

# Betti Numbers of a Class of Barely $G_2$ Manifolds

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**Abstract:** We calculate explicitly the Betti numbers of a class of barely  $G_2$  manifolds - that is,  $G_2$  manifolds that are realised as a product of a Calabi-Yau manifold and a circle, modulo an involution. The particular class which we consider are those spaces where the Calabi-Yau manifolds are complete intersections of hypersurfaces in products of complex projective spaces from which they inherit all their  $(1, 1)$ -cohomology and the involutions are free acting.

## 1. Introduction

One of the key concepts in String and M-theory is the concept of compactification - here the full 10- or 11-dimensional spacetime is considered to be of the form  $M_4 \times X$ , where  $M_4$  is the “large” 4-dimensional visible spacetime, while  $X$  is the “small” compact 6- or 7-dimensional Riemannian manifold. Due to considerations of supersymmetry, these compact manifolds have to satisfy certain conditions which place restrictions on the geometry. In the case of String theory, the 6-dimensional manifolds have to be Calabi-Yau manifolds - that is Kähler manifolds with vanishing first Chern class. The existence of Ricci-flat Kähler metrics for these manifolds has been proven by Yau in 1978 [1]. One of the first examples of a Calabi-Yau 3-fold (6 real dimensions) was the quintic - a degree 5 hypersurface in  $\mathbb{C}P^4$ . Later, Candelas et al. [2] found the first large class of Calabi-Yau manifolds - the Complete Intersection Calabi-Yau (CICY) manifolds, which are given by intersections of hypersurfaces in products of complex projective spaces. We review the details in Sect. 3. Since then even larger classes of Calabi-Yau manifolds have been constructed - such as Weighted Complete Intersection manifolds [3], and complete intersection manifolds in toric varieties [4]. So overall there is a very large pool of examples of Calabi-Yau manifolds, and it is in fact still an open question whether the number of topologically distinct Calabi-Yau 3-folds is finite or not. One of the great discoveries in the study of Calabi-Yau manifolds is Mirror Symmetry [5,6]. This symmetry first appeared in String Theory where evidence was found that

conformal field theories (CFTs) related to compactifications on a Calabi-Yau manifold with Hodge numbers  $(h_{1,1}, h_{2,1})$  are equivalent to CFTs on a Calabi-Yau manifold with Hodge numbers  $(h_{2,1}, h_{1,1})$ . Mirror symmetry is currently a powerful tool both for calculations in String Theory and in the study of the Calabi-Yau manifolds and their moduli spaces.

However if we go one dimension higher, and look at compactifications of  $M$ -theory, a natural analogue of a Calabi-Yau manifold in this setting is a 7-dimensional manifold with  $G_2$  holonomy. These manifolds are also Ricci-flat, but being odd-dimensional they are real manifolds. The first examples of  $G_2$  manifolds have been constructed by Joyce in [7]. While some work has been done both on the physical aspects of  $G_2$  compactifications (for example [8–11] among others) and on the structure and properties of the moduli space (for example [7, 12–15] among others), still very little is known about the overall structure of  $G_2$  moduli spaces. One of the problems is that there are relatively few examples of  $G_2$  manifolds, and for the ones that are known it is hard to do any calculations, because the examples are not very explicit. However there is a conjectured method of constructing  $G_2$  manifolds from Calabi-Yau manifolds, which could potentially yield many new examples of  $G_2$  manifolds. Here we take a Calabi-Yau 3-fold  $Y$  and let  $Z = (Y \times S^1)/\hat{\sigma}$ , where  $\hat{\sigma}$  acts as antiholomorphic involution on  $Y$  and acts as  $z \rightarrow -z$  on the  $S^1$ . In general, the result will have singularities, and it is still an unresolved question how to systematically resolve these singularities to obtain a smooth manifold with  $G_2$  holonomy. This construction has been suggested by Joyce in [7, 16]. A more basic approach is to only consider involutions without fixed points, so that the resulting manifold  $Z$  is smooth. Manifolds belonging to this class have been called *barely  $G_2$  manifolds* in [8]. Such manifolds do not have the full  $G_2$  holonomy, but rather only  $\mathbb{Z}_2 \times SU(3)$ . However, they do share many of the same properties as full  $G_2$  manifolds, so for many purposes they can play the same role as genuine  $G_2$  manifolds [8, 17]. In particular, if we consider a specific class of Calabi-Yau manifolds, such as CICY manifolds, we can construct a corresponding class of barely  $G_2$  manifolds rather explicitly. This is what we focus on in this paper. We first give an overview of  $G_2$  manifolds and CICY manifolds, and then describe the algorithm that was used to systematically calculate the Betti numbers of the barely  $G_2$  manifolds corresponding to the independent CICY manifolds. In order to work out how the involution acts on 2-forms, we need to know the structure of the second cohomology of the CICY manifold, and for this reason we limit our attention to those CICY manifolds which inherit all of their second cohomology from the ambient product space.

## 2. $G_2$ Manifolds

*2.1. Basics.* We will first review the basics of manifolds with  $G_2$  holonomy. The 14-dimensional exceptional Lie group  $G_2 \subset SO(7)$  is precisely the group of automorphisms of imaginary octonions, so it preserves the octonionic structure constants [18]. Suppose  $x^1, \dots, x^7$  are coordinates on  $\mathbb{R}^7$  and let  $e^{ijk} = dx^i \wedge dx^j \wedge dx^k$ . Then define  $\varphi_0$  to be the 3-form on  $\mathbb{R}^7$  given by

$$\varphi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}. \quad (2.1)$$

These precisely give the structure constants of the octonions, so  $G_2$  preserves  $\varphi_0$ . Since  $G_2$  preserves the standard Euclidean metric  $g_0$  on  $\mathbb{R}^7$ , it preserves the Hodge star, and hence the dual 4-form  $*\varphi_0$ , which is given by

$$*\varphi_0 = e^{4567} + e^{2367} + e^{2345} + e^{1357} - e^{1346} - e^{1256} - e^{1247}. \quad (2.2)$$

Now suppose  $X$  is a smooth, oriented 7-dimensional manifold. A  $G_2$  structure  $Q$  on  $X$  is a principal subbundle of the frame bundle  $F$ , with fibre  $G_2$ . However we can also uniquely define  $Q$  via 3-forms on  $X$ . Define a 3-form  $\varphi$  to be *positive* if we locally can choose coordinates such that  $\varphi$  is written in the form (2.1) - that is for every  $p \in X$  there is an isomorphism between  $T_p X$  and  $\mathbb{R}^7$  such that  $\varphi|_p = \varphi_0$ . Using this isomorphism, to each positive  $\varphi$  we can associate a metric  $g$  and a Hodge dual  $*\varphi$  which are identified with  $g_0$  and  $*\varphi_0$  under this isomorphism. It is shown in [16] that there is a 1 – 1 correspondence between positive 3-forms  $\varphi$  and  $G_2$  structures  $Q$  on  $X$ .

So given a positive 3-form  $\varphi$  on  $X$ , it is possible to define a metric  $g$  associated to  $\varphi$  and this metric then defines the Hodge star, which in turn gives the 4-form  $*\varphi$ . Thus although  $*\varphi$  looks linear in  $\varphi$ , it actually is not, so sometimes we will write  $\psi = *\varphi$  to emphasize that the relation between  $\varphi$  and  $*\varphi$  is very non-trivial.

It turns out that the holonomy group  $Hol(X, g) \subseteq G_2$  if and only if  $X$  has a torsion-free  $G_2$  structure [16]. In this case, the invariant 3-form  $\varphi$  satisfies

$$d\varphi = d*\varphi = 0 \tag{2.3}$$

and equivalently,  $\nabla\varphi = 0$  where  $\nabla$  is the Levi-Civita connection of  $g$ . So in fact, in this case  $\varphi$  is harmonic. Moreover, if  $Hol(X, g) \subseteq G_2$ , then  $X$  is Ricci-flat. The holonomy group is precisely  $G_2$  if and only if the fundamental group  $\pi_1(X)$  is finite. In particular, if  $Hol(X, g) = G_2$ , the first Betti number  $b_1$  vanishes. The reverse is however not true in general.

Special holonomy manifolds play a very important role in string and  $M$ -theory because of their relation to supersymmetry. In general, if we compactify string or  $M$ -theory on a manifold of special holonomy  $X$  the preservation of supersymmetry is related to existence of covariantly constant spinors (also known as parallel spinors). In fact, if all bosonic fields except the metric are set to zero, and a supersymmetric vacuum solution is sought, then in both string and  $M$ -theory, this gives precisely the equation

$$\nabla\xi = 0 \tag{2.4}$$

for a spinor  $\xi$ . As lucidly explained in [10], condition (2.4) on a spinor immediately implies special holonomy. Here  $\xi$  is invariant under parallel transport, and is hence invariant under the action of the holonomy group  $Hol(X, g)$ . This shows that the spinor representation of  $Hol(X, g)$  must contain the trivial representation. For  $Hol(X, g) = SO(n)$ , this is not possible since the spinor representation is reducible, so  $Hol(X, g) \subset SO(n)$ . In particular, Calabi-Yau 3-folds with  $SU(3)$  holonomy admit two covariantly constant spinors and  $G_2$  holonomy manifolds admit only one covariantly constant spinor. Hence eleven-dimensional supergravity compactified on a  $G_2$  holonomy manifold gives rise to a  $\mathcal{N} = 1$  effective theory. From [10, 11] and [9] we know that the deformations of the  $G_2$  3-form  $\varphi$  give  $b_3$  real moduli which combine with the deformations of the supergravity 3-form  $C$  to give  $b_3$  complex moduli. Together with modes of the gravitino, this gives  $b_3$  chiral multiplets. Decomposition of the  $C$ -field also gives  $b_2$  abelian gauge fields, which again combine with gravitino modes to give  $b_2$  vector multiplets. The structure of the moduli space has been studied in detail in [15].

Examples of compact  $G_2$  manifolds have been first constructed by Joyce [7] as resolutions of orbifolds  $T^7/\Gamma$  for a discrete group  $\Gamma$ . There  $\Gamma$  is taken to be a finite group of diffeomorphisms of  $T^7$  preserving the flat  $G_2$ -structure on  $T^7$ . The resulting orbifold will have a singular set coming from the fixed point of the action of  $\Gamma$ , and these singularities are resolved by gluing ALE spaces with holonomy  $SU(2)$  or  $SU(3)$ .

2.2. *G<sub>2</sub> manifolds from Calabi-Yau manifolds.* A simple way to construct a manifold with a torsion-free  $G_2$  structure is to consider  $X = Y \times S^1$ , where  $Y$  is a Calabi-Yau 3-fold. Define the metric and a 3-form on  $X$  as

$$g_X = d\theta^2 \times g_Y, \tag{2.5}$$

$$\varphi = d\theta \wedge \omega + \text{Re } \Omega, \tag{2.6}$$

where  $\theta$  is the coordinate on  $S^1$ ,  $\omega$  is the Kähler form on  $Y$  and  $\Omega$  is the holomorphic 3-form on  $Y$ . This then defines a torsion-free  $G_2$  structure, with

$$*\varphi = \frac{1}{2}\omega \wedge \omega - d\theta \wedge \text{Im } \Omega. \tag{2.7}$$

However, the holonomy of  $X$  in this case is  $SU(3) \subset G_2$ . From the Künneth formula we get the following relations between the Betti numbers of  $X$  and the Hodge numbers of  $Y$ :

$$b_1 = 1,$$

$$b_2 = h_{1,1},$$

$$b_3 = h_{1,1} + 2(h_{2,1} + 1).$$

In [7] and [16], Joyce describes a possible construction of a smooth manifold with holonomy equal to  $G_2$  from a Calabi-Yau manifold  $Y$ . So suppose  $Y$  is a Calabi-Yau 3-fold as above. Then suppose  $\sigma : Y \rightarrow Y$  is an antiholomorphic isometric involution on  $Y$ , that is,  $\chi$  preserves the metric on  $Y$  and satisfies

$$\sigma^2 = 1, \tag{2.8a}$$

$$\sigma^*(\omega) = -\omega, \tag{2.8b}$$

$$\sigma^*(\Omega) = \bar{\Omega}.$$

Such an involution  $\sigma$  is known as a *real structure* on  $Y$ . Define now a quotient given by

$$Z = (Y \times S^1) / \hat{\sigma}, \tag{2.9}$$

where  $\hat{\sigma} : Y \times S^1 \rightarrow Y \times S^1$  is defined by  $\hat{\sigma}(y, \theta) = (\sigma(y), -\theta)$ . The 3-form  $\varphi$  defined on  $Y \times S^1$  by (2.6) is invariant under the action of  $\hat{\sigma}$  and hence provides  $Z$  with a  $G_2$  structure. Similarly, the dual 4-form  $*\varphi$  given by (2.7) is also invariant. Generically, the action of  $\sigma$  on  $Y$  will have a non-empty fixed point set  $N$ , which is in fact a special Lagrangian submanifold on  $Y$  [16]. This gives rise to orbifold singularities on  $Z$ . The singular set is two copies of  $N$ . It is conjectured that if there exists a non-vanishing harmonic 1-form on  $N$ , then it is possible to resolve each singular point using an ALE 4-manifold with holonomy  $SU(2)$  in order to obtain a smooth manifold with holonomy  $G_2$ . The precise details of the proof of this conjecture are not yet available however. We will therefore consider only free-acting involutions, that is those without fixed points.

Manifolds defined by (2.9) with a freely acting involution were called *barely  $G_2$  manifolds* by Harvey and Moore in [8]. The cohomology of barely  $G_2$  manifolds is expressed in terms of the cohomology of the underlying Calabi-Yau manifold  $Y$ :

$$\begin{aligned} H^2(Z) &= H^2(Y)^+, \\ H^3(Z) &= H^2(Y)^- \oplus H^3(Y)^+. \end{aligned} \tag{2.10a}$$

Here the superscripts  $\pm$  refer to the  $\pm$  eigenspaces of  $\sigma^*$ . Thus  $H^2(Y)^+$  refers to two-forms on  $Y$  which are invariant under the action of involution  $\sigma$  and correspondingly  $H^2(Y)^-$  refers to two-forms which are odd under  $\sigma$ . Wedging an odd two-form on  $Y$  with  $d\theta$  gives an invariant 3-form on  $Y \times S^1$ , and hence these forms, together with the invariant 3-forms  $H^3(Y)^+$  on  $Y$ , give the three-forms on the quotient space  $Z$ . Also note that  $H^1(Z)$  vanishes, since the 1-form on  $S^1$  is odd under  $\hat{\sigma}$ .

Consider the action of  $\sigma$  on  $H^3(Y)$ . It sends  $H^{3,0}(Y)$  to  $H^{0,3}(Y)$  and  $H^{2,1}(Y)$  to  $H^{1,2}(Y)$ . Therefore the positive and negative eigenspaces are of equal dimension, so  $\dim H^3(Y)^+ = h_{2,1} + 1$ . Therefore the Betti numbers of  $Z$  in terms of Hodge numbers of  $Y$  are

$$\begin{aligned} b^1 &= 0, \\ b^2 &= h_{1,1}^+, \\ b^3 &= h_{1,1}^- + h_{2,1} + 1. \end{aligned} \tag{2.11a}$$

Hence in order to construct barely  $G_2$  manifolds we need to be able to find involutions of Calabi-Yau manifolds and determine the action of the involution on  $H^{1,1}(Y)$ . A relatively large class of Calabi-Yau manifolds for which this is not hard to do are the complete intersection Calabi-Yau manifolds. We review the properties of these manifolds in the next section.

### 3. Complete Intersection Calabi-Yau Manifolds

*3.1. Basics.* Complete intersection Calabi-Yau (CICY) manifolds were the first major class of Calabi-Yau manifolds which was discovered by Candelas et al. in [2]. Such a manifold  $M$  is defined as a complete intersection of  $K$  hypersurfaces in a product of  $m$  complex projective spaces  $W = \mathbb{C}\mathbb{P}^{n_1} \times \dots \times \mathbb{C}\mathbb{P}^{n_m}$ . Each hypersurface is defined as the zero set of a homogeneous holomorphic polynomial

$$f^a(z_r^\mu) = 0 \quad a = 1, \dots, K. \tag{3.12}$$

Each such polynomial is homogeneous of degree  $q_a^r$  with respect to the homogeneous coordinates of  $\mathbb{C}\mathbb{P}^{n_r}$ . By complete intersection it is meant that the  $K$ -form

$$\Theta = df^1 \wedge \dots \wedge df^K$$

does not vanish on  $M$ . This condition ensures that the resulting manifold is defined globally. In order for  $M$  to be a 3-fold, we obviously need

$$K = \sum_{i=1}^m n_i - 3. \tag{3.13}$$

The standard notation for a CICY manifold is a  $m \times (K + 1)$  array of the form

$$[n \parallel q], \tag{3.14}$$

where  $n$  is a column  $m$ -vector whose entries  $n_r$  are the dimensions of the  $\mathbb{C}\mathbb{P}^{n_r}$  factors, and  $q$  is a  $m \times K$  matrix with entries  $q_a^r$  which give the degrees of the polynomials in the coordinates of each of the  $\mathbb{C}\mathbb{P}^{n_r}$  factor. Each such array defining a CICY is known as a *configuration matrix*, while an equivalence class of configuration matrices under

permutation of all rows and all columns belonging to  $q$  is called a *configuration*. Clearly each such permutation defines exactly the same manifold.

As it was shown in [2], Chern classes can be computed directly from the defining quantities  $n$  and  $q$ . In particular, we immediately get the condition for a vanishing first Chern class:

$$n_r + 1 = \sum_{a=1}^K q_a^r \quad \forall r. \tag{3.15}$$

That is, the sum of entries of in each row of  $q$  must equal the dimension of the corresponding  $\mathbb{C}\mathbb{P}^{n_r}$  factors. This is hence precisely the condition for the complete intersection manifold to be Calabi-Yau. Moreover from the expressions for Chern classes, an expression for the Euler number is also obtained. This is given by

$$\chi_E(M) = \left[ \left( \sum_{r,s,t=1}^m c_3^{rst} x_r x_s x_t \right) \cdot \prod_{b=1}^K \left( \sum_{u=1}^m q_b^u x_u \right) \right]_{\text{coefficient of } \prod_{r=1}^m (x_r)^{n_r}}, \tag{3.16}$$

where

$$c_3^{rst} = \frac{1}{3} \left( (n_r + 1) \delta^{rst} - \sum_{a=1}^K q_a^r q_a^s q_a^t \right)$$

and  $\delta^{rst} = 1$  for  $r = s = t$  and vanishes otherwise.

Varying the coefficients of polynomials in a CICY configuration generally corresponds to complex structure deformations, but as it was shown in [19], there is no one to one correspondence. So it is said that each configuration corresponds to a partial deformation class. There are also various identities which relate different configurations, so not all configurations are independent. There are however 7868 independent configurations. A method for calculating Hodge numbers of the CICY manifolds has been found by Green and Hübsch in [19], and in [20] Green, Hübsch and Lütken calculated the Hodge numbers for each of the 7868 configurations. They found there were 265 unique pairs of Hodge numbers. Unfortunately, the original data with the CICY Hodge numbers has been lost, and the original computer code by Hübsch has been written in a curious mix of *C* and *Pascal* so the original code had to be rewritten in standard *C* in order to be able to recompile the list of Hodge numbers for CICY manifolds, which is necessary to be able to calculate the Betti numbers of corresponding barely  $G_2$  manifolds.

**3.2. Involutions.** Antiholomorphic involutions of projective spaces have been classified in [17], and here we briefly review their results. First consider involutions of a single projective space  $\mathbb{C}\mathbb{P}^n$ . Suppose we have homogeneous coordinates  $(z_0, z_1, \dots, z_n)$  on  $\mathbb{C}\mathbb{P}^n$ , then we can represent an anti-holomorphic involution  $\sigma$  by a matrix  $M$  which acts as

$$z_i \longrightarrow M_{ij} \bar{z}_j. \tag{3.17}$$

Without loss of generality we fix  $\det M = 1$  since multiplication by any non-zero complex number still gives the same involution. Moreover, involutions which differ only by a holomorphic change of basis can be regarded to be the same.

Also  $\sigma^2 = 1$  must be true projectively, so we get

$$M\bar{M} = \lambda I. \tag{3.18}$$

Taking the determinant of (3.18), we find that  $\lambda^{n+1} = 1$ , and taking the trace we see that  $\lambda$  is real. Thus  $\lambda = 1$  for  $n$  even and  $\lambda = \pm 1$  for  $n$  odd. The involution  $\sigma$  is required to be an isometry - that is, it must preserve the standard Fubini-Study metric of  $\mathbb{C}\mathbb{P}^n$ . Together with previous restrictions on  $M$ , this gives the condition

$$MM^\dagger = I. \tag{3.19}$$

Combining (3.18) and (3.19), we see that for  $\lambda = 1$  these equations imply that  $M$  is symmetric, and for  $\lambda = -1$  that  $M$  is antisymmetric. Moreover, due to (3.18), the real and imaginary parts of  $M$  commute, and so can be simultaneously brought into a canonical form - diagonal for  $\lambda = 1$  and block-diagonal for  $\lambda = -1$ . Another change of basis can be used to normalize the coefficients. Hence we get two distinct antiholomorphic involutions:

$$\begin{aligned} A : (z_0, z_1, \dots, z_n) &\longrightarrow (\bar{z}_0, \bar{z}_1, \dots, \bar{z}_n), \\ B : (z_0, z_1, \dots, z_{n-1}, z_n) &\longrightarrow (-\bar{z}_1, \bar{z}_0, \dots, -\bar{z}_n, \bar{z}_{n-1}). \end{aligned} \tag{3.20a}$$

The involution  $A$  corresponds to  $\lambda = +1$  and is defined for  $n$  both odd and even, whereas the involution  $B$  corresponds to  $\lambda = -1$  and is only defined for  $n$  odd. An important difference between the two involutions is that  $A$  has a fixed point set  $\{z_i = \bar{z}_i\}$ , whereas  $B$  acts freely without any fixed points.

So far we considered antiholomorphic involutions of a single projective space, but in general we are interested in products of projective spaces, so we should also consider involutions which mix different factors. As pointed out in [17], the only possibility for this is to exchange two identical projective factors  $\mathbb{C}\mathbb{P}^n$ , giving another involution  $C$ :

$$C : (\{y_i\}; \{z_i\}) \longrightarrow (\{\bar{z}_i\}; \{\bar{y}_i\}). \tag{3.21}$$

This involution clearly has a fixed point set  $\{y_i = \bar{z}_i\}$ .

Now that we have antiholomorphic involutions of projective spaces, we can use these to construct barely  $G_2$  manifolds from CICY manifolds, as in (2.9). In general we must either have an involution acting on each projective factor - either involutions  $A$  or  $B$  on single factors or involution  $C$  on a pair of identical projective factors.

Given a CICY configuration matrix, we will denote the resulting barely  $G_2$  manifold by the same configuration matrix, but indicating in the first column of the configuration matrix which involutions are acting on each projective factor. These actions will be

denoted by  $\bar{n}$ ,  $\hat{n}$  and  $\overset{n}{\underset{n}{\curvearrowright}}$  for involutions  $A$ ,  $B$  and  $C$ , respectively. For example, consider the configuration matrix:

$$\left[ \begin{array}{c} \hat{1} \\ \bar{1} \\ \bar{1} \\ \overset{2}{\underset{2}{\curvearrowright}} \\ \bar{3} \end{array} \middle\| \begin{array}{ccccc} 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \end{array} \right]^{1,39} \tag{3.22}$$

This denotes the barely  $G_2$  manifolds constructed from CICY with the same configuration matrix but with involution  $A$  acting on the  $\mathbb{C}P^2$  and  $\mathbb{C}P^3$  factors, involution  $B$  acting on the first remaining  $\mathbb{C}P^1$  factor and involution  $C$  acting on the remaining  $\mathbb{C}P^1 \times \mathbb{C}P^1$ . The superscripts  $(1, 39)$  give the Betti numbers  $b^2$  and  $b^3$  of the resulting 7-manifold. Note that since this example includes the action of involution  $B$  which has no fixed points, the full involution acting on the whole CICY is also free, so the resulting space is a smooth barely  $G_2$  manifold.

When the projective space involution restricts to the complete intersection space, conditions are imposed on the coefficients of the defining homogeneous equations. Thus the involutions must be compatible with the defining equations, and this may not always be possible. In particular, the invariance of the defining equations under the involution implies that the transformed equations must be equivalent to the original equations. Let us use the configuration matrix (3.22) to demonstrate this. Let  $u_i, v_i, w_i$  for  $i = 0, 1$  be the homogeneous coordinates on the  $\mathbb{C}P^1$  spaces, let  $y_j$  for  $j = 0, 1, 2$  be coordinates on  $\mathbb{C}P^2$  and  $z_k$  for  $k = 0, 1, 2, 3$  be the homogeneous coordinates on the  $\mathbb{C}P^3$  factor. Then the original defining equations are

$$\begin{cases} f_1(y, z) = f_2(y, z) = 0, \\ g_1(v, w, y) = g_2(v, w, z) = 0, \\ h(u, z) = 0, \end{cases} \tag{3.23}$$

where the  $f_i$  and  $g_i$  are polynomials homogeneous of degree 1 in their variable and  $h$  is a polynomial which is homogeneous of degree 2 in  $u_i$  and of degree 1 in  $z_k$ . Under the involution presented in (3.22), after taking the complex conjugates, these equations become

$$\begin{cases} \bar{f}_1(y, z) = \bar{f}_2(y, z) = 0, \\ \bar{g}_1(w, v, y) = \bar{g}_2(w, v, z) = 0, \\ \bar{h}(\hat{u}, z) = 0, \end{cases} \tag{3.24}$$

where  $\hat{u}_{2k} = -u_{2k+1}$  and  $\hat{u}_{2k+1} = u_{2k}$ . Then for some complex numbers  $\lambda_1, \lambda_2$  and  $\lambda_3$  we must have

$$g_1(v, w, y) = \lambda_1 \bar{g}_1(w, v, y), \tag{3.25a}$$

$$g_2(v, w, z) = \lambda_2 \bar{g}_2(w, v, z), \tag{3.25b}$$

$$h(u, z) = \lambda_3 \bar{h}(\hat{u}, z), \tag{3.25c}$$

and for some matrix  $M$  in  $GL(2, \mathbb{C})$  we must have

$$\text{and } \begin{pmatrix} f_1(y, z) \\ f_2(y, z) \end{pmatrix} = M \begin{pmatrix} \bar{f}_1(y, z) \\ \bar{f}_2(y, z) \end{pmatrix}. \tag{3.26}$$

For consistency in (3.25a) and (3.25b), we find that  $\lambda_1 \bar{\lambda}_1 = 1$  and  $\lambda_2 \bar{\lambda}_2 = 1$ . Without loss of generality, we can set  $\lambda_1 = \lambda_2 = 1$ . From (3.25c), we have

$$h(u, z) = \lambda_3 \bar{h}(\hat{u}, z) = \lambda_3 \bar{\lambda}_3 h(\hat{\hat{u}}, z) = \lambda_3 \bar{\lambda}_3 h(u, z). \tag{3.27}$$

Here we have used the fact that  $h(u, z)$  is of degree 2 in  $u_i$ , so even though  $\hat{\hat{u}} = -u$ , the minus sign cancels, and we get  $\lambda_3 \bar{\lambda}_3 = 1$ . So we can set  $\lambda_3 = 1$  without loss of generality. In order for (3.26) to be consistent, we find that we must have  $M\bar{M} = I$ , but



$M = I$  satisfies this condition and so fulfills the consistency criteria. We can see that all these conditions on the coefficients of the defining polynomials halve the number of possible choices for the coefficients. This also shows that not all combinations of involutions are possible. In particular, suppose if we wanted a  $B$  involution to act on the  $\mathbb{C}P^3$  factor. Then since  $\hat{z} = -z$ , and  $h(u, z)$  is of degree 1 in  $z$ , from (3.27) we would get that  $\lambda_3 \bar{\lambda}_3 = -1$ , which is clearly not possible. Also, the  $C$  involution is not always possible - the configuration must be invariant under the interchange of factors.

In order to construct all possible barely  $G_2$  manifolds from CICY manifolds, we must be able to find all possible involutions of a given CICY configuration. Since we want freely acting involutions, we only consider those combinations of involutions which contain a  $B$  involution.

The overall strategy is the following. We first find all possible combinations of  $C$  involutions, and then for each such combination we find the possible  $B$  involutions. The remaining factors which do not have any involutions acting on them get an  $A$  involution.

Suppose we have a configuration matrix with  $m$  rows and  $K$  columns - that is we have  $K$  hypersurfaces in a product of  $m$  projective factors. Let the coordinates be labelled by  $x^1, \dots, x^m$ , and let the homogeneous polynomials be  $f_1, \dots, f_K$ . So the intersection of hypersurfaces is given by

$$f_1 = f_2 = \dots = f_K = 0. \tag{3.28}$$

We want to check whether a  $C$  involution is possible on the first two factors. For this we assume that the two factors are of the same dimension, as this is a basic necessary condition for a  $C$  involution. Then we have to make sure that after the interchange of  $x^1$  and  $x^2$  the new set of homogeneous equations is equivalent to (3.28). This is true if and only if under the interchange of  $x^1$  and  $x^2$  the polynomials remain the same up to a change of ordering. In terms of the configuration matrix this means that under the interchange of two rows the matrix remains invariant up to a permutation of the columns. For more than one  $C$  involution acting on the same configuration matrix, we thus require that under the full set of row interchanges the matrix remains invariant up to a permutation of the columns.

To find all the possible  $C$  involutions for a given configuration matrix we do an exhaustive search of all possibilities. First we find all the possible combinations of pairs of rows that correspond to projective factors of equal dimensions. Then for each combination of pairs we check if under the interchange of rows in each pair the configuration matrix stays invariant up to a reordering of columns. If this is true, then it is possible to have  $C$  involutions acting on each of these pairs of rows. This procedure then gives us the full set  $\mathcal{C} = \{C_1, \dots, C_N\}$  of all possible combinations of  $C$  involutions acting on the configuration matrix.

Now given all the possible  $C$  involutions on a configuration matrix, for each such combination  $C_i \in \mathcal{C}$ , we need to find the possible  $B$  involutions. Suppose we have a configuration matrix as before, and we want to check whether a  $B$  involution is possible on the first projective factor. The basic necessary condition is that the dimension of this projective factor is odd. Then we need to make sure that the new set of homogeneous equations is equivalent to the old set. Let  $\mathcal{I}$  be the set of columns which have non-zero entries in the first row - or equivalently, the set of polynomials that involve  $x^1$ . First suppose that all columns in  $\mathcal{I}$  are distinct. Then for each  $i \in \mathcal{I}$  we require

$$f_i(z^1, \dots) = \lambda_i \bar{f}_i(\hat{z}^1, \dots) \tag{3.29}$$

for some constant  $\lambda_i \in \mathbb{C}$ . As in (3.27), we then have the consistency requirement

$$f_i(z^1, \dots) = \lambda_i \bar{f}_i(\hat{z}^1, \dots) = \lambda_i \bar{\lambda}_i f_i(\hat{\hat{z}}^1, \dots). \tag{3.30}$$

However,  $\hat{z}^1 = -z^1$ , but  $f_i$  is homogeneous of degree  $q_i^1$  in  $z^1$ , so  $f_i(\hat{\hat{z}}^1, \dots) = (-1)^{q_i^1} f_i(z^1, \dots)$ . Hence in order for (3.30) to be consistent,  $q_i^1$  needs to be even for each  $i$ . If this is true, then we can have a  $B$  involution on the first projective factor.

More generally, however, suppose that we have some identical columns in  $\mathcal{I}$ . In particular assume that columns  $k_1, \dots, k_r \in \mathcal{I}$  are all identical, and that the remaining columns in  $\mathcal{I}$  are distinct from these. These columns correspond to polynomials which have the same degrees in projective space coordinates. We can have an involution  $B$  if and only if

$$f_{k_1} = f_{k_2} = \dots = f_{k_r} = 0 \iff \hat{f}_{k_1} = \hat{f}_{k_2} = \dots = \hat{f}_{k_r} = 0.$$

So for some matrix  $M \in GL(r, \mathbb{C})$  we must have

$$\begin{pmatrix} f_{k_1}(z^1, \dots) \\ \vdots \\ f_{k_r}(z^1, \dots) \end{pmatrix} = M \begin{pmatrix} \bar{f}_{k_1}(\hat{z}^1, \dots) \\ \vdots \\ \bar{f}_{k_r}(\hat{z}^1, \dots) \end{pmatrix}. \tag{3.31}$$

From (3.31) we have the consistency condition

$$\begin{pmatrix} f_{k_1}(z^1, \dots) \\ \vdots \\ f_{k_r}(z^1, \dots) \end{pmatrix} = M \bar{M} \begin{pmatrix} f_{k_1}(\hat{\hat{z}}^1, \dots) \\ \vdots \\ f_{k_r}(\hat{\hat{z}}^1, \dots) \end{pmatrix} = (-1)^Q M \bar{M} \begin{pmatrix} f_{k_1}(z^1, \dots) \\ \vdots \\ f_{k_r}(z^1, \dots) \end{pmatrix}, \tag{3.32}$$

where  $Q = q_{k_1}^1 + \dots + q_{k_r}^1$ . If  $r$  is even, then we can always find a block-diagonal real matrix  $M$  such that  $M \bar{M} = M^2 = -I$ , so in this case the condition (3.32) is always consistent, independent of the parity of  $Q$ . For example for  $r = 2$  we could set  $M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ . However if  $r$  is odd, then it is not possible to find a matrix which satisfies  $M \bar{M} = -I$ , so we then cannot have  $Q$  odd.

To find all possible  $B$  involutions, we again proceed with an exhaustive search. We look for all possible combinations of  $B$  involutions for each combination of  $C$  involutions  $C_i \in \mathcal{C}$ . First we find the set  $\mathcal{R}$  of all possible combinations of rows such that the dimensions of the corresponding projective factors are odd, and such that these rows do not have a  $C$  involution from  $C_i$  acting on them. Given a combination  $R \in \mathcal{R}$ , we want to check if it is possible to have a  $B$  involution acting on each row in  $R$ . We look for the set  $\mathcal{I}$  of columns which have a non-zero entry in at least one of the rows in  $R$ . The set  $\mathcal{I}$  is then split into maximal subsets of identical columns. For each such subset we evaluate  $Q$  as above, and if for some subset of size  $r$ ,  $rQ$  is odd, then the consistency condition (3.32) is not fulfilled, and so the combination of rows  $R$  does not admit a  $B$  involution.

The above algorithm has been implemented in the programming language C. After running the algorithm, for each configuration matrix in the original list of 7868 CICY configurations we find the possible combinations of  $C$ -involutions, and for each combination of  $C$ -involution all the possible combinations of  $B$  involutions. Since we are

interested in manifolds with free-acting involutions, we are only concerned with those configuration that admit a  $B$ -involution. It turns out that a total of 4652 configurations do admit a  $B$ -involution, out of which 153 have unique pairs of Hodge numbers. The Hodge pairs for which there exist configurations that admit  $B$  involutions are listed in (3.33):

$h_{1,1}$	$h_{2,1}$	
1	65, 73, 89	
2	$50 + 2k$	for $k = 0, \dots, 13, 18$
3	$31 + 2k$	for $k = 0, 2, 3, \dots, 17, 19, 22$
4	$26 + 2k$	for $k = 0, 1, \dots, 19, 21$
5	$25 + 2k$	for $k = 0, 1, \dots, 18$
6	$24 + 2k$	for $k = 0, 1, \dots, 13, 15$
7	$23 + 2k$	for $k = 0, 1, \dots, 10, 12, 13$
8	$22 + 2k$	for $k = 0, \dots, 11$
9	$21 + 2k$	for $k = 0, \dots, 9$
10	$20 + 2k$	for $k = 0, \dots, 7$
11	$19 + 2k$	for $k = 0, \dots, 6$
12	$18 + 2k$	for $k = 0, \dots, 3, 5$
13	$17 + 2k$	for $k = 0, \dots, 4$
14	$16 + 2k$	for $k = 0, 1, 3$
15	15, 21	
16	20	
19	19	

(3.33)

As we can see there is a clear pattern - all these pairs of Hodge numbers have an even sum. In fact the only pairs of Hodge numbers that have an even sum but do not admit any  $B$  involutions are  $(2, 46)$ ,  $(2, 64)$ ,  $(3, 27)$  and  $(3, 33)$ .

#### 4. Barely $G_2$ Manifolds

*4.1. Betti numbers.* Now that we have found the CICY involutions, we can calculate the Betti numbers of the corresponding barely  $G_2$  manifolds. Thus we need to find the harmonic forms on these manifolds. As we know from Sect. 2.1, for this we only need to determine the stabilizer of the involution  $\sigma$  acting on the  $H^{1,1}(Y)$  of a CICY  $Y$ . In general, we can expect part of the cohomology group to come from  $H^{1,1}(W)$  (where  $W$  is the product of projective factors) and some of it may come from the embedding of the hypersurface. In fact, from [21] we have

$$\ker \left( j : H^{1,1}(W) \longrightarrow H^{1,1}(Y) \right) = H^1 \left( Y, \bigoplus_{a=1}^K E_a^* \right), \tag{4.1}$$

where  $E_a$  are the line bundles over  $W$ , the sections of which correspond to the polynomials  $P_a$ . The rank of this cohomology group can easily be calculated for CICY manifolds [19,22]. However, whenever the rank is non-zero, the configuration matrix can be reduced to an equivalent one for which the rank does indeed vanish [20, Cor. 2]. The simplest example of such a reduction is that a homogeneous hypersurface of degree 1 in  $\mathbb{C}P^1 \times \mathbb{C}P^1$  is again  $\mathbb{C}P^1$ . So in fact, the map  $j$  from  $H^{1,1}(W)$  to  $H^{1,1}(Y)$  may be taken to be injective. It turns out that all of the 7868 CICY configurations in [2] satisfy

this. There could still however be some elements of  $H^{1,1}(Y)$  that do not come from  $H^{1,1}(W)$ . The cycles corresponding to these cohomology classes are called *vanishing cycles*. However if  $h_{1,1}(Y) = h_{1,1}(W)$ , then  $j$  is in fact an isomorphism. We restrict our attention to this particular case, because otherwise we cannot say how the cohomology classes that correspond to vanishing cycles behave under the involution.

Since  $W$  is a product of complex projective factors, we have in fact that  $h_{1,1} = m$ , the number of complex projective factors in the given CICY. Then the harmonic  $(1, 1)$ -forms on  $Y$  are simply the pullbacks of the Kähler forms  $J_1, \dots, J_m$  on the corresponding complex projective factors. In the list of CICYs by Candelas et al., 4874 configurations satisfy this criterion, while the rest do not. The class of CICYs for which this holds have been referred to as *favourable* by Candelas and He [23].

Now suppose we have some involutions acting on  $Y \times S^1$ . First let us consider the case when there are no  $C$  involutions. In this case, no projective factors are mixed, and each of the Kähler forms is odd under the involution. Hence in this case,  $h_{1,1}^- = h_{1,1}$  and  $h_{1,1}^+ = 0$ . From (2.11), we thus have on the 7-dimensional quotient space that  $b_2 = 0$  and  $b_3 = h_{1,1} + h_{2,1} + 1$ .

Now consider the case when we have one  $C$  involution acting on  $Y$ . Without loss of generality assume that the  $C$  involution acts on the first two projective factors. Then  $J_1 + J_2$  is odd, while  $J_1 - J_2$  is even under this involution. The remaining Kähler forms remain odd as before. So in this case,  $h_{1,1}^- = h_{1,1} - 1$  and  $h_{1,1}^+ = 1$ , and so  $b_2 = 1$  and  $b_3 = h_{1,1} + h_{2,1}$ . When we have multiple  $C$  involutions,  $b_2$  correspondingly is equal to the number of  $C$  involutions:

$$b_2 = n_c, \tag{4.2a}$$

$$b_3 = h_{1,1} + h_{2,1} + 1 - n_c, \tag{4.2b}$$

where  $n_c$  is the number of  $C$  involutions acting on the base CICY manifold.

After doing all the calculations we find the following pairs of Betti numbers of the barely  $G_2$  manifolds:

$b_2$	$b_3$		
0	$31 + 2k$	for $k = 0, \dots, 22, 24, 29, 30$	
1	$30 + 2k$	for $k = 0, \dots, 19, 21$	
2	$29 + 2k$	for $k = 0, \dots, 10, 12, 13, 15$	(4.3)
3	$28 + 2k$	for $k = 0, \dots, 7, 9, 10$	
4	$27 + 2k$	for $k = 0, 1, 2, 3$	
5	26		

Thus we have a total of 76 distinct pairs of Betti numbers. All of these pairs have odd  $b_2 + b_3$ , and while most of Joyce’s examples of  $G_2$  holonomy manifolds have  $b_2 + b_3 \equiv 3 \pmod{4}$ , here we have a mix between  $b_2 + b_3 \equiv 1 \pmod{4}$  and  $b_2 + b_3 \equiv 3 \pmod{4}$ .

### 5. Concluding Remarks

We have obtained the Betti numbers of barely  $G_2$  manifolds obtained from Complete Intersection Calabi-Yau manifolds. This gives a class of manifolds that have an explicit description. One of the ways to use these examples is to try and understand the moduli spaces. On one hand we know the structure of the moduli space of the underlying CICY manifolds, but on the other hand, previous general results about the structure of  $G_2$  moduli spaces [14, 15] could be applied to these specific cases. In particular, quantities

like the Yukawa couplings and curvature could be calculated for these examples. This should then give a relationship between the corresponding Calabi-Yau quantities and the  $G_2$  quantities. This could then lead to much better understanding of  $G_2$  moduli spaces and their relationship to Calabi-Yau moduli spaces.

Another direction could be to construct barely  $G_2$  manifolds from some larger class of Calabi-Yau manifolds. In particular it is interesting to see what is the relationship between manifolds constructed from Calabi-Yau mirror pairs, and whether this could shed some light on possible  $G_2$  mirror symmetry.

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## References

1. Yau, S.-T.: On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I. *Comm. Pure Appl. Math.* **31**, 339–411 (1978)
2. Candelas P., Dale A.M., Lutken C.A., Schimmrigk R.: Complete Intersection Calabi-Yau Manifolds. *Nucl. Phys.* **B298**, 493 (1988)
3. Greene, B.R., Roan, S.S., Yau, S.-T.: Geometric singularities and spectra of Landau-Ginzburg models. *Commun. Math. Phys.* **142**, 245–260 (1991)
4. Batyrev, V.V., Borisov, L.A.: *On Calabi-Yau Complete Intersections in Toric Varieties*. <http://arXiv.org/abs/alg-geom/9412017v1>, 1994
5. Strominger, A., Yau, S.-T., Zaslow, E.: Mirror symmetry is T-duality. *Nucl. Phys.* **B479**, 243–259 (1996)
6. Hori, K., et.al.: *Mirror symmetry*. Providence, RI: Amer. Math. Soc., 2003
7. Joyce, D.D.: Compact Riemannian 7-manifolds with holonomy  $G_2$ . I, II. *J. Diff. Geom.* **43**(2), 291–328, 329–375 (1996)
8. Harvey, J.A., Moore, G.W.: *Superpotentials and membrane instantons*. <http://arXiv.org/abs/hep-th/9907026v1>, 1999
9. Gutowski, J., Papadopoulos, G.: Moduli spaces and brane solitons for M theory compactifications on holonomy  $G(2)$  manifolds. *Nucl. Phys.* **B615**, 237–265 (2001)
10. Acharya, B.S., Gukov, S.: M theory and Singularities of Exceptional Holonomy Manifolds. *Phys. Rept.* **392**, 121–189 (2004)
11. Beasley, C., Witten, E.: A note on fluxes and superpotentials in M-theory compactifications on manifolds of  $G(2)$  holonomy. *JHEP* **0207**, 046 (2002)
12. Lee, J.-H., Leung, N.C.: *Geometric structures on  $G(2)$  and  $Spin(7)$ -manifolds*. <http://arXiv.org/abs/math/0202045v2> [math.DG], 2007
13. Karigiannis, S.: Flows of  $G_2$  Structure, I. *Quart. J. Math.* **60**, 487–522 (2009)
14. Karigiannis, S., Leung, N.C.: Hodge theory for  $G_2$ -manifolds: Intermediate Jacobians and Abel-Jacobi maps. *Proc. London Math. Soc.* **99**(3), 297–325 (2009)
15. Grigorian, S., Yau, S.-T.: Local geometry of the  $G_2$  moduli space. *Commun. Math. Phys.* **287**, 459–488 (2009)
16. Joyce, D.D.: *Compact manifolds with special holonomy*. Oxford Mathematical Monographs. Oxford: Oxford University Press, 2000
17. Partouche, H., Pioline, B.: Rolling among  $G(2)$  vacua. *JHEP* **0103**, 005 (2001)
18. Baez, J.: The Octonions. *Bull. Amer. Math. Soc. (N.S.)* **39**, 145–205 (2002)
19. Green, P., Hübsch, T.: Polynomial deformations and cohomology of Calabi-Yau manifolds. *Commun. Math. Phys.* **113**, 505 (1987)
20. Green, P.S., Hübsch, T., Lutken, C.A.: All Hodge Numbers of All Complete Intersection Calabi-Yau Manifolds. *Class. Quant. Grav.* **6**, 105–124 (1989)
21. Green, P.S., Hübsch, T.:  $(1, 1)^3$  Couplings in Calabi-Yau Threefolds. *Class. Quant. Grav.* **6**, 311 (1989)
22. Hübsch, T.: *Calabi-Yau manifolds: A Bestiary for physicists*. Singapore: World Scientific, 1992
23. He, A.-M., Candelas, P.: On the number of complete intersection Calabi-Yau manifolds. *Commun. Math. Phys.* **135**, 193–200 (1990)