charge $Q_1(\xi)$ and the Hamiltonian $H_1(\xi)$.

More generally, it has been proved in ref. [50] that for asymptotically AdS spacetimes in $D = 2n + 1$ dimensions, the ambiguity in the definition of a Wald Hamiltonian can be written as

$$H'(\xi) = H(\xi) + \int_{\Sigma_\infty} d^{2n-1}y \sqrt{\sigma} u_i H_j^i \xi^j, \quad (\text{D.13})$$

in the notation of the ADM foliation (3.12). The tensor H_j^i has conformal weight $2n$ and its trace is proportional to the Weyl anomaly of the holographic stress tensor. The best known example is the four-dimensional one, where H_j^i is quadratic in the curvature [50–52].

In addition to Einstein-Hilbert action, TMG contains the terms

$$I_2 = \frac{1}{32\pi G \mu} \int_M (L_{\text{CS}}(\omega) + 2\lambda_A T^A). \quad (\text{D.14})$$

For the present discussion on Wald Hamiltonians, we will restrict ourselves to the asymptotically AdS sector, where $\lambda_A = e_A/\ell^2$. Precise boundary conditions suitable for Log Gravity and Warped AdS asymptotics are currently under investigation.

In this case, the contribution to the surface term of the action (D.14) is

$$\begin{aligned}\Theta_2 &= \frac{1}{32\pi G \mu} \left(\delta \omega^{AB} \omega_{BA} + \frac{2}{\ell^2} \delta e^A e_A \right) \\ &= d^2x \frac{1}{16\pi G \mu} \left(-\delta K_i^k K_{kj} - K_i^k K_j^l e_{al} \delta e_k^a + \frac{1}{\ell^2} \delta e_i^a e_{aj} \right) \epsilon^{ij},\end{aligned}\quad (\text{D.15})$$

where e_i^a is the boundary zweibein of the induced metric $h_{ij} = \eta_{ab} e_i^a e_j^b$. The above variation is not expressible in terms of variations of the metric tensor, but adopts the form

$$\Theta_2 = d^2x \tilde{H}_a^i \delta e_{(0)i}^a, \quad (\text{D.16})$$

where

$$\tilde{H}_a^i = \frac{1}{8\pi G \ell^2 \mu} g_{(1)j}^k e_{(0)ak} \epsilon^{ij} \tag{D.17}$$

is a covariantly conserved tensor with vanishing trace. Indeed, the tensor $\tilde{H}^{ij} = 2\tilde{H}_a^i e_{(0)}^{aj}$ has physical relevance, because it is the same as the holographic stress tensor associated to the action (D.14), whose antisymmetric part carries the gravitational anomaly [20]

$$\tilde{H}^{ij} \epsilon_{ij} = \frac{1}{8\pi G \mu} \mathcal{R}_{(0)}. \tag{D.18}$$

In an analogous way as for the Einstein-Hilbert action, where Weyl anomaly is nonvanishing, one could expect an ambiguity in the Wald Hamiltonian definition due to the existence of gravitational anomaly.

References

- [1] S. Deser, R. Jackiw and S. Templeton, *Three-Dimensional Massive Gauge Theories*, *Phys. Rev. Lett.* **48** (1982) 975 [SPIRES]; *Topologically massive gauge theories*, *Ann. Phys.* **140** (1982) 372 [SPIRES].
- [2] S. Carlip, S. Deser, A. Waldron and D.K. Wise, *Cosmological Topologically Massive Gravitons and Photons*, *Class. Quant. Grav.* **26** (2009) 075008 [arXiv:0803.3998] [SPIRES].
- [3] S. Deser, *A note on positive energy of topologically massive gravity*, arXiv:0907.4135 [SPIRES].
- [4] W. Li, W. Song and A. Strominger, *Chiral Gravity in Three Dimensions*, *JHEP* **04** (2008) 082 [arXiv:0801.4566] [SPIRES].
- [5] E. Witten, *Three-Dimensional Gravity Revisited*, arXiv:0706.3359 [SPIRES].
- [6] A. Maloney and E. Witten, *Quantum Gravity Partition Functions in Three Dimensions*, arXiv:0712.0155 [SPIRES].
- [7] D. Grumiller and N. Johansson, *Instability in cosmological topologically massive gravity at the chiral point*, *JHEP* **07** (2008) 134 [arXiv:0805.2610] [SPIRES].
- [8] M.-i. Park, *Constraint Dynamics and Gravitons in Three Dimensions*, *JHEP* **09** (2008) 084 [arXiv:0805.4328] [SPIRES].
- [9] S. Carlip, S. Deser, A. Waldron and D.K. Wise, *Topologically Massive AdS Gravity*, *Phys. Lett. B* **666** (2008) 272 [arXiv:0807.0486] [SPIRES].
- [10] D. Grumiller, R. Jackiw and N. Johansson, *Canonical analysis of cosmological topologically massive gravity at the chiral point*, arXiv:0806.4185 [SPIRES].
- [11] S. Carlip, *The Constraint Algebra of Topologically Massive AdS Gravity*, *JHEP* **10** (2008) 078 [arXiv:0807.4152] [SPIRES].
- [12] G. Giribet, M. Kleban and M. Porrati, *Topologically Massive Gravity at the Chiral Point is Not Chiral*, *JHEP* **10** (2008) 045 [arXiv:0807.4703] [SPIRES].
- [13] M. Blagojevic and B. Cvetkovic, *Canonical structure of topologically massive gravity with a cosmological constant*, *JHEP* **05** (2009) 073 [arXiv:0812.4742] [SPIRES].

- [14] D. Grumiller and N. Johansson, *Consistent boundary conditions for cosmological topologically massive gravity at the chiral point*, *Int. J. Mod. Phys. D* **17** (2009) 2367 [[arXiv:0808.2575](#)] [[SPIRES](#)].
- [15] M. Henneaux, C. Martinez and R. Troncoso, *Asymptotically anti-de Sitter spacetimes in topologically massive gravity*, *Phys. Rev. D* **79** (2009) 081502 [[arXiv:0901.2874](#)] [[SPIRES](#)].
- [16] A. Garbarz, G. Giribet and Y. Vasquez, *Asymptotically AdS₃ Solutions to Topologically Massive Gravity at Special Values of the Coupling Constants*, *Phys. Rev. D* **79** (2009) 044036 [[arXiv:0811.4464](#)] [[SPIRES](#)].
- [17] A. Maloney, W. Song and A. Strominger, *Chiral Gravity, Log Gravity and Extremal CFT*, [arXiv:0903.4573](#) [[SPIRES](#)].
- [18] V. Balasubramanian and P. Kraus, *A stress tensor for anti-de Sitter gravity*, *Commun. Math. Phys.* **208** (1999) 413 [[hep-th/9902121](#)] [[SPIRES](#)].
- [19] P. Kraus and F. Larsen, *Holographic gravitational anomalies*, *JHEP* **01** (2006) 022 [[hep-th/0508218](#)] [[SPIRES](#)].
- [20] S.N. Solodukhin, *Holography with Gravitational Chern-Simons Term*, *Phys. Rev. D* **74** (2006) 024015 [[hep-th/0509148](#)] [[SPIRES](#)].
- [21] K. Skenderis, M. Taylor and B.C. van Rees, *Topologically Massive Gravity and the AdS/CFT Correspondence*, *JHEP* **09** (2009) 045 [[arXiv:0906.4926](#)] [[SPIRES](#)].
- [22] S. Deser and B. Tekin, *Energy in topologically massive gravity*, *Class. Quant. Grav.* **20** (2003) L259 [[gr-qc/0307073](#)] [[SPIRES](#)].
- [23] A. Bouchareb and G. Clement, *Black hole mass and angular momentum in topologically massive gravity*, *Class. Quant. Grav.* **24** (2007) 5581 [[arXiv:0706.0263](#)] [[SPIRES](#)].
- [24] S. Deser and X. Xiang, *Canonical formulations of full nonlinear topologically massive gravity*, *Phys. Lett. B* **263** (1991) 39 [[SPIRES](#)].
- [25] S. Carlip, *Inducing Liouville theory from topologically massive gravity*, *Nucl. Phys. B* **362** (1991) 111 [[SPIRES](#)].
- [26] O. Mišković and R. Olea, *On boundary conditions in three-dimensional AdS gravity*, *Phys. Lett. B* **640** (2006) 101 [[hep-th/0603092](#)] [[SPIRES](#)].
- [27] R. Olea, *Regularization of odd-dimensional AdS gravity: Kounterterms*, *JHEP* **04** (2007) 073 [[hep-th/0610230](#)] [[SPIRES](#)].
- [28] V. Iyer and R.M. Wald, *Some properties of Noether charge and a proposal for dynamical black hole entropy*, *Phys. Rev. D* **50** (1994) 846 [[gr-qc/9403028](#)] [[SPIRES](#)].
- [29] M. Bañados, C. Teitelboim and J. Zanelli, *The Black hole in three-dimensional space-time*, *Phys. Rev. Lett.* **69** (1992) 1849 [[hep-th/9204099](#)] [[SPIRES](#)].
- [30] M. Bañados, M. Henneaux, C. Teitelboim and J. Zanelli, *Geometry of the (2+1) black hole*, *Phys. Rev. D* **48** (1993) 1506 [[gr-qc/9302012](#)] [[SPIRES](#)].
- [31] A.A. Garcia, F.W. Hehl, C. Heinicke and A. Macias, *Exact vacuum solution of a (1+2)-dimensional Poincaré gauge theory: BTZ solution with torsion*, *Phys. Rev. D* **67** (2003) 124016 [[gr-qc/0302097](#)] [[SPIRES](#)].
- [32] K.A. Moussa, G. Clement and C. Leygnac, *The black holes of topologically massive gravity*, *Class. Quant. Grav.* **20** (2003) L277 [[gr-qc/0303042](#)] [[SPIRES](#)].

- [33] S. Deser, I. Kanik and B. Tekin, *Conserved charges of higher D Kerr-AdS spacetimes*, *Class. Quant. Grav.* **22** (2005) 3383 [[gr-qc/0506057](#)] [[SPIRES](#)].
- [34] S. Olmez, O. Sarioglu and B. Tekin, *Mass and angular momentum of asymptotically AdS or flat solutions in the topologically massive gravity*, *Class. Quant. Grav.* **22** (2005) 4355 [[gr-qc/0507003](#)] [[SPIRES](#)].
- [35] M. Blagojević and B. Cvetković, *Asymptotic structure of topologically massive gravity in spacelike stretched AdS sector*, *JHEP* **09** (2009) 006 [[arXiv:0907.0950](#)] [[SPIRES](#)].
- [36] M.-I. Park, *BTZ black hole with gravitational Chern-Simons: Thermodynamics and statistical entropy*, *Phys. Rev. D* **77** (2008) 026011 [[hep-th/0608165](#)] [[SPIRES](#)].
- [37] Y. Tachikawa, *Black hole entropy in the presence of Chern-Simons terms*, *Class. Quant. Grav.* **24** (2007) 737 [[hep-th/0611141](#)] [[SPIRES](#)].
- [38] O. Mišković and R. Olea, *Topological regularization and self-duality in four-dimensional anti-de Sitter gravity*, *Phys. Rev. D* **79** (2009) 124020 [[arXiv:0902.2082](#)] [[SPIRES](#)].
- [39] E. Ayon-Beato and M. Hassaine, *pp waves of conformal gravity with self-interacting source*, *Ann. Phys.* **317** (2005) 175 [[hep-th/0409150](#)] [[SPIRES](#)]; *Exploring AdS waves via nonminimal coupling*, *Phys. Rev. D* **73** (2006) 104001 [[hep-th/0512074](#)] [[SPIRES](#)].
- [40] G. Clement, *Black holes with a null Killing vector in new massive gravity in three dimensions*, [arXiv:0905.0553](#) [[SPIRES](#)].
- [41] G. Barnich and F. Brandt, *Covariant theory of asymptotic symmetries, conservation laws and central charges*, *Nucl. Phys. B* **633** (2002) 3 [[hep-th/0111246](#)] [[SPIRES](#)];
G. Barnich and G. Compere, *Surface charge algebra in gauge theories and thermodynamic integrability*, *J. Math. Phys.* **49** (2008) 042901 [[arXiv:0708.2378](#)] [[SPIRES](#)].
- [42] S. Detournay, private communication.
- [43] D. Anninos, W. Li, M. Padi, W. Song and A. Strominger, *Warped AdS₃ Black Holes*, *JHEP* **03** (2009) 130 [[arXiv:0807.3040](#)] [[SPIRES](#)].
- [44] G. Compere and S. Detournay, *Semi-classical central charge in topologically massive gravity*, *Class. Quant. Grav.* **26** (2009) 012001 [*Erratum ibid.* **26** (2009) 139801] [[arXiv:0808.1911](#)] [[SPIRES](#)].
- [45] E. Sezgin and Y. Tani, *Witten-Nester Energy in Topologically Massive Gravity*, *Class. Quant. Grav.* **26** (2009) 235005 [[arXiv:0903.3779](#)] [[SPIRES](#)].
- [46] G. Compere and S. Detournay, *Boundary conditions for spacelike and timelike warped AdS₃ spaces in topologically massive gravity*, *JHEP* **08** (2009) 092 [[arXiv:0906.1243](#)] [[SPIRES](#)].
- [47] C. Fefferman and C.R. Graham, *Conformal invariants*, in *The mathematical heritage of Elie Cartan*, Lyon (1984), Astérisque, Numero Hors Serie, 95 (1985).
- [48] E.W. Mielke and P. Baekler, *Topological gauge model of gravity with torsion*, *Phys. Lett. A* **156** (1991) 399 [[SPIRES](#)].
- [49] R. Olea, *Mass, angular momentum and thermodynamics in four-dimensional Kerr-AdS black holes*, *JHEP* **06** (2005) 023 [[hep-th/0504233](#)] [[SPIRES](#)].
- [50] I. Papadimitriou and K. Skenderis, *Thermodynamics of asymptotically locally AdS spacetimes*, *JHEP* **08** (2005) 004 [[hep-th/0505190](#)] [[SPIRES](#)].

- [51] A. Ashtekar and S. Das, *Asymptotically anti-de Sitter space-times: Conserved quantities*, *Class. Quant. Grav.* **17** (2000) L17 [[hep-th/9911230](#)] [[SPIRES](#)].
- [52] S. Hollands, A. Ishibashi and D. Marolf, *Comparison between various notions of conserved charges in asymptotically AdS-spacetimes*, *Class. Quant. Grav.* **22** (2005) 2881 [[hep-th/0503045](#)] [[SPIRES](#)].