

ELLIPTIC HYPERGEOMETRY OF SUPERSYMMETRIC DUALITIES

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ABSTRACT. We give a full list of known $\mathcal{N} = 1$ supersymmetric quantum field theories related by the Seiberg duality conjectures for the $SU(N)$, $SP(2N)$ and G_2 gauge groups. Many of the presented dualities are new, not considered earlier in the literature. For all these theories we construct superconformal indices and express them in terms of elliptic hypergeometric integrals. This gives a systematic extension of the related Römelsberger and Dolan-Osborn results. Equality of indices in dual theories leads to various identities for elliptic hypergeometric integrals. About half of them was proven earlier, and another half represents new challenging conjectures. In particular, we conjecture a dozen of new elliptic beta integrals on root systems, extending the univariate elliptic beta integral discovered by the first author.

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1. INTRODUCTION

The main goal of this work consists in merging of two fields of recent active research in mathematical physics – the Seiberg duality in supersymmetric field theories [70, 71] and the theory of elliptic hypergeometric functions [80]. Seiberg duality is an electric-magnetic duality of certain four dimensional quantum field theories with the symmetry group $G_{st} \times G \times F$, where the superconformal group $G_{st} = SU(2, 2|1)$ describes properties of the space-time, G is a local gauge invariance group, and F is a global symmetry flavor group. Conjecturally, such theories are equivalent to each other at their infrared fixed points, existence of which follows from a deeply nontrivial nonperturbative dynamics [43, 74].

The simplest topological characteristics of supersymmetric theories is the Witten index [92]. Its highly nontrivial superconformal generalization was proposed recently by Römelsberger [67, 68] (for $\mathcal{N} = 1$ theories) and Kinney et al [45] (for extended supersymmetric theories). These superconformal indices describe the structure of BPS states protected by one supercharge and its conjugate. They can be considered as a kind of partition functions in the corresponding space. Starting from early work [75, 88], it is known that such partition functions are described by matrix integrals over the classical groups. The central conjecture of Römelsberger [68] is the equality of superconformal indices in the Seiberg dual theories. In an interesting work [23], Dolan and Osborn have found explicit form of these indices for a number of theories and discovered that they coincide with particular examples of the elliptic hypergeometric integrals [84]. This identification allowed them to prove Römelsberger's conjecture for several dualities either on the basis of known exact computability of these integrals or on the existence of non-trivial symmetry transformations for them.

The general notion of elliptic hypergeometric integrals was introduced by the first author in [76, 78]. First example of such integrals, discovered in [76], formed a new class of exactly computable integrals of hypergeometric type called elliptic beta integrals. Such a name was chosen because these integrals can be considered as a top level generalization of the well-known Euler beta integral [1]:

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \quad \text{Re } \alpha, \text{ Re } \beta > 0. \quad (1.1)$$

Elliptic hypergeometric functions generalize known plain hypergeometric functions and their q -analogues [1]. Moreover, their properties have clarified the origins of many old notions of the hypergeometric world [77]. The limits of the elliptic hypergeometric integrals (or of the elliptic hypergeometric series hidden behind them), corresponding to degenerations of the elliptic curve, brought to light new types of q -hypergeometric functions as well [62, 63, 8].

In the present work (which was started in August of 2008 when the first author has occasionally known on [23]), we extend systematically the Römelsberger and Dolan-Osborn results. More precisely, we present a full list of known $\mathcal{N} = 1$ superconformal field theories related by the duality conjecture for simple gauge groups $G = SU(N), SP(2N), G_2$. For all of them we express superconformal indices in terms of the elliptic hypergeometric integrals. Using Seiberg dualities established earlier in the literature (see references below) we come to a large number of identities for elliptic hypergeometric integrals. About half of them were proven earlier, which yields a justification of the corresponding dualities. A part of the appearing relations for indices were described in [23], and we prove equalities of superconformal indices for many other dualities. Another half of the constructed identities represents new challenging conjectures requiring rigorous mathematical proof. We give some indications on how some of them can be proved with the help of “hypergeometric” techniques.

Moreover, from some known relations for elliptic hypergeometric integrals we find many new dualities not considered earlier in the literature. Thus, we describe both new elliptic hypergeometric identities and new $\mathcal{N} = 1$ supersymmetric theories obeying electric-magnetic duality. In particular, we conjecture more than ten new elliptic beta integrals on root systems, extending the univariate elliptic beta integral of [76].

Analyzing general structure of all relations for integrals in this paper, we formulate two conjectures. Namely, we argue that for the existence of a non-trivial identity for elliptic hypergeometric integrals it is necessary and sufficient to construct the so-called totally elliptic hypergeometric terms [77, 81] (equal in the examples below to ratios of the kernels of elliptic hypergeometric integrals). The second conjecture claims that the same total ellipticity (and related modular invariance) is responsible for the validity of ’t Hooft anomaly matching conditions [36], which are fulfilled for all our dualities (the old and new ones).

The detailed consideration of the multiple duality phenomenon for the $G = SP(2N)$ group case and a brief announcement of other results of this work were given in paper [86]. Our results were reported also at IV-th Sakharov conference on physics (Lebedev Institute, Moscow, May 2009), Conformal field theory workshop (Landau Institute, Chernogolovka, June 2009), XVI-th International congress on mathematical physics (Prague, August 2009), and about ten seminars at different institutes. We thank the organizers of these meetings and seminars for invitations and kind hospitality.

2. GENERAL STRUCTURE OF THE ELLIPTIC HYPERGEOMETRIC INTEGRALS

We start our consideration from reviewing the general structure of the elliptic hypergeometric integrals. For any $x \in \mathbb{C}$ and the base $p \in \mathbb{C}, |p| < 1$, we define the infinite product

$$(x; p)_\infty = \prod_{j=0}^{\infty} (1 - xp^j).$$

Then the theta function is defined as

$$\theta(x; p) = (x; p)_\infty (px^{-1}; p)_\infty,$$

where $x \in \mathbb{C}^*$. This function has symmetry properties

$$\theta(x^{-1}; p) = \theta(px; p) = -x^{-1}\theta(x; p).$$

It is related to the standard theta series by the Jacobi triple product identity

$$\sum_{n \in \mathbb{Z}} p^{n(n-1)/2} x^n = (p; p)_\infty \theta(-x; p).$$

For arbitrary $q \in \mathbb{C}$ and $n \in \mathbb{Z}$, we introduce the elliptic shifted factorials

$$\theta(x; q; p)_n := \begin{cases} \prod_{j=0}^{n-1} \theta(xq^j; p), & \text{for } n > 0 \\ \prod_{j=1}^{-n} \frac{1}{\theta(xq^{-j}; p)}, & \text{for } n < 0 \end{cases}$$

with the normalization $\theta(x; q; p)_0 = 1$. For $p = 0$ we have $\theta(x; 0) = 1 - x$ and

$$\theta(x; q; 0)_n = (x; q)_n = (1 - x)(1 - qx) \dots (1 - q^{n-1}x),$$

the standard q -Pochhammer symbol [1].

We use the conventions

$$\theta(x_1, \dots, x_k; p) = \prod_{j=1}^k \theta(x_j; p), \quad \theta(tx^{\pm 1}; p) = \theta(tx, tx^{-1}; p).$$

The addition law for θ -functions can be written now as

$$\theta(xw^{\pm 1}, yz^{\pm 1}; p) - \theta(xz^{\pm 1}, yw^{\pm 1}; p) = yw^{-1}\theta(xy^{\pm 1}, wz^{\pm 1}; p),$$

where $x, y, w, z \in \mathbb{C}^*$.

For arbitrary $m \in \mathbb{Z}$, we have the quasiperiodicity relations

$$\begin{aligned} \theta(p^m x; p) &= (-x)^{-m} p^{-\frac{m(m-1)}{2}} \theta(x; p), \\ \theta(p^m x; q; p)_k &= (-x)^{-mk} q^{-\frac{mk(k-1)}{2}} p^{-\frac{km(m-1)}{2}} \theta(x; q; p)_k, \\ \theta(x; p^m q; p)_k &= (-x)^{-\frac{mk(k-1)}{2}} q^{-\frac{mk(k-1)(2k-1)}{6}} p^{-\frac{mk(k-1)}{4} \left(\frac{m(2k-1)}{3} - 1\right)} \theta(x; q; p)_k. \end{aligned}$$

We relate bases p, q and r with three complex numbers $\omega_{1,2,3} \in \mathbb{C}$ in the following way

$$q = e^{2\pi i \frac{\omega_1}{\omega_2}}, \quad p = e^{2\pi i \frac{\omega_3}{\omega_2}}, \quad r = e^{2\pi i \frac{\omega_3}{\omega_1}}.$$

Their modular transformed ($\tau \rightarrow -1/\tau$) partners are

$$\tilde{q} = e^{-2\pi i \frac{\omega_2}{\omega_1}}, \quad \tilde{p} = e^{-2\pi i \frac{\omega_2}{\omega_3}}, \quad \tilde{r} = e^{-2\pi i \frac{\omega_1}{\omega_3}}.$$

Elliptic gamma functions are defined as appropriate meromorphic solutions of the following finite difference equation

$$f(u + \omega_1) = \theta(e^{2\pi i u/\omega_2}; p) f(u). \quad (2.1)$$

Its particular solution, called the (standard) elliptic gamma function, is

$$f(u) = \Gamma(e^{2\pi i u/\omega_2}; p, q), \quad \Gamma(z; p, q) = \prod_{j,k=0}^{\infty} \frac{1 - z^{-1} p^{j+1} q^{k+1}}{1 - z p^j q^k}, \quad (2.2)$$

where $|q|, |p| < 1, z \in \mathbb{C}^*$ (note that the equation itself does not demand $|q| < 1$). For incommensurate $\omega_{1,2,3}$, it can be defined uniquely as the meromorphic solution of (2.1) satisfying simultaneously two more equations:

$$f(u + \omega_2) = f(u), \quad f(u + \omega_3) = \theta(e^{2\pi i u/\omega_2}; q) f(u)$$

with the normalization condition $f(\sum_{k=1}^3 \omega_k/2) = 1$.

The modified elliptic gamma function has the form

$$G(u; \omega) = \Gamma(e^{2\pi i \frac{u}{\omega_2}}; p, q) \Gamma(re^{-2\pi i \frac{u}{\omega_1}}; \tilde{q}, r). \quad (2.3)$$

It defines the unique simultaneous solution of equation (2.1) and two other equations:

$$f(u + \omega_2) = \theta(e^{2\pi i u/\omega_1}; r) f(u), \quad f(u + \omega_3) = \frac{\theta(e^{2\pi i \frac{u}{\omega_2}}; q)}{\theta(e^{-2\pi i \frac{u}{\omega_1}}; \tilde{q})} f(u)$$

with the same normalization condition $f(\sum_{k=1}^3 \omega_k/2) = 1$. Here the third equation can be simplified using the modular transformation for theta functions

$$\theta(e^{-2\pi i \frac{u}{\omega_1}}; \tilde{q}) = e^{\pi i B_{2,2}(u|\omega_1, \omega_2)} \theta(e^{2\pi i \frac{u}{\omega_2}}; q), \quad (2.4)$$

where

$$B_{2,2}(u|\omega_1, \omega_2) = \frac{1}{\omega_1 \omega_2} \left(u^2 - (\omega_1 + \omega_2)u + \frac{\omega_1^2 + \omega_2^2}{6} + \frac{\omega_1 \omega_2}{2} \right),$$

is the second Bernoulli polynomial. These statements are based on the Jacobi theorem stating that if a meromorphic $\varphi(u)$ satisfies the system of equations

$$\varphi(u + \omega_1) = \varphi(u + \omega_2) = \varphi(u + \omega_3) = \varphi(u)$$

for $\omega_{1,2,3} \in \mathbb{C}$ linearly independent over \mathbb{Z} , then $\varphi(u) = \text{const.}$ The restricted values of bases $p^n = q^m$, $n, m \in \mathbb{Z}$ (or, equivalently, $r^n = \tilde{q}^m$ or $\tilde{r}^n = \tilde{p}^m$) may be called the torsion points, since the Jacobi theorem fails for them.

The function

$$G(u; \omega) = e^{-\frac{\pi i}{3} B_{3,3}(u|\omega)} \Gamma(e^{-2\pi i \frac{u}{\omega_3}}; \tilde{r}, \tilde{p}), \quad (2.5)$$

where $|\tilde{p}|, |\tilde{r}| < 1$,

$$\begin{aligned} B_{3,3}(u|\omega_1, \omega_2, \omega_3) &= \frac{1}{\omega_1 \omega_2 \omega_3} \left(u^3 - \frac{3u^2}{2} \sum_{k=1}^3 \omega_k \right. \\ &\quad \left. + \frac{u}{2} \left(\sum_{k=1}^3 \omega_k^2 + 3 \sum_{j < k} \omega_j \omega_k \right) - \frac{1}{4} \left(\sum_{k=1}^3 \omega_k \right) \sum_{j < k} \omega_j \omega_k \right). \end{aligned}$$

is the third Bernoulli polynomial, satisfies the same three equations and normalization as (2.3). Hence, they coincide and this fact yields one of the $SL(3; \mathbb{Z})$ -group modular transformation laws for the elliptic gamma function. From the expression (2.5) it is easy to see that $G(u; \omega)$ is a meromorphic function of u for $\omega_1/\omega_2 > 0$, i.e. when $|q| = 1$. The region $|q| > 1$ is similar to $|q| < 1$, it can be reached by a symmetry transformation.

In this picture one has 3 elliptic curves with the modular parameters $\tau_1 = \omega_1/\omega_2$, $\tau_2 = \omega_3/\omega_2$, $\tau_3 = \omega_3/\omega_1$, satisfying the constraint $\tau_3 = \tau_2/\tau_1$. The theory of generalized gamma functions was built by Barnes [2]. Implicitly, the function $\Gamma(z; p, q)$ appeared in the free energy per site of Baxter's eight vertex model [3] (see also [89] and [25]) – exactly in the form which will be used below in the superconformal indices context. Systematic investigation of its properties was launched in [69]. Its relation to the $SL(3, \mathbb{Z})$ -group of modular transformation was described in [25]. The modified (“unit circle”) elliptic gamma function $G(u; \omega)$ was introduced in [78] (see also [19]). Both elliptic gamma functions are directly related to the Barnes multiple gamma function of the third order [28, 78].

In terms of the $\Gamma(z; p, q)$ -function one can write

$$\theta(x; q; p)_n = \frac{\Gamma(xq^n; p, q)}{\Gamma(x; p, q)}.$$

The short-hand conventions

$$\begin{aligned} \Gamma(t_1, \dots, t_k; p, q) &:= \Gamma(t_1; p, q) \cdots \Gamma(t_k; p, q), \\ \Gamma(tz^{\pm 1}; p, q) &:= \Gamma(tz; p, q) \Gamma(tz^{-1}; p, q), \quad \Gamma(z^{\pm 2}; p, q) := \Gamma(z^2; p, q) \Gamma(z^{-2}; p, q) \end{aligned}$$

are used below. The simplest properties of $\Gamma(z; p, q)$ are:

- the symmetry $\Gamma(z; p, q) = \Gamma(z; q, p)$,

- the finite difference equations of the first order

$$\Gamma(qz; p, q) = \theta(z; p)\Gamma(z; p, q), \quad \Gamma(pz; p, q) = \theta(z; q)\Gamma(z; p, q),$$

- the reflection equation

$$\Gamma(z; p, q)\Gamma(pq/z; p, q) = 1,$$

- the duplication formula

$$\Gamma(z^2; p, q) = \Gamma(z, -z, q^{1/2}z, -q^{1/2}z, p^{1/2}z, -p^{1/2}z, (pq)^{1/2}z, -(pq)^{1/2}z; p, q),$$

- and the limiting relations

$$\lim_{p \rightarrow 0} \Gamma(z; p, q) = \frac{1}{(z; q)_\infty}, \quad \lim_{z \rightarrow 1} (1-z)\Gamma(z; p, q) = \frac{1}{(p; p)_\infty (q; q)_\infty}.$$

Definition 1. [77] *A meromorphic function $f(x_1, \dots, x_n; p)$ of n variables $x_j \in \mathbb{C}^*$, which together with p compose all indeterminates of this function, is called totally p -elliptic if*

$$f(px_1, \dots, x_n; p) = \dots = f(x_1, \dots, px_n; p) = f(x_1, \dots, x_n; p).$$

Positions of zeros and poles of all elliptic functions are considered as indeterminates, that is the arguments of $f(x_1, \dots, x_n; p)$ include these variables. Note that there are no totally elliptic functions of one or two indeterminates x_i .

Consider n -dimensional integrals

$$I(y_1, \dots, y_m) = \int_{x \in D} \Delta(x_1, \dots, x_n; y_1, \dots, y_m) \prod_{j=1}^n \frac{dx_j}{x_j},$$

where $D \subset \mathbb{C}^n$ is some domain of integration and $\Delta(x_1, \dots, x_n; y_1, \dots, y_m)$ is a meromorphic function of x_j, y_k , and y_k denote the “external” parameters.

Definition 2. [78] *The integral $I(y_1, \dots, y_m; p, q)$ is called the elliptic hypergeometric integral if there are two distinguished complex parameters p and q such that I 's kernel $\Delta(x_1, \dots, x_n; y_1, \dots, y_m; p, q)$ satisfies the following system of linear first order q -difference equations in the integration variables x_j :*

$$\frac{\Delta(\dots qx_j \dots; y_1, \dots, y_m; p, q)}{\Delta(x_1, \dots, x_n; y_1, \dots, y_m; p, q)} = h_j(x_1, \dots, x_n; y_1, \dots, y_m; q; p),$$

where h_j are some p -elliptic functions of the variables x_j ,

$$h_j(\dots px_i \dots; y_1, \dots, y_m; q; p) = h_j(x_1, \dots, x_n; y_1, \dots, y_m; q; p).$$

The kernel Δ is called then the elliptic hypergeometric term, and the functions $h_j(x_1, \dots, x_n; y_1, \dots, y_m; q; p)$ – the certificates.

This definition is not the most general possible one, but it is sufficient for the purposes of the present paper. The elliptic hypergeometric series can be introduced as sums of residues of particular sequences of poles in the elliptic hypergeometric integrals kernels [17] and, because of the convergence difficulties, are less general than the integrals. In the one-dimensional case, $n = 1$, the structure of admissible elliptic hypergeometric terms Δ can be described explicitly. Indeed, any meromorphic p -elliptic function $f(px) = f(x)$ can be written in the form

$$f_p(x) = z \prod_{k=1}^N \frac{\theta(t_k x; p)}{\theta(w_k x; p)}, \quad \prod_{k=1}^N t_k = \prod_{k=1}^N w_k,$$

where $z, t_1, \dots, t_N, w_1, \dots, w_N$ are arbitrary complex parameters. The positive integer N is called the order of the elliptic function, and the linear constraint on parameters – the balancing condition. From the identity

$$z = \frac{\theta(zx, px; p)}{\theta(pzx, x; p)}$$

we see that z is not a distinguished parameter – it can be obtained from t_k and w_k by appropriate reduction without spoiling the balancing condition. Therefore we set $z = 1$.

Now, for $|q| < 1$, the general solution of the equation $\Delta(qx) = f_p(x)\Delta(x)$ is

$$\Delta(x) = \varphi(x) \prod_{k=1}^N \frac{\Gamma(t_k x; p, q)}{\Gamma(w_k x; p, q)}, \quad \varphi(x) = \prod_{k=1}^M \frac{\theta(a_k x; q)}{\theta(b_k x; q)}, \quad \prod_{k=1}^M a_k = \prod_{k=1}^M b_k,$$

where $\varphi(qx) = \varphi(x)$ is an arbitrary q -elliptic function. However, we can write

$$\varphi(x) = \prod_{k=1}^M \frac{\Gamma(pa_k x, b_k x; p, q)}{\Gamma(a_k x, pb_k x; p, q)}$$

and see that such a function can be obtained after replacing N by $N + 2M$ appropriate specification of the original parameters t_k and w_k with the balancing condition preserved. Therefore we can drop $\varphi(x)$ function and find that the general elliptic hypergeometric term for $n = 1$ has the form:

$$\Delta(x; t_1, \dots, t_N, w_1, \dots, w_N; p, q) = \prod_{k=1}^N \frac{\Gamma(t_k x; p, q)}{\Gamma(w_k x; p, q)}, \quad \prod_{k=1}^N \frac{t_k}{w_k} = 1.$$

This functions is symmetric in p and q , i.e. we can repeat the above considerations with these parameters permuted. Note that for incommensurate p and q (i.e., when $p^j \neq q^k$, $j, k \in \mathbb{Z}$) the equations

$$\Delta(qx) = f_p(x)\Delta(x), \quad \Delta(px) = f_q(x)\Delta(x)$$

determine $\Delta(x)$ up to a multiplicative constant.

For $|q| > 1$,

$$\Delta(x; t_1, \dots, t_N, w_1, \dots, w_N; p, q) = \prod_{k=1}^N \frac{\Gamma(q^{-1}w_k x; p, q^{-1})}{\Gamma(q^{-1}t_k x; p, q^{-1})}, \quad \prod_{k=1}^N \frac{t_k}{w_k} = 1.$$

For $|q| = 1$, the requirement of meromorphicity in x is too strong. In this case one has to use the modified elliptic gamma function $G(u; \omega)$, or modular transformations, which we skip for brevity.

In analogy with the series case, considered in [77], it is natural to extend the notion of total ellipticity to elliptic hypergeometric terms entering integrals [78].

Definition 3. *An elliptic hypergeometric integral*

$$I(y_1, \dots, y_m; p, q) = \int_{x \in D} \Delta(x_1, \dots, x_n; y_1, \dots, y_m; p, q) \prod_{j=1}^n \frac{dx_j}{x_j}$$

is called *totally elliptic* if all its kernel's certificates $h_j(x_1, \dots, x_n; y_1, \dots, y_m; q; p)$, $j = 1, \dots, n+m$, are totally elliptic functions, i.e. they are p -elliptic in all variables $x_1, \dots, x_n, y_1, \dots, y_m$ and q . In particular,

$$h_j(x_1, \dots, x_n; y_1, \dots, y_m; pq; p) = h_j(x_1, \dots, x_n; y_1, \dots, y_m; q; p).$$

Theorem 1 (Rains, Spiridonov, 2004). *Given maps $\epsilon(m^{(a)}) = \epsilon(m_1^{(a)}, \dots, m_n^{(a)}) : \mathbb{Z}^n \rightarrow \mathbb{Z}$, $a = 1, \dots, M$, with finite support, define the meromorphic function*

$$\Delta(x_1, \dots, x_n; p, q) = \prod_{a=1}^M \Gamma(x_1^{m_1^{(a)}} x_2^{m_2^{(a)}} \dots x_n^{m_n^{(a)}}; p, q)^{\epsilon(m^{(a)})}. \quad (2.6)$$

Suppose Δ is a totally elliptic hypergeometric term, i.e. its certificates are p -elliptic functions of q and x_1, \dots, x_n . Then these certificates are also modular invariant.

The proof is elementary. The certificates have the explicit form

$$h_i(x; q; p) = \frac{\Delta(\dots qx_i \dots; p, q)}{\Delta(x_1, \dots, x_n; p, q)} = \prod_{a=1}^M \theta(x^{m^{(a)}}; q; p)_{m_i^{(a)}}^{\epsilon(m^{(a)})}.$$

The conditions for h_i to be elliptic in x_j yield the constraints

$$\sum_{a=1}^M \epsilon(m^{(a)}) m_i^{(a)} m_j^{(a)} m_k^{(a)} = 0, \quad (2.7)$$

$$\sum_{a=1}^M \epsilon(m^{(a)}) m_i^{(a)} m_j^{(a)} = 0 \quad (2.8)$$

for $1 \leq i, j, k \leq n$. The conditions of ellipticity in q add one more constraint

$$\sum_{a=1}^M \epsilon(m^{(a)}) m_i^{(a)} = 0. \quad (2.9)$$

The latter equation guarantees that h_i have an equal number of theta functions in their numerators and denominators. The modular invariance of h_i follows then automatically from the transformation property (2.4). Such a direct relation between total ellipticity and modularity was conjectured to be true in general in [77].

The simplest known nontrivial totally elliptic hypergeometric term corresponds to $n = 6$, $M = 29$ and has the form [81]:

$$\Delta(x; t_1, \dots, t_6; p, q) = \frac{\prod_{j=1}^6 \Gamma(t_j x^{\pm 1}; p, q)}{\Gamma(x^{\pm 2}; p, q) \prod_{1 \leq i < j \leq 6} \Gamma(t_i t_j; p, q)}, \quad \prod_{j=1}^6 t_j = pq,$$

or, after plugging in $t_6 = pq / \prod_{i=1}^5 t_i$,

$$\Delta(x; t_1, \dots, t_5; p, q) = \frac{\prod_{j=1}^5 \Gamma(t_j x^{\pm 1}, t_j^{-1} \prod_{i=1}^5 t_i; p, q)}{\Gamma(x^{\pm 2}, \prod_{i=1}^5 t_i x^{\pm 1}; p, q) \prod_{1 \leq i < j \leq 5} \Gamma(t_i t_j; p, q)}.$$

Theorem 2. [76] **Elliptic beta integral.** *For $|p|, |q|, |t_j| < 1$,*

$$\frac{(p; p)_\infty (q; q)_\infty}{4\pi i} \int_{\mathbb{T}} \Delta(x; t_1, \dots, t_6; p, q) \frac{dx}{x} = 1, \quad (2.10)$$

where \mathbb{T} is the unit circle with positive orientation.

At the bottom of this relation one finds the Euler beta integral (1.1). It served as an entry ticket to the large class of new exactly computable integrals discussed in [17, 18, 19, 61, 78, 87], which is essentially extended by the conjectures presented in this paper.

In [78, 80, 82] the integral standing on the left hand side of (2.10) was generalized to an elliptic analogue of the Gauss hypergeometric function obeying many classical properties. It

also admits generalizations to elliptic hypergeometric functions of higher orders and integrals of higher dimensions on root systems (for a list, see [84]).

Two totally elliptic hypergeometric terms associated with the multidimensional elliptic beta integrals of type I on root systems BC_n [18] and A_n [78] were constructed in [81]. One more similar example for the root system A_n was built in [87]. Some time ago, using the combination of tricks introduced in [81] and [65], the first author has further generalized the former two terms to arbitrary number of parameters [85]. For instance, define the kernel

$$\Delta_n(z, t; p, q) = \prod_{1 \leq i < j \leq n} \frac{1}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \prod_{j=1}^n \frac{\prod_{i=1}^{2n+2m+4} \Gamma(t_i z_j^{\pm 1}; p, q)}{\Gamma(z_j^{\pm 2}; p, q)}$$

and the type I multivariable elliptic hypergeometric integral for the BC_n root system:

$$I_n^{(m)}(t_1, \dots, t_{2n+2m+4}) = \frac{(p; p)_\infty^n (q; q)_\infty^n}{2^n n! (2\pi i)^n} \int_{\mathbb{T}^n} \Delta_n(z, t; p, q) \prod_{j=1}^n \frac{dz_j}{z_j},$$

where $|t_j| < 1$ and $\prod_{j=1}^{2n+2m+4} t_j = (pq)^{m+1}$.

Theorem 3. [61] For $|pq|^{1/2} < |t_j| < 1$, the integrals $I_n^{(m)}$ satisfy the relation

$$I_n^{(m)}(t_1, \dots, t_{2n+2m+4}) = \prod_{1 \leq r < s \leq 2n+2m+4} \Gamma(t_r t_s; p, q) I_m^{(n)} \left(\frac{\sqrt{pq}}{t_1}, \dots, \frac{\sqrt{pq}}{t_{2n+2m+4}} \right).$$

This is an elliptic analogue of the symmetry transformation for some plain hypergeometric integrals established by Dixon in [21].

Theorem 4. [85] The ratio

$$\rho(z, y; t; p, q) = \prod_{1 \leq r < s \leq 2n+2m+4} \Gamma(t_r t_s; p, q)^{-1} \frac{\Delta_n(z; t; p, q)}{\Delta_m(y/\sqrt{pq}; \sqrt{pq}/t; p, q)}$$

is the totally elliptic hypergeometric term. I.e., all ratios $\rho(\dots, qv, \dots)/\rho(\dots, v, \dots)$ for $v \in \{z_1, \dots, z_n, y_1, \dots, y_m, t_1, \dots, t_{2n+2m+4}\}$ are p -elliptic functions of all variables z_i, y_k, t_l , and q .

This term $\rho(z, y; t; p, q)$ contains elliptic gamma functions with the non-removable integer powers of pq in the arguments. Therefore the ansatz (2.6) does not cover all interesting totally elliptic hypergeometric terms. As we shall show below, there are also examples of terms having fractional powers of pq . For them the total ellipticity condition is slightly modified: it is necessary to consider dilatations of the parameter q by appropriate powers of p . Introducing the variable $x_0 = (pq)^{1/K}$, $K = 1, 2, \dots$ and adding to arguments of the elliptic gamma functions in (2.6) the terms with x_0 in integer powers $m_0^{(a)}$, it is not difficult to find the general form of constraints on integers $m_j^{(a)}$ and $\epsilon(m^{(a)})$ guaranteeing total ellipticity (with special p^K -ellipticity condition for the variable q). However, these constraints look much less beautiful than the Diophantine equations described above. Moreover, at the moment it is not clear which part of the modular transformation group survives because of the presence of fractional parts of modular variables in the arguments of respective elliptic functions-certificates.

In the present work, we have checked that *all* nontrivial relations for elliptic hypergeometric integrals described below define totally elliptic hypergeometric terms through the ratios of the corresponding integral kernels. Namely, we have verified this property for relations

- the initial Seiberg dualities (4.6) and (4.7); (5.1) and (5.2);
- multiple dualities for $SP(2N)$ gauge group (6.1), (6.2), (6.3) and (6.4);

- duality for $SP(2N)$ case (7.1) and (7.2);
- multiple dualities for SU gauge group (8.1), (8.2), (8.3) and (8.4); (9.1), (9.2), (9.3) and (9.4);
- KS type of dualities for SU gauge group (10.2) and (10.3) (see Appendix D for a detailed consideration of this case); (10.5) and (10.6); (10.8) and (10.9); (10.11) and (10.12); (10.14) and (10.15); (10.17) and (10.18); (10.20) and (10.21); (10.23) and (10.24);
- KS type of dualities for SP gauge group (11.2) and (11.3); (11.5) and (11.6); (11.8) and (11.9); (11.11) and (11.12);
- confinement for SU theories (13.1) and (13.2); (13.6) and (13.7); (13.8) and (13.9); (13.10) and (13.11); (13.12) and (13.13); (13.30) and (13.31); (13.32) and (13.33); (13.34) and (13.35); (13.36) and (13.37); (13.38) and (13.39); (13.40) and (13.41); (13.42) and (13.43);
- confinement for SP theories (13.44) and (13.45); (13.46) and (13.47); (13.48) and (13.49);
- dualities for G_2 gauge group (14.1) and (14.2); (14.3).

On the basis of this large amount of computational work (our auxiliary file with its details takes more than 100 pages), we put forward the following

Conjecture. *The condition of total ellipticity for the elliptic hypergeometric terms is necessary and sufficient for the existence of the exact integration formulas for elliptic beta integrals or of the nontrivial Weyl group symmetry transformations for the elliptic hypergeometric integrals.*

It is known that behind each elliptic hypergeometric integral there is a terminating elliptic hypergeometric series appearing from the residue calculus for restricted values of parameters [17]. The above conjecture has a natural meaning in terms of such series – it simply demands that the summation or transformation identities for them involve ratios of Jacobi forms with appropriate quasiperiodicity and modular properties in the sense of Eichler and Zagier [24]. Already this fact is sufficient (when there are no fractional powers of pq) for the confirmation of the series identities to rather high powers of small $\log q$ expansions [17].

It should be noted that for a given interesting elliptic hypergeometric integral there may exist more than one totally elliptic hypergeometric term. In the examples of [81, 87] the totally elliptic hypergeometric terms were supplemented with particular difference equations with the totally elliptic function coefficients. Therefore analysis of the sufficiency condition looks much more neat – it should address the non-uniqueness questions and the list of admissible technical tools.

3. SUPERCONFORMAL INDEX

3.1. $\mathcal{N} = 1$ Superconformal Algebra. In 4 dimensions the *conformal* algebra $SO(4, 2)$ is formed by the generators of translations P_a , generators of special conformal transformations K_a , generators of the Lorentz group $SO(3, 1)$, $M_{ab} = -M_{ba}$, and the generator of dilations H . The commutation relations have the form

$$\begin{aligned}
 [M_{ab}, P_c] &= i(\eta_{ac}P_b - \eta_{bc}P_a), & [M_{ab}, K_c] &= i(\eta_{ac}K_b - \eta_{bc}K_a), \\
 [M_{ab}, M_{cd}] &= i(\eta_{ac}M_{bd} - \eta_{bc}M_{ad} - \eta_{ad}M_{bc} + \eta_{bd}M_{ac}), \\
 [H, P_a] &= P_a, & [H, K_a] &= -K_a, & [K_a, P_b] &= -2iM_{ab} - 2\eta_{ab}H,
 \end{aligned}
 \tag{3.1}$$

where $\eta_{ab} = \text{diag}(-1, 1, 1, 1)$ and all indices take values $a = 0, 1, 2, 3$. In terms of the matrix M_{AB}

$$M_{AB} = \begin{pmatrix} M_{ab} & -\frac{1}{2}(P_a - K_a) & -\frac{1}{2}(P_a + K_a) \\ \frac{1}{2}(P_b - K_b) & 0 & H \\ \frac{1}{2}(P_b + K_b) & -H & 0 \end{pmatrix}, \quad (3.2)$$

where $A, B = 0, 1, 2, 3, 4, 5$, the commutation relations are rewritten in simpler form

$$[M_{AB}, M_{CD}] = i(\eta_{AC}M_{BD} - \eta_{BC}M_{AD} - \eta_{AD}M_{BC} + \eta_{BD}M_{AC}), \quad (3.3)$$

with $\eta_{AB} = \text{diag}(-1, 1, 1, 1, 1, -1)$.

In the spinorial basis one defines

$$\begin{aligned} P_{\alpha\dot{\alpha}} &= (\sigma^a)_{\alpha\dot{\alpha}}P_a, & K^{\dot{\alpha}\alpha} &= (\bar{\sigma}^a)^{\dot{\alpha}\alpha}K_a, \\ M_{\alpha}^{\beta} &= -\frac{i}{4}(\sigma^a\bar{\sigma}^b)_{\alpha}^{\beta}M_{ab}, & \bar{M}_{\dot{\beta}}^{\dot{\alpha}} &= -\frac{i}{4}(\bar{\sigma}^a\sigma^b)^{\dot{\alpha}}_{\dot{\beta}}M_{ab}, \end{aligned} \quad (3.4)$$

where

$$\sigma^a = (I, \sigma^i), \quad \bar{\sigma}^a = (I, -\sigma^i)$$

and σ^i are the usual Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (3.5)$$

Using the standard angular momentum generators, we set

$$M_{\alpha}^{\beta} = \begin{pmatrix} J_3 & J_+ \\ J_- & -J_3 \end{pmatrix}, \quad \bar{M}_{\dot{\beta}}^{\dot{\alpha}} = \begin{pmatrix} \bar{J}_3 & \bar{J}_+ \\ \bar{J}_- & -\bar{J}_3 \end{pmatrix},$$

with

$$[J_+, J_-] = 2J_3, \quad [\bar{J}_+, \bar{J}_-] = 2\bar{J}_3.$$

Then the tensor M_{ab} is expressed through these operators as

$$M_{ab} = \begin{pmatrix} 0 & \frac{i}{2}(\bar{J}_+ + \bar{J}_- - J_+ - J_-) & \frac{1}{2}(J_+ + \bar{J}_- - \bar{J}_+ - J_-) & \frac{i(\bar{J}_3 - J_3)}{2} \\ -\frac{i}{2}(\bar{J}_+ + \bar{J}_- - J_+ - J_-) & 0 & -(J_3 + \bar{J}_3) & \frac{i}{2}(J_+ + \bar{J}_+ - J_- - \bar{J}_-) \\ -\frac{i}{2}(J_+ + \bar{J}_+ - \bar{J}_- - J_-) & (J_3 + \bar{J}_3) & 0 & -\frac{i}{2}(J_+ + J_- + \bar{J}_+ + \bar{J}_-) \\ -i(\bar{J}_3 - J_3) & -\frac{i}{2}(J_+ + \bar{J}_+ - \bar{J}_- - J_-) & \frac{1}{2}(J_+ + J_- + \bar{J}_+ + \bar{J}_-) & 0 \end{pmatrix}.$$

The conformal algebra (3.1) can be rewritten now as

$$\begin{aligned} [M_{\alpha}^{\beta}, M_{\gamma}^{\delta}] &= \delta_{\gamma}^{\beta}M_{\alpha}^{\delta} - \delta_{\alpha}^{\delta}M_{\gamma}^{\beta}, & [\bar{M}_{\dot{\beta}}^{\dot{\alpha}}, \bar{M}_{\dot{\delta}}^{\dot{\gamma}}] &= \delta_{\dot{\delta}}^{\dot{\beta}}\bar{M}_{\dot{\alpha}}^{\dot{\gamma}} - \delta_{\dot{\alpha}}^{\dot{\delta}}\bar{M}_{\dot{\gamma}}^{\dot{\beta}} \\ [M_{\alpha}^{\beta}, P_{\gamma\dot{\delta}}] &= \delta_{\gamma}^{\beta}P_{\alpha\dot{\delta}} - \frac{1}{2}\delta_{\alpha}^{\beta}P_{\gamma\dot{\delta}}, & [\bar{M}_{\dot{\beta}}^{\dot{\alpha}}, P_{\gamma\dot{\delta}}] &= -\delta_{\dot{\delta}}^{\dot{\alpha}}P_{\gamma\dot{\beta}} + \frac{1}{2}\delta_{\dot{\beta}}^{\dot{\alpha}}P_{\gamma\dot{\delta}} \\ [M_{\alpha}^{\beta}, K^{\dot{\gamma}\delta}] &= -\delta_{\alpha}^{\delta}K^{\dot{\gamma}\beta} + \frac{1}{2}\delta_{\alpha}^{\beta}K^{\dot{\gamma}\delta}, & [\bar{M}_{\dot{\beta}}^{\dot{\alpha}}, K^{\dot{\gamma}\delta}] &= \delta_{\dot{\beta}}^{\dot{\alpha}}K^{\dot{\gamma}\delta} - \frac{1}{2}\delta_{\dot{\beta}}^{\dot{\alpha}}K^{\dot{\gamma}\delta} \\ [M_{\alpha}^{\beta}, H] &= 0, & [\bar{M}_{\dot{\beta}}^{\dot{\alpha}}, H] &= 0 \\ [H, P_{\alpha\dot{\beta}}] &= P_{\alpha\dot{\beta}}, & [H, K^{\dot{\alpha}\beta}] &= -K^{\dot{\alpha}\beta}. \end{aligned} \quad (3.6)$$

In four dimensions the conformal group can be extended by introducing supercharges $Q_{\alpha}, \bar{Q}_{\dot{\alpha}}$ and their superconformal partners $S^{\alpha}, \bar{S}^{\dot{\alpha}}$, where $\alpha, \dot{\alpha} = 1, 2$. Supercharges satisfy the relations [90]

$$\{Q_{\alpha}, \bar{Q}_{\dot{\alpha}}\} = 2P_{\alpha\dot{\alpha}}, \quad \{Q_{\alpha}, Q_{\beta}\} = \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0 \quad (3.7)$$

while their superconformal partners obey

$$\{\bar{S}^{\dot{\alpha}}, S^{\alpha}\} = 2K^{\dot{\alpha}\alpha}, \quad \{\bar{S}^{\dot{\alpha}}, \bar{S}^{\dot{\beta}}\} = \{S^{\alpha}, S^{\beta}\} = 0. \quad (3.8)$$

The cross-anti-commutators of the Q_α and S_α have the form

$$\{Q_\alpha, \bar{S}^{\dot{\alpha}}\} = 0, \quad \{S^\alpha, \bar{Q}_{\dot{\alpha}}\} = 0, \quad (3.9)$$

while

$$\begin{aligned} \{Q_\alpha, S^\beta\} &= 4 \left(M_\alpha^\beta + \frac{1}{2} \delta_\alpha^\beta H + \frac{3}{4} \delta_\alpha^\beta R \right), \\ \{\bar{S}^{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} &= 4 \left(\bar{M}_{\dot{\beta}}^{\dot{\alpha}} - \frac{1}{2} \delta_{\dot{\beta}}^{\dot{\alpha}} H + \frac{3}{4} \delta_{\dot{\beta}}^{\dot{\alpha}} R \right), \end{aligned} \quad (3.10)$$

where R is the R -charge generating $U(1)_R$ -symmetry group.

The bosonic and fermionic generators cross-commute as

$$\begin{aligned} [M_\alpha^\beta, Q_\gamma] &= \delta_\gamma^\beta Q_\alpha - \frac{1}{2} \delta_\alpha^\beta Q_\gamma, & [M_\alpha^\beta, \bar{Q}_{\dot{\gamma}}] &= 0, \\ [M_\alpha^\beta, S^\gamma] &= -\delta_\alpha^\gamma S^\beta + \frac{1}{2} \delta_\alpha^\beta S^\gamma, & [M_\alpha^\beta, \bar{S}^{\dot{\gamma}}] &= 0, \\ [\bar{M}_{\dot{\beta}}^{\dot{\alpha}}, Q_\gamma] &= 0, & [\bar{M}_{\dot{\beta}}^{\dot{\alpha}}, \bar{Q}_{\dot{\gamma}}] &= -\delta_{\dot{\gamma}}^{\dot{\alpha}} \bar{Q}_{\dot{\beta}} + \frac{1}{2} \delta_{\dot{\beta}}^{\dot{\alpha}} \bar{Q}_{\dot{\gamma}}, \\ [\bar{M}_{\dot{\beta}}^{\dot{\alpha}}, S^\gamma] &= 0, & [\bar{M}_{\dot{\beta}}^{\dot{\alpha}}, \bar{S}^{\dot{\gamma}}] &= \delta_{\dot{\beta}}^{\dot{\gamma}} \bar{S}^{\dot{\alpha}} - \frac{1}{2} \delta_{\dot{\beta}}^{\dot{\alpha}} \bar{S}^{\dot{\gamma}}, \\ [P_{\alpha\dot{\beta}}, S^\gamma] &= \delta_\alpha^\gamma \bar{Q}_{\dot{\beta}}, & [P_{\alpha\dot{\beta}}, \bar{S}^{\dot{\gamma}}] &= \delta_{\dot{\beta}}^{\dot{\gamma}} Q_\alpha, \\ [K^{\dot{\alpha}\beta}, Q_\gamma] &= \delta_\gamma^{\dot{\alpha}} \bar{S}^{\dot{\beta}}, & [K^{\dot{\alpha}\beta}, \bar{Q}_{\dot{\gamma}}] &= \delta_{\dot{\gamma}}^{\dot{\alpha}} S^\beta, \\ [H, Q_\alpha] &= \frac{1}{2} Q_\alpha, & [H, \bar{Q}_{\dot{\alpha}}] &= \frac{1}{2} \bar{Q}_{\dot{\alpha}}, \\ [H, S^\alpha] &= -\frac{1}{2} S^\alpha, & [H, \bar{S}^{\dot{\alpha}}] &= -\frac{1}{2} \bar{S}^{\dot{\alpha}}. \end{aligned} \quad (3.11)$$

The R -charge commutes with all bosonic generators and has non-trivial commutators only with the supercharges and their superconformal partners

$$\begin{aligned} [R, Q_\alpha] &= -Q_\alpha, & [R, \bar{Q}_{\dot{\alpha}}] &= \bar{Q}_{\dot{\alpha}}, \\ [R, S^\alpha] &= S^\alpha, & [R, \bar{S}^{\dot{\alpha}}] &= -\bar{S}^{\dot{\alpha}}. \end{aligned} \quad (3.12)$$

Let us now simplify the shape of the $\mathcal{N} = 1$ superconformal algebra by introducing the notations

$$\mathcal{M}_A^B = \begin{pmatrix} M_\alpha^\beta + \frac{1}{2} \delta_\alpha^\beta H & \frac{1}{2} P_{\alpha\dot{\beta}} \\ \frac{1}{2} K^{\dot{\alpha}\beta} & \bar{M}_{\dot{\beta}}^{\dot{\alpha}} - \frac{1}{2} \delta_{\dot{\beta}}^{\dot{\alpha}} H \end{pmatrix}, \quad \mathcal{Q}_A = \begin{pmatrix} Q_\alpha \\ \bar{S}^{\dot{\alpha}} \end{pmatrix}, \quad \bar{\mathcal{Q}}^B = (S^\beta \quad \bar{Q}_{\dot{\beta}}) \quad (3.13)$$

Then the (anti)commutators (3.6),(3.7),(3.8),(3.9),(3.11),(3.12) combine to

$$\begin{aligned} [\mathcal{M}_A^B, \mathcal{M}_C^D] &= \delta_C^B \mathcal{M}_A^D - \delta_A^D \mathcal{M}_C^B, \\ [\mathcal{M}_A^B, \mathcal{Q}_C] &= \delta_C^B \mathcal{Q}_A - \frac{1}{4} \delta_A^B \mathcal{Q}_C, & [\mathcal{M}_A^B, \bar{\mathcal{Q}}^C] &= -\delta_A^C \bar{\mathcal{Q}}^B + \frac{1}{4} \delta_A^B \bar{\mathcal{Q}}^C, \\ [R, \mathcal{Q}_A] &= -\mathcal{Q}_A, & [R, \bar{\mathcal{Q}}^B] &= \bar{\mathcal{Q}}^B, \\ \{\mathcal{Q}_A, \bar{\mathcal{Q}}^B\} &= 4\mathcal{M}_A^B + 3\delta_A^B R, & \{\mathcal{Q}_A, \mathcal{Q}_B\} &= 0, & \{\bar{\mathcal{Q}}^A, \bar{\mathcal{Q}}^B\} &= 0, \end{aligned} \quad (3.14)$$

where

$$\delta_A^B = \begin{pmatrix} \delta_\alpha^\beta & 0 \\ 0 & \delta_{\dot{\beta}}^{\dot{\alpha}} \end{pmatrix}.$$

3.2. **The index.** Suppose there exist a supercharge Q and its Hermitean conjugate Q^\dagger satisfying the relation

$$\{Q, Q\} = 0, \quad \{Q^\dagger, Q^\dagger\} = 0, \quad \{Q, Q^\dagger\} = 2H, \quad (3.15)$$

where H is the Hamiltonian of a taken system ($= P_0$). This is a universal situation valid down to non-relativistic quantum mechanics. The Witten index [92] defined as

$$Tr(-1)^F$$

tells (under certain conditions) whether the supersymmetry is broken spontaneously or not. By definition the operator $(-1)^F$ is

$$(-1)^F = \exp(2\pi i \tilde{J}_3), \quad \{Q, (-1)^F\} = 0, \quad (3.16)$$

where in the spinorial basis $\tilde{J}_3 = -(J_3 + \bar{J}_3)$. It distinguishes bosonic states $|b\rangle$ from the fermionic ones $|f\rangle$,

$$(-1)^F |b\rangle = |b\rangle, \quad (-1)^F |f\rangle = -|f\rangle.$$

Because of the cancellation of contributions of states with positive energies to $Tr(-1)^F$, this trace formally can be evaluated using the zero-energy states

$$Tr(-1)^F = n_B^{E=0} - n_F^{E=0}, \quad (3.17)$$

where $n_B^{E=0}$ and $n_F^{E=0}$ are the numbers of bosonic and fermionic ground states. Therefore, if $Tr(-1)^F \neq 0$, supersymmetry is not broken. However, because of the presence of infinitely many states one needs a regulator commuting with Q (to save cancellations). Then the regularized Witten index is defined as

$$Ind = Tr((-1)^F e^{-\beta H}), \quad (3.18)$$

and formally it does not depend on the parameter β .

As to $\mathcal{N} = 1$ superconformal theories, there are different possibilities to realize relation (3.15), because of the presence of the operators $S^\alpha, \bar{S}^{\dot{\alpha}}$ – superconformal partners of supercharges $Q_\alpha, \bar{Q}_{\dot{\alpha}}$. Namely, one takes a pair of generators \mathcal{Q} with adjoint \mathcal{Q}^\dagger , such that

$$\{\mathcal{Q}, \mathcal{Q}^\dagger\} = 2\mathcal{H}, \quad (3.19)$$

where \mathcal{H} does not coincide with the Hamiltonian. Still, one can consider the subspace of the Hilbert space composed of the states $|\psi\rangle$ annihilated by \mathcal{H} , $\mathcal{H}|\psi\rangle = 0$, and define the Witten index $Ind = Tr((-1)^F e^{-\beta \mathcal{H}})$. However, the space of BPS states $|\psi\rangle$ is infinite dimensional and one has to introduce other regulators, which leads to nontrivial generalization of the index itself.

In $\mathcal{N} = 1$ superconformal algebra there are four non-trivial possibilities to choose supercharges $\mathcal{Q}, \mathcal{Q}^\dagger$ for constructing the superconformal index

$$\begin{aligned} \{Q_1, S^1\} &= 2 \left(H + 2J_3 + \frac{3}{2}R \right), \\ \{Q_2, S^2\} &= 2 \left(H - 2J_3 + \frac{3}{2}R \right), \\ \{\bar{Q}_1, -\bar{S}^{\dot{1}}\} &= 2 \left(H - 2\bar{J}_3 - \frac{3}{2}R \right), \\ \{\bar{Q}_2, -\bar{S}^{\dot{2}}\} &= 2 \left(H + 2\bar{J}_3 - \frac{3}{2}R \right). \end{aligned} \quad (3.20)$$

The generators commuting with the corresponding pair of supercharges written above for each case are

$$\begin{aligned} \overline{M}_{\dot{\beta}}^{\dot{\alpha}}, H + \frac{1}{2}R, P_{2\dot{\alpha}}, K^{\dot{\alpha}2}, \\ \overline{M}_{\dot{\beta}}^{\dot{\alpha}}, H + \frac{1}{2}R, P_{1\dot{\alpha}}, K^{\dot{\alpha}1}, \\ M_{\alpha}^{\beta}, H - \frac{1}{2}R, P_{\alpha\dot{2}}, K^{\dot{2}\alpha}, \end{aligned}$$

and for the last pair

$$M_{\alpha}^{\beta}, H - \frac{1}{2}R, P_{\alpha 1}, K^{1\alpha},$$

respectively (see the commutation relations (3.14)).

Let us stick to the choice of generators

$$\mathcal{Q} = \overline{Q}_i, \quad \mathcal{Q}^\dagger = -\overline{S}^i, \quad \mathcal{H} = H - 2\overline{J}_3 - \frac{3}{2}R.$$

Now to simplify the commutation relations for the algebra the following matrix is introduced

$$\mathcal{M}_A^B = \begin{pmatrix} M_{\alpha}^{\beta} + \frac{1}{2}\delta_{\alpha}^{\beta}\mathcal{R} & \mathcal{P}_{\alpha} \\ \overline{\mathcal{P}}^{\beta} & -\mathcal{R} + \frac{1}{2}\mathcal{H} \end{pmatrix}, \quad (3.21)$$

where $\mathcal{P}_{\alpha} = \frac{1}{2}P_{\alpha\dot{2}}$, $\overline{\mathcal{P}}^{\beta} = \frac{1}{2}K^{\dot{2}\beta}$, and

$$\mathcal{R} = H - \frac{1}{2}R.$$

We obtain the $SU(2,1)$ Lie algebra with the relations

$$[\mathcal{M}_A^B, \mathcal{M}_C^D] = \delta_C^B \mathcal{M}_A^D - \delta_A^D \mathcal{M}_C^B. \quad (3.22)$$

To regularize the trace over the infinite dimensional space of zero modes of \mathcal{H} , one needs regulators commuting with the distinguished supercharges \mathcal{Q} and \mathcal{Q}^\dagger . In the described situation an additional regulator is $t^{\mathcal{R}}$ for some arbitrary complex variable t restricted as $|t| < 1$ to ensure damping. Since M_{α}^{β} commutes with \overline{Q}_i and \overline{S}^i , there is one more regulator x^{2J_3} , $|x| < 1$, resolving the degeneracy ensured by M_{α}^{β} . Finally, one has [67, 68]

$$\text{ind}(t, x) = \text{Tr}(-1)^F x^{2J_3} t^{\mathcal{R}}. \quad (3.23)$$

This index depends on the chemical potentials x and t , in difference from the chemical potential β of the omitted regulator $\exp(-\beta\mathcal{H})$.

In the presence of internal symmetries, one can introduce more regulators to resolve the degeneracies. For $U(1)_1 \times U(1)_2 \times \dots \times U(1)_k$ global symmetry group, one introduces chemical potentials $2\mu_1, 2\mu_2, \dots, 2\mu_k$ and extends the superconformal index in the following way

$$\text{ind}(t, x, \mu_i) = \text{Tr}(-1)^F x^{2J_3} t^{\mathcal{R}} e^{2\sum_{i=1}^k \mu_i q_i}, \quad (3.24)$$

where q_i is the generator of $U(1)_i$'s group. For a non-abelian local gauge invariance group G with generators $G_j, j = 1, \dots, \text{rank } G$, and a non-abelian flavor group F with the generators $F_j, j = 1, \dots, \text{rank } F$, the superconformal index is

$$\text{ind}(t, x, z, y) = \text{Tr} \left((-1)^F x^{2J_3} t^{\mathcal{R}} e^{\sum_{i=1}^{\text{rank } G} g_i G^i} e^{\sum_{i=1}^{\text{rank } F} f_i F^i} \right), \quad (3.25)$$

where g_i and f_i are the chemical potentials for groups G and F correspondingly. For brevity we shall assume that abelian $U(1)$ -factors enter the flavor group contributions as well. From the representation theory it is known that $\text{Tr} \exp(\sum_{i=1}^{\text{rank } G} g_i G^i) = \chi_G(z)$ is the character of the corresponding representation of the gauge group G , where z is the set of complex eigenvalues of

matrices realizing G . The same is valid for the flavor group F : $Tr \exp(\sum_{i=1}^{rank M} f_i F^i) = \chi_F(y)$ is the character of the representations forming the space of free fields states, and y is the set of complex eigenvalues of matrices realizing F .

Since all physical observables are gauge invariant, one is interested in the index for gauge singlet operators. Therefore formula (3.25) is averaged over the gauge group, which yields the matrix integral

$$I(t, x, y) = \int_G d\mu(g) Tr \left((-1)^F x^{2J_3} t^{\mathcal{R}} e^{\sum_{i=1}^{rank G} g_i G^i} e^{\sum_{i=1}^{rank F} f_i F^i} \right), \quad (3.26)$$

where $d\mu(g)$ is the G -invariant matrix measure. This is the superconformal index – the key object for our purposes. By construction, it has the meaning of a particular $SU(2, 2|1) \times G \times F$ group character naturally restricted to the space of BPS states and integrated over the gauge group.

3.3. Calculation of the index. Explicit computation of the superconformal index was performed by Römelsberger [68]. According to his prescription one should first compute the trace in index (3.25) over the single particle states, which yields the formula¹

$$\begin{aligned} \text{ind}(t, x, z, y) &= \frac{2t^2 - t(x + x^{-1})}{(1 - tx)(1 - tx^{-1})} \chi_{adj}(g) \\ &+ \sum_i \frac{t^{2r_i} \chi_{R_F, i}(f) \chi_{R_G, i}(g) - t^{2-2r_i} \chi_{\bar{R}_F, i}(f) \chi_{\bar{R}_G, i}(g)}{(1 - tx)(1 - tx^{-1})}, \end{aligned} \quad (3.27)$$

where the first term represents contribution of the gauge fields belonging to the adjoint representation of the group G , and the sum over i corresponds to the chiral matter fields φ_i transforming as the gauge group representations $R_{G, i}$ and flavor symmetry representations $R_{F, i}$. The functions $\chi_{adj}(g)$, $\chi_{R_F, i}(f)$ and $\chi_{R_G, i}(g)$ are the corresponding characters – their general forms for major classical groups are described in the Appendix A.

For the $U(1)_R$ -group generated by the R -charge the terms proportional to t^{2r_i} and t^{2-2r_i} result from a chiral scalar field with the R -charge $2r_i$ and the fermion partner of the conjugate anti-chiral fields whose R -charge is $-2r_i$. For several $U(1)$ groups $U(1)_1 \times \dots \times U(1)_k \times U(1)_R$ variables r_i should have the form

$$r_i = R_i + \sum_{j=1}^k q_{ij} \mu_j,$$

where $2R_i$ is the R -charge of the i -th matter field, q_{ij} are the normalized hypercharges of the i -th matter field for the $U(1)_j$ group and $2\mu_j$ is the chemical potential for the latter $U(1)_j$ group.

In order to obtain the full superconformal index, this single particle index is inserted into the “plethystic” exponential with the subsequent averaging over the gauge group:

$$I(t, x, y) = \int_G d\mu(g) \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} \text{ind}(t^n, x^n, z^n, y^n) \right). \quad (3.28)$$

Similar objects appeared in computation of partition functions of different statistical mechanics models and quantum field theories, see, e.g., [75, 88, 54, 45, 53, 4, 26, 22].

¹In Romelsberger’s prescription in the denominator there stands the term $(1 - t\chi_{SU(2)_R, f}(\gamma) + t^2)$, where $\chi_{SU(2)_R, f}(\gamma)$ is the character for the fundamental representation of the $SU(2)$ group discussed above. If we parameterize this group characters by x then we have formula (3.27).

Clearly, there are qualitatively different contributions to superconformal indices – from the matter fields and the gauge fields. The generic form of a matter field single particle states contribution to the index $\text{ind}(t, x, z, y)$ in the presence of some global $U(1)$ symmetry group looks as follows

$$i_S^{part}(p, q, y) = \frac{t^{2r} z - t^{2-2r} z^{-1}}{(1-tx)(1-tx^{-1})}, \quad (3.29)$$

where t, x are the same variables as in (3.27) and $z = e^{2\mu}$ is the chemical potential for the $U(1)$ group. It is convenient to introduce new variables

$$p = tx, \quad q = tx^{-1}, \quad y = t^{2r} z,$$

where p and q are in general complex variables satisfying the constraints $|q|, |p| < 1$. As a result, $i_S^{part}(p, q, y) = (y - pq/y)/((1-p)(1-q))$. Then the described index building algorithm yields the elliptic gamma function [68] (cf. [3])

$$\Gamma(y; p, q) = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} i_S^{part}(p^n, q^n, y^n)\right) = \prod_{j,k=0}^{\infty} \frac{1 - y^{-1} p^{j+1} q^{k+1}}{1 - y p^j q^k}. \quad (3.30)$$

For the gauge field part one can set

$$i_V(p, q) = \frac{2t^2 - t(x + x^{-1})}{(1-tx)(1-tx^{-1})} \chi_{adj}(g) = \left(-\frac{p}{1-p} - \frac{q}{1-q}\right) \chi_{adj}(g).$$

For the $SU(2)$ group one has $\chi_{adj}(g) = z^2 + z^{-2} + 1$. Substituting pieces of this expression in the corresponding places of the index, we have the following characteristic building blocks

$$\begin{aligned} \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \left(-\frac{p^n}{1-p^n} - \frac{q^n}{1-q^n}\right) (z^{2n} + z^{-2n})\right) &= \frac{\theta(z^2; p)\theta(z^2; q)}{(1-z^2)^2} \\ &= \frac{1}{(1-z^2)(1-z^{-2})\Gamma(z^{\pm 2}; p, q)}, \\ \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \left(-\frac{p^n}{1-p^n} - \frac{q^n}{1-q^n}\right)\right) &= (p; p)_{\infty} (q; q)_{\infty}. \end{aligned}$$

Similar expressions are found for the higher rank gauge groups.

4. SEIBERG DUALITY FOR UNITARY GAUGE GROUPS

First we consider the usual $\mathcal{N} = 1$ supersymmetric quantum chromodynamics (SQCD) as an electric theory with the internal symmetry groups [71]

$$G = SU(N), \quad F = SU(N_f) \times SU(N_f) \times U(1)_B,$$

where $U(1)_B$ is generated by the baryon number charge (the $U(1)_R$ group generated by the R -charge enters the superconformal group). For such supersymmetric versions of QCD there are two chiral scalar multiplets Q and \tilde{Q} belonging to the fundamental f and anti-fundamental \bar{f} representations of $SU(N_c)$ respectively, each carrying a baryon number, and the vector multiplet V in the adjoint representation. The field content of the electric theory is given by the following table

	$SU(N)$	$SU(N_f)$	$SU(N_f)$	$U(1)_B$	$U(1)_R$
Q	f	f	1	$q_B = 1$	$2R_Q = \tilde{N}/N_f$
\tilde{Q}	\bar{f}	1	\bar{f}	$\tilde{q}_B = -1$	$2R_{\tilde{Q}} = \tilde{N}/N_f$
V	adj	1	1	0	$2R_V = 1$

Here q_B, \tilde{q}_B denote the baryonic charge and $2R_Q, 2R_{\tilde{Q}}, 2R_V$ are R -charges of the fields. The dual magnetic theory has the symmetry groups

$$G = SU(\tilde{N}), \quad F = SU(N_f) \times SU(N_f) \times U(1)_B,$$

where $\tilde{N} = N_f - N$. The field content of the dual theory is fixed in the table

Field	$SU(\tilde{N})$	$SU(N_f)$	$SU(N_f)$	$U(1)_B$	$U(1)_R$
q	f	\bar{f}	1	$q'_B = N/\tilde{N}$	$2R_q = N/N_f$
\tilde{q}	\bar{f}	1	f	$\tilde{q}'_B = -N/\tilde{N}$	$2R_{\tilde{q}} = N/N_f$
\tilde{V}	adj	1	1	0	$2R_V = 1$
M	1	f	\bar{f}	0	$2R_M = 2\tilde{N}/N_f$

It should be stressed here that this duality works only in the so-called conformal window

$$\frac{3}{2}N < N_f < 3N.$$

The first inequality is obtained from the condition that the magnetic theory is asymptotically free in one-loop approximation, and the other inequality is a consequence of the demand that the electric theory has the asymptotic freedom².

We define the r -charges for the R -current and the baryonic $U(1)_B$ current in the electric theory

$$r_Q = R_Q + q_B x, \quad r_{\tilde{Q}} = R_{\tilde{Q}} + \tilde{q}_B x,$$

Where x is the $U(1)_B$ -group chemical potential. In the magnetic theory we set

$$r_q = R_q + q'_B x, \quad r_{\tilde{q}} = R_{\tilde{q}} + \tilde{q}'_B x, \quad r_M = R_M.$$

Then the single particle states index for the electric theory is

$$\begin{aligned} i_E(p, q, z, s, t) = & - \left(\frac{p}{1-p} + \frac{q}{1-q} \right) \chi_{SU(N), \text{adj}}(z) \\ & + \frac{1}{(1-p)(1-q)} \left((pq)^{r_Q} \chi_{SU(N_f), f}(s) \chi_{SU(N), f}(z) - (pq)^{1-r_Q} \chi_{SU(N_f), \bar{f}}(s) \chi_{SU(N), \bar{f}}(z) \right) \\ & + \frac{1}{(1-p)(1-q)} \left((pq)^{r_{\tilde{Q}}} \chi_{SU(N_f), \bar{f}}(t) \chi_{SU(N), \bar{f}}(z) - (pq)^{1-r_{\tilde{Q}}} \chi_{SU(N_f), f}(t) \chi_{SU(N), f}(z) \right). \end{aligned} \quad (4.1)$$

For the magnetic theory we have

$$\begin{aligned} i_M(p, q, z, s, t) = & - \left(\frac{p}{1-p} + \frac{q}{1-q} \right) \chi_{SU(\tilde{N}), \text{adj}}(z) \\ & + \frac{1}{(1-p)(1-q)} \left((pq)^{r_q} \chi_{SU(N_f), \bar{f}}(s) \chi_{SU(\tilde{N}), f}(z) - (pq)^{1-r_q} \chi_{SU(N_f), f}(s) \chi_{SU(\tilde{N}), \bar{f}}(z) \right) \\ & + \frac{1}{(1-p)(1-q)} \left((pq)^{r_{\tilde{q}}} \chi_{SU(N_f), f}(t) \chi_{SU(\tilde{N}), \bar{f}}(z) - (pq)^{1-r_{\tilde{q}}} \chi_{SU(N_f), \bar{f}}(t) \chi_{SU(\tilde{N}), f}(z) \right) \\ & + \frac{1}{(1-p)(1-q)} \left((pq)^{r_M} \chi_{SU(N_f), f}(s) \chi_{SU(N_f), \bar{f}}(t) - (pq)^{1-r_M} \chi_{SU(N_f), \bar{f}}(s) \chi_{SU(N_f), f}(t) \right). \end{aligned} \quad (4.2)$$

²The one-loop beta function for the gauge coupling [52] is given by $\beta_g = -\frac{g^3}{16\pi^2} \left(\frac{11}{3}T(\mathbf{adj}) - \frac{2}{3}T(F) - \frac{1}{3}T(S) \right)$, where $T(F)$ is the sum of coefficients $T(\mathbf{r})$ (see the Appendix C for more details) over all fermions, $T(S)$ is the sum over all scalars and $T(\mathbf{adj})$ is $T(\mathbf{r})$ for the adjoint representation.

The superconformal indices take the form (see the invariant measures in the Appendix B)

$$I_E = \frac{(p; p)_\infty^{N-1} (q; q)_\infty^{N-1}}{N!} \times \int_{\mathbb{T}^{N-1}} \frac{\prod_{i=1}^{N_f} \prod_{j=1}^N \Gamma((pq)^{r_Q} s_i z_j, (pq)^{\tilde{r}_Q} t_i^{-1} z_j^{-1}; p, q)}{\prod_{1 \leq i < j \leq N} \Gamma(z_i z_j^{-1}, z_i^{-1} z_j; p, q)} \prod_{j=1}^{N-1} \frac{dz_j}{2\pi i z_j} \quad (4.3)$$

and

$$I_M = \frac{(p; p)_\infty^{\tilde{N}-1} (q; q)_\infty^{\tilde{N}-1}}{\tilde{N}!} \prod_{1 \leq i, j \leq N_f} \Gamma((pq)^{r_M} s_i t_j^{-1}; p, q) \times \int_{\mathbb{T}^{\tilde{N}-1}} \frac{\prod_{i=1}^{N_f} \prod_{j=1}^{\tilde{N}} \Gamma((pq)^{r_q} s_i^{-1} z_j, (pq)^{r_{\tilde{q}}} t_i z_j^{-1}; p, q)}{\prod_{1 \leq i < j \leq \tilde{N}} \Gamma(z_i z_j^{-1}, z_i^{-1} z_j; p, q)} \prod_{j=1}^{\tilde{N}-1} \frac{dz_j}{2\pi i z_j}, \quad (4.4)$$

Let us renormalize the variables

$$s_i \rightarrow (pq)^{-r_Q} s_i, \quad t_i^{-1} \rightarrow (pq)^{-r_{\tilde{Q}}} t_i^{-1}, \quad i = 1, \dots, N_f. \quad (4.5)$$

Then the superconformal indices are rewritten as

$$I_E = \frac{(p; p)_\infty^{N-1} (q; q)_\infty^{N-1}}{N!} \times \int_{\mathbb{T}^{N-1}} \frac{\prod_{i=1}^{N_f} \prod_{j=1}^N \Gamma(s_i z_j, t_i^{-1} z_j^{-1}; p, q)}{\prod_{1 \leq i < j \leq N} \Gamma(z_i z_j^{-1}, z_i^{-1} z_j; p, q)} \prod_{j=1}^{N-1} \frac{dz_j}{2\pi i z_j}, \quad (4.6)$$

and

$$I_M = \frac{(p; p)_\infty^{\tilde{N}-1} (q; q)_\infty^{\tilde{N}-1}}{\tilde{N}!} \prod_{1 \leq i, j \leq N_f} \Gamma(s_i t_j^{-1}; p, q) \times \int_{\mathbb{T}^{\tilde{N}-1}} \frac{\prod_{i=1}^{N_f} \prod_{j=1}^{\tilde{N}} \Gamma(S^{1/\tilde{N}} s_i^{-1} z_j, T^{-1/\tilde{N}} t_i z_j^{-1}; p, q)}{\prod_{1 \leq i < j \leq \tilde{N}} \Gamma(z_i z_j^{-1}, z_i^{-1} z_j; p, q)} \prod_{j=1}^{\tilde{N}-1} \frac{dz_j}{2\pi i z_j}, \quad (4.7)$$

where

$$S = \prod_{i=1}^{N_f} s_i, \quad T^{-1} = \prod_{i=1}^{N_f} t_i^{-1}$$

and the balancing condition reads

$$ST^{-1} = (pq)^{N_f - N}.$$

As discussed by Dolan and Osborn [23], the equality $I_E = I_M$ follows from the $A_n \leftrightarrow A_m$ root systems symmetry transformation³ established by Rains [61]. For $N = \tilde{N} = 2$ this identity is a simple consequence of the symmetry transformation for an elliptic analogue of the Gauss hypergeometric function discovered earlier by the first author in [78].

To be rigorous, it should be mentioned that the needed equality between elliptic hypergeometric integrals takes place only under certain constraints on the parameters. Namely, the

³It should be stressed here that this transformation is valid for any N_f while the Seiberg duality is expected to exist only in the conformal window, where we have appropriate R -charges yielding an anomaly free theory. One cannot extrapolate this duality outside of this window except of the boundary points $N_f = \frac{3}{2}N$ and $N_f = 3N$, although the relation between integrals remains true. We thank A. Schwimmer and S. Theisen for a discussion on this point.

kernels of the integrals are meromorphic functions of integration variables $z_j \in \mathbb{C}^*$. There are two qualitatively different geometric sequences of poles of these kernels – some of them converge to zero $z_j = 0$ and others go to infinity. So, the equality $I_E = I_M$ with the integration contours \mathbb{T} on both sides is true provided \mathbb{T} separates these two types of pole sequences. In the present situation, this is guaranteed for $|S|^{1/\tilde{N}} < |s_j| < 1$ and $1 < |t_j| < T^{1/\tilde{N}}$. All the relations for superconformal indices described below have similar constraints on the parameters, but we shall not describe them in detail for brevity, assuming that the separability conditions for pole sequences are satisfied by the contour \mathbb{T} .

5. SYMPLECTIC GAUGE GROUPS

The electric theory we consider [39] has the overall symmetry group

$$G = SP(2N), \quad F = SU(2N_f) \times U(1)_R,$$

and the following field content

Field	$SP(2N)$	$SU(2N_f)$	$U(1)_R$
Q	f	f	$1 - (N + 1)/N_f$
V	adj	1	1

The dual magnetic theory has the overall symmetry group

$$G = SP(2\tilde{N}), \quad F = SU(2N_f) \times U(1)_R,$$

where $\tilde{N} = N_f - N - 2$. And the field content is described in the table

Field	$SP(2N)$	$SU(2N_f)$	$U(1)_R$
q	f	\bar{f}	$(N + 1)/N_f$
\tilde{V}	adj	1	1
M	1	T_A	$2(\tilde{N} + 1)/N_f$

The conformal window for this duality is

$$\frac{3}{2}(N + 1) < N_f < 3(N + 1).$$

For these theories we have the following indices (in the renormalized variables) [23]

$$I_E = \frac{(p; p)_\infty^N (q; q)_\infty^N}{2^N N!} \times \int_{\mathbb{T}^N} \frac{\prod_{i=1}^{2N_f} \prod_{j=1}^N \Gamma(t_i z_j^{\pm 1}; p, q)}{\prod_{1 \leq i < j \leq N} \Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q) \prod_{j=1}^N \Gamma(z_j^{\pm 2}; p, q)} \prod_{j=1}^N \frac{dz_j}{2\pi i z_j} \quad (5.1)$$

and

$$I_M = \frac{(p; p)_\infty^{\tilde{N}} (q; q)_\infty^{\tilde{N}}}{2^{\tilde{N}} \tilde{N}!} \prod_{1 \leq i < j \leq 2N_f} \Gamma(t_i t_j; p, q) \times \int_{\mathbb{T}^{\tilde{N}}} \frac{\prod_{i=1}^{2N_f} \prod_{j=1}^{\tilde{N}} \Gamma((pq)^{1/2} t_i^{-1} z_j^{\pm 1}; p, q)}{\prod_{1 \leq i < j \leq \tilde{N}} \Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q) \prod_{j=1}^{\tilde{N}} \Gamma(z_j^{\pm 2}; p, q)} \prod_{j=1}^{\tilde{N}} \frac{dz_j}{2\pi i z_j}, \quad (5.2)$$

with the balancing condition

$$\prod_{i=1}^{2N_f} t_i = (pq)^{N_f - N - 1}.$$

For $N = \tilde{N} = 1$, equality $I_E = I_M$ is a consequence of the symmetry transformation established in [78]. Arbitrary ranks N, \tilde{N} identity was proven by Rains in [61].

6. MULTIPLE DUALITY FOR $SP(2N)$ GAUGE GROUP

There exists a multiple duality phenomenon, when one theory has many dual theories for different gauge groups. In this chapter we describe briefly such a situation for theories with $SP(2N)$ gauge group. The key group responsible for the corresponding multiple duality is $W(E_7)$ – the Weyl group for the exceptional root system E_7 . However, here we skip description of its structure, for corresponding details see our previous paper [86].

Let us take $\mathcal{N} = 1$ SQCD electric theory with the overall internal symmetry group $G \times F$, where

$$G = SP(2N), \quad F = SU(8) \times U(1).$$

This theory has one chiral scalar multiplet Q belonging to the fundamental representations of G and F , a vector multiplet V in the adjoint representation, and the antisymmetric $SP(2N)$ -tensor field X . The field content of the theory is fixed in the table

	$SP(2N)$	$SU(8)$	$U(1)$	$U(1)_R$
Q	f	f	$-\frac{N-1}{4}$	$\frac{1}{2}$
X	T_A	1	1	0
V	adj	1	0	1

For $N = 1$ the field X is absent and $U(1)$ -group is completely decoupled.

This electric theory and its particular magnetic dual (with $N > 1$) were considered in [11]. However, as we described in [86] there are other dualities. In a special section below we show that the 't Hooft anomaly matching conditions are fulfilled for all our new dualities.

The electric superconformal index is

$$I_E = \frac{(p; p)_\infty^N (q; q)_\infty^N}{2^N N!} \Gamma((pq)^s; p, q)^{N-1} \int_{\mathbb{T}^N} \prod_{1 \leq i < j \leq N} \frac{\Gamma((pq)^s z_i^{\pm 1} z_j^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \times \prod_{j=1}^N \frac{\prod_{i=1}^8 \Gamma((pq)^{r_Q} y_i z_j^{\pm 1}; p, q)}{\Gamma(z_j^{\pm 2}; p, q)} \frac{dz_j}{2\pi i z_j}, \quad (6.1)$$

where

$$r_Q = R_Q + e_Q s, \quad r_X = e_X s,$$

and $2R_Q = 1/2$ is the R -charge of the Q -field, $e_Q = -(N-1)/4$ and $e_X = 1$ are the $U(1)$ -group hypercharges with s being its chemical potential.

The first (new) class of magnetic theories has the symmetry groups

$$G = SP(2N), \quad F = SU(4) \times SU(4) \times U(1)_B \times U(1).$$

It contains two chiral scalar multiplets q and \tilde{q} belonging to the fundamental representations of $SP(2N)$, gauge field in the adjoint representation \tilde{V} , the anti-symmetric tensor representation Y , and the singlets M_J and \tilde{M}_J .

	$SP(2N)$	$SU(4)$	$SU(4)$	$U(1)_B$	$U(1)$	$U(1)_R$
q	f	f	1	-1	$-\frac{N-1}{4}$	$\frac{1}{2}$
\tilde{q}	f	1	f	1	$-\frac{N-1}{4}$	$\frac{1}{2}$
Y	T_A	1	1	0	1	0
M_J	1	T_A	1	2	$\frac{2J-N+1}{2}$	1
\tilde{M}_J	1	1	T_A	-2	$\frac{2J-N+1}{2}$	1
\tilde{V}	adj	1	1	0	0	1

In this and other tables of this section describing properties of the fields the capital index J takes values $0, \dots, N-1$, which is not mentioned for brevity.

The superconformal index in this magnetic theory is

$$\begin{aligned}
 I_M^{(1)} &= \prod_{J=0}^{N-1} \prod_{1 \leq i < j \leq 4} \Gamma((pq)^{r_{M_J}} y_i y_j; p, q) \prod_{5 \leq i < j \leq 8} \Gamma((pq)^{r_{\tilde{M}_J}} y_i y_j; p, q) \\
 &\times \Gamma((pq)^s; p, q)^{N-1} \frac{(p; p)_\infty^N (q; q)_\infty^N}{2^N N!} \int_{\mathbb{T}^N} \prod_{1 \leq i < j \leq N} \frac{\Gamma((pq)^s z_i^{\pm 1} z_j^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \\
 &\times \prod_{j=1}^N \frac{\prod_{i=1}^4 \Gamma((pq)^{r_q} v^{-2} y_i z_j^{\pm 1}; p, q) \prod_{i=5}^8 \Gamma((pq)^{r_{\tilde{q}}} v^2 y_i z_j^{\pm 1}; p, q)}{\Gamma(z_j^{\pm 2}; p, q)} \frac{dz_j}{2\pi i z_j}, \quad (6.2)
 \end{aligned}$$

where

$$\begin{aligned}
 r_q &= R_q - \frac{N-1}{4}s, \quad r_{\tilde{q}} = R_{\tilde{q}} - \frac{N-1}{4}s, \quad r_Y = s, \\
 r_{M_J} &= R_{M_J} - \frac{1}{2}(N-1-2J)s, \quad r_{\tilde{M}_J} = R_{\tilde{M}_J} - \frac{1}{2}(N-1-2J)s.
 \end{aligned}$$

and $v = \sqrt[4]{y_1 y_2 y_3 y_4}$.

The second (new) class of dual magnetic theories has the same symmetries as in the previous case, but different representation content. Their field content is described in the following table

	$SP(2N)$	$SU(4)$	$SU(4)$	$U(1)_B$	$U(1)$	$U(1)_R$
q	f	f	1	1	$-\frac{N-1}{4}$	$\frac{1}{2}$
\tilde{q}	f	1	f	-1	$-\frac{N-1}{4}$	$\frac{1}{2}$
Y	T_A	1	1	0	1	0
M_J	1	f	f	0	$\frac{2J-N+1}{2}$	1
\tilde{V}	adj	1	1	0	0	1

The index for this magnetic theory is given by

$$\begin{aligned}
 I_M^{(2)} &= \Gamma((pq)^s; p, q)^{N-1} \prod_{J=0}^{N-1} \prod_{i=1}^4 \prod_{j=5}^8 \Gamma((pq)^{r_{M_J}} y_i y_j; p, q) \\
 &\times \frac{(p; p)_\infty^N (q; q)_\infty^N}{2^N N!} \int_{\mathbb{T}^N} \prod_{1 \leq i < j \leq N} \frac{\Gamma((pq)^s z_i^{\pm 1} z_j^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \\
 &\times \prod_{j=1}^N \frac{\prod_{i=1}^4 \Gamma((pq)^{r_q} v^2 y_i^{-1} z_j^{\pm 1}; p, q) \prod_{i=5}^8 \Gamma((pq)^{r_{\tilde{q}}} v^{-2} y_i^{-1} z_j^{\pm 1}; p, q)}{\Gamma(z_j^{\pm 2}; p, q)} \frac{dz_j}{2\pi i z_j}, \quad (6.3)
 \end{aligned}$$

where

$$r_q = r_{\tilde{q}} = \frac{1}{4} - \frac{N-1}{4}s, \quad r_Y = s, \quad r_{M_J} = \frac{1}{2} - \frac{1}{2}(N-1-2J)s.$$

Finally, the third type of magnetic theories, which was constructed originally in [11], has the symmetry groups

$$G = SP(2N), \quad F = SU(8) \times U(1),$$

and its fields content is

	$SP(2N)$	$SU(8)$	$U(1)$	$U(1)_R$
q	f	\bar{f}	$-\frac{N-1}{4}$	$\frac{1}{2}$
Y	T_A	1	1	0
M_J	1	T_A	$\frac{2J-N+1}{2}$	1
\tilde{V}	adj	1	0	1

The magnetic superconformal index has the form

$$\begin{aligned}
I_M^{(3)} &= \Gamma((pq)^{r_Y}; p, q)^{N-1} \prod_{J=0}^{N-1} \prod_{1 \leq i < j \leq 8} \Gamma((pq)^{r_{M_J}} y_i y_j; p, q) \\
&\times \frac{(p; p)_\infty^N (q; q)_\infty^N}{2^N N!} \int_{\mathbb{T}^N} \prod_{1 \leq i < j \leq N} \frac{\Gamma((pq)^{r_Y} z_i^{\pm 1} z_j^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \\
&\times \prod_{j=1}^N \frac{\prod_{i=1}^8 \Gamma((pq)^{r_q} y_i^{-1} z_j^{\pm 1}; p, q)}{\Gamma(z_j^{\pm 2}; p, q)} \frac{dz_j}{2\pi i z_j},
\end{aligned} \tag{6.4}$$

where

$$r_q = \frac{1 - s(N-1)}{4}, \quad r_Y = s, \quad r_{M_J} = sJ + \frac{1 - s(N-1)}{2}.$$

The $SP(2)$ gauge group case can be obtained from the tables above by substituting $N = 1$ and deleting fields X in the electric theory and Y in the magnetic theories, which decouple completely. The number of mesons in dual theories is reduced as well. Equality of superconformal indices for $N = 1$ follows from the results of [78], and the needed identities for elliptic hypergeometric integrals for $N > 1$ were established in [61]. As argued in [86], there should be in total 72 theories dual to each other – this number equals to the dimension of the coset group $W(E_7)/S_8$ responsible for the dualities (in this respect, see also [51]).

7. A NEW $SP(2N) \leftrightarrow SP(2M)$ GROUPS DUALITY

We take as the electric theory SQCD based on the symmetry groups

$$G = SP(2M), \quad F = SU(4) \times SP(2(M+N)) \times U(1)$$

with the fields content fixed in the table below

	$SP(2M)$	$SU(4)$	$SP(2(M+N))$	$U(1)$	$U(1)_R$
Q_1	f	\bar{f}	1	$-\frac{M-N-2}{4}$	0
Q_2	f	1	f	$-\frac{1}{2}$	1
X	T_A	1	1	1	0
V	adj	1	1	0	1

The dual magnetic theory has the symmetries

$$G = SP(2N), \quad F = SU(4) \times SP(2(M+N)) \times U(1)$$

and the fields content is described below

	$SP(2N)$	$SU(4)$	$SP(2(M+N))$	$U(1)$	$U(1)_R$
q_1	f	f	1	$\frac{M-N+2}{4}$	0
q_2	f	1	f	$-\frac{1}{2}$	1
M	1	\overline{f}	f	$-\frac{M-N}{4}$	1
N_j	1	\overline{T}_A	1	$j - \frac{M-N-2}{2}$	0
Y	T_A	1	1	1	0
\widetilde{V}	adj	1	1	0	1

where $j = 0, \dots, M - N - 1$. We assume also that either $N = M$ (in this case the fields N_j are absent) or $M > N$. Then a simple analysis shows that the conformal window reads now as $N \leq M < N + 2$.

The superconformal indices for these theories are easily found from the group-representation content described in the tables

$$\begin{aligned}
 I_E &= \frac{(p; p)_\infty^M (q; q)_\infty^M}{2^M M!} \Gamma(t; p, q)^M \int_{\mathbb{T}^M} \prod_{1 \leq i < j \leq M} \frac{\Gamma(tz_i^{\pm 1} z_j^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \\
 &\times \prod_{i=1}^M \frac{\prod_{k=1}^4 \Gamma(tt_k^{-1} z_i^{\pm 1}; p, q) \prod_{j=1}^{N+M} \Gamma(s_j z_i^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 2}; p, q) \prod_{j=1}^{N+M} \Gamma(ts_j z_i^{\pm 1}; p, q)} \frac{dz_i}{2\pi i z_i}
 \end{aligned} \tag{7.1}$$

and

$$\begin{aligned}
 I_M &= \frac{(p; p)_\infty^N (q; q)_\infty^N}{2^N N!} \Gamma(t; p, q)^N \prod_{k=1}^4 \prod_{j=1}^{M+N} \frac{\Gamma(ts_j t_k^{-1}; p, q)}{\Gamma(t_k s_j; p, q)} \\
 &\times \prod_{i=0}^{M-N-1} \prod_{1 \leq k < r \leq 4} \Gamma(t^{i+2} t_k^{-1} t_r^{-1}; p, q) \\
 &\times \int_{\mathbb{T}^N} \prod_{1 \leq i < j \leq N} \frac{\Gamma(tz_i^{\pm 1} z_j^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \prod_{i=1}^N \frac{\prod_{k=1}^4 \Gamma(t_k z_i^{\pm 1}; p, q) \prod_{j=1}^{N+M} \Gamma(s_j z_i^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 2}; p, q) \prod_{j=1}^{N+M} \Gamma(ts_j z_i^{\pm 1}; p, q)} \frac{dz_i}{2\pi i z_i},
 \end{aligned} \tag{7.2}$$

where the balancing condition is $\prod_{r=1}^4 t_r = t^{2+M-N}$.

We have checked that the anomalies of these two theories match (see below), which is a very strong indication that the theories are dual to each other. This is another new duality that we describe in this paper. The elliptic hypergeometric identity $I_E = I_M$, which would give another argument supporting duality, is a special case of the Rains Conjecture 1 in [64]. We have found also the duality corresponding to general form of this conjecture, but its analysis is not complete yet.

8. MULTIPLE DUALITY FOR $SU(4)$ GAUGE GROUP

The electric theory of interest was considered in [14]. It has the symmetries

$$G = SU(4), \quad F = SU(2) \times SU(4) \times SU(4) \times U(1)_1 \times U(1)_2.$$

The field content is presented in the table

	$SU(4)$	$SU(2)$	$SU(4)$	$SU(4)$	$U(1)_1$	$U(1)_2$	$U(1)_R$
Q	f	1	f	1	1	-1	$\frac{1}{2}$
\widetilde{Q}	\overline{f}	1	1	f	-1	-1	$\frac{1}{2}$
A	T_A	f	1	1	0	2	0
V	adj	1	1	1	0	0	1

Its superconformal index is (in the renormalized variables)

$$I_E = \frac{(p; p)_\infty^3 (q; q)_\infty^3}{4!} \int_{\mathbb{T}^3} \prod_{1 \leq k < l \leq 4} \frac{\prod_{j=1}^2 \Gamma(s_j z_k z_l; p, q)}{\Gamma(z_k^{-1} z_l, z_k z_l^{-1}; p, q)} \quad (8.1)$$

$$\times \prod_{k,j=1}^4 \Gamma(t_k z_j, u_k z_j^{-1}; p, q) \prod_{j=1}^3 \frac{dz_j}{2\pi i z_j},$$

where

$$S = \prod_{j=1}^2 s_j, \quad T = \prod_{j=1}^4 t_j, \quad U = \prod_{j=1}^4 u_j$$

and the balancing condition is

$$S^2 T U = (pq)^2.$$

The first duality corresponds to the theory described in the table below

	$SU(4)$	$SU(2)$	$SU(4)$	$SU(4)$	$U(1)_1$	$U(1)_2$	$U(1)_R$
q	f	1	f	1	-1	-1	$\frac{1}{2}$
\tilde{q}	f	1	1	f	1	-1	$\frac{1}{2}$
a	T_A	f	1	1	0	2	0
B	1	f	T_A	1	-2	0	1
\tilde{B}	1	f	1	T_A	-2	0	1
\tilde{V}	adj	1	1	1	0	0	1

The resulting superconformal index is given by the integral

$$I_M^{(1)} = \frac{(p; p)_\infty^3 (q; q)_\infty^3}{4!} \prod_{j=1}^2 \prod_{1 \leq k < l \leq 4} \Gamma(s_j t_k t_l, s_j u_k u_l; p, q) \int_{\mathbb{T}^3} \prod_{1 \leq k < l \leq 4} \frac{\prod_{j=1}^2 \Gamma(s_j z_k z_l; p, q)}{\Gamma(z_k^{-1} z_l, z_k z_l^{-1}; p, q)}$$

$$\times \prod_{k,j=1}^4 \Gamma(\sqrt[4]{U/T} t_k z_j, \sqrt[4]{T/U} u_k z_j^{-1}; p, q) \prod_{j=1}^3 \frac{dz_j}{2\pi i z_j}. \quad (8.2)$$

The second dual theory is described in the table

	$SU(4)$	$SU(2)$	$SU(4)$	$SU(4)$	$U(1)_1$	$U(1)_2$	$U(1)_R$
q	f	1	f	1	1	-1	$\frac{1}{2}$
\tilde{q}	f	1	1	f	-1	-1	$\frac{1}{2}$
a	T_A	f	1	1	0	2	0
M_0	1	1	f	f	0	-2	1
M_2	1	1	f	f	0	2	1
\tilde{V}	adj	1	1	1	0	0	1

with the superconformal index

$$I_M^{(2)} = \frac{(p; p)_\infty^3 (q; q)_\infty^3}{4!} \prod_{k,l=1}^4 \Gamma(t_k u_l, S t_k u_l; p, q) \int_{\mathbb{T}^3} \prod_{1 \leq k < l \leq 4} \frac{\prod_{j=1}^2 \Gamma(s_j z_k z_l; p, q)}{\Gamma(z_k^{-1} z_l, z_k z_l^{-1}; p, q)}$$

$$\times \prod_{k,j=1}^4 \Gamma(\sqrt{T} t_k^{-1} z_j, \sqrt{U} u_k^{-1} z_j^{-1}; p, q) \prod_{j=1}^3 \frac{dz_j}{2\pi i z_j}. \quad (8.3)$$

The third dual theory has the field content

	$SU(4)$	$SU(2)$	$SU(4)$	$SU(4)$	$U(1)_1$	$U(1)_2$	$U(1)_R$
q	f	1	f	1	-1	-1	$\frac{1}{2}$
\tilde{q}	\bar{f}	1	1	f	1	-1	$\frac{1}{2}$
a	T_A	f	1	1	0	2	0
M_0	1	1	f	f	0	-2	1
M_2	1	1	f	f	0	2	1
B	1	f	T_A	1	2	0	1
\bar{B}	1	f	1	T_A	-2	0	1
\tilde{V}	adj	1	1	1	0	0	1

and the superconformal index

$$\begin{aligned}
 I_M^{(3)} &= \frac{(p;p)_\infty^3 (q;q)_\infty^3}{4!} \prod_{k,l=1}^4 \Gamma(t_k u_l, S t_k u_l; p, q) \prod_{j=1}^2 \prod_{1 \leq k < l \leq 4} \Gamma(s_j t_k t_l, s_j u_k u_l; p, q) \\
 &\times \int_{\mathbb{T}^3} \prod_{1 \leq k < l \leq 4} \frac{\prod_{j=1}^2 \Gamma(s_j z_k z_l; p, q)}{\Gamma(z_k^{-1} z_l, z_k z_l^{-1}; p, q)} \prod_{k,j=1}^4 \Gamma(\sqrt[4]{TU} t_k^{-1} z_j, \sqrt[4]{TU} u_k^{-1} z_j^{-1}; p, q) \prod_{j=1}^3 \frac{dz_j}{2\pi i z_j}.
 \end{aligned} \tag{8.4}$$

The equalities $I_E = I_M^{(1)} = I_M^{(2)} = I_M^{(3)}$ are new transformation identities requiring a rigorous proof. Note that we do not have here the structure of the full $W(E_7)$ group, but instead the set of its discrete Weyl reflections without S_8 -permutational symmetry.

9. MULTIPLE DUALITY FOR ARBITRARY RANK $SU(2N)$ GAUGE GROUP

In this section the multiple $SU(4)$ dualities are generalized to the higher rank $G = SU(2N)$ gauge group which was also considered in [14]. The overall flavor symmetry group of the theories is rather unusual

$$F = SU(4) \times SU(4) \times U(1)_1 \times U(1)_2 \times U(1)_3.$$

The field content of the electric theory is shown in the table

	$SU(2N)$	$SU(4)$	$SU(4)$	$U(1)_1$	$U(1)_2$	$U(1)_3$	$U(1)_R$
Q	f	f	1	0	1	$2N - 2$	$\frac{1}{2}$
\tilde{Q}	\bar{f}	1	f	0	-1	$2N - 2$	$\frac{1}{2}$
A	T_A	1	1	1	0	-4	0
\bar{A}	\bar{T}_A	1	1	-1	0	-4	0
V	adj	1	1	0	0	0	1

The corresponding superconformal index is

$$\begin{aligned}
 I_E &= \frac{(p;p)_\infty^{2N-1} (q;q)_\infty^{2N-1}}{(2N)!} \int_{\mathbb{T}^{2N-1}} \prod_{1 \leq j < k \leq 2N} \frac{\Gamma(U z_j z_k, V z_j^{-1} z_k^{-1}; p, q)}{\Gamma(z_j^{-1} z_k, z_j z_k^{-1}; p, q)} \\
 &\times \prod_{j=1}^{2N} \prod_{k=1}^4 \Gamma(s_k z_j, t_k z_j^{-1}; p, q) \prod_{j=1}^{2N-1} \frac{dz_j}{2\pi i z_j},
 \end{aligned} \tag{9.1}$$

where

$$S = \prod_{k=1}^4 s_k, \quad T = \prod_{k=1}^4 t_k,$$

and the balancing condition reads

$$(UV)^{2N-2} ST = (pq)^2.$$

This is the two-parameter (higher order) extension of the type II elliptic beta integral for the root system A_{2N-1} introduced in [78].

Magnetic dual theories have the same gauge and global symmetry groups. The field content of the first theory is given in the table

	$SU(2N)$	$SU(4)$	$SU(4)$	$U(1)_1$	$U(1)_2$	$U(1)_3$	$U(1)_R$
a	T_A	1	1	1	0	-4	0
\bar{a}	\bar{T}_A	1	1	-1	0	-4	0
q	f	f	1	0	-1	$2N-2$	$\frac{1}{2}$
\bar{q}	\bar{f}	1	f	0	1	$2N-2$	$\frac{1}{2}$
H_m	1	T_A	1	0	2	$4N-4-8m$	1
\bar{H}_m	1	1	T_A	0	-2	$4N-4-8m$	1
\bar{V}	adj	1	1	0	0	0	1

where $m = 0, \dots, N-1$. This leads to the magnetic index

$$\begin{aligned}
I_M^{(1)} &= \frac{(p; p)_\infty^{2N-1} (q; q)_\infty^{2N-1}}{(2N)!} \prod_{m=0}^{N-1} \prod_{1 \leq k < l \leq 4} \Gamma((UV)^m s_k s_l, (UV)^m t_k t_l; p, q) \\
&\times \int_{\mathbb{T}^{2N-1}} \prod_{1 \leq j < k \leq 2N} \frac{\Gamma(U z_j z_k, V z_j^{-1} z_k^{-1}; p, q)}{\Gamma(z_j^{-1} z_k, z_j z_k^{-1}; p, q)} \\
&\times \prod_{j=1}^{2N} \prod_{k=1}^4 \Gamma(\sqrt[4]{T/S} s_k z_j, \sqrt[4]{S/T} t_k z_j^{-1}; p, q) \prod_{j=1}^{2N-1} \frac{dz_j}{2\pi i z_j}.
\end{aligned} \tag{9.2}$$

Second dual theory is described in the table

	$SU(2N)$	$SU(4)$	$SU(4)$	$U(1)_1$	$U(1)_2$	$U(1)_3$	$U(1)_R$
a	T_A	1	1	1	0	-4	0
\bar{a}	\bar{T}_A	1	1	-1	0	-4	0
q	f	\bar{f}	1	0	1	$2N-2$	$\frac{1}{2}$
\bar{q}	\bar{f}	1	f	0	-1	$2N-2$	$\frac{1}{2}$
M_k	1	f	f	0	0	$4N-4-8k$	1
\bar{V}	adj	1	1	0	0	0	1

where $k = 0, \dots, N-1$. Its superconformal index has the form

$$\begin{aligned}
I_M^{(2)} &= \frac{(p; p)_\infty^{2N-1} (q; q)_\infty^{2N-1}}{(2N)!} \prod_{m=0}^{N-1} \prod_{k,l=1}^4 \Gamma((UV)^m s_k t_l; p, q) \\
&\times \int_{\mathbb{T}^{2N-1}} \prod_{1 \leq j < k \leq 2N} \frac{\Gamma(U z_j z_k, V z_j^{-1} z_k^{-1}; p, q)}{\Gamma(z_j^{-1} z_k, z_j z_k^{-1}; p, q)} \\
&\times \prod_{j=1}^{2N} \prod_{k=1}^4 \Gamma(\sqrt{S} s_k^{-1} z_j, \sqrt{T} t_k^{-1} z_j^{-1}; p, q) \prod_{j=1}^{2N-1} \frac{dz_j}{2\pi i z_j}.
\end{aligned} \tag{9.3}$$

Third duality corresponds to the theory

	$SU(2N)$	$SU(4)$	$SU(4)$	$U(1)_1$	$U(1)_2$	$U(1)_3$	$U(1)_R$
a	T_A	1	1	1	0	-4	0
\bar{a}	$\overline{T_A}$	1	1	-1	0	-4	0
q	f	\overline{f}	1	0	-1	$2N - 2$	$\frac{1}{2}$
\bar{q}	\overline{f}	1	\overline{f}	0	1	$2N - 2$	$\frac{1}{2}$
M_k	1	f	\overline{f}	0	0	$4N - 4 - 8k$	1
H_m	1	T_A	1	0	2	$4N - 4 - 8m$	1
$\overline{H_m}$	1	1	T_A	0	-2	$4N - 4 - 8m$	1
\overline{V}	adj	1	1	0	0	0	1

where $k, m = 0, \dots, N - 1$. The corresponding index is

$$\begin{aligned}
 I_M^{(3)} &= \frac{(p; p)_\infty^{2N-1} (q; q)_\infty^{2N-1}}{(2N)!} \prod_{m=0}^{N-1} \prod_{k,l=1}^4 \Gamma((UV)^m s_k t_l; p, q) \\
 &\quad \times \prod_{m=0}^{N-1} \prod_{1 \leq k < l \leq 4} \Gamma((UV)^m s_k s_l, (UV)^m t_k t_l; p, q) \\
 &\quad \times \int_{\mathbb{T}^{2N-1}} \prod_{1 \leq j < k \leq 2N} \frac{\Gamma(U z_j z_k, V z_j^{-1} z_k^{-1}; p, q)}{\Gamma(z_j^{-1} z_k, z_j z_k^{-1}; p, q)} \\
 &\quad \times \prod_{j=1}^{2N} \prod_{k=1}^4 \Gamma(\sqrt[4]{ST} s_k^{-1} z_j, \sqrt[4]{ST} t_k^{-1} z_j^{-1}; p, q) \prod_{j=1}^{2N-1} \frac{dz_j}{2\pi i z_j}.
 \end{aligned} \tag{9.4}$$

From the duality analysis for field theories we conjecture that $I_E = I_M^{(1)} = I_M^{(2)} = I_M^{(3)}$ under certain constraints on the parameters of the integrals. These are new identities for the described elliptic hypergeometric integrals.

Note that similarly constructed candidates of dual theories for the $SU(2N + 1)$ gauge group fail to match the global anomalies, see [14]. This does not close, however, the opportunity to find similar identities for integrals defined over $SU(2N + 1)$ -group as well.

10. KUTASOV-SCHWIMMER TYPE DUALITIES FOR UNITARY GAUGE GROUP

Now we would like to discuss generalizations of the Seiberg dualities for SU and SP groups discovered by Kutasov and Schwimmer (KS) [47, 48] and studied in detail in [49] and other papers. For brevity we skip separate global symmetry group descriptions since it can be read off straightforwardly from the field contents of the theories given in the tables. The first column in the tables describes types of gauge group representations for the fields, while other columns, except of the last one, describe field representations and the hypercharges for the subgroups of the flavor group F . Also, we skip the detailed description of single particle states indices writing out directly the integrals for the superconformal indices together with the balancing condition, if there is any. In this section we describe such dualities for the unitary $SU(N)$ gauge group.

10.1. $SU(N)$ gauge group with the adjoint field. The following electric-magnetic duality is described in [48]. The field content of the electric theory is given in the table

	$SU(N)$	$SU(N_f)$	$SU(N_f)$	$U(1)_B$	$U(1)_R$
Q	f	f	1	1	$2r = 1 - \frac{2N}{(K+1)N_f}$
\tilde{Q}	\bar{f}	1	\bar{f}	-1	$2\tilde{r} = 1 - \frac{2N}{(K+1)N_f}$
X	adj	1	1	0	$2s = \frac{2}{K+1}$
V	adj	1	1	0	1

And the magnetic theory ingredients are described in the table

	$SU(\tilde{N})$	$SU(N_f)$	$SU(N_f)$	$U(1)_B$	$U(1)_R$
q	f	\bar{f}	1	N/\tilde{N}	$2r' = 1 - \frac{2\tilde{N}}{(K+1)N_f}$
\tilde{q}	\bar{f}	1	f	$-N/\tilde{N}$	$2\tilde{r}' = 1 - \frac{2\tilde{N}}{(K+1)N_f}$
Y	adj	1	1	0	$2s = \frac{2}{K+1}$
$M_j, j = 1, \dots, K$	1	f	\bar{f}	0	$2r_{M_j} = 2 - \frac{4N}{(K+1)N_f} + \frac{2}{K+1}(j-1)$
\tilde{V}	adj	1	1	0	1

Here the dual gauge group dimension is

$$\tilde{N} = KN_f - N, \quad K = 1, 2, \dots, \quad (10.1)$$

with the constraint $N_f \geq 3N/(K+1)$.

Defining $U = (pq)^s = (pq)^{1/(K+1)}$, for these theories we find the following indices

$$I_E = \frac{(p; p)_\infty^{N-1} (q; q)_\infty^{N-1}}{N!} \Gamma(U; p, q)^{N-1} \int_{\mathbb{T}^{N-1}} \prod_{1 \leq i < j \leq N} \frac{\Gamma(Uz_i z_j^{-1}, Uz_i^{-1} z_j; p, q)}{\Gamma(z_i z_j^{-1}, z_i^{-1} z_j; p, q)} \\ \times \prod_{i=1}^{N_f} \prod_{j=1}^N \Gamma(s_i z_j, t_i^{-1} z_j^{-1}; p, q) \prod_{j=1}^{N-1} \frac{dz_j}{2\pi i z_j} \quad (10.2)$$

and

$$I_M = \frac{(p; p)_\infty^{\tilde{N}-1} (q; q)_\infty^{\tilde{N}-1}}{\tilde{N}!} \Gamma(U; p, q)^{\tilde{N}-1} \prod_{l=1}^K \prod_{1 \leq i, j \leq N_f} \Gamma(U^{l-1} s_i t_j^{-1}; p, q) \quad (10.3) \\ \times \int_{\mathbb{T}^{\tilde{N}-1}} \prod_{1 \leq i < j \leq \tilde{N}} \frac{\Gamma(Uz_i z_j^{-1}, Uz_i^{-1} z_j; p, q)}{\Gamma(z_i z_j^{-1}, z_i^{-1} z_j; p, q)} \\ \times \prod_{i=1}^{N_f} \prod_{j=1}^{\tilde{N}} \Gamma(U(ST)^{\frac{K}{2N}} s_i^{-1} z_j, U(ST)^{-\frac{K}{2N}} t_i z_j^{-1}; p, q) \prod_{j=1}^{\tilde{N}-1} \frac{dz_j}{2\pi i z_j},$$

where

$$S = \prod_{i=1}^{N_f} s_i, \quad T^{-1} = \prod_{i=1}^{N_f} t_i^{-1},$$

and the balancing condition is

$$U^{2N} S T^{-1} = (pq)^{N_f}.$$

An important fact is that these theories contain matter fields in the adjoint representation of the gauge group. The conjecture that $I_E = I_M$ (under appropriate contour-separability constraints mentioned earlier) represents a new type of the elliptic hypergeometric identities, not described earlier [84]. Therefore we present in the Appendix D the analysis of the total ellipticity condition hidden behind this identity.

10.2. **Two adjoint matter fields case.** This duality was considered in [6, 7]. The electric theory is

	$SU(N)$	$SU(N_f)$	$SU(N_f)$	$U(1)_B$	$U(1)_R$
Q	f	f	1	1	$1 - \frac{N}{N_f(K+1)}$
\tilde{Q}	\bar{f}	1	\bar{f}	-1	$1 - \frac{N}{N_f(K+1)}$
X	adj	1	1	0	$\frac{2}{K+1}$
Y	adj	1	1	0	$\frac{K}{K+1}$
V	adj	1	1	0	1

The magnetic theory has the following field content

	$SU(\tilde{N})$	$SU(N_f)$	$SU(N_f)$	$U(1)_B$	$U(1)_R$
q	f	\bar{f}	1	$\frac{N}{\tilde{N}}$	$1 - \frac{\tilde{N}}{N_f(K+1)}$
\tilde{q}	\bar{f}	1	f	$-\frac{N}{\tilde{N}}$	$1 - \frac{N}{N_f(K+1)}$
X	adj	1	1	0	$\frac{2}{K+1}$
Y	adj	1	1	0	$\frac{K}{K+1}$
M_{LJ}	1	f	\bar{f}	0	$2 - \frac{2N}{N_f(K+1)} + \frac{2L+KJ}{K+1}$
\tilde{V}	adj	1	1	0	1

where

$$\tilde{N} = 3KN_f - N, \quad (10.4)$$

K is odd, $0 \leq L \leq K-1$, and $J = 0, 1, 2$ with the constraint $N_f > N/(3K-1)$.

The corresponding superconformal indices are

$$I_E = \frac{(p; p)_\infty^{N-1} (q; q)_\infty^{N-1}}{N!} \int_{\mathbb{T}^{N-1}} \prod_{1 \leq i < j \leq N} \frac{\Gamma(Uz_i z_j^{-1}, Uz_i^{-1} z_j; p, q) \Gamma(U^{K/2} z_i z_j^{-1}, U^{K/2} z_i^{-1} z_j; p, q)}{\Gamma(z_i z_j^{-1}, z_i^{-1} z_j; p, q)} \\ \times \prod_{i=1}^{N_f} \prod_{j=1}^N \Gamma(s_i z_j, t_i^{-1} z_j^{-1}; p, q) \prod_{j=1}^{N-1} \frac{dz_j}{2\pi i z_j}, \quad (10.5)$$

and

$$I_M = \frac{(p; p)_\infty^{N-1} (q; q)_\infty^{N-1}}{N!} \prod_{L=0}^{K-1} \prod_{J=0}^2 \prod_{i,j=1}^{N_f} \Gamma(U^{L+KJ/2} s_i t_j^{-1}; p, q) \\ \times \int_{\mathbb{T}^{\tilde{N}-1}} \prod_{1 \leq i < j \leq \tilde{N}} \frac{\Gamma(Uz_i z_j^{-1}, Uz_i^{-1} z_j; p, q) \Gamma(U^{K/2} z_i z_j^{-1}, U^{K/2} z_i^{-1} z_j; p, q)}{\Gamma(z_i z_j^{-1}, z_i^{-1} z_j; p, q)} \\ \times \prod_{i=1}^{N_f} \prod_{j=1}^{\tilde{N}} \Gamma(U^{(-K+2)/2} (ST)^{\frac{3K}{2N}} s_i^{-1} z_j, U^{(-K+2)/2} (ST)^{-\frac{3K}{2N}} t_i z_j^{-1}; p, q) \prod_{j=1}^{\tilde{N}-1} \frac{dz_j}{2\pi i z_j}, \quad (10.6)$$

where $U = (pq)^{\frac{1}{K+1}}$ and the balancing condition reads

$$U^N ST^{-1} = (pq)^{N_f}.$$

Again, the conjectured equality $I_E = I_M$ is a new type of identities requiring a rigorous proof.

10.3. **Generalized KS type dualities.** These dualities were considered in paper [38].

10.3.1. *First pair of dual theories.* Electric theory:

	$SU(N)$	$SU(N_f)$	$SU(N_f)$	$U(1)$	$U(1)_B$	$U(1)_R$
Q	f	f	1	0	$\frac{1}{N}$	$2r = 1 - \frac{N+2K}{(K+1)N_f}$
\tilde{Q}	\bar{f}	1	f	0	$-\frac{1}{N}$	$2r = 1 - \frac{N+2K}{(K+1)N_f}$
X	T_A	1	1	1	$\frac{2}{N}$	$2s = \frac{1}{K+1}$
\tilde{X}	\bar{T}_A	1	1	-1	$-\frac{2}{N}$	$2s = \frac{1}{K+1}$
V	adj	1	1	0	0	1

Magnetic theory:

	$SU(\tilde{N})$	$SU(N_f)$	$SU(N_f)$	$U(1)$	$U(1)_B$	$U(1)_R$
q	f	\bar{f}	1	$\frac{K(N_f-2)}{\tilde{N}}$	$\frac{1}{\tilde{N}}$	$2r' = 1 - \frac{\tilde{N}+2K}{(K+1)N_f}$
\tilde{q}	\bar{f}	1	\bar{f}	$-\frac{K(N_f-2)}{\tilde{N}}$	$-\frac{1}{\tilde{N}}$	$2r' = 1 - \frac{\tilde{N}+2K}{(K+1)N_f}$
Y	T_A	1	1	$\frac{N-N_f}{\tilde{N}}$	$\frac{2}{\tilde{N}}$	$2s = \frac{1}{K+1}$
\tilde{Y}	\bar{T}_A	1	1	$-\frac{N-N_f}{\tilde{N}}$	$-\frac{2}{\tilde{N}}$	$2s = \frac{1}{K+1}$
M_j	1	f	f	0	0	$\frac{\tilde{N}-N+(2j+1)N_f}{N_f(K+1)}$
P_r	1	T_A	1	-1	0	$\frac{N-N+(2r+2)N_f}{N_f(K+1)}$
\tilde{P}_r	1	1	T_A	1	0	$\frac{\tilde{N}-N+(2r+2)N_f}{N_f(K+1)}$
V	adj	1	1	0	0	1

Here

$$\tilde{N} = (2K+1)N_f - 4K - N, \quad K = 0, 1, 2, \dots, \quad (10.7)$$

$j = 0, \dots, K$ and $r = 0, \dots, K-1$.

The electric index is

$$I_E = \frac{(p; p)_\infty^{N-1} (q; q)_\infty^{N-1}}{N!} \int_{\mathbb{T}^{N-1}} \prod_{1 \leq i < j \leq N} \frac{\Gamma(U z_i z_j, U^{-1} (pq)^{\frac{1}{K+1}} z_i^{-1} z_j^{-1}; p, q)}{\Gamma(z_i^{-1} z_j, z_i z_j^{-1}; p, q)} \\ \times \prod_{j=1}^N \prod_{k=1}^{N_f} \Gamma(s_k z_j, t_k z_j^{-1}; p, q) \prod_{j=1}^{N-1} \frac{dz_j}{2\pi i z_j}. \quad (10.8)$$

The magnetic index is

$$I_M = \frac{(p; p)_\infty^{\tilde{N}-1} (q; q)_\infty^{\tilde{N}-1}}{\tilde{N}!} \prod_{j=0}^K \prod_{k, l=1}^{N_f} \Gamma((pq)^{\frac{j}{K+1}} s_k t_l; p, q) \\ \times \prod_{r=0}^{K-1} \prod_{1 \leq k < l \leq N_f} \Gamma(U^{-1} (pq)^{\frac{r+1}{K+1}} s_k s_l, U (pq)^{\frac{r}{K+1}} t_k t_l; p, q) \\ \times \int_{\mathbb{T}^{\tilde{N}-1}} \prod_{1 \leq i < j \leq \tilde{N}} \frac{\Gamma(\tilde{U} z_i z_j, \tilde{U}^{-1} (pq)^{\frac{1}{K+1}} z_i^{-1} z_j^{-1}; p, q)}{\Gamma(z_i^{-1} z_j, z_i z_j^{-1}; p, q)} \\ \times \prod_{j=1}^{\tilde{N}} \prod_{k=1}^{N_f} \Gamma((U\tilde{U})^{\frac{1}{2}} s_k^{-1} z_j, (U\tilde{U})^{-\frac{1}{2}} (pq)^{\frac{1}{K+1}} t_k^{-1} z_j^{-1}; p, q) \prod_{j=1}^{\tilde{N}-1} \frac{dz_j}{2\pi i z_j}, \quad (10.9)$$

where $S = \prod_{j=1}^{N_f} s_j$, $T = \prod_{j=1}^{N_f} t_j$, the balancing condition looks as

$$ST = (pq)^{N_f - \frac{N+2K}{K+1}}$$

and $\tilde{U} = U^{\frac{N-F}{\tilde{N}}}(ST^{-1})^{\frac{1}{\tilde{N}}}(pq)^{\frac{\tilde{N}-N+F}{2\tilde{N}(K+1)}}$.

10.3.2. *Second pair of dual theories.* Electric theory:

	$SU(N)$	$SU(N_f)$	$SU(N_f)$	$U(1)$	$U(1)_B$	$U(1)_R$
Q	f	f	1	0	$\frac{1}{N}$	$2r = 1 - \frac{N-2K}{(K+1)N_f}$
\tilde{Q}	\bar{f}	1	f	0	$-\frac{1}{N}$	$2r = 1 - \frac{N-2K}{(K+1)N_f}$
X	T_S	1	1	1	$\frac{2}{N}$	$2s = \frac{1}{K+1}$
\tilde{X}	\bar{T}_S	1	1	-1	$-\frac{2}{N}$	$2s = \frac{1}{K+1}$
V	adj	1	1	0	0	1

Magnetic theory:

	$SU(\tilde{N})$	$SU(N_f)$	$SU(N_f)$	$U(1)$	$U(1)_B$	$U(1)_R$
q	f	\bar{f}	1	$\frac{K(N_f+2)}{\tilde{N}}$	$\frac{1}{\tilde{N}}$	$2r' = 1 - \frac{N-2K}{(K+1)N_f}$
\tilde{q}	\bar{f}	1	\bar{f}	$-\frac{K(N_f+2)}{\tilde{N}}$	$-\frac{1}{\tilde{N}}$	$2r' = 1 - \frac{N-2K}{(K+1)N_f}$
Y	T_S	1	1	$\frac{N-N_f}{\tilde{N}}$	$\frac{2}{\tilde{N}}$	$2s = \frac{1}{K+1}$
\tilde{Y}	\bar{T}_S	1	1	$-\frac{N-N_f}{\tilde{N}}$	$-\frac{2}{\tilde{N}}$	$2s = \frac{1}{K+1}$
M_j	1	f	f	0	0	$\frac{\tilde{N}-N+(2j+1)N_f}{N_f(K+1)}$
P_r	1	T_S	1	-1	0	$\frac{\tilde{N}-N+(2r+2)N_f}{N_f(K+1)}$
\tilde{P}_r	1	1	T_S	1	0	$\frac{\tilde{N}-N+(2r+2)N_f}{N_f(K+1)}$
V	adj	1	1	0	0	1

Here

$$\tilde{N} = (2K+1)N_f + 4K - N, \quad K = 0, 1, 2, \dots, \quad (10.10)$$

$j = 0, \dots, K$ and $r = 0, \dots, K-1$.

The electric index is given by the integral

$$I_E = \frac{(p; p)_\infty^{N-1} (q; q)_\infty^{N-1}}{N!} \int_{\mathbb{T}^{N-1}} \prod_{1 \leq i < j \leq N} \frac{\Gamma(U z_i z_j, U^{-1} (pq)^{\frac{1}{K+1}} z_i^{-1} z_j^{-1}; p, q)}{\Gamma(z_i^{-1} z_j, z_i z_j^{-1}; p, q)} \quad (10.11)$$

$$\times \prod_{j=1}^N \Gamma(U z_j^2, U^{-1} (pq)^{\frac{1}{K+1}} z_j^{-2}; p, q) \prod_{k=1}^{N_f} \Gamma(s_k z_j, t_k z_j^{-1}; p, q) \prod_{j=1}^{N-1} \frac{dz_j}{2\pi i z_j}.$$

The magnetic index is given by the integral

$$\begin{aligned}
I_M &= \frac{(p; p)_\infty^{\tilde{N}-1} (q; q)_\infty^{\tilde{N}-1}}{\tilde{N}!} \prod_{j=0}^{K-1} \prod_{k,l=1}^{N_f} \Gamma((pq)^{\frac{j}{K+1}} s_k t_l; p, q) \\
&\times \prod_{r=0}^{K-1} \prod_{1 \leq k < l \leq N_f} \Gamma(U^{-1}(pq)^{\frac{r+1}{K+1}} s_k s_l, U(pq)^{\frac{r}{K+1}} t_k t_l; p, q) \\
&\times \prod_{r=0}^{K-1} \prod_{k=1}^{N_f} \Gamma(U^{-1}(pq)^{\frac{r+1}{K+1}} s_k^2, U(pq)^{\frac{r}{K+1}} t_k^2; p, q) \\
&\times \int_{\mathbb{T}^{\tilde{N}-1}} \prod_{1 \leq i < j \leq \tilde{N}} \frac{\Gamma(\tilde{U} z_i z_j, \tilde{U}^{-1}(pq)^{\frac{1}{K+1}} z_i^{-1} z_j^{-1}; p, q)}{\Gamma(z_i^{-1} z_j, z_i z_j^{-1}; p, q)} \prod_{j=1}^{\tilde{N}} \Gamma(\tilde{U} z_j^2, \tilde{U}^{-1}(pq)^{\frac{1}{K+1}} z_j^{-2}; p, q) \\
&\times \prod_{j=1}^{\tilde{N}} \prod_{k=1}^{N_f} \Gamma((U\tilde{U})^{\frac{1}{2}} s_k^{-1} z_j, (U\tilde{U})^{-\frac{1}{2}} (pq)^{\frac{1}{K+1}} t_k^{-1} z_j^{-1}; p, q) \prod_{j=1}^{\tilde{N}-1} \frac{dz_j}{2\pi i z_j},
\end{aligned} \tag{10.12}$$

where $S = \prod_{j=1}^{N_f} s_j$, $T = \prod_{j=1}^{N_f} t_j$, the balancing condition looks as

$$ST = (pq)^{N_f - \frac{N-2K}{K+1}},$$

and $\tilde{U} = U^{\frac{N-F}{N}} (ST^{-1})^{\frac{1}{N}} (pq)^{\frac{\tilde{N}-N+F}{2\tilde{N}(K+1)}}$.

10.3.3. *Third pair of dual theories.* In comparison with the dualities described in previous two subsections, this case involves non-abelian flavor subgroups of different ranks.

The electric theory:

	$SU(N)$	$SU(N_f)$	$SU(N_f - 8)$	$U(1)$	$U(1)_B$	$U(1)_R$
Q	f	f	1	$-(2K+1) + \frac{2(4K+3)}{N_f}$	$\frac{1}{N}$	$2r = 1 - \frac{N+2(4K+3)}{2(K+1)N_f}$
\tilde{Q}	\bar{f}	1	f	$2K+1 + \frac{2(4K+3)}{N_f-8}$	$-\frac{1}{N}$	$2\tilde{r} = 1 - \frac{N-2(4K+3)}{2(K+1)(N_f-8)}$
X	T_A	1	1	1	$\frac{2}{N}$	$2s = \frac{1}{2(K+1)}$
\tilde{X}	\bar{T}_S	1	1	-1	$-\frac{2}{N}$	$2s = \frac{1}{2(K+1)}$
V	adj	1	1	0	0	1

The magnetic theory:

	$SU(\tilde{N})$	$SU(N_f)$	$SU(N_f - 8)$	$U(1)$	$U(1)_B$	$U(1)_R$
q	f	\bar{f}	1	$2K + 1 - \frac{2(4K+3)}{N_f}$	$\frac{1}{N}$	$2r' = 1 - \frac{N+2(4K+3)}{2(K+1)N_f}$
\tilde{q}	\bar{f}	1	\bar{f}	$-(2K + 1) - \frac{2(4K+3)}{N_f-8}$	$-\frac{1}{N}$	$2\tilde{r}' = 1 - \frac{\tilde{N}-2(4K+3)}{2(K+1)(N_f-8)}$
Y	T_A	1	1	-1	$\frac{2}{N}$	$2s = \frac{1}{2(K+1)}$
\tilde{Y}	\bar{T}_S	1	1	1	$-\frac{2}{N}$	$2s = \frac{1}{2(K+1)}$
M_J	1	f	f	$2(4K + 3)(\frac{1}{N_f} + \frac{1}{N_f-8})$	0	$2(r + \tilde{r}) + \frac{J}{K+1}$
P_{2L}	1	T_S	1	$-4K - 3 + \frac{4(4K+3)}{N_f}$	0	$4r + \frac{4L+1}{2(K+1)}$
P_{2M+1}	1	T_A	1	$-4K - 3 + \frac{4(4K+3)}{N_f}$	0	$4r + \frac{4M+3}{2(K+1)}$
\tilde{P}_{2L}	1	1	T_A	$4K + 3 + \frac{2(4K+3)}{N_f-8}$	0	$4\tilde{r} + \frac{4L+1}{2(K+1)}$
\tilde{P}_{2M+1}	1	1	T_S	$4K + 3 + \frac{2(4K+3)}{N_f-8}$	0	$4\tilde{r} + \frac{4M+3}{2(K+1)}$
\tilde{V}	adj	1	1	0	0	1

Here $J = 0, \dots, 2K + 1$, $L = 0, \dots, K$, $M = 0, \dots, K - 1$, and

$$\tilde{N} = (4K + 3)(N_f - 4) - N, \quad K = 0, 1, 2, \dots \quad (10.13)$$

The electric index is

$$I_E = \frac{(p; p)_\infty^{N-1} (q; q)_\infty^{N-1}}{N!} \int_{\mathbb{T}^{N-1}} \prod_{1 \leq i < j \leq N} \frac{\Gamma(U z_i z_j, U^{-1}(pq)^{\frac{1}{2(K+1)}} z_i^{-1} z_j^{-1}; p, q)}{\Gamma(z_i^{-1} z_j, z_i z_j^{-1}; p, q)} \quad (10.14)$$

$$\times \prod_{j=1}^N \Gamma(U^{-1}(pq)^{\frac{1}{2(K+1)}} z_j^{-2}; p, q) \prod_{k=1}^{N_f} \Gamma(s_k z_j; p, q) \prod_{l=1}^{N_f-8} \Gamma(t_l z_j^{-1}; p, q) \prod_{j=1}^{N-1} \frac{dz_j}{2\pi i z_j}.$$

The magnetic index is

$$I_M = \frac{(p; p)_\infty^{\tilde{N}-1} (q; q)_\infty^{\tilde{N}-1}}{\tilde{N}!} \prod_{J=0}^{2K+1} \prod_{i=1}^{N_f} \prod_{j=1}^{N_f-8} \Gamma((pq)^{\frac{J}{2(K+1)}} s_i t_j; p, q) \quad (10.15)$$

$$\times \prod_{l=0}^{2K} \prod_{1 \leq i < j \leq N_f} \Gamma((pq)^{\frac{2l+1}{4(K+1)}} u^{-1} v^{-1} s_i s_j; p, q) \prod_{l=0}^K \prod_{i=1}^{N_f} \Gamma((pq)^{\frac{4l+1}{4(K+1)}} u^{-1} v^{-1} s_i^2; p, q)$$

$$\times \prod_{m=0}^{2K} \prod_{1 \leq i < j \leq N_f-8} \Gamma((pq)^{\frac{2m+1}{4(K+1)}} u v t_i t_j; p, q) \prod_{m=0}^{K-1} \prod_{i=1}^{N_f-8} \Gamma((pq)^{\frac{4m+3}{4(K+1)}} u v t_i^2; p, q)$$

$$\times \int_{\mathbb{T}^{\tilde{N}-1}} \prod_{1 \leq i < j \leq \tilde{N}} \frac{\Gamma(\tilde{U} z_i z_j, \tilde{U}^{-1}(pq)^{\frac{1}{2(K+1)}} z_i^{-1} z_j^{-1}; p, q)}{\Gamma(z_i^{-1} z_j, z_i z_j^{-1}; p, q)}$$

$$\times \prod_{j=1}^{\tilde{N}} \Gamma(\tilde{U}^{-1}(pq)^{\frac{1}{2(K+1)}} z_j^{-2}; p, q) \prod_{j=1}^{\tilde{N}} \prod_{k=1}^{N_f} \Gamma((U\tilde{U})^{\frac{1}{2}} s_k^{-1} z_j; p, q)$$

$$\times \prod_{j=1}^{\tilde{N}} \prod_{l=1}^{N_f-8} \Gamma((U\tilde{U})^{-\frac{1}{2}} (pq)^{\frac{1}{2(K+1)}} t_l^{-1} z_j^{-1}; p, q) \prod_{j=1}^{\tilde{N}-1} \frac{dz_j}{2\pi i z_j},$$

where

$$U = uv(pq)^{\frac{1}{4(K+1)}}, \quad \tilde{U} = u^{-1} v^{\frac{N}{N}} (pq)^{\frac{1}{4(K+1)}}.$$

$S = \prod_{j=1}^{N_f} s_j$, $T = \prod_{j=1}^{N_f-8} t_j$ and the balancing condition is

$$STu^{-4}v^{-4} = (pq)^{N_f-4-\frac{N}{2(K+1)}}.$$

10.4. Adjoint, symmetric and conjugate symmetric tensor matter fields. This duality was constructed by Brodie and Strassler [7]. The electric theory is

	$SU(N)$	$SU(N_f)$	$SU(N_f)$	$U(1)$	$U(1)_B$	$U(1)_R$
Q	f	f	1	0	$\frac{1}{N}$	$1 - \frac{N-2}{N_f(K+1)}$
\tilde{Q}	\bar{f}	1	\bar{f}	0	$-\frac{1}{N}$	$1 - \frac{N-2}{N_f(K+1)}$
X	adj	1	1	0	0	$\frac{2}{K+1}$
Y	T_S	1	1	1	$\frac{2}{N}$	$\frac{K}{K+1}$
\tilde{Y}	\bar{T}_S	1	1	-1	$-\frac{2}{N}$	$\frac{K}{K+1}$
V	adj	1	1	0	0	1

The magnetic theory is

	$SU(\tilde{N})$	$SU(N_f)$	$SU(N_f)$	$U(1)$	$U(1)_B$	$U(1)_R$
q	f	\bar{f}	1	$\frac{KN_f+2}{\tilde{N}}$	$\frac{1}{\tilde{N}}$	$1 - \frac{\tilde{N}-2}{N_f(K+1)}$
\tilde{q}	\bar{f}	1	f	$-\frac{KN_f+2}{\tilde{N}}$	$-\frac{1}{\tilde{N}}$	$1 - \frac{\tilde{N}-2}{N_f(K+1)}$
X	adj	1	1	0	0	$\frac{2}{K+1}$
Y	T_S	1	1	$\frac{N-KN_f}{\tilde{N}}$	$\frac{2}{\tilde{N}}$	$\frac{K}{K+1}$
\bar{Y}	\bar{T}_S	1	1	$-\frac{N-KN_f}{\tilde{N}}$	$-\frac{2}{\tilde{N}}$	$\frac{K}{K+1}$
N_I	1	f	\bar{f}	0	0	$\frac{2I}{K+1} + \frac{2K}{K+1} + 2 - 2\frac{N-2}{N_f(K+1)}$
M_I	1	f	\bar{f}	0	0	$\frac{2I}{K+1} + 2 - 2\frac{N-2}{N_f(K+1)}$
P_{2J+1}	1	T_A	1	-1	0	$2\frac{2J+1}{K+1} + \frac{K}{K+1} + 2 - 2\frac{N-2}{N_f(K+1)}$
P_{2J}	1	T_S	1	-1	0	$2\frac{2J}{K+1} + \frac{K}{K+1} + 2 - 2\frac{N-2}{N_f(K+1)}$
\tilde{P}_{2J+1}	1	1	\bar{T}_A	1	0	$2\frac{2J+1}{K+1} + \frac{K}{K+1} + 2 - 2\frac{N-2}{N_f(K+1)}$
\tilde{P}_{2J}	1	1	\bar{T}_S	1	0	$2\frac{2J}{K+1} + \frac{K}{K+1} + 2 - 2\frac{N-2}{N_f(K+1)}$
\tilde{V}	adj	1	1	0	0	1

Here

$$\tilde{N} = 3KN_f + 4 - N, \quad (10.16)$$

K is odd, $I = 0, 1, \dots, K-1$, $J = 0, 1, \dots, \frac{K-1}{2}$ and $2J+1 \neq K$.

The indices are

$$\begin{aligned}
I_E &= \frac{(p;p)_\infty^{N-1}(q;q)_\infty^{N-1}}{N!} \int_{\mathbb{T}^{N-1}} \prod_{1 \leq i < j \leq N} \frac{\Gamma(Uz_i z_j^{-1}, Uz_i^{-1} z_j; p, q)}{\Gamma(z_i z_j^{-1}, z_i^{-1} z_j; p, q)} \\
&\times \prod_{1 \leq i < j \leq N} \Gamma(U^{K/2} XY z_i z_j, U^{K/2} (XY)^{-1} z_i^{-1} z_j^{-1}; p, q) \\
&\times \prod_{i=1}^N \Gamma(U^{K/2} XY z_i^2, U^{K/2} (XY)^{-1} z_i^{-2}; p, q) \\
&\times \prod_{i=1}^{N_f} \prod_{j=1}^N \Gamma(s_i z_j, t_i^{-1} z_j^{-1}; p, q) \prod_{j=1}^{N-1} \frac{dz_j}{2\pi i z_j},
\end{aligned} \quad (10.17)$$

and

$$\begin{aligned}
 I_M &= \frac{(p; p)_\infty^{\tilde{N}-1} (q; q)_\infty^{\tilde{N}-1}}{\tilde{N}!} \prod_{L=0}^{K-1} \prod_{i,j=1}^{N_f} \Gamma(U^{L+K} s_i t_j^{-1}, U^L s_i t_j^{-1}; p, q) \\
 &\quad \times \prod_{J=0}^{K-1} \prod_{1 \leq i < j \leq N_f} \Gamma((XY)^{-1} U^{J+K/2} s_i s_j, XY U^{J+K/2} t_i^{-1} t_j^{-1}; p, q) \\
 &\quad \times \prod_{J=0}^{\frac{K-1}{2}} \prod_{i=1}^{N_f} \Gamma((XY)^{-1} U^{2J+K/2} s_i^2, XY U^{2J+K/2} t_i^{-2}; p, q) \\
 &\quad \times \int_{\mathbb{T}^{\tilde{N}-1}} \prod_{1 \leq i < j \leq \tilde{N}} \frac{\Gamma(U z_i z_j^{-1}, U z_i^{-1} z_j; p, q)}{\Gamma(z_i z_j^{-1}, z_i^{-1} z_j; p, q)} \\
 &\quad \times \prod_{1 \leq i < j \leq \tilde{N}} \Gamma(U^{K/2} X^{\frac{N-KN_f}{N}} Y^{\frac{N}{N}} z_i z_j, U^{K/2} (X^{\frac{N-KN_f}{N}} Y^{\frac{N}{N}})^{-1} z_i^{-1} z_j^{-1}; p, q) \\
 &\quad \times \prod_{i=1}^{\tilde{N}} \Gamma(U^{K/2} X^{\frac{N-KN_f}{N}} Y^{\frac{N}{N}} z_i^2, U^{K/2} (X^{\frac{N-KN_f}{N}} Y^{\frac{N}{N}})^{-1} z_i^{-2}; p, q) \tag{10.18} \\
 &\quad \times \prod_{i=1}^{N_f} \prod_{j=1}^{\tilde{N}} \Gamma(U^{(-K+2)/2} X^{\frac{KN_f+2}{N}} Y^{\frac{3KN_f+4}{2N}} s_i^{-1} z_j; p, q) \\
 &\quad \times \prod_{i=1}^{N_f} \prod_{j=1}^{\tilde{N}} \Gamma(U^{(-K+2)/2} X^{-\frac{KN_f+2}{N}} Y^{-\frac{3KN_f+4}{2N}} t_i z_j^{-1}; p, q) \prod_{j=1}^{\tilde{N}-1} \frac{dz_j}{2\pi i z_j},
 \end{aligned}$$

where $U = (pq)^{1/(K+1)}$, $Y = (ST)^{1/N_f}$, $S = \prod_{i=1}^{N_f} s_i$, $T^{-1} = \prod_{i=1}^{N_f} t_i^{-1}$, X is an arbitrary chemical potential associated with the $U(1)$ -group and the balancing condition reads

$$U^{N-2} S T^{-1} = (pq)^{N_f}.$$

10.5. Adjoint, anti-symmetric and conjugate anti-symmetric tensor matter fields. This duality was considered in [7]. The electric theory is

	$SU(N)$	$SU(N_f)$	$SU(N_f)$	$U(1)$	$U(1)_B$	$U(1)_R$
Q	f	f	1	0	$\frac{1}{N}$	$1 - \frac{N+2}{N_f(K+1)}$
\tilde{Q}	\bar{f}	1	\bar{f}	0	$-\frac{1}{N}$	$1 - \frac{N+2}{N_f(K+1)}$
X	adj	1	1	0	0	$\frac{2}{K+1}$
Y	T_A	1	1	1	$\frac{2}{N}$	$\frac{K}{K+1}$
\tilde{Y}	\bar{T}_A	1	1	-1	$-\frac{2}{N}$	$\frac{K}{K+1}$
V	adj	1	1	0	0	1

The magnetic theory is

	$SU(\tilde{N})$	$SU(N_f)$	$SU(N_f)$	$U(1)$	$U(1)_B$	$U(1)_R$
q	f	\bar{f}	1	$\frac{KN_f-2}{\tilde{N}}$	$\frac{1}{\tilde{N}}$	$1 - \frac{N+2}{N_f(K+1)}$
\tilde{q}	\bar{f}	1	f	$-\frac{KN_f-2}{\tilde{N}}$	$-\frac{1}{\tilde{N}}$	$1 - \frac{N+2}{N_f(K+1)}$
X	adj	1	1	0	0	$\frac{2}{K+1}$
Y	T_A	1	1	$\frac{N-KN_f}{\tilde{N}}$	$\frac{2}{\tilde{N}}$	$\frac{K}{K+1}$
\tilde{Y}	\bar{T}_A	1	1	$-\frac{N-KN_f}{\tilde{N}}$	$-\frac{2}{\tilde{N}}$	$\frac{K}{K+1}$
N_I	1	f	\bar{f}	0	0	$\frac{2I}{K+1} + \frac{2K}{K+1} + 2 - 2\frac{N+2}{N_f(K+1)}$
M_I	1	f	\bar{f}	0	0	$\frac{2I}{K+1} + 2 - 2\frac{N+2}{N_f(K+1)}$
P_{2J+1}	1	T_S	1	-1	0	$2\frac{2J+1}{K+1} + \frac{K}{K+1} + 2 - 2\frac{N+2}{N_f(K+1)}$
P_{2J}	1	T_A	1	-1	0	$2\frac{2J}{K+1} + \frac{K}{K+1} + 2 - 2\frac{N+2}{N_f(K+1)}$
\tilde{P}_{2J+1}	1	1	\bar{T}_S	1	0	$2\frac{2J+1}{K+1} + \frac{K}{K+1} + 2 - 2\frac{N+2}{N_f(K+1)}$
\tilde{P}_{2J}	1	1	\bar{T}_A	1	0	$2\frac{2J}{K+1} + \frac{K}{K+1} + 2 - 2\frac{N+2}{N_f(K+1)}$
\tilde{V}	adj	1	1	0	0	1

Here

$$\tilde{N} = 3KN_f - 4 - N, \quad (10.19)$$

K is odd, $I = 0, 1, \dots, K-1$, $J = 0, 1, \dots, \frac{K-1}{2}$ and $2J+1 \neq K$.

The superconformal indices are

$$\begin{aligned}
I_E &= \frac{(p; p)_\infty^{N-1} (q; q)_\infty^{N-1}}{N!} \int_{\mathbb{T}^{N-1}} \prod_{1 \leq i < j \leq N} \frac{\Gamma(Uz_i z_j^{-1}, Uz_i^{-1} z_j; p, q)}{\Gamma(z_i z_j^{-1}, z_i^{-1} z_j; p, q)} \\
&\quad \times \prod_{1 \leq i < j \leq N} \Gamma(U^{K/2} XY z_i z_j, U^{K/2} (XY)^{-1} z_i^{-1} z_j^{-1}; p, q) \\
&\quad \times \prod_{i=1}^{N_f} \prod_{j=1}^N \Gamma(s_i z_j, t_i^{-1} z_j^{-1}; p, q) \prod_{j=1}^{N-1} \frac{dz_j}{2\pi i z_j}, \quad (10.20)
\end{aligned}$$

and

$$\begin{aligned}
 I_M = & \frac{(p; p)_\infty^{\tilde{N}-1} (q; q)_\infty^{\tilde{N}-1}}{\tilde{N}!} \prod_{L=0}^{K-1} \prod_{i,j=1}^{N_f} \Gamma(U^{L+K} s_i t_j^{-1}, U^L s_i t_j^{-1}; p, q) \\
 & \times \prod_{J=0}^{K-1} \prod_{1 \leq i < j \leq N_f} \Gamma((XY)^{-1} U^{J+K/2} s_i s_j, XY U^{J+K/2} t_i^{-1} t_j^{-1}; p, q) \\
 & \times \prod_{J=0}^{\frac{K-3}{2}} \prod_{i=1}^{N_f} \Gamma((XY)^{-1} U^{2J+1+K/2} s_i^2, XY U^{2J+1+K/2} t_i^{-2}; p, q) \\
 & \times \int_{\mathbb{T}^{\tilde{N}-1}} \prod_{1 \leq i < j \leq \tilde{N}} \frac{\Gamma(U z_i z_j^{-1}, U z_i^{-1} z_j; p, q)}{\Gamma(z_i z_j^{-1}, z_i^{-1} z_j; p, q)} \\
 & \times \prod_{1 \leq i < j \leq \tilde{N}} \Gamma(U^{K/2} X^{\frac{N-KN_f}{N}} Y^{\frac{N}{N}} z_i z_j, U^{K/2} (X^{\frac{N-KN_f}{N}} Y^{\frac{N}{N}})^{-1} z_i^{-1} z_j^{-1}; p, q) \\
 & \times \prod_{i=1}^{N_f} \prod_{j=1}^{\tilde{N}} \Gamma(U^{(-K+2)/2} X^{\frac{KN_f-2}{N}} Y^{\frac{3KN_f-4}{2N}} s_i^{-1} z_j; p, q) \\
 & \times \prod_{i=1}^{N_f} \prod_{j=1}^{\tilde{N}} \Gamma(U^{(-K+2)/2} X^{-\frac{KN_f-2}{N}} Y^{-\frac{3KN_f-4}{2N}} t_i z_j^{-1}; p, q) \prod_{j=1}^{\tilde{N}-1} \frac{dz_j}{2\pi i z_j},
 \end{aligned} \tag{10.21}$$

where $U = (pq)^{1/(K+1)}$, $Y = (ST)^{1/N_f}$, $S = \prod_{i=1}^{N_f} s_i$, $T^{-1} = \prod_{i=1}^{N_f} t_i^{-1}$, X is an arbitrary parameter and the balancing condition reads

$$U^{N+2} S T^{-1} = (pq)^{N_f}.$$

10.6. Adjoint, anti-symmetric and conjugate symmetric tensor matter fields. This duality was discussed in [7]. The electric theory is⁴

	$SU(N)$	$SU(N_f)$	$SU(N_f - 8)$	$U(1)$	$U(1)_B$	$U(1)_R$
Q	f	f	1	$\frac{6}{N_f} - 1$	$\frac{1}{N}$	$1 - \frac{N+6K}{N_f(K+1)}$
\tilde{Q}	\bar{f}	1	f	$\frac{6}{N_f-8} + 1$	$-\frac{1}{N}$	$1 - \frac{N-6K}{(N_f-8)(K+1)}$
X	adj	1	1	0	0	$\frac{2}{K+1}$
Y	T_A	1	1	1	$\frac{2}{N}$	$\frac{K}{K+1}$
\tilde{Y}	\bar{T}_S	1	1	-1	$-\frac{2}{N}$	$\frac{K}{K+1}$
V	adj	1	1	0	0	1

The magnetic theory is

⁴In the original paper [7] there were misprints for the values of $U(1)$ -group hypercharges which were corrected in [46].

	$SU(\tilde{N})$	$SU(N_f)$	$SU(N_f - 8)$	$U(1)$	$U(1)_B$	$U(1)_R$
q	f	\bar{f}	1	$1 - \frac{6}{N_f}$	$\frac{1}{\tilde{N}}$	$1 - \frac{N+6K}{N_f(K+1)}$
\tilde{q}	\bar{f}	1	\bar{f}	$-1 - \frac{6}{N_f-8}$	$-\frac{1}{\tilde{N}}$	$1 - \frac{N-6K}{(N_f-8)(K+1)}$
X	adj	1	1	0	0	$\frac{2}{K+1}$
Y	T_A	1	1	1	$\frac{2}{\tilde{N}}$	$\frac{K}{K+1}$
\tilde{Y}	\bar{T}_S	1	1	-1	$-\frac{2}{\tilde{N}}$	$\frac{K}{K+1}$
N_J	1	f	f	$x_1 + x_2$	0	$\frac{2J}{K+1} + \frac{2K}{K+1} + 2r_1 + 2r_2$
M_J	1	f	f	$x_1 + x_2$	0	$\frac{2J}{K+1} + 2r_1 + 2r_2$
P_J	1	T_S	1	$2x_1 - 1$	0	$\frac{2J}{K+1} + \frac{K}{K+1} + 2 - 2\frac{N+6K}{N_f(K+1)}$
\tilde{P}_J	1	1	T_A	$2x_2 + 1$	0	$\frac{2J}{K+1} + \frac{K}{K+1} + 2 - 2\frac{N-6K}{(N_f-8)(K+1)}$
V	adj	1	1	0	0	1

Here $J = 0, 1, \dots, K - 1$ and

$$\tilde{N} = 3K(N_f - 4) - N. \quad (10.22)$$

The superconformal indices are

$$\begin{aligned}
I_E &= \frac{(p; p)_\infty^{N-1} (q; q)_\infty^{N-1}}{N!} \int_{\mathbb{T}^{N-1}} \prod_{1 \leq i < j \leq N} \frac{\Gamma(U z_i z_j^{-1}, U z_i^{-1} z_j; p, q)}{\Gamma(z_i z_j^{-1}, z_i^{-1} z_j; p, q)} \\
&\quad \prod_{1 \leq i < j \leq N} \Gamma(U^{K/2} X Y z_i z_j, U^{K/2} (X Y)^{-1} z_i^{-1} z_j^{-1}; p, q) \\
&\quad \times \prod_{i=1}^N \Gamma(U^{K/2} (X Y)^{-1} z_i^{-2}; p, q) \\
&\quad \times \prod_{j=1}^N \prod_{i=1}^{N_f} \Gamma(s_i z_j; p, q) \prod_{k=1}^{N_f-8} \Gamma(t_k z_j^{-1}; p, q) \prod_{j=1}^{N-1} \frac{dz_j}{2\pi i z_j} \quad (10.23)
\end{aligned}$$

and

$$\begin{aligned}
 I_M &= \frac{(p; p)_{\infty}^{\tilde{N}-1} (q; q)_{\infty}^{\tilde{N}-1}}{\tilde{N}!} \prod_{L=0}^{K-1} \prod_{i=1}^{N_f} \prod_{j=1}^{N_f-8} \Gamma(U^{L+K} s_i t_j, U^L s_i t_j; p, q) \\
 &\quad \times \prod_{J=0}^{K-1} \prod_{1 \leq i < j \leq N_f} \Gamma((XY)^{-1} U^{J+K/2} s_i s_j; p, q) \\
 &\quad \times \prod_{J=0}^{K-1} \prod_{1 \leq i < j \leq N_f-8} \Gamma(XY U^{J+K/2} t_i t_j; p, q) \\
 &\quad \times \prod_{J=0}^{K-1} \prod_{i=1}^{N_f} \Gamma((XY)^{-1} U^{J+K/2} s_i^2; p, q) \\
 &\quad \times \int_{\mathbb{T}^{\tilde{N}-1}} \prod_{1 \leq i < j \leq \tilde{N}} \frac{\Gamma(U z_i z_j^{-1}, U z_i^{-1} z_j; p, q)}{\Gamma(z_i z_j^{-1}, z_i^{-1} z_j; p, q)} \\
 &\quad \times \prod_{1 \leq i < j \leq \tilde{N}} \Gamma(U^{K/2} XY^{\frac{N}{\tilde{N}}} z_i z_j, U^{K/2} (XY^{\frac{N}{\tilde{N}}})^{-1} z_i^{-1} z_j^{-1}; p, q) \\
 &\quad \times \prod_{i=1}^{\tilde{N}} \Gamma(U^{K/2} (XY^{\frac{N}{\tilde{N}}})^{-1} z_i^{-2}; p, q) \\
 &\quad \times \prod_{j=1}^{\tilde{N}} \prod_{i=1}^{N_f} \Gamma(U^{(-K+2)/2} Y^{\frac{3K(N_f-4)}{2\tilde{N}}} s_i^{-1} z_j; p, q) \\
 &\quad \times \prod_{j=1}^{\tilde{N}} \prod_{k=1}^{N_f-8} \Gamma(U^{(-K+2)/2} Y^{-\frac{3K(N_f-4)}{2\tilde{N}}} t_k^{-1} z_j^{-1}; p, q) \prod_{j=1}^{\tilde{N}-1} \frac{dz_j}{2\pi i z_j},
 \end{aligned} \tag{10.24}$$

where $U = (pq)^{1/(K+1)}$, $S = \prod_{i=1}^{N_f} s_i$, $T = \prod_{i=1}^{N_f-8} t_i$,

$$Y = \left(ST^{-1} X^{2N_f-8} (pq)^{\frac{2(K-2)}{K+1}} \right)^{\frac{1}{N_f-4}}$$

and the balancing condition reads

$$U^N X^{-4} Y^{-4} S T = (pq)^{N_f-4}.$$

The equalities $I_E = I_M$ for all the dualities described in this section require a rigorous mathematical confirmation. For the moment we have only one justifying argument coming from the total ellipticity condition associated with the kernels of the corresponding pairs of integrals.

11. KS TYPE DUALITIES FOR SYMPLECTIC GAUGE GROUPS

11.1. The anti-symmetric tensor matter field. For $SP(2N)$ group the following electric-magnetic duality was discovered in [37]. The electric theory:

	$SP(2N)$	$SU(2N_f)$	$U(1)_R$
Q	f	f	$2r = 1 - \frac{2(N+K)}{(K+1)N_f}$
X	T_A	1	$2s = \frac{2}{K+1}$
V	adj	1	1

The magnetic theory:

	$SP(2\tilde{N})$	$SU(2N_f)$	$U(1)_R$
q	f	\bar{f}	$2\tilde{r} = 1 - \frac{2(\tilde{N}_c+k)}{(K+1)N_f}$
Y	T_A	1	$2s = \frac{2}{K+1}$
$M_j, j = 1 \dots K$	1	T_A	$2r_j = 2\frac{K+j}{K+1} - 4\frac{\tilde{N}_c+K}{(K+1)N_f}$
\tilde{V}	adj	1	1

Here

$$\tilde{N} = K(N_f - 2) - N, \quad K = 1, 2, \dots, \quad (11.1)$$

with the constraint $N_f > N/K$.

Defining $U = (pq)^s = (pq)^{\frac{1}{K+1}}$, for these theories we find the following indices

$$I_E = \frac{(p; p)_\infty^N (q; q)_\infty^N}{2^N N!} \Gamma(U; p, q)^{N-1} \int_{\mathbb{T}^N} \prod_{1 \leq i < j \leq N} \frac{\Gamma(U z_i^{\pm 1} z_j^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \\ \times \prod_{j=1}^N \frac{\prod_{i=1}^{2N_f} \Gamma(s_i z_j^{\pm 1}; p, q)}{\Gamma(z_j^{\pm 2}; p, q)} \prod_{j=1}^N \frac{dz_j}{2\pi i z_j} \quad (11.2)$$

and

$$I_M = \frac{(p; p)_\infty^{\tilde{N}} (q; q)_\infty^{\tilde{N}}}{2^{\tilde{N}} \tilde{N}!} \Gamma(U; p, q)^{\tilde{N}-1} \prod_{l=1}^K \prod_{1 \leq i < j \leq 2N_f} \Gamma(U^{l-1} s_i s_j; p, q) \\ \times \int_{\mathbb{T}^{\tilde{N}}} \prod_{1 \leq i < j \leq \tilde{N}} \frac{\Gamma(U z_i^{\pm 1} z_j^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \prod_{j=1}^{\tilde{N}} \frac{\prod_{i=1}^{2N_f} \Gamma(U s_i^{-1} z_j^{\pm 1}; p, q)}{\Gamma(z_j^{\pm 2}; p, q)} \prod_{j=1}^{\tilde{N}} \frac{dz_j}{2\pi i z_j}, \quad (11.3)$$

where $S = \prod_{i=1}^{2N_f} s_i$ and the balancing condition reads

$$U^{2(N+K)} S = (pq)^{N_f}.$$

11.2. Symmetric tensor matter field. Another electric-magnetic duality is described in [50]. The electric theory:

	$SP(2N)$	$SU(2N_f)$	$U(1)_R$
Q	f	f	$2r = 1 - \frac{N+1}{(K+1)N_f}$
X	$adj = T_S$	1	$2s = \frac{1}{K+1}$
V	adj	1	1

The magnetic theory:

	$SP(2\tilde{N})$	$SU(2N_f)$	$U(1)_R$
q	f	\bar{f}	$2\tilde{r} = 1 - \frac{N+1}{(K+1)N_f}$
Y	adj	1	$2s = \frac{1}{K+1}$
$M_{2j}, j = 0, \dots, K$	1	T_A	$2r_{2j} = 2 - \frac{2(N+1)-2jN_f}{(K+1)N_f}$
$M_{2j+1}, j = 0, \dots, K-1$	1	T_S	$2r_{2j+1} = 2 - \frac{2(N+1)-(2j+1)N_f}{(K+1)N_f}$
\tilde{V}	adj	1	1

Here

$$\tilde{N} = (2K+1)N_f - N - 2, \quad K = 0, 1, 2, \dots, \quad (11.4)$$

with the constraint $N_f \geq (N+1)/(2K+1)$.

Defining $U = (pq)^s = (pq)^{\frac{1}{2(K+1)}}$, we have the following superconformal indices for these theories

$$\begin{aligned} I_E &= \frac{(p; p)_\infty^N (q; q)_\infty^N}{2^N N!} \Gamma(U; p, q)^N \int_{\mathbb{T}^N} \prod_{1 \leq i < j \leq N} \frac{\Gamma(U z_i^{\pm 1} z_j^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \\ &\times \prod_{j=1}^N \frac{\Gamma(U z_j^{\pm 2}; p, q)}{\Gamma(z_j^{\pm 2}; p, q)} \prod_{i=1}^{2N_f} \prod_{j=1}^N \Gamma(s_i z_j^{\pm 1}; p, q) \prod_{j=1}^N \frac{dz_j}{2\pi i z_j} \end{aligned} \quad (11.5)$$

and

$$\begin{aligned} I_M &= \frac{(p; p)_\infty^{\tilde{N}} (q; q)_\infty^{\tilde{N}}}{2^{\tilde{N}} \tilde{N}!} \Gamma(U; p, q)^{\tilde{N}} \prod_{l=0}^{2K} \prod_{1 \leq i < j \leq 2N_f} \Gamma(U^l s_i s_j; p, q) \\ &\times \prod_{l=0}^{K-1} \prod_{i=1}^{2N_f} \Gamma(U^{2l+1} s_i^2; p, q) \int_{\mathbb{T}^{\tilde{N}}} \prod_{1 \leq i < j \leq \tilde{N}} \frac{\Gamma(U z_i^{\pm 1} z_j^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \\ &\times \prod_{j=1}^{\tilde{N}} \frac{\Gamma(U z_j^{\pm 2}; p, q)}{\Gamma(z_j^{\pm 2}; p, q)} \prod_{i=1}^{2N_f} \prod_{j=1}^{\tilde{N}} \Gamma(U s_i^{-1} z_j^{\pm 1}; p, q) \prod_{j=1}^{\tilde{N}} \frac{dz_j}{2\pi i z_j}, \end{aligned} \quad (11.6)$$

where $S = \prod_{i=1}^{2N_f} s_i$ and the balancing condition reads

$$U^{2(N+1)} S = (pq)^{N_f}.$$

11.3. Two anti-symmetric tensor matter fields. This duality was investigated in [7]. The electric theory:

	$SP(2N)$	$SU(2N_f)$	$U(1)_R$
Q	f	f	$1 - \frac{N+2K+1}{(K+1)N_f}$
X	T_A	1	$\frac{2}{K+1}$
Y	T_A	1	$\frac{K}{K+1}$
V	adj	1	1

The magnetic theory:

	$SP(2\tilde{N})$	$SU(2N_f)$	$U(1)_R$
q	f	\bar{f}	$1 - \frac{N+2K+1}{(K+1)N_f}$
\tilde{X}	T_A	1	$\frac{2}{K+1}$
\tilde{Y}	T_A	1	$\frac{K}{K+1}$
$M_{J_0}, J = 0, \dots, K-1$	1	T_A	$2 - \frac{N+2K+1}{(K+1)N_f} + \frac{2J}{K+1}$
$M_{2J+1}, J = 0, \dots, \frac{K-1}{2}$	1	T_A	$2 - \frac{N+2K+1}{(K+1)N_f} + \frac{2(2J)}{K+1} + \frac{K}{K+1}$
$M_{2J+1}, J = 0, \dots, \frac{K-3}{2}$	1	T_S	$2 - \frac{N+2K+1}{(K+1)N_f} + \frac{2(2J+1)}{K+1} + \frac{K}{K+1}$
$M_{J_2}, J = 0, \dots, K-1$	1	T_A	$2 - \frac{N+2K+1}{(K+1)N_f} + \frac{2J}{K+1} + \frac{2K}{K+1}$
\tilde{V}	adj	1	1

Here K is odd and

$$\tilde{N} = 3KN_f - 4K - 2 - N. \quad (11.7)$$

For these theories we have the following superconformal indices

$$I_E = \frac{(p; p)_\infty^N (q; q)_\infty^N}{2^N N!} \Gamma(U, U^{\frac{K}{2}}; p, q)^{N-1} \quad (11.8)$$

$$\times \int_{\mathbb{T}^N} \prod_{1 \leq i < j \leq N} \frac{\Gamma(U z_i^{\pm 1} z_j^{\pm 1}, U^{\frac{K}{2}} z_i^{\pm 1} z_j^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \prod_{i=1}^{2N_f} \prod_{j=1}^N \frac{\Gamma(s_i z_j^{\pm 1}; p, q)}{\Gamma(z_j^{\pm 2}; p, q)} \prod_{j=1}^N \frac{dz_j}{2\pi i z_j}$$

and

$$I_M = \frac{(p; p)_\infty^{\tilde{N}} (q; q)_\infty^{\tilde{N}}}{2^{\tilde{N}} \tilde{N}!} \Gamma(U, U^{\frac{K}{2}}; p, q)^{\tilde{N}-1} \quad (11.9)$$

$$\times \prod_{J=0}^{K-1} \prod_{L=0}^2 \prod_{1 \leq i < j \leq 2N_f} \Gamma(U^{J+\frac{KL}{2}} s_i s_j; p, q) \prod_{J=0}^{\frac{K-3}{2}} \prod_{j=1}^{2N_f} \Gamma(U^{2J+1+\frac{K}{2}} s_j^2; p, q)$$

$$\times \int_{\mathbb{T}^{\tilde{N}}} \prod_{1 \leq i < j \leq \tilde{N}} \frac{\Gamma(U z_i^{\pm 1} z_j^{\pm 1}, U^{\frac{K}{2}} z_i^{\pm 1} z_j^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \prod_{i=1}^{2N_f} \prod_{j=1}^{\tilde{N}} \frac{\Gamma(U^{1-\frac{K}{2}} s_i^{-1} z_j^{\pm 1}; p, q)}{\Gamma(z_j^{\pm 2}; p, q)} \prod_{j=1}^{\tilde{N}} \frac{dz_j}{2\pi i z_j},$$

where $S = \prod_{i=1}^{2N_f} s_i$, $U = (pq)^{\frac{1}{K+1}}$ and the balancing condition reads

$$U^{N+2K+1} S = (pq)^{N_f}.$$

11.4. Symmetric and anti-symmetric tensor matter fields. This duality was found in [7]. The electric theory:

	$SP(2N)$	$SU(2N_f)$	$U(1)_R$
Q	f	f	$1 - \frac{N+2K-1}{(K+1)N_f}$
X	T_A	1	$\frac{2}{K+1}$
Y	T_S	1	$\frac{K}{K+1}$
V	adj	1	1

The magnetic theory:

	$SP(2\tilde{N})$	$SU(2N_f)$	$U(1)_R$
q	f	\bar{f}	$1 - \frac{N+2K-1}{(K+1)N_f}$
\tilde{X}	T_A	1	$\frac{2}{K+1}$
\tilde{Y}	T_S	1	$\frac{K}{K+1}$
$M_{J_0}, J = 0, \dots, K-1$	1	T_A	$2 - \frac{N+2K+1}{(K+1)N_f} + \frac{2J}{K+1}$
$M_{2J-1}, J = 0, \dots, \frac{K-1}{2}$	1	T_S	$2 - \frac{N+2K+1}{(K+1)N_f} + \frac{2(2J)}{K+1} + \frac{K}{K+1}$
$M_{2J+1}, J = 0, \dots, \frac{K-3}{2}$	1	T_A	$2 - \frac{N+2K+1}{(K+1)N_f} + \frac{2(2J+1)}{K+1} + \frac{K}{K+1}$
$M_{J_2}, J = 0, \dots, K-1$	1	T_A	$2 - \frac{N+2K+1}{(K+1)N_f} + \frac{2J}{K+1} + \frac{2K}{K+1}$
\tilde{V}	adj	1	1

Here K is odd and

$$\tilde{N} = 3KN_f - 4K + 2 - N. \quad (11.10)$$

For these theories we have the following superconformal indices

$$I_E = \frac{(p; p)_\infty^N (q; q)_\infty^N}{2^N N!} \Gamma(U; p, q)^{N-1} \Gamma(U^{\frac{K}{2}}; p, q)^N \quad (11.11)$$

$$\times \int_{\mathbb{T}^N} \prod_{1 \leq i < j \leq N} \frac{\Gamma(U z_i^{\pm 1} z_j^{\pm 1}, U^{\frac{K}{2}} z_i^{\pm 1} z_j^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \prod_{i=1}^{2N_f} \prod_{j=1}^N \frac{\Gamma(s_i z_j^{\pm 1}, U^{\frac{K}{2}} z_j^{\pm 2}; p, q)}{\Gamma(z_j^{\pm 2}; p, q)} \prod_{j=1}^N \frac{dz_j}{2\pi i z_j}$$

and

$$I_M = \frac{(p; p)_\infty^{\tilde{N}} (q; q)_\infty^{\tilde{N}}}{2^{\tilde{N}} \tilde{N}!} \Gamma(U; p, q)^{\tilde{N}-1} \Gamma(U^{\frac{K}{2}}; p, q)^{\tilde{N}} \prod_{J=0}^{K-1} \prod_{L=0}^2 \prod_{1 \leq i < j \leq 2N_f} \Gamma(U^{J+\frac{KL}{2}} s_i s_j; p, q)$$

$$\times \prod_{J=0}^{\frac{K-1}{2}} \prod_{j=1}^{2N_f} \Gamma(U^{2J+\frac{K}{2}} s_j^2; p, q) \int_{\mathbb{T}^{\tilde{N}}} \prod_{1 \leq i < j \leq \tilde{N}} \frac{\Gamma(U z_i^{\pm 1} z_j^{\pm 1}, U^{\frac{K}{2}} z_i^{\pm 1} z_j^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \quad (11.12)$$

$$\times \prod_{i=1}^{2N_f} \prod_{j=1}^{\tilde{N}} \frac{\Gamma(U^{1-\frac{K}{2}} s_i^{-1} z_j^{\pm 1}, U^{\frac{K}{2}} z_j^{\pm 2}; p, q)}{\Gamma(z_j^{\pm 2}; p, q)} \prod_{j=1}^{\tilde{N}} \frac{dz_j}{2\pi i z_j},$$

where $S = \prod_{i=1}^{2N_f} s_i$, $U = (pq)^{\frac{1}{K+1}}$ and the balancing condition reads

$$U^{N+2K-1} S = (pq)^{N_f}.$$

The equalities $I_E = I_M$ for all the dualities described in this section represent new elliptic hypergeometric identities requiring a rigorous mathematical confirmation.

12. SOME OTHER NEW DUALITIES

Let us denote

$$I_{A_N}(t_1, \dots, t_{N+3}, u_1, \dots, u_{N+3}; p, q) = \frac{(p; p)_\infty^N (q; q)_\infty^N}{(N+1)!} \quad (12.1)$$

$$\times \int_{\mathbb{T}^N} \frac{\prod_{i=1}^{N+1} \prod_{r=1}^{N+3} \Gamma(t_r z_i, u_r z_i^{-1}; p, q)}{\prod_{1 \leq i < j \leq N+1} \Gamma(z_i z_j^{-1}, z_i^{-1} z_j; p, q)} \prod_{j=1}^N \frac{dz_j}{2\pi i z_j},$$

with the balancing condition $\prod_{i=1}^{N+3} t_i u_i = (pq)^2$, and

$$I_{BC_N}(t_1, \dots, t_{2N+6}; p, q) = \frac{(p; p)_\infty^N (q; q)_\infty^N}{2^N N!} \quad (12.2)$$

$$\times \int_{\mathbb{T}^N} \frac{\prod_{i=1}^N \prod_{r=1}^{2N+6} \Gamma(t_r z_i^{\pm 1}; p, q)}{\prod_{1 \leq i < j \leq N} \Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q) \prod_{j=1}^N \Gamma(z_j^{\pm 2}; p, q)} \prod_{j=1}^N \frac{dz_j}{2\pi i z_j}$$

with the balancing condition $\prod_{r=1}^{2N+6} t_r = (pq)^2$.

12.1. $SU \leftrightarrow SP$ groups duality. For the first new duality the electric gauge group is $G = SU(N+1)$, but the dual gauge group is of the different type $G = SP(2N)$. The flavor symmetry group in both cases is

$$F = SU(N+3) \times SU(N+3) \times U(1)_B.$$

The field content of dual theories is given in the tables below

	$SU(N+1)$	$SU(N+3)$	$SU(N+3)$	$U(1)_B$	$U(1)_R$
Q_1	f	f	1	2	$\frac{2}{N+3}$
Q_2	\bar{f}	1	f	-2	$\frac{2}{N+3}$
V	adj	1	1	0	1

	$SP(2N)$	$SU(N+3)$	$SU(N+3)$	$U(1)_B$	$U(1)_R$
q_1	f	f	1	$-(N+1)$	$\frac{2}{N+3}$
q_2	\bar{f}	1	f	$N+1$	$\frac{2}{N+3}$
X_1	1	\bar{T}_A	1	$2(N+1)$	$2\frac{N+1}{N+3}$
X_2	1	1	T_A	$-2(N+1)$	$2\frac{N+1}{N+3}$
V	adj	1	1	0	1

The superconformal indices are

$$I_E = I_{A_N}(t_1, \dots, t_{N+3}, u_1, \dots, u_{N+3}; p, q), \quad (12.3)$$

$$I_M = \prod_{1 \leq i < j \leq N+3} \Gamma(T/t_i t_j, U/u_i u_j; p, q) I_{BC_N}(\dots (U/T)^{1/4} t_i \dots, \dots (T/U)^{1/4} u_i \dots; p, q),$$

where

$$T = \prod_{1 \leq i \leq N+3} t_i, \quad U = \prod_{1 \leq i \leq N+3} u_i$$

The equality $I_E = I_M$ represents the mixed elliptic hypergeometric integrals transformation proven in [61]. We used this identity as a starting point for finding the described new Seiberg pair of field theories.

12.2. $SU \leftrightarrow SU$ groups duality. Again, we use consequences of the mixed transformations derived in [61]. Corresponding dualities have the flavor symmetry groups

$$F = SU(K) \times SU(N+3-K) \times U(1)_1 \times SU(K) \times SU(N+3-K) \times U(1)_1 \times U(1)_B,$$

for arbitrary $0 \leq K \leq N+3$. The field content of the initial field theory is given in the table

	$SU(N+1)$	$SU(N+3)$	$SU(N+3)$	$U(1)_B$	$U(1)_R$
Q_1	f	f	1	1	$\frac{2}{N+3}$
Q_2	\bar{f}	1	f	-1	$\frac{2}{N+3}$
V	adj	1	1	0	1

In order to verify the 't Hooft anomalies matching conditions for relevant flavor symmetry subgroups, it is useful to rewrite the latter table as

	$SU(N+1)$	$SU(K)$	$SU(M)$	$U(1)_1$	$SU(K)$	$SU(M)$	$U(1)_2$	$U(1)_B$	$U(1)_R$
q_1	f	f	1	M	1	1	0	1	$\frac{2}{N+3}$
q_2	f	1	f	$-K$	1	1	0	1	$\frac{2}{N+3}$
q_3	\bar{f}	1	1	0	f	1	M	-1	$\frac{2}{N+3}$
q_4	\bar{f}	1	1	0	1	f	$-K$	-1	$\frac{2}{N+3}$
V	adj	1	1	0	1	1	0	0	1

where $M = N + 3 - K$. The dual theory content is described in the following table

	$SU(N+1)$	$SU(K)$	$SU(M)$	$U(1)_1$	$SU(K)$	$SU(M)$	$U(1)_2$	$U(1)_B$	$U(1)_R$
q_1	\bar{f}	f	1	$\frac{K(K-2)}{(N+1)} - K + M$	1	1	$\frac{MK}{(N+1)}$	$1 - M$	$\frac{2}{N+3}$
q_2	f	1	f	$-\frac{K(K-2)}{N+1}$	1	1	$\frac{-MK}{(N+1)}$	$1 - K$	$\frac{2}{N+3}$
q_3	f	1	1	$\frac{MK}{(N+1)}$	f	1	$\frac{K(K-2)}{(N+1)} - K + M$	$M - 1$	$\frac{2}{N+3}$
q_4	\bar{f}	1	1	$\frac{-MK}{(N+1)}$	1	f	$-\frac{K(K-2)}{N+1}$	$-(1 - K)$	$\frac{2}{N+3}$
X_1	1	f	1	M	1	f	$-K$	0	$\frac{4}{N+3}$
X_2	1	1	f	$-K$	f	1	M	0	$\frac{4}{N+3}$
Y_1	1	\bar{f}	\bar{f}	$K - M$	1	1	0	$N + 1$	$2\frac{N+1}{N+3}$
Y_2	1	1	1	0	\bar{f}	\bar{f}	$K - M$	$-(N + 1)$	$2\frac{N+1}{N+3}$
V	adj	1	1	0	1	1	0	0	1

The superconformal indices have the form

$$I_E = I_{A_N}(t_1, \dots, t_{N+3}, u_1, \dots, u_{N+3}; p, q), \quad (12.4)$$

$$I_M = \prod_{1 \leq r < K, K \leq s \leq N+3} \Gamma(t_r u_s, t_s u_r, T/t_s t_r, U/u_r u_s) I_{A_N}(t'_1, \dots, t'_{N+3}, u'_1, \dots, u'_{N+3}; p, q),$$

where

$$\begin{aligned} t'_r &= (T/U)^{\frac{N+1-K}{2(N+1)}} (T_K/U_K)^{1/(N+1)} u_r, \quad 1 \leq r < K + 1, \\ t'_r &= (U/T)^{\frac{K}{2(N+1)}} (T_K/U_K)^{1/(N+1)} t_r, \quad K + 1 \leq r \leq N + 3, \\ u'_r &= (U/T)^{\frac{N+1-K}{2(N+1)}} (U_K/T_K)^{1/(N+1)} t_r, \quad 1 \leq r < K + 1, \\ u'_r &= (T/U)^{\frac{K}{2(N+1)}} (U_K/T_K)^{1/(N+1)} u_r, \quad K + 1 \leq r \leq N + 3, \end{aligned}$$

$$T = \prod_{r=1}^{N+3} t_r, \quad U = \prod_{r=1}^{N+3} u_r, \quad T_K = \prod_{r=1}^K t_r, \quad U_K = \prod_{r=1}^K u_r.$$

The equality $I_E = I_M$ for $K = 1$ was suggested in [78] and the general relation with the complete proof for arbitrary K is given in [61].

13. S-CONFINEMENT

Following [12, 13, 71] by s -confinement we mean smooth confinement without chiral symmetry breaking and with a non-vanishing confining superpotential. We call a theory confining when its infrared physics can be described completely in terms of gauge invariant composite fields and their interactions. This description has to be valid everywhere on the moduli space of vacua. The definition of s -confinement requires also that the theory dynamically generates a confining superpotential. Furthermore, the phase without chiral symmetry breaking implies that the origin of the classical moduli space serves also as a vacuum in the quantum theory. In this vacuum all the global symmetries present in the ultraviolet regime remain unbroken. Finally, the confining superpotential is a holomorphic function of the confined degrees of freedom and couplings, which describes all the interactions in the extreme infrared. From the point of view

of elliptic hypergeometric functions the s -confinement means that the dual theory possesses the trivial gauge group and the integrals describing superconformal indices are computable exactly, defining highly non-trivial elliptic beta integrals [76].

13.1. $SU(N)$ gauge group. In this section we present known examples of the confining theories with the unitary gauge group.

For brevity we combine the electric and magnetic theories in a single table separating them by the double line. The magnetic theory fields are denoted using the conventions of [12].

13.1.1. $SU(N)$ with $(N+1)(f+\bar{f})$. [71]

	$SU(N)$	$SU(N+1)$	$SU(N+1)$	$U(1)$	$U(1)_R$
Q	f	f	1	1	$\frac{1}{N+1}$
\tilde{Q}	\bar{f}	1	f	-1	$\frac{1}{N+1}$
V	adj	1	1	0	1
$Q\tilde{Q}$		f	f	0	$\frac{2}{N+1}$
Q^N		\bar{f}	1	N	$\frac{N}{N+1}$
\tilde{Q}^N		1	\bar{f}	$-N$	$\frac{N}{N+1}$

The superconformal indices for this theory are equal to, after appropriate renormalization of the parameters (as explained at the beginning of this paper and in the next section)

$$\begin{aligned}
I_E &= \frac{(p; p)_\infty^{N-1} (q; q)_\infty^{N-1}}{N!} \int_{\mathbb{T}^{N-1}} \prod_{1 \leq j < k \leq N} \frac{1}{\Gamma(z_i z_j^{-1}, z_i^{-1} z_j; p, q)} \\
&\quad \times \prod_{j=1}^N \prod_{m=1}^{N+1} \Gamma(s_m z_j, t_m z_j^{-1}; p, q) \prod_{j=1}^{N-1} \frac{dz_j}{2\pi i z_j},
\end{aligned} \tag{13.1}$$

and

$$I_M = \prod_{m=1}^{N+1} \Gamma(S s_m^{-1}, T t_m^{-1}; p, q) \prod_{k,m=1}^{N+1} \Gamma(s_k t_m; p, q), \tag{13.2}$$

where

$$S = \prod_{m=1}^{N+1} s_m, \quad T = \prod_{m=1}^{N+1} t_m$$

and the balancing condition reads $ST = pq$.

The exact computation of the integral $I_E = I_M$ was conjectured and partially confirmed in [78]. Its complete proofs are given in [61, 81]. In the simplest $p \rightarrow 0$ limit it is reduced to one of the Gustafson integrals [31].

13.1.2. $SU(2N)$ with $T_A + 2N\bar{f} + 4f$. The theory with $G = SU(2N)$ gauge group and flavor group

$$F = SU(2N) \times SU(4) \times U(1)_1 \times U(1)_2$$

was considered in [58, 60]. The field content of both theories is described in the table below

	$SU(2N)$	$SU(2N)$	$SU(4)$	$U(1)_1$	$U(1)_2$	$U(1)_R$
Q	f	1	f	$-2N$	$-2N + 2$	$\frac{1}{2}$
\tilde{Q}	\bar{f}	f	1	4	$-2N + 2$	0
A	T_A	1	1	0	$2N + 4$	0
V	adj	1	1	0	0	1
$Q\tilde{Q}$		f	f	$4 - 2N$	$-4N + 4$	$\frac{1}{2}$
$A\tilde{Q}^2$		T_A	1	8	$-2N + 8$	0
A^N		1	1	0	$2N^2 + 4N$	0
$A^{N-1}Q^2$		1	T_A	$-4N$	$2N^2 - 2N$	1
$A^{N-1}Q^4$		1	1	$-8N$	$2N^2 - 8N$	2
\tilde{Q}^{2N}		1	1	$8N$	$-4N^2 + 4N$	0

This theory was found to be s -confining. We would like to describe computation of the corresponding indices in more detail. As explained at the beginning of the paper, first we should write the single particle states index containing hypercharges of the fields for the $U(1)_R$ and other $U(1)$ -groups. In the present case we write

$$r_Q = R_Q + q_{1Q}x + q_{2Q}y,$$

where $q_{1Q} = -2N$, $q_{2Q} = -2N + 2$ and x and y are chemical potentials for the $U(1)_1$ and $U(1)_2$ groups. Using this prescription, from the general formulas (3.27) and (3.28), we find the full superconformal index of the electric theory

$$I_E = \frac{(p; p)_\infty^{2N-1} (q; q)_\infty^{2N-1}}{(2N)!} \int_{\mathbb{T}^{2N-1}} \prod_{1 \leq j < k \leq 2N} \frac{\Gamma((pq)^{(N+2)y} z_j z_k; p, q)}{\Gamma(z_i^{-1} z_j, z_i z_j^{-1}; p, q)} \quad (13.3)$$

$$\times \prod_{j=1}^{2N} \prod_{k=1}^{2N} \Gamma((pq)^{2x-(N-1)y} t_k z_j^{-1}; p, q) \prod_{l=1}^4 \Gamma((pq)^{-Nx-(N-1)y+\frac{1}{4}} s_l z_j; p, q) \prod_{j=1}^{2N-1} \frac{dz_j}{2\pi i z_j}$$

The index in the magnetic side is written as

$$I_M = \prod_{j=1}^{2N} \prod_{k=1}^4 \Gamma((pq)^{(2-N)x-2(N-1)y+\frac{1}{4}} t_j s_k; p, q) \prod_{1 \leq j < k \leq 2N} \Gamma((pq)^{4x-(N-4)y} t_j t_k; p, q)$$

$$\times \prod_{1 \leq j < k \leq 4} \Gamma((pq)^{-2Nx+N(N-1)y+\frac{1}{2}} s_j s_k; p, q) \Gamma((pq)^{N(N+2)y}; p, q) \quad (13.4)$$

$$\times \Gamma((pq)^{1-4N+N(N-4)y}; p, q) \Gamma((pq)^{4Nx-2N(N-1)y}; p, q).$$

Now change of variables

$$t_j \rightarrow (pq)^{-2x+(N-1)y} t_j, \quad j = 1, \dots, 2N, \quad (13.5)$$

$$s_j \rightarrow (pq)^{Nx+(N-1)y-\frac{1}{4}} s_j, \quad j = 1, \dots, 2N,$$

$$t = (pq)^{(N+2)y},$$

$$T = \prod_{j=1}^{2N} t_j = ((pq)^{2x-(N-1)y})^{2N},$$

$$S = \prod_{j=1}^4 s_j = ((pq)^{-Nx-(N-1)y+\frac{1}{4}})^4.$$

Using these relations, we come to the following integrals

$$I_E = \frac{(p; p)_\infty^{2N-1} (q; q)_\infty^{2N-1}}{(2N)!} \int_{\mathbb{T}^{2N-1}} \prod_{1 \leq j < k \leq 2N} \frac{\Gamma(tz_j z_k; p, q)}{\Gamma(z_j z_k^{-1}, z_j^{-1} z_k; p, q)} \quad (13.6)$$

$$\times \prod_{j=1}^{2N} \prod_{k=1}^{2N} \Gamma(t_k z_j^{-1}; p, q) \prod_{i=1}^4 \Gamma(s_i z_j; p, q) \prod_{j=1}^{2N-1} \frac{dz_j}{2\pi i z_j}$$

and

$$I_M = \prod_{1 \leq j < k \leq 2N} \Gamma(tt_j t_k; p, q) \prod_{k=1}^{2N} \prod_{i=1}^4 \Gamma(t_k s_i; p, q) \frac{\Gamma(t^N, T; p, q)}{\Gamma(t^N T; p, q)} \quad (13.7)$$

$$\times \prod_{1 \leq i < m \leq 4} \Gamma(t^{\frac{2N-2}{2}} s_i s_m; p, q),$$

where the balancing condition reads

$$t^{2N-2} S T = pq.$$

Equality $I_E = I_M$ defines the elliptic beta integral introduced in [78]. It represents an elliptic extension of the Gustafson-Rakha q -beta integral for odd number of integration variables [33].

13.1.3. $SU(2N+1)$ with $T_A + (2N+1)\bar{f} + 4f$. [58, 60]

	$SU(2N+1)$	$SU(2N+1)$	$SU(4)$	$U(1)_1$	$U(1)_2$	$U(1)_R$
Q	f	1	f	$-2N-1$	$-2N+1$	$\frac{1}{2}$
\tilde{Q}	\bar{f}	f	1	4	$-2N+1$	0
A	T_A	1	1	0	$2N+5$	0
V	adj	1	1	0	0	1
$Q\tilde{Q}$		f	f	$3-2N$	$-4N+2$	$\frac{1}{2}$
$A\tilde{Q}^2$		T_A	1	8	$-2N+7$	0
$A^N Q$		1	f	$-2N-1$	$2N^2+3N+1$	$\frac{1}{2}$
$A^{N-1} Q^3$		1	f	$-6N-3$	$2N^2-3N-2$	$\frac{3}{2}$
\tilde{Q}^{2N+1}		1	1	$8N+4$	$-4N^2+1$	0

The indices have the form

$$I_E = \frac{(p; p)_\infty^{2N} (q; q)_\infty^{2N}}{(2N+1)!} \int_{\mathbb{T}^{2N}} \prod_{1 \leq j < k \leq 2N+1} \frac{\Gamma(tz_j z_k; p, q)}{\Gamma(z_j z_k^{-1}, z_j^{-1} z_k; p, q)} \quad (13.8)$$

$$\times \prod_{j=1}^{2N+1} \prod_{k=1}^{2N+1} \Gamma(t_k z_j^{-1}; p, q) \prod_{i=1}^4 \Gamma(s_i z_j; p, q) \prod_{j=1}^{2N} \frac{dz_j}{2\pi i z_j}$$

and

$$I_M = \prod_{1 \leq j < k \leq 2N+1} \Gamma(tt_j t_k; p, q) \prod_{k=1}^{2N+1} \prod_{i=1}^4 \Gamma(t_k s_i; p, q) \Gamma(T; p, q) \prod_{i=1}^4 \frac{\Gamma(t^N s_i; p, q)}{\Gamma(t^N T s_i; p, q)}, \quad (13.9)$$

where

$$T = \prod_{k=1}^{2N+1} t_k, \quad S = \prod_{k=1}^4 s_k$$

and the balancing condition reads

$$t^{2N-1}ST = pq.$$

The equality $I_E = I_M$ was also suggested in [78] as an elliptic extension of the Gustafson-Rakha q -beta integral with an even number of integrations.

13.1.4. $SU(2N+1)$ with $T_A + \bar{T}_A + 3\bar{f} + 3f$. Models [13]:

	$SU(2N+1)$	$SU(3)$	$SU(3)$	$U(1)_1$	$U(1)_2$	$U(1)_3$	$U(1)_R$
Q	f	f	1	0	1	$2N-1$	$\frac{1}{3}$
\tilde{Q}	\bar{f}	1	f	0	-1	$2N-1$	$\frac{1}{3}$
A	T_A	1	1	1	0	-3	0
\tilde{A}	\bar{T}_A	1	1	-1	0	-3	0
V	adj	1	1	0	0	0	1
$Q(A\tilde{A})^k\tilde{Q}$		f	f	0	0	$4N-2-6k$	$\frac{2}{3}$
$\tilde{A}(A\tilde{A})^kQ^2$		T_A	1	-1	2	$4N-5-6k$	$\frac{2}{3}$
$A(A\tilde{A})^k\tilde{Q}^2$		1	T_A	1	-2	$4N-5-6k$	$\frac{2}{3}$
$A^N Q$		f	1	N	1	$-N-1$	$\frac{1}{3}$
$\tilde{A}^N \tilde{Q}$		1	f	$-N$	-1	$-N-1$	$\frac{1}{3}$
$A^{N-1}Q^3$		1	1	$N-1$	3	$3N$	1
$\tilde{A}^{N-1}\tilde{Q}^3$		1	1	$-N+1$	-3	$3N$	1
$(A\tilde{A})^m$		1	1	0	0	$-6m$	0

where $k = 0, \dots, N-1$ and $m = 1, \dots, N$.

The superconformal indices are written as

$$\begin{aligned}
 I_E &= \frac{(p; p)_\infty^{2N} (q; q)_\infty^{2N}}{(2N+1)!} \int_{\mathbb{T}^{2N}} \prod_{1 \leq i < j \leq 2N+1} \frac{\Gamma(tz_i z_j, sz_i^{-1} z_j^{-1}; p, q)}{\Gamma(z_i z_j^{-1}, z_i^{-1} z_j; p, q)} \\
 &\quad \times \prod_{j=1}^{2N+1} \prod_{k=1}^3 \Gamma(t_k z_j, s_k z_j^{-1}; p, q) \prod_{j=1}^{2N} \frac{dz_j}{2\pi i z_j}
 \end{aligned} \tag{13.10}$$

and

$$\begin{aligned}
 I_M &= \prod_{i=1}^3 \Gamma(t^N t_i, s^N s_i; p, q) \Gamma(t^{N-1} t_1 t_2 t_3, s^{N-1} s_1 s_2 s_3; p, q) \\
 &\quad \times \prod_{j=1}^N \Gamma((ts)^j; p, q) \prod_{i,k=1}^3 \Gamma((ts)^{j-1} t_i s_k; p, q) \prod_{1 \leq i < k \leq 3} \Gamma(t^{j-1} s^j t_i t_k, t^j s^{j-1} s_i s_k; p, q),
 \end{aligned} \tag{13.11}$$

where the balancing condition reads

$$(ts)^{2N-1} \prod_{k=1}^3 t_k s_k = pq.$$

The equality $I_E = I_M$ was derived in [78] by purely algebraic means as a consequence of other elliptic beta integrals. In the simplest $p \rightarrow 0$ limit it reduces to a Gustafson's q -beta integral for the root system A_{2N} [32].

13.1.5. $SU(2N)$ with $T_A + \bar{T}_A + 3\bar{f} + 3f$. Models [13]:

	$SU(2N)$	$SU(3)$	$SU(3)$	$U(1)_1$	$U(1)_2$	$U(1)_3$	$U(1)_R$
Q	f	f	1	0	1	$2N - 2$	$\frac{1}{3}$
\tilde{Q}	\bar{f}	1	f	0	-1	$2N - 2$	$\frac{1}{3}$
A	T_A	1	1	1	0	-3	0
\tilde{A}	\bar{T}_A	1	1	-1	0	-3	0
V	adj	1	1	0	0	0	1
$Q(A\tilde{A})^k\tilde{Q}$		f	f	0	0	$4N - 2 - 6k$	$\frac{2}{3}$
$\tilde{A}(A\tilde{A})^kQ^2$		T_A	1	-1	2	$4N - 5 - 6k$	$\frac{2}{3}$
$A(A\tilde{A})^k\tilde{Q}^2$		1	T_A	1	-2	$4N - 5 - 6k$	$\frac{2}{3}$
A^N		f	1	N	1	$-N - 1$	$\frac{1}{3}$
\tilde{A}^N		1	f	$-N$	-1	$-N - 1$	$\frac{1}{3}$
$A^{N-1}Q^2$		1	1	$N - 1$	3	$3N$	1
$\tilde{A}^{N-1}\tilde{Q}^2$		1	1	$-N + 1$	-3	$3N$	1
$(A\tilde{A})^n$		1	1	0	0	$-6m$	0

where $k = 0, \dots, N - 1$, $m = 0, \dots, N - 2$ and $n = 1, \dots, N - 1$.

The expressions for the superconformal indices are

$$I_E = \frac{(p; p)_\infty^{2N-1} (q; q)_\infty^{2N-1}}{(2N)!} \int_{\mathbb{T}^{2N-1}} \prod_{1 \leq i < j \leq 2N} \frac{\Gamma(tz_i z_j, sz_i^{-1} z_j^{-1}; p, q)}{\Gamma(z_i z_j^{-1}, z_i^{-1} z_j; p, q)} \quad (13.12)$$

$$\times \prod_{j=1}^{2N} \prod_{k=1}^3 \Gamma(t_k z_j, s_k z_j^{-1}; p, q) \prod_{j=1}^{2N-1} \frac{dz_j}{2\pi i z_j}$$

and

$$I_M = \Gamma(t^N s^N; p, q) \prod_{1 \leq i < k \leq 3} \Gamma(t^{N-1} t_i t_k, s^{N-1} s_i s_k; p, q) \prod_{j=1}^N \prod_{i,k=1}^3 \Gamma((ts)^{j-1} t_i s_k; p, q) \quad (13.13)$$

$$\times \prod_{j=1}^{N-1} \left(\Gamma((ts)^j; p, q) \prod_{1 \leq i < k \leq 3} \Gamma(t^{j-1} s^j t_i t_k, t^j s^{j-1} s_i s_k; p, q) \right),$$

where the balancing condition reads

$$(ts)^{2N-2} \prod_{k=1}^3 t_k s_k = pq.$$

The equality $I_E = I_M$ was also derived in [78] as a consequence of some other elliptic beta integrals. It reduces to one of Gustafson's integrals for the root system A_{2N-1} [32] in the simplest $p \rightarrow 0$ limit.

13.1.6. $SU(KN_f - 1)$ with $N_f f + N_f \bar{f} + 1adj$. Taking $N = KN_f - 1$ in (10.1) (or, $\tilde{N} = 1$), we find the s -confining dual theory discussed in [16]. The field content of these theories is easily found from the tables given in that section. Namely, in the electric theory one should fix N as described. On the magnetic side one should keep all the mesons and baryons, and set $\tilde{N} = 1$

in the gauge group part. Therefore for this case we write directly the superconformal indices

$$I_E = \frac{(p; p)_\infty^{N-1} (q; q)_\infty^{N-1}}{N!} \Gamma(U; p, q)^{N-1} \int_{\mathbb{T}^{N-1}} \prod_{1 \leq i < j \leq N} \frac{\Gamma(U z_i z_j^{-1}, U z_j z_i^{-1}; p, q)}{\Gamma(z_i z_j^{-1}, z_i^{-1} z_j; p, q)} \quad (13.14)$$

$$\times \prod_{i=1}^{N_f} \prod_{j=1}^N \Gamma(s_i z_j, t_i^{-1} z_j^{-1}; p, q) \prod_{j=1}^{N-1} \frac{dz_j}{2\pi i z_j},$$

and

$$I_M = \prod_{l=1}^K \prod_{1 \leq i, j \leq N_f} \Gamma(U^{l-1} s_i t_j^{-1}; p, q) \prod_{i=1}^{N_f} \Gamma(U(ST)^{\frac{K}{2}} s_i^{-1}, U(ST)^{-\frac{K}{2}} t_i; p, q), \quad (13.15)$$

where $U = (pq)^{\frac{1}{K+1}}$ and the balancing condition reads

$$U^{2KN_f-2} \prod_{i=1}^{N_f} s_i t_i^{-1} = (pq)^{N_f}.$$

For $K = 1$ one obtains the known A_N -root systems integral of type I from section 13.1.1. The conjecture $I_E = I_M$ for $K > 1$ represents a new elliptic beta integral requiring rigorous mathematical justification.

13.1.7. $SU(3KN_f - 1)$ with $N_f f + N_f \bar{f} + 2adj$. If we take $N = 3KN_f - 1$ in (10.4) we obtain the s -confinement in the magnetic theory discussed in [46]. The superconformal indices look as follows

$$I_E = \frac{(p; p)_\infty^{N-1} (q; q)_\infty^{N-1}}{N!} \int_{\mathbb{T}^{N-1}} \prod_{1 \leq i < j \leq N} \frac{\Gamma(U z_i z_j^{-1}, U z_i^{-1} z_j; p, q) \Gamma(U^{K/2} z_i z_j^{-1}, U^{K/2} z_i^{-1} z_j; p, q)}{\Gamma(z_i z_j^{-1}, z_i^{-1} z_j; p, q)}$$

$$\times \prod_{i=1}^{N_f} \prod_{j=1}^N \Gamma(s_i z_j, t_i^{-1} z_j^{-1}; p, q) \prod_{j=1}^{N-1} \frac{dz_j}{2\pi i z_j}, \quad (13.16)$$

$$I_M = \prod_{L=0}^{K-1} \prod_{J=0}^2 \Gamma(U^{L+KJ/2} s_i t_j^{-1}; p, q)$$

$$\times \prod_{i=1}^{N_f} \Gamma(U^{(-K+2)/2} (ST)^{\frac{3K}{2N}} s_i^{-1}, U^{(-K+2)/2} (ST)^{-\frac{3K}{2N}} t_i; p, q), \quad (13.17)$$

where $U = (pq)^{\frac{1}{K+1}}$ and the balancing condition reads

$$U^N ST^{-1} = (pq)^{N_f}.$$

The equality $I_E = I_M$ is a new conjectural elliptic beta integral.

13.1.8. $SU((2K+1)N_f - 4K - 1)$ with $N_f f + N_f \bar{f} + 2T_A$. If we take $N = (2K+1)N_f - 4K - 1$ in (10.7) we obtain the s -confinement in the magnetic theory discussed in [46]. The corresponding

superconformal indices are

$$\begin{aligned}
I_E &= \frac{(p; p)_\infty^{N-1} (q; q)_\infty^{N-1}}{N!} \int_{\mathbb{T}^{N-1}} \prod_{1 \leq i < j \leq N} \frac{\Gamma(U z_i z_j, U^{-1} (pq)^{\frac{1}{K+1}} z_i^{-1} z_j^{-1}; p, q)}{\Gamma(z_i^{-1} z_j, z_i z_j^{-1}; p, q)} \\
&\times \prod_{j=1}^N \prod_{k=1}^{N_f} \Gamma(s_k z_j, t_k z_j^{-1}; p, q) \prod_{j=1}^{N-1} \frac{dz_j}{2\pi i z_j}, \tag{13.18}
\end{aligned}$$

where $S = \prod_{j=1}^{N_f} s_j$, $T = \prod_{j=1}^{N_f} t_j$ and the balancing condition is

$$ST = (pq)^{N_f - \frac{N+2K}{K+1}}.$$

The magnetic index is found to be

$$\begin{aligned}
I_M &= \prod_{j=0}^K \prod_{k, l=1}^{N_f} \Gamma((pq)^{\frac{j}{K+1}} s_k t_l; p, q) \\
&\times \prod_{r=0}^{K-1} \prod_{1 \leq k < l \leq N_f} \Gamma(U^{-1} (pq)^{\frac{r+1}{K+1}} s_k s_l, U (pq)^{\frac{r}{K+1}} t_k t_l; p, q) \\
&\times \prod_{k=1}^{N_f} \Gamma((U\tilde{U})^{\frac{1}{2}} s_k^{-1}, (U\tilde{U})^{-\frac{1}{2}} (pq)^{\frac{1}{K+1}} t_k^{-1}; p, q), \tag{13.19}
\end{aligned}$$

where $\tilde{U} = U^{2KN_f - 4K - 1} ST^{-1} (pq)^{\frac{1 - KN_f + 2K}{K+1}}$ and U is an arbitrary parameter.

For $K = 0$ and any parameter U one obtains the integral discussed in section 13.1.1, while the general conjecture $I_E = I_M$ represents another new elliptic beta integral.

13.1.9. $SU((2K+1)N_f + 4K - 1)$ with $N_f f + N_f \bar{f} + 2T_S$. If we take $N = (2K+1)N_f + 4K - 1$ in (10.10) we obtain again the s -confinement in magnetic theory [46]. Corresponding electric index is

$$\begin{aligned}
I_E &= \frac{(p; p)_\infty^{N-1} (q; q)_\infty^{N-1}}{N!} \int_{\mathbb{T}^{N-1}} \prod_{1 \leq i < j \leq N} \frac{\Gamma(U z_i z_j, U^{-1} (pq)^{\frac{1}{K+1}} z_i^{-1} z_j^{-1}; p, q)}{\Gamma(z_i^{-1} z_j, z_i z_j^{-1}; p, q)} \\
&\times \prod_{j=1}^N \Gamma(U z_j^2, U^{-1} (pq)^{\frac{1}{K+1}} z_j^{-2}; p, q) \prod_{k=1}^{N_f} \Gamma(s_k z_j, t_k z_j^{-1}; p, q) \prod_{j=1}^{N-1} \frac{dz_j}{2\pi i z_j}. \tag{13.20}
\end{aligned}$$

The magnetic index is

$$\begin{aligned}
 I_M &= \Gamma(\tilde{U}, \tilde{U}^{-1}(pq)^{\frac{1}{K+1}}; p, q) \prod_{j=0}^K \prod_{k,l=1}^{N_f} \Gamma((pq)^{\frac{j}{K+1}} s_k t_l; p, q) \\
 &\times \prod_{r=0}^{K-1} \prod_{1 \leq k < l \leq N_f} \Gamma(U^{-1}(pq)^{\frac{r+1}{K+1}} s_k s_l, U(pq)^{\frac{r}{K+1}} t_k t_l; p, q) \\
 &\times \prod_{r=0}^{K-1} \prod_{k=1}^{N_f} \Gamma(U^{-1}(pq)^{\frac{r+1}{K+1}} s_k^2, U(pq)^{\frac{r}{K+1}} t_k^2; p, q) \\
 &\times \prod_{k=1}^{N_f} \Gamma((U\tilde{U})^{\frac{1}{2}} s_k^{-1}, (U\tilde{U})^{-\frac{1}{2}} (pq)^{\frac{1}{K+1}} t_k^{-1}; p, q),
 \end{aligned} \tag{13.21}$$

where $S = \prod_{j=1}^{N_f} s_j$, $T = \prod_{j=1}^{N_f} t_j$, the balancing condition reads

$$ST = (pq)^{N_f - \frac{N-2K}{K+1}},$$

and $\tilde{U} = U^{2KN_f+4K-1} S T^{-1} (pq)^{\frac{1-KN_f-2K}{K+1}}$.

Presently, the conjecture $I_E = I_M$ is confirmed only for $K = 0$ and any U , which reduces again to the integral of section 13.1.1.

13.1.10. $SU((4K+3)(N_f-4)-1)$ with $N_f f + (N_f-8)\bar{f} + T_A + T_S$. If we take $N = (4K+3)(N_f-4)-1$ in (10.13) we obtain the s -confinement in magnetic theory [46]. The electric index is

$$\begin{aligned}
 I_E &= \frac{(p; p)_\infty^{N-1} (q; q)_\infty^{N-1}}{N!} \int_{\mathbb{T}^{N-1}} \prod_{1 \leq i < j \leq N} \frac{\Gamma(U z_i z_j, U^{-1}(pq)^{\frac{1}{2(K+1)}} z_i^{-1} z_j^{-1}; p, q)}{\Gamma(z_i^{-1} z_j, z_i z_j^{-1}; p, q)} \\
 &\times \prod_{j=1}^N \Gamma(U^{-1}(pq)^{\frac{1}{2(K+1)}} z_j^{-2}; p, q) \prod_{k=1}^{N_f} \Gamma(s_k z_j; p, q) \prod_{l=1}^{N_f-8} \Gamma(t_l z_j^{-1}; p, q) \prod_{j=1}^{N-1} \frac{dz_j}{2\pi i z_j}.
 \end{aligned} \tag{13.22}$$

The magnetic index is

$$\begin{aligned}
 I_M &= \prod_{J=0}^{2K+1} \prod_{i=1}^{N_f} \prod_{j=1}^{N_f-8} \Gamma((pq)^{\frac{J}{2(K+1)}} s_i t_j; p, q) \\
 &\times \prod_{l=0}^{2K} \prod_{1 \leq i < j \leq N_f} \Gamma((pq)^{\frac{2l+1}{4(K+1)}} (uv)^{-1} s_i s_j; p, q) \prod_{l=0}^K \prod_{i=1}^{N_f} \Gamma((pq)^{\frac{4l+1}{4(K+1)}} (uv)^{-1} s_i^2; p, q) \\
 &\times \prod_{m=0}^{2K} \prod_{1 \leq i < j \leq N_f-8} \Gamma((pq)^{\frac{2m+1}{4(K+1)}} u v t_i t_j; p, q) \prod_{m=0}^{K-1} \prod_{i=1}^{N_f-8} \Gamma((pq)^{\frac{4m+3}{4(K+1)}} u v t_i^2; p, q) \\
 &\times \Gamma(\tilde{U}^{-1}(pq)^{\frac{1}{2(K+1)}}; p, q) \prod_{k=1}^{N_f} \Gamma((U\tilde{U})^{\frac{1}{2}} s_k^{-1}; p, q) \\
 &\times \prod_{l=1}^{N_f-8} \Gamma((U\tilde{U})^{-\frac{1}{2}} (pq)^{\frac{1}{2(K+1)}} t_l^{-1}; p, q),
 \end{aligned} \tag{13.23}$$

where $S = \prod_{j=1}^{N_f} s_j$, $T = \prod_{j=1}^{M_f} t_j$,

$$U = uv(pq)^{\frac{1}{4(K+1)}}, \quad \tilde{U} = u^{-1}v^{(4K+3)(N_f-4)-1}(pq)^{\frac{1}{4(K+1)}},$$

and the balancing condition reads

$$u^{-4}v^{-4}ST = (pq)^{N_f-4-\frac{1}{2(K+1)}}.$$

The equality $I_E = I_M$ is the conjecture defining one more new elliptic beta integral.

13.1.11. $SU(3KN_f+3)$ with $N_f\bar{f} + N_f\bar{f} + adj + T_S + \bar{T}_S$. If we take $N = 3KN_f+3$ for K -odd in (10.16) we obtain again the s -confinement in magnetic theory [46]. The electric index is

$$\begin{aligned} I_E &= \frac{(p;p)_\infty^{N-1}(q;q)_\infty^{N-1}}{N!} \int_{\mathbb{T}^{N-1}} \prod_{1 \leq i < j \leq N} \frac{\Gamma(Uz_i z_j^{-1}, Uz_i^{-1} z_j; p, q)}{\Gamma(z_i z_j^{-1}, z_i^{-1} z_j; p, q)} \\ &\quad \times \prod_{1 \leq i < j \leq N} \Gamma(U^{K/2} XY z_i z_j, U^{K/2} (XY)^{-1} z_i^{-1} z_j^{-1}; p, q) \\ &\quad \times \prod_{i=1}^N \Gamma(U^{K/2} XY z_i^2, U^{K/2} (XY)^{-1} z_i^2; p, q) \\ &\quad \times \prod_{i=1}^{N_f} \prod_{j=1}^N \Gamma(s_i z_j, t_i^{-1} z_j^{-1}; p, q) \prod_{j=1}^{N-1} \frac{dz_j}{2\pi i z_j} \end{aligned} \quad (13.24)$$

and

$$\begin{aligned} I_M &= \prod_{L=0}^{K-1} \prod_{i,j=1}^{N_f} \Gamma(U^{L+K} s_i t_j^{-1}, U^L s_i t_j^{-1}; p, q) \\ &\quad \times \prod_{J=0}^{K-1} \prod_{1 \leq i < j \leq N_f} \Gamma((XY)^{-1} U^{J+K/2} s_i s_j, XY U^{J+K/2} t_i^{-1} t_j^{-1}; p, q) \\ &\quad \times \prod_{J=0}^{\frac{K-1}{2}} \prod_{i=1}^{N_f} \Gamma((XY)^{-1} U^{2J+K/2} s_i^2, XY U^{2J+K/2} t_i^{-2}; p, q) \\ &\quad \times \Gamma(U^{K/2} X^{N-KN_f} Y^N, U^{K/2} (X^{N-KN_f} Y^N)^{-1}; p, q) \\ &\quad \times \prod_{i=1}^{N_f} \Gamma(U^{(-K+2)/2} X^{KN_f+2} Y^{\frac{3KN_f+4}{2}} s_i^{-1}; p, q) \\ &\quad \times \prod_{i=1}^{N_f} \Gamma(U^{(-K+2)/2} X^{-(KN_f+2)} Y^{-\frac{3KN_f+4}{2}} t_i; p, q), \end{aligned} \quad (13.25)$$

where $U = (pq)^{1/(K+1)}$, $Y = (ST)^{1/N_f}$, $S = \prod_{i=1}^{N_f} s_i$, $T^{-1} = \prod_{i=1}^{N_f} t_i^{-1}$, X is an arbitrary parameter and the balancing condition reads

$$U^{N-2}ST^{-1} = (pq)^{N_f}.$$

Again, the proof of the general equality $I_E = I_M$ is absent.

13.1.12. $SU(3KN_f - 5)$ with $N_f f + N_f \bar{f} + adj + T_A + \bar{T}_A$. If we take $N = 3KN_f - 5$ for K -odd in (10.19) we obtain again the s -confinement in magnetic theory [46]. The electric index is

$$\begin{aligned}
 I_E &= \frac{(p; p)_\infty^{N-1} (q; q)_\infty^{N-1}}{N!} \int_{\mathbb{T}^{N-1}} \prod_{1 \leq i < j \leq N} \frac{\Gamma(U z_i z_j^{-1}, U z_i^{-1} z_j; p, q)}{\Gamma(z_i z_j^{-1}, z_i^{-1} z_j; p, q)} \\
 &\quad \times \prod_{1 \leq i < j \leq N} \Gamma(U^{K/2} X Y z_i z_j, U^{K/2} (XY)^{-1} z_i^{-1} z_j^{-1}; p, q) \\
 &\quad \times \prod_{i=1}^{N_f} \prod_{j=1}^N \Gamma(s_i z_j, t_i^{-1} z_j^{-1}; p, q) \prod_{j=1}^{N-1} \frac{dz_j}{2\pi i z_j}
 \end{aligned} \tag{13.26}$$

and

$$\begin{aligned}
 I_M &= \prod_{L=0}^{K-1} \prod_{i,j=1}^{N_f} \Gamma(U^{L+K} s_i t_j^{-1}, U^L s_i t_j^{-1}; p, q) \\
 &\quad \times \prod_{J=0}^{K-1} \prod_{1 \leq i < j \leq N_f} \Gamma((XY)^{-1} U^{J+K/2} s_i s_j, XY U^{J+K/2} t_i^{-1} t_j^{-1}; p, q) \\
 &\quad \times \prod_{J=0}^{\frac{K-3}{2}} \prod_{i=1}^{N_f} \Gamma((XY)^{-1} U^{2J+1+K/2} s_i^2, XY U^{2J+1+K/2} t_i^{-2}; p, q) \\
 &\quad \times \prod_{i=1}^{N_f} \Gamma(U^{(-K+2)/2} X^{KN_f-2} Y^{\frac{3KN_f-4}{2}} s_i^{-1}; p, q) \\
 &\quad \times \prod_{i=1}^{N_f} \Gamma(U^{(-K+2)/2} X^{-(KN_f-2)} Y^{-\frac{3KN_f-4}{2}} t_i; p, q),
 \end{aligned} \tag{13.27}$$

where $U = (pq)^{1/(K+1)}$, $Y = (ST)^{1/N_f}$, $S = \prod_{i=1}^{N_f} s_i$, $T^{-1} = \prod_{i=1}^{N_f} t_i^{-1}$, X is an arbitrary parameter and the balancing condition reads

$$U^{N-2} S T^{-1} = (pq)^{N_f}.$$

No proof of the equality $I_E = I_M$ is known at present.

13.1.13. $SU(N)$ with $N_f f + (N_f - 8)\bar{f} + adj + T_A + \bar{T}_S$. If we take $N = 3K(N_f - 4) - 1$ in (10.22) we obtain again the s -confinement in magnetic theory [46]. The superconformal indices

are

$$\begin{aligned}
I_E &= \frac{(p; p)_\infty^{N-1} (q; q)_\infty^{N-1}}{N!} \int_{\mathbb{T}^{N-1}} \prod_{1 \leq i < j \leq N} \frac{\Gamma(U z_i z_j^{-1}, U z_i^{-1} z_j; p, q)}{\Gamma(z_i z_j^{-1}, z_i^{-1} z_j; p, q)} \\
&\quad \times \prod_{i=1}^N \Gamma(U^{K/2} (XY)^{-1} z_i^2; p, q) \\
&\quad \times \prod_{1 \leq i < j \leq N} \Gamma(U^{K/2} XY z_i z_j, U^{K/2} (XY)^{-1} z_i^{-1} z_j^{-1}; p, q) \\
&\quad \times \prod_{i=1}^N \Gamma(U^{K/2} (XY)^{-1} z_i^2; p, q) \\
&\quad \times \prod_{j=1}^N \prod_{i=1}^{N_f} \Gamma(s_i z_j; p, q) \prod_{k=1}^{N_f-8} \Gamma(t_k z_j^{-1}; p, q) \prod_{j=1}^{N-1} \frac{dz_j}{2\pi i z_j} \tag{13.28}
\end{aligned}$$

and

$$\begin{aligned}
I_M &= \prod_{L=0}^{K-1} \prod_{i=1}^{N_f} \prod_{j=1}^{N_f-8} \Gamma(U^{L+K} s_i t_j, U^L s_i t_j; p, q) \\
&\quad \times \prod_{J=0}^{K-1} \prod_{1 \leq i < j \leq N_f} \Gamma((XY)^{-1} U^{J+K/2} s_i s_j; p, q) \\
&\quad \times \prod_{J=0}^{K-1} \prod_{1 \leq i < j \leq N_f-8} \Gamma(XY U^{J+K/2} t_i t_j; p, q) \\
&\quad \times \prod_{J=0}^{K-1} \prod_{i=1}^{N_f} \Gamma((XY)^{-1} U^{J+K/2} s_i^2; p, q) \tag{13.29} \\
&\quad \times \Gamma(U^{K/2} (XY^N)^{-1}; p, q) \\
&\quad \times \prod_{i=1}^{N_f} \Gamma(U^{(-K+2)/2} Y^{\frac{3K(N_f-4)}{2}} s_i^{-1}; p, q) \\
&\quad \times \prod_{k=1}^{N_f-8} \Gamma(U^{(-K+2)/2} Y^{-\frac{3K(N_f-4)}{2}} t_k^{-1}; p, q),
\end{aligned}$$

where $U = (pq)^{1/(K+1)}$, $S = \prod_{i=1}^{N_f} s_i$, $T = \prod_{i=1}^{N_f-8} t_i$, the balancing condition reads $U^N X^{-4} Y^{-4} S T = (pq)^{N_f-4}$ and

$$Y = \left(S T^{-1} X^{2N_f-8} (pq)^{\frac{2(K-2)}{K+1}} \right)^{\frac{1}{N_f-4}}.$$

Equality of indices defines one more unproven elliptic beta integral.

13.1.14. *New confining duality.* Let us take the electric and magnetic $\mathcal{N} = 1$ superconformal field theories described by the table below

	$SU(N+1)$	$SP(2N)$	$SU(N+3)$	$U(1)$	$U(1)_R$
Q_1	\bar{f}	1	f	1	0
Q_2	f	f	1	$-\frac{N+3}{2}$	1
X	\bar{T}_A	1	1	$N+3$	0
V	adj	1	1	0	1
$q_1 = Q_1^{N+1}$		1	\bar{T}_A	$N+1$	0
$q_2 = Q_1 Q_2$		f	f	$-\frac{N+1}{2}$	1

The dynamically generated potential in this case is

$$W_{dyn} = \frac{1}{\Lambda^{4N+1}} Q_1^{N+1} (Q_1 Q_2)^2.$$

The indices read

$$I_E = \frac{(p; p)_\infty^N (q; q)_\infty^N}{(N+1)!} \int_{\mathbb{T}^N} \prod_{1 \leq i < j \leq N+1} \frac{\Gamma(Sz_i^{-1} z_j^{-1}; p, q)}{\Gamma(z_i z_j^{-1}, z_i^{-1} z_j; p, q)} \\ \times \prod_{j=1}^{N+1} \frac{\prod_{k=1}^N \Gamma(t_k z_j; p, q) \prod_{m=1}^{N+3} \Gamma(s_m z_j^{-1}; p, q)}{\prod_{k=1}^N \Gamma(St_k z_j^{-1}; p, q)} \prod_{j=1}^N \frac{dz_j}{2\pi i z_j} \quad (13.30)$$

and

$$I_M = \prod_{k=1}^N \prod_{m=1}^{N+3} \frac{\Gamma(t_k s_m; p, q)}{\Gamma(St_k s_m^{-1}; p, q)} \prod_{1 \leq l < m \leq N+3} \Gamma(Ss_l^{-1} s_m^{-1}; p, q), \quad (13.31)$$

with the balancing condition

$$S = \prod_{m=1}^{N+3} s_m.$$

The elliptic beta integral described by the equality $I_E = I_M$ was discovered by the first author and Warnaar in [87]. Here it defines a new pair of $\mathcal{N} = 1$ supersymmetric quantum field theories dual to each other, which was not considered earlier in the literature. Conjecturally, there should exist a symmetry transformation for a higher order generalization of I_E depending on the bigger number of parameters. Correspondingly, there should exist a more complicated Seiberg duality as well.

13.2. Exceptional cases for the unitary gauge group.

13.2.1. $SU(4)$ with $3f + 3\bar{f}$. The dual theories are described in the table below [13].

	$SU(4)$	$SU(2)$	$SU(3)$	$SU(3)$	$U(1)_1$	$U(1)_2$	$U(1)_R$
Q	f	1	f	1	1	2	$\frac{1}{3}$
\tilde{Q}	\bar{f}	1	1	f	-1	2	$\frac{1}{3}$
A	T_A	f	1	1	0	-3	0
V	adj	1	1	1	0	0	1
$M_0 = Q\tilde{Q}$		1	f	f	0	4	$\frac{2}{3}$
$M_2 = QA^2\tilde{Q}$		1	f	f	0	-2	$\frac{2}{3}$
$H = AQ^2$		f	\bar{f}	1	2	1	$\frac{2}{3}$
$\tilde{H} = A\tilde{Q}^2$		f	1	\bar{f}	-2	1	$\frac{2}{3}$
$T = A^2$		T_S	1	1	0	-6	1

The superconformal indices are given by formulas

$$I_E = \frac{(p; p)_\infty^3 (q; q)_\infty^3}{4!} \int_{\mathbb{T}^3} \prod_{1 \leq i < j \leq 4} \frac{\prod_{k=1}^2 \Gamma(s_k z_i z_j; p, q)}{\Gamma(z_i z_j^{-1}, z_i^{-1} z_j; p, q)} \quad (13.32)$$

$$\times \prod_{j=1}^4 \prod_{k=1}^3 \Gamma(t_k z_j, u_k z_j^{-1}; p, q) \prod_{j=1}^3 \frac{dz_j}{2\pi i z_j}$$

and

$$I_M = \Gamma(S; p, q) \prod_{j=1}^2 \Gamma(s_j^2; p, q) \prod_{k, l=1}^3 \Gamma(t_k u_l, S t_k u_l; p, q) \quad (13.33)$$

$$\times \prod_{k=1}^2 \prod_{l=1}^3 \Gamma(T s_k t_l^{-1}, U s_k u_l^{-1}; p, q),$$

where

$$S = \prod_{k=1}^2 s_k, \quad T = \prod_{k=1}^3 t_k, \quad U = \prod_{k=1}^3 u_k$$

and the balancing condition reads

$$S^2 T U = pq.$$

The duality conjecture requires that $I_E = I_M$, which defines a new elliptic beta integral requiring a rigorous proof. This relation and others given below carry an exceptional character because they are not generalizable to arbitrary rank gauge groups.

13.2.2. $SU(6)$ with $4f + 4\bar{f}$. The following pair of models was constructed in [13]:

	$SU(6)$	$SU(4)$	$SU(4)$	$U(1)_1$	$U(1)_2$	$U(1)_R$
Q	f	f	1	1	3	1
\bar{Q}	\bar{f}	1	f	-1	3	1
A	T_{3A}	1	1	0	-4	-1
V	adj	1	1	0	0	1
$M_0 = Q\bar{Q}$		f	f	0	6	2
$M_2 = QA^2\bar{Q}$		f	f	0	-2	0
$B_1 = AQ^3$		f	1	3	5	2
$\bar{B}_1 = A\bar{Q}^3$		1	\bar{f}	-3	5	2
$B_3 = A^3Q^3$		f	1	3	-3	0
$\bar{B}_3 = A^3\bar{Q}^3$		1	\bar{f}	-3	-3	0
T		1	1	0	-16	4

Their superconformal indices are

$$I_E = \frac{(p; p)_\infty^5 (q; q)_\infty^5}{6!} \int_{\mathbb{T}^5} \prod_{1 \leq i < j < k \leq 6} \frac{\Gamma(U z_i z_j z_k; p, q)}{\Gamma(z_i z_j^{-1}, z_i^{-1} z_j; p, q)} \quad (13.34)$$

$$\times \prod_{j=1}^6 \prod_{k=1}^4 \Gamma(s_k z_j, t_k z_j^{-1}; p, q) \prod_{j=1}^5 \frac{dz_j}{2\pi i z_j}$$

and

$$\begin{aligned}
 I_M &= \prod_{k, l=1}^4 \Gamma(s_k t_l, U^2 s_k t_l; p, q) \\
 &\quad \times \prod_{k=1}^4 \Gamma(SU s_k^{-1}, SU^3 s_k^{-1}, TU t_k^{-1}, TU^3 t_k^{-1}; p, q),
 \end{aligned} \tag{13.35}$$

where

$$S = \prod_{k=1}^4 s_k, \quad T = \prod_{k=1}^4 t_k$$

and the balancing condition reads

$$STU^6 = pq.$$

We come once again to a new conjectured computable elliptic hypergeometric integral: $I_E = I_M$.

13.2.3. $SU(5)$ with $3T_A + 3\bar{f}$. Models [13]:

	$SU(5)$	$SU(3)$	$SU(3)$	$U(1)$	$U(1)_R$
Q	f	1	\bar{f}	-3	$\frac{2}{3}$
A	T_A	f	1	1	0
V	adj	1	1	0	1
AQ^2		f	\bar{f}	-5	$\frac{4}{3}$
A^3Q		T_{AS}	f	0	$\frac{2}{3}$
A^5		T_S	1	5	0

Indices:

$$\begin{aligned}
 I_E &= \frac{(p; p)_\infty^4 (q; q)_\infty^4}{5!} \int_{\mathbb{T}^4} \prod_{1 \leq i < j \leq 5} \frac{\prod_{k=1}^3 \Gamma(s_k z_i z_j; p, q)}{\Gamma(z_i z_j^{-1}, z_i^{-1} z_j; p, q)} \\
 &\quad \times \prod_{j=1}^5 \prod_{k=1}^3 \Gamma(t_k z_j^{-1}; p, q) \prod_{j=1}^4 \frac{dz_j}{2\pi i z_j}
 \end{aligned} \tag{13.36}$$

and

$$\begin{aligned}
 I_M &= \prod_{k, l=1}^3 \Gamma(T s_k t_l^{-1}; p, q) \prod_{1 \leq j < k \leq 3} \Gamma(S s_j s_k; p, q) \\
 &\quad \times \prod_{j=1}^3 \Gamma(S s_j^2; p, q) \prod_{k, j, l=1; k \neq j}^3 \Gamma(s_k^2 s_j t_l; p, q) \prod_{l=1}^3 \Gamma(S t_l; p, q)^2,
 \end{aligned} \tag{13.37}$$

where

$$S = \prod_{k=1}^3 s_k, \quad T = \prod_{k=1}^3 t_k$$

and the balancing condition reads

$$S^3 T = pq.$$

The equality $I_E = I_M$ yields a new elliptic beta integral, which is conjectured to hold for parameter values guaranteeing that only sequences of poles of the integrand in I_E converging to zero are located inside the contour \mathbb{T} .

13.2.4. $SU(5)$ with $2T_A + 4\bar{f} + 2f$. Models [13]:

	$SU(5)$	$SU(2)$	$SU(4)$	$SU(2)$	$U(1)_1$	$U(1)_2$	$U(1)_R$
Q	f	1	1	f	-2	1	$\frac{1}{3}$
\tilde{Q}	\bar{f}	1	f	1	1	1	$\frac{1}{3}$
A	T_A	f	1	1	0	-1	0
V	adj	1	1	1	0	0	1
$Q\tilde{Q}$		1	f	f	-1	2	$\frac{2}{3}$
$A\tilde{Q}^2$		f	T_A	1	2	1	$\frac{2}{3}$
A^2Q		T_S	1	f	-2	-1	$\frac{1}{3}$
$A^3\tilde{Q}$		f	f	1	1	-2	$\frac{1}{3}$
$A^2Q^2\tilde{Q}$		1	f	1	-3	1	1

Indices:

$$I_E = \frac{(p; p)_\infty^4 (q; q)_\infty^4}{5!} \int_{\mathbb{T}^4} \prod_{1 \leq i < j \leq 5} \frac{\prod_{k=1}^2 \Gamma(s_k z_i z_j; p, q)}{\Gamma(z_i z_j^{-1}, z_i^{-1} z_j; p, q)} \quad (13.38)$$

$$\times \prod_{j=1}^5 \prod_{k=1}^4 \Gamma(t_k z_j^{-1}, u_k z_j; p, q) \prod_{j=1}^4 \frac{dz_j}{2\pi i z_j}$$

and

$$I_M = \prod_{k=1}^4 \Gamma(SU t_k; p, q) \prod_{k=1}^4 \prod_{l=1}^2 \Gamma(t_k u_l, St_k s_l; p, q) \prod_{k=1}^2 \Gamma(Su_k; p, q) \quad (13.39)$$

$$\prod_{k,l=1}^2 \Gamma(s_l^2 u_k; p, q) \prod_{k=1}^2 \prod_{1 \leq l < m \leq 4} \Gamma(s_k t_l t_m; p, q),$$

where

$$S = \prod_{k=1}^2 s_k, \quad T = \prod_{k=1}^4 t_k, \quad U = \prod_{k=1}^2 u_k,$$

and the balancing condition reads

$$S^3 T U = pq.$$

The equality $I_E = I_M$ defines a new elliptic beta integral requiring a rigorous proof.

13.2.5. $SU(6)$ with $2T_A + f + 5\bar{f}$. Models [13]:

	$SU(6)$	$SU(2)$	$SU(5)$	$U(1)_1$	$U(1)_2$	$U(1)_R$
Q	f	1	1	-5	-4	0
\tilde{Q}	\bar{f}	1	f	1	-4	0
A	T_A	f	1	0	3	$\frac{1}{4}$
V	adj	1	1	0	0	1
$Q\tilde{Q}$		1	f	-4	-8	0
$A\tilde{Q}^2$		f	T_A	2	-5	$\frac{1}{4}$
A^3		T_{3S}	1	0	9	$\frac{3}{4}$
$A^3 Q\tilde{Q}$		f	f	-4	1	$\frac{3}{4}$
$A^4 \tilde{Q}^2$		1	T_A	2	4	1

Indices:

$$\begin{aligned}
 I_E &= \frac{(p; p)_\infty^5 (q; q)_\infty^5}{6!} \int_{\mathbb{T}^5} \prod_{1 \leq i < j \leq 6} \frac{\Gamma(Uz_i z_j; p, q)}{\Gamma(z_i z_j^{-1}, z_i^{-1} z_j; p, q)} \\
 &\quad \times \prod_{l=1}^2 \prod_{1 \leq j < k \leq 6} \Gamma(s_l z_j z_k; p, q) \prod_{j=1}^6 \prod_{k=1}^5 \Gamma(t_k z_j^{-1}; p, q) \prod_{j=1}^5 \frac{dz_j}{2\pi i z_j}
 \end{aligned} \tag{13.40}$$

and

$$\begin{aligned}
 I_M &= \prod_k^5 \Gamma(Ut_k; p, q) \prod_{k=1}^2 \prod_{j=1}^5 \Gamma(SU s_k t_j; p, q) \prod_{k=1}^2 \prod_{1 \leq j < l \leq 5} \Gamma(s_k t_j t_l; p, q) \\
 &\quad \times \prod_{1 \leq j < k \leq 5} \Gamma(S^2 t_j t_k; p, q) \prod_{j=1}^2 \Gamma(s_j^3, S s_j; p, q),
 \end{aligned} \tag{13.41}$$

where

$$S = \prod_{k=1}^2 s_k, \quad T = \prod_{k=1}^5 t_k$$

and the balancing condition reads

$$S^4 T U = pq.$$

The unproven equality $I_E = I_M$ defines a new elliptic beta integral.

13.2.6. $SU(7)$ with $2T_A + 6\bar{f}$. Models [13]:

	$SU(7)$	$SU(2)$	$SU(6)$	$U(1)$	$U(1)_R$
Q	\bar{f}	1	f	-5	$\frac{1}{3}$
A	T_A	f	1	3	0
V	adj	1	1	0	1
AQ^2		f	T_A	-7	$\frac{2}{3}$
$A^4 Q$		T_S	f	7	$\frac{1}{3}$

Indices:

$$\begin{aligned}
 I_E &= \frac{(p; p)_\infty^6 (q; q)_\infty^6}{7!} \int_{\mathbb{T}^6} \prod_{1 \leq i < j \leq 7} \frac{1}{\Gamma(z_i z_j^{-1}, z_i^{-1} z_j; p, q)} \\
 &\quad \times \prod_{k=1}^2 \prod_{1 \leq i < j \leq 7} \Gamma(s_k z_i z_j; p, q) \prod_{k=1}^6 \prod_{j=1}^7 \Gamma(t_k z_j^{-1}; p, q) \prod_{j=1}^6 \frac{dz_j}{2\pi i z_j}
 \end{aligned} \tag{13.42}$$

and

$$I_M = \prod_{k=1}^6 \Gamma(S^2 t_k; p, q) \prod_{k=1}^6 \prod_{l=1}^2 \Gamma(S s_l^2 t_k; p, q) \prod_{k=1}^2 \prod_{1 \leq l < m \leq 6} \Gamma(s_k t_l t_m; p, q), \tag{13.43}$$

where

$$S = \prod_{k=1}^2 s_k, \quad T = \prod_{k=1}^6 t_k$$

and the balancing condition reads

$$S^5 T = pq.$$

The conjecture $I_E = I_M$ gives us a new exact integration formula.

13.3. Symplectic gauge group.

13.3.1. $SP(2N)$ with $(2N + 4)f$. Models [39]:

	$SP(2N)$	$SU(2N + 4)$	$U(1)_R$
Q	f	f	$2r = \frac{1}{N+2}$
V	adj	1	1
Q^2		T_A	$2r = \frac{2}{N+2}$

Indices:

$$I_E = \frac{(p; p)_\infty^N (q; q)_\infty^N}{2^N N!} \int_{\mathbb{T}^N} \prod_{1 \leq i < j \leq N} \frac{1}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \quad (13.44)$$

$$\times \prod_{j=1}^N \frac{\prod_{m=1}^{2N+4} \Gamma(t_m z_j^{\pm 1}; p, q)}{\Gamma(z_j^{\pm 2}; p, q)} \prod_{j=1}^N \frac{dz_j}{2\pi i z_j}$$

and

$$I_M = \prod_{1 \leq m < s \leq 2N+4} \Gamma(t_m t_s; p, q), \quad (13.45)$$

where the balancing condition reads

$$\prod_{m=1}^{2N+4} t_m = pq.$$

The equality $I_E = I_M$ defines a BC_N elliptic integral of type I representing an elliptic analogue of the computable Dixon integral [21]. It was introduced and partially justified by van Diejen and the first author in [17] and completely proven in [61] and [78]. Its simplest $p \rightarrow 0$ limit yields one of the Gustafson q -beta integrals [31].

13.3.2. $SP(2N)$ with $6f$ and T_A . This duality was considered in [9, 11]. The flavor symmetry group is

$$F = SU(6) \times U(1)$$

and the field content is

	$SP(2N)$	$SU(6)$	$U(1)$	$U(1)_R$
Q	f	f	$N - 1$	$2r = \frac{1}{3}$
A	T_A	1	-3	0
V	adj	1	0	1
A^k		1	$-3k$	0
QA^mQ		T_A	$2(N - 1) - 3m$	$\frac{2}{3}$

where $k = 2, \dots, N$ and $m = 0, \dots, N - 1$.

The electric superconformal index is given by the integral

$$I_E = \frac{(p; p)_\infty^N (q; q)_\infty^N}{2^N N!} \Gamma(t; p, q)^{N-1} \int_{\mathbb{T}^N} \prod_{1 \leq i < j \leq N} \frac{\Gamma(t z_i^{\pm 1} z_j^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \quad (13.46)$$

$$\times \prod_{j=1}^N \frac{\prod_{m=1}^6 \Gamma(t_m z_j^{\pm 1}; p, q)}{\Gamma(z_j^{\pm 2}; p, q)} \prod_{j=1}^N \frac{dz_j}{2\pi i z_j}$$

and the magnetic index is

$$I_M = \prod_{j=2}^N \Gamma(t^j; p, q) \prod_{j=1}^N \prod_{1 \leq m < s \leq 6} \Gamma(t^{j-1} t_m t_s; p, q), \quad (13.47)$$

where the balancing condition reads

$$t^{2N-2} \prod_{m=1}^6 t_m = pq.$$

The equality $I_E = I_M$ is the elliptic Selberg integral introduced by van Diejen and the first author [17] and proven in [18] as a consequence of the BC_n -elliptic beta integral of type I (its direct proof is given also in [61]). The Selberg integral plays a fundamental role in mathematics and mathematical physics because of a large number of applications [27]. Note that for $N_f = 3$, $K = N$ this exactly computable integral gives a confirmation of the KS duality for these special values of parameters.

13.3.3. $SP(2M) + 4\bar{f} + 2Mf + T_A$. This new confining duality is obtained from the results of section 7 by formal setting $N = 0$. The models are described in the table

	$SP(2M)$	$SU(4)$	$SP(2M)$	$U(1)$	$U(1)_R$
Q_1	f	\bar{f}	1	$-\frac{M-2}{4}$	0
Q_2	f	1	f	$-\frac{1}{2}$	1
X	T_A	1	1	1	0
V	adj	1	1	0	1
$M = Q_1 Q_2$		\bar{f}	f	$-\frac{M}{4}$	1
$N_j = Q_1^2 X^j$		\bar{T}_A	1	$j - \frac{M-2}{2}$	0

where $j = 0, \dots, M-1$.

Conjecturally equal superconformal indices look as

$$I_E = \frac{(p; p)_\infty^M (q; q)_\infty^M \Gamma(t; p, q)^M}{2^M M!} \int_{\mathbb{T}^M} \prod_{1 \leq i < j \leq M} \frac{\Gamma(t z_i^{\pm 1} z_j^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \times \prod_{i=1}^M \frac{\prod_{k=1}^4 \Gamma(t t_k^{-1} z_i^{\pm 1}; p, q) \prod_{j=1}^M \Gamma(s_j z_i^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 2}; p, q) \prod_{j=1}^M \Gamma(t s_j z_i^{\pm 1}; p, q)} \frac{dz_i}{2\pi i z_i} \quad (13.48)$$

and

$$I_M = \prod_{k=1}^4 \prod_{j=1}^M \frac{\Gamma(t s_j t_k^{-1}; p, q)}{\Gamma(t_k s_j; p, q)} \prod_{i=0}^{M-1} \prod_{1 \leq k < r \leq 4} \Gamma(t^{i+2} t_k^{-1} t_r^{-1}; p, q), \quad (13.49)$$

where the balancing condition is $\prod_{k=1}^4 t_k = t^{2+M}$.

13.3.4. $SP(2K(N_f - 2))$ with $N_f f + T_A$. This duality was considered in [16, 46]. Looking at (11.1) we see that the choice $N = K(N_f - 2)$ yields $\tilde{N} = 0$, and the dual theory is s -confining. The field content of the electric and magnetic theories is easily found from the table given in

that section. For brevity we present only the superconformal indices

$$\begin{aligned}
I_E &= \frac{(p; p)_\infty^{K(N_f-2)} (q; q)_\infty^{K(N_f-2)}}{2^{K(N_f-2)} (K(N_f-2))!} \Gamma(U; p, q)^{K(N_f-2)-1} \\
&\quad \times \int_{\mathbb{T}^{K(N_f-2)}} \prod_{1 \leq i < j \leq K(N_f-2)} \frac{\Gamma(U z_i^{\pm 1} z_j^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \\
&\quad \times \prod_{j=1}^{K(N_f-2)} \frac{\prod_{i=1}^{2N_f} \Gamma(s_i z_j^{\pm 1}; p, q)}{\Gamma(z_j^{\pm 2}; p, q)} \prod_{j=1}^{K(N_f-2)} \frac{dz_j}{2\pi i z_j}, \tag{13.50}
\end{aligned}$$

where $U = (pq)^{\frac{1}{K+1}}$ and

$$I_M = \Gamma(U; p, q)^{-1} \prod_{l=1}^K \prod_{1 \leq i < j \leq 2N_f} \Gamma(U^{l-1} s_i s_j; p, q), \tag{13.51}$$

where the balancing condition reads

$$U^{2KN_f-2K} \prod_{i=1}^{2N_f} s_i = (pq)^{N_f}.$$

The conjecture $I_E = I_M$ is a new elliptic beta integral. For $K = 1$ it reduces to the special case discussed in section 13.3.1.

13.3.5. $SP(2(N_f-2+2KN_f))$ with $N_f f + T_S$. Looking at (11.4) and fixing $N = N_f - 2 + 2KN_f$ we obtain the s -confining theory which was considered in [16, 46]. The superconformal indices are

$$\begin{aligned}
I_E &= \frac{(p; p)_\infty^N (q; q)_\infty^N}{2^N N!} \Gamma(U; p, q)^N \int_{\mathbb{T}^N} \prod_{1 \leq i < j \leq N} \frac{\Gamma(U z_i^{\pm 1} z_j^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \prod_{1 \leq i \leq N} \Gamma(U z_i^{\pm 2}; p, q) \\
&\quad \times \frac{\prod_{i=1}^{2N_f} \prod_{1 \leq j \leq N_f-2+2KN_f} \Gamma(s_i z_j^{\pm 1}; p, q)}{\prod_{1 \leq j \leq N} \Gamma(z_j^{\pm 2}; p, q)} \prod_{1 \leq j \leq N} \frac{dz_j}{2\pi i z_j}, \tag{13.52}
\end{aligned}$$

$$\begin{aligned}
I_M &= \prod_{l=0}^K \prod_{1 \leq i < j \leq 2N_f} \Gamma(U^l s_i s_j; p, q) \\
&\quad \times \prod_{l=0}^{K-1} \prod_{1 \leq i < j \leq 2N_f} \Gamma(U^{(2l+1)/2} s_i s_j; p, q) \prod_{i=1}^{2N_f} \Gamma(U^{(2l+1)/2} s_i^{\pm 2}; p, q), \tag{13.53}
\end{aligned}$$

where $U = (pq)^{\frac{1}{K+1}}$ and the balancing condition reads

$$U^{2N_f-2+4KN_f} \prod_{i=1}^{2N_f} s_i = (pq)^{N_f}.$$

The equality $I_E = I_M$ represents the final new conjecture for elliptic beta integrals for arbitrary rank symplectic groups which we were able to find.

13.3.6. $SP(2(3KN_f - 4K - 2))$ with $N_f f + 2T_A$. Looking at (11.7) and fixing $N = 3KN_f - 4K - 2$ for K odd we obtain the s -confining theory which was considered in [46]. The superconformal indices are

$$I_E = \frac{(p; p)_\infty^N (q; q)_\infty^N}{2^N N!} \Gamma(U, U^{\frac{K}{2}}; p, q)^{N-1} \quad (13.54)$$

$$\times \int_{\mathbb{T}^N} \prod_{1 \leq i < j \leq N} \frac{\Gamma(U z_i^{\pm 1} z_j^{\pm 1}, U^{\frac{K}{2}} z_i^{\pm 1} z_j^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \prod_{i=1}^{2N_f} \prod_{j=1}^N \frac{\Gamma(s_i z_j^{\pm 1}; p, q)}{\Gamma(z_j^{\pm 2}; p, q)} \prod_{j=1}^N \frac{dz_j}{2\pi i z_j}$$

and

$$I_M = \Gamma(U, U^{\frac{K}{2}}; p, q)^{-1} \prod_{J=0}^{K-1} \prod_{L=0}^2 \prod_{1 \leq i < j \leq 2N_f} \Gamma(U^{J+\frac{KL}{2}} s_i s_j; p, q) \prod_{J=0}^{\frac{K-1}{2}} \prod_{j=1}^{2N_f} \Gamma(U^{2J+1+\frac{K}{2}} s_j^2; p, q), \quad (13.55)$$

where $S = \prod_{i=1}^{2N_f} s_i$, $U = (pq)^{\frac{1}{K+1}}$ and the balancing condition reads

$$U^{3KN_f - 2K - 1} S = (pq)^{N_f}.$$

The conjectural equality $I_E = I_M$ is an exceptional relation, similar to the ones we considered above for unitary groups.

13.3.7. $SP(2(3KN_f - 4K + 2))$ with $N_f f + T_S + T_A$. Looking at (11.10) and fixing $N = 3KN_f - 4K + 2$ for K odd we obtain the s -confining theory which was considered in [46]. The superconformal indices are

$$I_E = \frac{(p; p)_\infty^N (q; q)_\infty^N}{2^N N!} \Gamma(U; p, q)^{N-1} \Gamma(U^{\frac{K}{2}}; p, q)^N \quad (13.56)$$

$$\times \int_{\mathbb{T}^N} \prod_{1 \leq i < j \leq N} \frac{\Gamma(U z_i^{\pm 1} z_j^{\pm 1}, U^{\frac{K}{2}} z_i^{\pm 1} z_j^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \prod_{i=1}^{2N_f} \prod_{j=1}^N \frac{\Gamma(s_i z_j^{\pm 1}, U^{\frac{K}{2}} z_j^{\pm 2}; p, q)}{\Gamma(z_j^{\pm 2}; p, q)} \prod_{j=1}^N \frac{dz_j}{2\pi i z_j}$$

and

$$I_M = \Gamma(U; p, q)^{-1} \prod_{J=0}^{K-1} \prod_{L=0}^2 \prod_{1 \leq i < j \leq 2N_f} \Gamma(U^{J+\frac{KL}{2}} s_i s_j; p, q) \prod_{J=0}^{\frac{K-1}{2}} \prod_{j=1}^{2N_f} \Gamma(U^{2J+\frac{K}{2}} s_j^2; p, q), \quad (13.57)$$

where $S = \prod_{i=1}^{2N_f} s_i$, $U = (pq)^{\frac{1}{K+1}}$ and the balancing condition reads

$$U^{3KN_f - 2K + 1} S = (pq)^{N_f}.$$

The conjectural equality $I_E = I_M$ is our last new elliptic beta integral.

14. EXCEPTIONAL G_2 GROUP

G_2 with 5 flavors. This duality was discussed in [30, 56]. The upper table describes the electric theory, its magnetic dual, presented in the lower table, does not have local gauge symmetry (s -confinement).

	G_2	$SU(5)$	$U(1)_R$
Q	7	f	$2r = \frac{1}{5}$
V	adj	1	1
Q^2		T_S	$\frac{2}{5}$
Q^3		T_A	$\frac{3}{5}$
Q^4		f	$\frac{4}{5}$

The superconformal indices are

$$I_E = \frac{(p; p)_\infty^2 (q; q)_\infty^2}{2^2 3} \prod_{m=1}^5 \Gamma(t_m; p, q) \int_{\mathbb{T}^2} \frac{\prod_{k=1}^3 \prod_{m=1}^5 \Gamma(t_m z_k^{\pm 1}; p, q)}{\prod_{1 \leq j < k \leq 3} \Gamma(z_j^{\pm 1} z_k^{\pm 1}; p, q)} \prod_{j=1}^2 \frac{dz_j}{2\pi i z_j} \quad (14.1)$$

and

$$I_M = \prod_{m=1}^5 \frac{\Gamma(t_m^2; p, q)}{\Gamma((pq)^{1/2} t_m; p, q)} \prod_{1 \leq l < m \leq 5} \frac{\Gamma(t_l t_m; p, q)}{\Gamma((pq)^{1/2} t_l t_m; p, q)}, \quad (14.2)$$

where

$$z_1 z_2 z_3 = 1, \quad |t_m| < 1,$$

and the balancing condition reads

$$\prod_{m=1}^5 t_m = (pq)^{1/2}.$$

The conjecture $I_E = I_M$ describes the first elliptic beta integral for an exceptional root systems (it was mentioned in [86] and proposed also earlier by M. Ito).

G_2 with $5 < N_f < 12$ flavors. This duality was discovered in [57]. The electric theory has gauge group G_2 , but its magnetic dual has $SU(N_f - 3)$ gauge group. Their field content is presented in the tables below.

	G_2	$SU(N_f)$	$U(1)_R$
Q	7	f	$2r = 1 - \frac{4}{N_f}$
V	adj	1	1
	$SU(N_f - 3)$	$SU(N_f)$	$U(1)_R$
q	f	f	$2r_q = \frac{3}{N_f} (1 - \frac{1}{N_f - 3})$
q_0	\bar{f}	1	$2r_{q_0} = 1 - \frac{1}{N_f - 3}$
s	T_S	1	$2r_s = \frac{2}{N_f - 3}$
M	1	T_S	$2r_M = 2 - \frac{8}{N_f}$
V	adj	1	1

Corresponding superconformal indices are described by the integrals

$$I_E = \frac{(p; p)_\infty^2 (q; q)_\infty^2}{2^2 3} \prod_{m=1}^{N_f} \Gamma(t_m; p, q) \int_{\mathbb{T}^2} \frac{\prod_{k=1}^3 \prod_{m=1}^{N_f} \Gamma(t_m z_k^{\pm 1}; p, q)}{\prod_{1 \leq j < k \leq 3} \Gamma(z_j^{\pm 1} z_k^{\pm 1}; p, q)} \prod_{k=1}^2 \frac{dz_k}{2\pi i z_k}, \quad (14.3)$$

$$I_M = \frac{(p; p)_\infty^{N_f - 4} (q; q)_\infty^{N_f - 4}}{(N_f - 3)!} \prod_{1 \leq j < k \leq N_f} \Gamma(t_j t_k; p, q) \prod_{j=1}^{N_f} \Gamma(t_j^2; p, q) \quad (14.4)$$

$$\times \int_{\mathbb{T}^{N_f - 4}} \prod_{1 \leq j < k \leq N_f - 3} \frac{\Gamma((pq)^{r_s} z_j z_k; p, q)}{\Gamma(z_j^{-1} z_k, z_j z_k^{-1}; p, q)}$$

$$\times \prod_{j=1}^{N_f - 3} \Gamma((pq)^{r_s} z_j^2; p, q) \prod_{j=1}^{N_f - 3} \Gamma((pq)^{(1 - r_s)/2} z_j^{-1}; p, q)$$

$$\times \prod_{k=1}^{N_f} \Gamma((pq)^{(1 - r_s)/2} t_k^{-1} z_j^{-1}; p, q) \prod_{j=1}^{N_f - 4} \frac{dz_j}{2\pi i z_j},$$

where

$$z_1 z_2 z_3 = 1, \quad |t_m| < 1,$$

and the balancing condition reads

$$\prod_{m=1}^{N_f} t_m = (pq)^{(N_f-4)/2}.$$

The equality $I_E = I_M$ represents a new symmetry transformation formula for general elliptic hypergeometric integral on the G_2 root system. Independently it was also considered earlier by F. A. Dolan.

For $N_f = 5$ the integral I_M takes the form

$$\begin{aligned} I_M = & \frac{(p; p)_\infty (q; q)_\infty}{2} \prod_{1 \leq j < k \leq 5} \Gamma(t_j t_k; p, q) \prod_{j=1}^5 \Gamma(t_j^2; p, q) \\ & \times \int_{\mathbb{T}} \frac{\Gamma((pq)^{1/4} z_j^{\pm 1}; p, q) \prod_{k=1}^5 \Gamma((pq)^{1/4} t_k^{-1} z_j^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}; p, q)} \frac{dz}{2\pi i z}. \end{aligned} \quad (14.5)$$

Using the univariate elliptic beta integral, one can compute this I_M and find the index coinciding with I_M described at the beginning of this section. As to the proof of the general G_2 -transformation $I_E = I_M$, it should be a subcase of the original Seiberg duality relation for the $SU(3)$ -gauge group. Indeed, let us take $N = 3$ and $t_i^{-1} = s_i$ in the electric index (4.6). If we set then $s_{N_f} = pq$, we obtain the G_2 -group electric index (14.3) with N_f and t_i replaced by $N_f - 1$ and s_i , respectively. Therefore it is expected that G_2 -magnetic index can be obtained after appropriate restrictions in I_M (4.6). The difficult part consists in computing the limit $s_{N_f} \rightarrow pq$, since it leads to a diverging integral multiplied by a vanishing coefficient. This limit is currently under investigation.

15. 'T HOOFT ANOMALY MATCHING CONDITIONS

In this section we check the standard 't Hooft anomaly matching conditions [36, 90] for some of new dualities. The needed Casimir operators for SU and SP groups can be found in the Appendix C. There are also the discrete anomalies matching conditions [16], but we skipped their consideration in the present work.

Multiple $SP(2N)$ duality. Let us begin with the multiple duality for $SP(2N)$ gauge group discussed in section 6. To check the matching of anomalies we should do it for the smaller flavor groups present in all theories. The subgroup $SU(4) \times SU(4) \times U(1)_B \times U(1) \times U(1)_R$ of the electric theory has the following anomalies

$$\begin{array}{ll} SU^3(4)_L & 2N \\ SU^2(4)_L \times U(1)_R & -2N^2 + 1 \\ SU^2(4)_L \times U(1) & \frac{3N^2 - 2N - 1}{2} \\ SU^2(4)_L \times U(1)_B & 2N \\ U(1)_R & -(2N^2 + 7N + 1) \\ U(1)_R^3 & -(2N^2 + N + 1) \\ U(1)_B^2 \times U(1)_R & 0 \\ U(1)^2 \times U(1)_R & -\frac{N^3 - 1}{2} \end{array} \quad (15.1)$$

We have verified that all three dual magnetic theories have the same anomalies. Also it is easy to check that the real anomaly is equal to zero in the electric and magnetic theories. The calculation in the electric theory is

$$2N + 2 - \frac{1}{2}8 - (2N - 2) = 0.$$

$SP \leftrightarrow SP$ groups duality. Here we discuss the duality of section 7. In the electric theory we have found the following anomalies for $SU(4) \times SP(2(M + N)) \times U(1) \times U(1)_R$ global symmetry group

$$\begin{array}{ll}
SU^3(4) & -2M \\
SU^2(4) \times U(1) & -\frac{1}{2}M(M - N - 2) \\
SP^2(2(M + N)) \times U(1) & -M \\
SU^2(4) \times U(1)_R & -2M \\
SP^2(2(M + N)) \times U(1)_R & 0 \\
U(1)_R & 1 - 6M \\
U(1)_R^3 & 1 - 6M \\
U(1)^2 \times U(1)_R & \frac{1}{2}(-M^3 + 2NM^2 - MN^2 - 4MN - 2M + 2)
\end{array} \quad (15.2)$$

coinciding with the anomalies in the magnetic duals.

Calculation of the real anomaly yields

$$-4 - (2M - 2) + 2M + 2 = 0.$$

New confining duality. Now we discuss the duality of section 13.1.14. In the electric theory we have found the following anomalies for $SP(2N) \times SU(N + 3) \times U(1)_1 \times U(1)_2 \times U(1)_R$ global symmetry group

$$\begin{array}{ll}
SU^3(N + 3) & N + 1 \\
SP^2(2N) \times U(1)_R & 0 \\
SU^2(N + 3) \times U(1)_R & -(N + 1) \\
SP^2(2N) \times U(1) & -\frac{(N + 1)(N + 3)}{2} \\
SU^2(N + 3) \times U(1) & N + 1 \\
U(1)_R & -\frac{1}{2}(N + 2)(N + 3) \\
U(1)_R^3 & -\frac{1}{2}(N + 2)(N + 3) \\
U(1)^2 \times U(1)_R & -\frac{1}{2}(N + 1)^2(N + 2)(N + 3)
\end{array} \quad (15.3)$$

and the same picture holds for the magnetic partner.

Calculation of the real anomaly yields

$$-(N + 3) - (N - 1) + 2(N + 1) = 0.$$

$SU \leftrightarrow SP$ groups duality. The anomalies matching for the common global group $SU(N+3) \times SU(N+3) \times U(1)_B \times U(1)_R$ of the duality described in section 12.1 is checked and yields:

$$\begin{aligned}
 SU^3(N+3)_L & N+1 \\
 SU^2(N+3)_L \times U(1)_R & -\frac{(N+1)^2}{N+3} \\
 SU^2(N+3)_L \times U(1)_B & 2(N+1) \\
 U(1)_R & -(N^2+2N+2) \\
 U(1)_R^3 & -\frac{N^4-9N^2-10N+2}{(N+3)^2} \\
 U(1)_B^2 \times U(1)_R & -8(N+1)^2.
 \end{aligned} \tag{15.4}$$

$SU \leftrightarrow SU$ groups duality. Here we consider the dualities presented in section 12.2. The anomalies matching is checked for the global group $SU(K)_L \times SU(N+3-K)_L \times U(1)_1 \times SU(K)_R \times SU(N+3-K)_R \times U(1)_2 \times U(1)_B \times U(1)_R$ yielding

$$\begin{aligned}
 SU^3(K)_L & N+1 \\
 SU^2(K)_L \times U(1)_R & -\frac{(N+1)^2}{N+3} \\
 SU^2(K)_L \times U(1)_B & (N+1) \\
 SU^2(K)_L \times U(1)_1 & (N+1)(N+3-K) \\
 U(1)_R & -(N^2+2N+2) \\
 U(1)_R^3 & -\frac{N^4-9N^2-10N+2}{(N+3)^2} \\
 U(1)_B^2 \times U(1)_R & -2(N+1)^2.
 \end{aligned} \tag{15.5}$$

Comparing the 't Hooft anomaly matching conditions for all dualities described in this paper and the analysis of total ellipticity of the elliptic hypergeometric terms lying behind the equalities of superconformal indices, we come to the

Conjecture. *The condition of total ellipticity for an elliptic hypergeometric term is necessary and sufficient for validity of the 't Hooft anomaly matching conditions for dual superconformal field theories whose superconformal indices are determined by this term.*

For proving this hypothesis it is necessary to take formal mathematical definition of anomalies as cocycles of the gauge groups (see, e.g., [66]). For dual theories we have two, in general different, gauge groups. Therefore anomaly matching condition looks as an equality of Chern classes of dual theories, and the conditions of total ellipticity – as a condition of vanishing of the combined Chern classes. These questions will be discussed in more detail separately.

16. CONCLUSION

To be clear, this paper does not contain a description of all known dual superconformal theories. We have limited ourselves only to *simple* gauge groups $G = SU(N), SP(2N), G_2$. First, there are other simple groups $G = SO(N), F_4, E_6, E_7, E_8$ consideration of which we skipped. The situation with the dualities for the exceptional groups [20, 44] is not clear in general (except of the G_2 -cases described above) due to the complexity of the invariants of these groups [10, 59]. There are very many dualities involving orthogonal groups $SO(N)$. Originally we hoped to tackle them as well, but their amount is very big, and it was decided to

consider them separately. It is known that many group-theoretical structures for $SO(N)$ group can be obtained as reductions of the $SP(2N)$ -group structures. Some of such reductions were considered by Dolan and Osborn at the level of superconformal indices [23]. However, there are much more cases that they have considered. Many elliptic hypergeometric integrals for the B_N (i.e., groups $SO(2N+1)$) and D_N (i.e., groups $SO(2N)$) root systems can be obtained by special restriction of the BC_N -integrals (cf. the forms of the corresponding invariant measures given in the Appendix B). However, it is not clear at the moment whether superconformal indices of *all* known $SO(N)$ -group theories and their duals can be obtained in this way.

Second, we deliberately skipped consideration of the superconformal indices for extended $\mathcal{N} > 1$ supersymmetric field theories [45]. The best known examples correspond to the Seiberg-Witten $\mathcal{N} = 2$ theories [72, 73]. Consider the following electric and magnetic theories

	$SO(3)$	$SU(3)$	$U(1)_R$		$SO(4)$	$SU(3)$	$U(1)_R$
Q	f	f	$\frac{2}{3}$	q	f	f	$\frac{1}{3}$
V	adj	1	1	M	1	T_S	$\frac{4}{3}$
				\tilde{V}	adj	1	1

As discussed by Intriligator and Seiberg [40, 41, 42] (see also [29]), the $SO(3)$ Seiberg duality electric model becomes the $SU(2)$ group $\mathcal{N} = 4$ super-Yang-Mills theory in the infrared region after taking the tree level superpotential $W_{tree} = \sqrt{2} \det Q$ in the electric theory. The superconformal indices then are

$$I_E = \frac{(p; p)_\infty (q; q)_\infty}{2} \prod_{j=1}^3 \Gamma((pq)^{1/3} s_j; p, q) \int_{\mathbb{T}} \frac{\prod_{j=1}^3 \Gamma((pq)^{1/3} s_j z^{\pm 1}; p, q)}{\Gamma(z^{\pm 1}; p, q)} \frac{dz}{2\pi i z}. \quad (16.1)$$

and

$$I_M = \frac{(p; p)_\infty^2 (q; q)_\infty^2}{4} \prod_{1 \leq i < j \leq 3} \Gamma((pq)^{\frac{2}{3}} s_i s_j; p, q) \prod_{i=1}^3 \Gamma((pq)^{\frac{2}{3}} s_i^2; p, q) \\ \times \int_{\mathbb{T}^2} \prod_{j=1}^2 \frac{\prod_{i=1}^3 \Gamma((pq)^{1/6} s_i^{-1} z_j^{\pm 1}; p, q)}{\Gamma(z_1^{\pm 1} z_2^{\pm 1}; p, q)} \frac{dz_j}{2\pi i z_j}, \quad (16.2)$$

By a change of integration variables in I_M , one of the integrations can be taken with the help of univariate elliptic beta integral, which shows that (16.2) is equal to (16.1). This equality can be obtained as a reduction of the BC_N -relations as well [23]. We suppose therefore that it is necessary to consider first all possible $SP(2N)$ -group identities for integrals and then try to reduce them to the relations for superconformal indices of extended supersymmetric dual theories.

Third, we skipped the quiver gauge group cases, when there are more than one simple gauge group (or the deconfinement phenomenon [5]). It is expected that equalities of the superconformal indices for them are mere consequences of the so-called Bailey-type chains (forming a tree) of symmetry transformations discovered by the first author in [79] and extended in [87] to root systems. Within this context the duality transformation acquires a simple meaning of the integral transformation for functions obeying many properties of the classical Fourier transformation (see [87]).

Let us list some other possible applications of our results. Counting of the gauge invariant operators for a number of supersymmetric gauge theories was considered in detail in [34, 35]. It is not difficult to see that the corresponding generating functions are obtained from our superconformal indices by taking the limits $p, q \rightarrow 0$. To take the limit $p \rightarrow 0$ one needs

first to get rid off the balancing conditions by multiplying a number of parameters by integer powers of p and application of the reflection formula for the elliptic gamma function, see [84]. However, in this work we have a much larger list of theories where this gauge invariant operators counting technique is applicable (in particular, this concerns the theories described in chapters 7, 8, 9, 10.2-10.6, 11.2-11.4, 12, 13 and 14). The limit $p \rightarrow 0$ in the simplest cases leads to q -hypergeometric functions, the meaning of which is not clarified yet from the superconformal index point of view. The subsequent limit $q \rightarrow 0$ can be replaced by $q \rightarrow 1$, which yields the plain hypergeometric functions, which also should have thus some meaning within the gauge field theories. Similar clarification is needed for the situations when elliptic hypergeometric integrals are reduced to terminating elliptic hypergeometric series by some special choices of the parameters.

In [78, 83] the first author has constructed biorthogonal functions associated with the elliptic beta integral. Naturally, it was conjectured there that some multivariable biorthogonal functions exist for all known elliptic beta integrals (which serve as the orthogonality measures). First family of such functions was constructed by Rains in [61, 62]. So, the expected number of such families of biorthogonal functions has now increased essentially.

In [82] it was shown that some of the BC_n elliptic hypergeometric integrals can be associated with the relativistic Calogero-Sutherland type models, and it was conjectured that other models of such type can be built out of all other existing elliptic beta integrals and their appropriate generalizations. Because we have now interpretation of the elliptic hypergeometric integrals as superconformal indices of supersymmetric field theories, we, naturally, come to the conjecture that behind each $\mathcal{N} = 1$ superconformal field theory there is a Calogero-Sutherland type model for which these integrals serve either as the topological indices or the wave functions normalizations, respectively. In particular, we would like to mention in this context appearance of the usual elliptic Calogero-Sutherland models within the $\mathcal{N} = 2$ Seiberg-Witten theories [55].

The group-theoretical interpretation of the elliptic hypergeometric integrals discussed in [68, 23, 86] and this paper opens possibilities for general structural theorems on the integrals themselves. Namely, it may play a key role in the classification of such integrals on root systems. All the questions mentioned above deserve detailed investigation either in relation to the supersymmetric dualities or on their own. As to the proofs of many new hypergeometric identities conjectured in this paper we refer to known methods described in [17, 18, 61, 65, 76, 78, 79, 87] (or indicated above in some cases) which are available for their treating. We plan to consider them case by case depending on their tractability.

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APPENDIX A. CHARACTERS OF REPRESENTATIONS OF CLASSICAL GROUPS

Here we present general results for characters of the Lie groups discussed in the paper.

For $SU(N)$ group the characters, depending on $x = (x_1, \dots, x_n)$ subject to the constraint $\prod_{i=1}^n x_i = 1$, are the well known Schur polynomials

$$s_{\underline{\lambda}}(x) = s_{(\lambda_1, \dots, \lambda_n)}(x) = \frac{\det [x_i^{\lambda_j + n - j}]}{\det [x_i^{n - j}]}, \quad (\text{A.1})$$

where $\underline{\lambda}$ is the partition ordered so that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. They obey the property $s_{(\lambda_1, \dots, \lambda_n)}(x) = s_{(\lambda_1 + c, \dots, \lambda_n + c)}(x)$, where $c \in \mathbb{Z}$. Therefore one can assume that $\lambda_n = 0$ without loss of generality.

Characters of the fundamental and antifundamental representations of $SU(N)$ group are

$$\chi_{SU(N),f}(x) = s_{(1,0,\dots,0)}(x) = \sum_{i=1}^N x_i, \quad \chi_{SU(N),\bar{f}} = s_{(1,\dots,1,0)}(x) = \chi_{SU(N),f}(x^{-1}).$$

The character for the adjoint representation is

$$\chi_{SU(N),adj}(x) = s_{(2,1,\dots,1,0)}(x) = \sum_{1 \leq i, j \leq N} x_i x_j^{-1} - 1.$$

The character for the absolutely anti-symmetric tensor representation of rank two for $SU(N)$ group is

$$\chi_{SU(N),T_A}(x) = s_{(1,1,0,\dots,0)}(x) = \sum_{1 \leq i < j \leq N} x_i x_j, \quad \chi_{SU(N),\bar{T}_A} = \chi_{SU(N),T_A}(x^{-1}).$$

The characters for symmetric representations of $SU(N)$ are

$$\chi_{SU(N),T_S}(x) = s_{(2,0,\dots,0)}(x) = \sum_{1 \leq i < j \leq N} x_i x_j + \sum_{i=1}^N x_i^2, \quad \chi_{SU(N),\bar{T}_S}(x) = \chi_{SU(N),T_S}(x^{-1}).$$

The character for the absolutely anti-symmetric tensor representation of rank three for $SU(N)$ group is

$$\chi_{SU(N),T_{3A}}(x) = s_{(1,1,1,0,\dots,0)}(x) = \sum_{1 \leq i < j < k \leq N} x_i x_j x_k.$$

For the absolutely symmetric tensor representation of rank three for $SU(N)$ the character has the form

$$\chi_{SU(N),T_{3S}}(x) = s_{(3,0,\dots,0)}(x) = \sum_{1 \leq i < j < k \leq N} x_i x_j x_k + \sum_{i, j=1, i \neq j}^N x_i^2 x_j + \sum_{i=1}^N x_i^3.$$

In the mixed case, we have

$$\chi_{SU(N),T_{AS}}(x) = s_{(2,1,0,\dots,0)}(x) = 2 \sum_{1 \leq i < j < k \leq N} x_i x_j x_k + \sum_{i, j=1; i \neq j}^N x_i^2 x_j.$$

The Weyl characters for $SP(2N)$ group are given by the determinant

$$\tilde{s}_{(\lambda_1, \dots, \lambda_n)}(x) = \frac{\det [x_i^{\lambda_j + n - j + 1} - x_i^{-\lambda_j - n + j - 1}]}{\det [x_i^{n - j + 1} - x_i^{-n + j - 1}]}, \quad (\text{A.2})$$

with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$. Characters of the fundamental and antifundamental representations of $SP(2N)$ group have the form

$$\chi_{SP(2N),f}(x) = \chi_{SP(2N),\bar{f}}(x) = \tilde{s}_{(1,0,\dots,0)}(x) = \sum_{i=1}^N (x_i + x_i^{-1}),$$

The character for the adjoint $SP(2N)$ representation

$$\begin{aligned} \chi_{SP(2N),adj}(x) &= \tilde{s}_{(2,0,\dots,0)}(x) \\ &= \sum_{1 \leq i < j \leq N} (x_i x_j + x_i x_j^{-1} + x_i^{-1} x_j + x_i^{-1} x_j^{-1}) + \sum_{i=1}^N (x_i^2 + x_i^{-2}) + N. \end{aligned}$$

For the absolutely anti-symmetric representation of $SP(2N)$ we have the character

$$\chi_{SP(2N),T_A}(x) = \tilde{s}_{(1,1,0,\dots,0)}(x) = \sum_{1 \leq i < j \leq N} (x_i x_j + x_i x_j^{-1} + x_i^{-1} x_j + x_i^{-1} x_j^{-1}) + (N - 1).$$

As to the exceptional G_2 group, its fundamental representation has the character

$$\chi(z_1, z_2, z_3) = 1 + \sum_{i=1}^3 (z_i + z_i^{-1}),$$

where $z_1 z_2 z_3 = 1$. The character for the adjoint representation of G_2 group is

$$\chi(z_1, z_2, z_3) = 2 + \sum_{1 \leq i < j \leq 3} (z_i z_j + z_i^{-1} z_j + z_i z_j^{-1} + z_i^{-1} z_j^{-1}),$$

where, again, $z_1 z_2 z_3 = 1$.

APPENDIX B. INVARIANT MATRIX GROUP MEASURES

Here we would like to present the invariant measures for the integrals over classical Lie groups and over the exceptional group G_2 . The invariant measure for the unitary group $SU(N)$ with any symmetric function $f(z)$, where $z = (z_1, \dots, z_N)$, has the following form

$$\int_{SU(N)} d\mu(z) f(z) = \frac{1}{N!} \int_{\mathbb{T}^{N-1}} \Delta(z) \Delta(z^{-1}) f(z) \prod_{i=1}^{N-1} \frac{dz_i}{2\pi i z_i}, \quad (\text{B.1})$$

where $\Delta(z)$ is the Vandermonde determinant

$$\Delta(z) = \prod_{1 \leq i < j \leq N} (z_i - z_j).$$

The invariant measure for the symplectic group $SP(2N)$ with any symmetric function $f(z)$, $z = (z_1, \dots, z_N)$, has the form

$$\int_{SP(2N)} d\mu(z) f(z) = \frac{(-1)^N}{2^N N!} \int_{\mathbb{T}^N} \prod_{i=1}^N (z_i - z_i^{-1})^2 \Delta(z + z^{-1})^2 f(z) \prod_{i=1}^N \frac{dz_i}{2\pi i z_i}, \quad (\text{B.2})$$

For the invariant measures over the orthogonal group $SO(N)$ and any symmetric function $f(z)$, $z = (z_1, \dots, z_N)$, one has to distinguish the cases of odd and even N :

$$\int_{SO(2N)} d\mu(z) f(z) = \frac{1}{2^{N-1} N!} \int_{\mathbb{T}^N} \Delta(z + z^{-1})^2 f(z) \prod_{i=1}^N \frac{dz_i}{2\pi i z_i}, \quad (\text{B.3})$$

and

$$\int_{SO(2N+1)} d\mu(z) f(z) = \frac{(-1)^N}{2^N N!} \int_{\mathbb{T}^N} \prod_{j=1}^N \left(z_j^{\frac{1}{2}} - z_j^{-\frac{1}{2}} \right)^2 \Delta(z + z^{-1})^2 f(z) \prod_{i=1}^N \frac{dz_i}{2\pi i z_i}, \quad (\text{B.4})$$

For the invariant measure for the exceptional group G_2 and any symmetric function $f(z)$, $z = (z_1, z_2, z_3)$, where $z_1 z_2 z_3 = 1$, we have

$$\int_{G_2} d\mu(z) f(z) = \frac{1}{2^2 3} \int_{\mathbb{T}^2} \Delta(z + z^{-1})^2 f(z) \prod_{i=1}^2 \frac{dz_i}{2\pi i z_i}, \quad (\text{B.5})$$

APPENDIX C. RELEVANT CASIMIR OPERATORS

Commutators of the generators T^a of some classical Lie group are defined with the help of structure constants f^{abc}

$$[T^a, T^b] = i f^{abc} T^c. \quad (\text{C.1})$$

Then it is straightforward to obtain the Casimir operators [90]

$$\begin{aligned} \sum_{a,l} (T_{\mathbf{r}}^a)_l^m (T_{\mathbf{r}}^a)_n^l &= C_2(\mathbf{r}) \delta_n^m, \\ \sum_{n,m} (T_{\mathbf{r}}^a)_n^m (T_{\mathbf{r}}^b)_m^n &= T(\mathbf{r}) \delta^{ab}, \end{aligned} \quad (\text{C.2})$$

where \mathbf{r} is some irreducible representation. The Casimir operators and the dimension of the representation $d(\mathbf{r})$ are connected through the adjoint representation **adj**

$$d(\mathbf{r}) C_2(\mathbf{r}) = d(\mathbf{adj}) T(\mathbf{r}). \quad (\text{C.3})$$

For checking the t'Hooft anomaly matching conditions we need the triple Casimir operator which comes from

$$A^{abc} = Tr[T^a \{T^b, T^c\}]. \quad (\text{C.4})$$

Then it is convenient to define this operator $A(\mathbf{r})$ relative to the fundamental one

$$A^{abc}(\mathbf{r}) = A(\mathbf{r}) \mathbf{A}^{abc}, \quad (\text{C.5})$$

where $\mathbf{A}^{abc} = Tr[T_F^a \{T_F^b, T_F^c\}]$ and T_F^a are the generators in the fundamental representation.

In the tables below we give the dimensions $d(\mathbf{r})$, the Casimir operators $2T(\mathbf{r})$ and $A(\mathbf{r})$ for the unitary group⁵ and the dimensions $d(\mathbf{r})$, the Casimir operators $T(\mathbf{r})$ for symplectic and G_2 groups.

$SU(N)$ group

Irrep \mathbf{r}	$d(\mathbf{r})$	$2T(\mathbf{r})$	$A(\mathbf{r})$
f	N	1	1
adj	$N^2 - 1$	$2N$	0
T_A	$\frac{1}{2}N(N-1)$	$N-2$	$N-4$
T_S	$\frac{1}{2}N(N+1)$	$N+2$	$N+4$
T_{3A}	$\frac{1}{6}N(N-1)(N-2)$	$\frac{1}{2}(N-3)(N-2)$	$\frac{1}{2}(N-6)(N-3)$
T_{3S}	$\frac{1}{6}N(N+1)(N+2)$	$\frac{1}{2}(N+3)(N+2)$	$\frac{1}{2}(N+6)(N+3)$
T_{AS}	$\frac{1}{3}N(N-1)(N+1)$	$N^2 - 3$	$N^2 - 9$

⁵Note that in verification of the anomaly matchings for unitary groups we use $2T(\mathbf{r})$

$SP(2N)$ group		
Irrep \mathbf{r}	$d(\mathbf{r})$	$T(\mathbf{r})$
f	$2N$	1
$\mathbf{adj} = T_S$	$N(2N + 1)$	$2N + 2$
T_A	$N(2N - 1) - 1$	$2N - 2$

G_2 group		
Irrep \mathbf{r}	$d(\mathbf{r})$	$T(\mathbf{r})$
f	7	2
\mathbf{adj}	14	8

APPENDIX D. TOTAL ELLIPTICITY FOR THE KS DUALITY INTEGRALS

In order to illustrate the work hidden behind our first conjecture we briefly describe in this Appendix verification of the total ellipticity for the transformation identity for elliptic hypergeometric integrals associated with the Kutasov-Schwimmer duality from section 10.1.

First, we change the variables \underline{z} in (10.3) to $\underline{z} = U^{-1}(ST)^{-\frac{K}{2\tilde{N}}}\underline{y}$. Then the equality of integrals (10.2) and (10.3) is rewritten in the following form

$$\kappa_N \int_{\mathbb{T}^{N-1}} \Delta_E(\underline{z}, \underline{t}, \underline{s}) \prod_{j=1}^{N-1} \frac{dz_j}{2\pi i z_j} = \kappa_{\tilde{N}} \int_{\mathbb{T}^{\tilde{N}-1}} \Delta_M(\underline{y}, \underline{t}, \underline{s}) \prod_{j=1}^{\tilde{N}-1} \frac{dy_j}{2\pi i y_j}, \quad (\text{D.1})$$

where $\tilde{N} = KN_f - N$ and

$$\kappa_N = \frac{(p; p)_\infty^{N-1} (q; q)_\infty^{N-1}}{N!} \Gamma(U; p, q)^{N-1}.$$

The kernels for the elliptic hypergeometric integrals are

$$\begin{aligned} \Delta_E(\underline{z}, \underline{t}, \underline{s}) &= \prod_{1 \leq i < j \leq N} \frac{\Gamma(U z_i z_j^{-1}, U z_i^{-1} z_j; p, q)}{\Gamma(z_i z_j^{-1}, z_i^{-1} z_j; p, q)} \prod_{i=1}^{N_f} \prod_{j=1}^N \Gamma(s_i z_j, t_i^{-1} z_j^{-1}; p, q), \quad (\text{D.2}) \\ \Delta_M(\underline{y}, \underline{t}, \underline{s}) &= \prod_{l=1}^K \prod_{i,j=1}^{N_f} \Gamma(U^{l-1} s_i t_j^{-1}; p, q) \prod_{1 \leq i < j \leq \tilde{N}} \frac{\Gamma(U y_i y_j^{-1}, U y_i^{-1} y_j; p, q)}{\Gamma(y_i y_j^{-1}, y_i^{-1} y_j; p, q)} \\ &\quad \times \prod_{i=1}^{N_f} \prod_{j=1}^{\tilde{N}} \Gamma(s_i^{-1} y_j, U^2 t_i y_j^{-1}; p, q). \end{aligned}$$

The balancing condition reads now as

$$U^{2N} ST^{-1} = (pq)^{N_f},$$

together with other constraints $\prod_{i=1}^N z_i = 1$ and $\prod_{i=1}^{\tilde{N}} y_i = U^{\tilde{N}+NK} (pq)^{-\frac{N_f K}{2}} S^K$.

Theorem 5. *The function*

$$\rho(\underline{z}, \underline{y}, \underline{t}, \underline{s}) = \frac{\Delta_E(\underline{z}, \underline{t}, \underline{s})}{\Delta_M(\underline{y}, \underline{t}, \underline{s})}$$

is the totally elliptic hypergeometric kernel.

Ellipticity of the z -variables certificates. As described in the first section, we should consider the ratios

$$\begin{aligned} h_a^z(\underline{z}, \underline{y}, \underline{t}, \underline{s}, q) &= \frac{\rho(\underline{z}, \underline{y}, \underline{t}, \underline{s})|_{z_a \rightarrow qz_a, z_N \rightarrow q^{-1}z_N}}{\rho(\underline{z}, \underline{y}, \underline{t}, \underline{s})} \quad (\text{D.3}) \\ &= \prod_{j=1, j \neq a}^{N-1} \frac{\theta(Uz_a z_j^{-1}, Uz_j z_N^{-1}, q^{-1}z_a^{-1}z_j, q^{-1}z_j^{-1}z_N; p)}{\theta(Uq^{-1}z_a^{-1}z_j, Uq^{-1}z_j^{-1}z_N, z_a z_j^{-1}, z_j z_N^{-1}; p)} \\ &\quad \times \frac{\theta(Uqz_a z_N^{-1}, Uz_a z_N^{-1}, q^{-2}z_a^{-1}z_N, q^{-1}z_a^{-1}z_N; p)}{\theta(Uq^{-1}z_a^{-1}z_N, Uq^{-2}z_a^{-1}z_N, qz_a z_N^{-1}, z_a z_N^{-1}; p)} \prod_{i=1}^{N_f} \frac{\theta(s_i z_a; p)}{\theta(q^{-1}s_i z_N; p)} \frac{\theta(t_i^{-1} z_N^{-1}; p)}{\theta(q^{-1}t_i^{-1} z_a^{-1}; p)} \end{aligned}$$

and check that these are totally elliptic functions. Indeed, $h_a^z(\underline{z}, \underline{y}, \underline{t}, \underline{s}, q)$ functions are automatically invariant under the transformations 1) $s_b \rightarrow ps_b, s_{N_f} \rightarrow p^{-1}s_{N_f}$, 2) $t_b \rightarrow pt_b, t_{N_f} \rightarrow p^{-1}t_{N_f}$, 3) $y_b \rightarrow py_b, y_{\tilde{N}} \rightarrow p^{-1}y_{\tilde{N}}$. Whereas the invariance with respect to the substitutions 4) $z_c \rightarrow pz_c, z_N \rightarrow p^{-1}z_N$ for $c \neq a$ and 5) $z_a \rightarrow pz_a, z_N \rightarrow p^{-1}z_N$ uses the balancing condition. Similarly, one checks the invariance with respect to the mixed transformations $s_b \rightarrow ps_b, t_c^{-1} \rightarrow p^{-1}t_c^{-1}$ and $y_d \rightarrow p^K y_d$.

The most complicated part of the work consists in establishing ellipticity in the variable q . The nontrivial fact is that we have fractional powers of q entering (D.3) through the variable U . Therefore one should scale q by such a power of p that there will be a match with the periods of elliptic function $h_a^z(\underline{z}, \underline{y}, \underline{t}, \underline{s}, q)$. Simultaneously, we should preserve the balancing condition and all other constraints on the parameters we have. This is reached by the following transformation of the parameters

$$6) \quad q \rightarrow p^{K+1}q, \quad t_{N_f}^{-1} \rightarrow p^{(K+1)N_f - 2N}t_{N_f}^{-1}, \quad y_{\tilde{N}} \rightarrow p^{\tilde{N} + NK - N_f K(K+1)/2}y_{\tilde{N}}, \quad (\text{D.4})$$

which guarantees that $U \rightarrow pU$, as required. It is a matter of a neat computation (at the intermediate steps there appears a very cumbersome expression) to show that $h_a^z(\underline{z}, \underline{y}, \underline{t}, \underline{s}, q)$ do not change under these substitutions.

Ellipticity of the y -variables certificates. Now we consider the ratios

$$\begin{aligned} h_a^y(\underline{z}, \underline{y}, \underline{t}, \underline{s}, q) &= \frac{\rho(\underline{z}, \underline{y}, \underline{t}, \underline{s})|_{y_a \rightarrow qy_a, y_{\tilde{N}} \rightarrow q^{-1}y_{\tilde{N}}}}{\rho(\underline{z}, \underline{y}, \underline{t}, \underline{s})} \quad (\text{D.5}) \\ &= \prod_{j=1, j \neq a}^{\tilde{N}-1} \frac{\theta(Uq^{-1}y_a^{-1}y_j, Uq^{-1}y_j^{-1}y_{\tilde{N}}, y_a y_j^{-1}, y_j y_{\tilde{N}}^{-1}; p)}{\theta(Uy_a y_j^{-1}, Uy_j y_{\tilde{N}}^{-1}, q^{-1}y_a^{-1}y_j, q^{-1}y_j^{-1}y_{\tilde{N}}; p)} \\ &\quad \times \frac{\theta(Uq^{-1}y_a^{-1}y_{\tilde{N}}, Uq^{-2}y_a^{-1}y_{\tilde{N}}, qy_a y_{\tilde{N}}^{-1}, y_a y_{\tilde{N}}^{-1}; p)}{\theta(Uqy_a y_{\tilde{N}}^{-1}, Uy_a y_{\tilde{N}}^{-1}, q^{-2}y_a^{-1}y_{\tilde{N}}, q^{-1}y_a^{-1}y_{\tilde{N}}; p)} \prod_{i=1}^{N_f} \frac{\theta(q^{-1}s_i^{-1}y_{\tilde{N}}; p)}{\theta(s_i^{-1}y_a; p)} \frac{\theta(U^2 q^{-1}t_i y_a^{-1}; p)}{\theta(U^2 t_i y_{\tilde{N}}^{-1}; p)}. \end{aligned}$$

Again these are totally elliptic functions. They are automatically invariant under the transformations 1) $s_b \rightarrow ps_b, s_{N_f} \rightarrow p^{-1}s_{N_f}$, 2) $t_b \rightarrow pt_b, t_{N_f} \rightarrow p^{-1}t_{N_f}$, 3) $z_b \rightarrow pz_b, z_N \rightarrow p^{-1}z_N$. The invariance with respect to the substitutions 4) $y_b \rightarrow py_b, y_{\tilde{N}} \rightarrow p^{-1}y_{\tilde{N}}$, $b \neq a$, and 5) $y_a \rightarrow py_a, y_{\tilde{N}} \rightarrow p^{-1}y_{\tilde{N}}$ uses the balancing condition. Again the most difficult part is the verification of the invariance with respect to the transformations

$$6) \quad q \rightarrow p^{K+1}q, \quad U \rightarrow pU, \quad t_{N_f}^{-1} \rightarrow p^{(K+1)N_f - 2N}t_{N_f}^{-1}, \quad y_{\tilde{N}} \rightarrow p^{\tilde{N} + NK - N_f K(K+1)/2}y_{\tilde{N}}.$$

Ellipticity of the t -parameters certificates. Now we need to investigate the functions

$$\begin{aligned}
 h_a^t(\underline{z}, \underline{y}, \underline{t}, \underline{s}, q) &= \frac{\rho(\underline{z}, \underline{y}, \underline{t}, \underline{s})|_{t_a \rightarrow qt_a, t_{N_f} \rightarrow q^{-1}t_{N_f}}}{\rho(\underline{z}, \underline{y}, \underline{t}, \underline{s})} \\
 &= \prod_{l=1}^K \prod_{i=1}^{N_f} \frac{\theta(U^{l-1}q^{-1}s_i t_a^{-1}; p)}{\theta(U^{l-1}s_i t_{N_f}^{-1}; p)} \prod_{j=1}^N \frac{\theta(t_{N_f}^{-1}z_j^{-1}; p)}{\theta(q^{-1}t_a^{-1}z_j^{-1}; p)} \prod_{i=1}^{\tilde{N}} \frac{\theta(U^2q^{-1}t_{N_f}y_j^{-1}; p)}{\theta(U^2t_a y_j^{-1}; p)}
 \end{aligned} \tag{D.6}$$

and show that they are totally elliptic. Again, invariance under 1) $y_b \rightarrow py_b, y_{\tilde{N}} \rightarrow p^{-1}y_{\tilde{N}}$, 2) $s_b \rightarrow ps_b, s_{N_f} \rightarrow p^{-1}s_{N_f}$, and 3) $z_b \rightarrow pz_b, z_N \rightarrow p^{-1}z_N$ transformations is automatic. The balancing condition is needed for symmetries 4) $t_c \rightarrow pt_c, t_{N_f} \rightarrow p^{-1}t_{N_f}$, $c \neq a$, and 5) $t_a \rightarrow pt_a, t_{N_f} \rightarrow p^{-1}t_{N_f}$. The computations for the transformations

$$6) \quad q \rightarrow p^{K+1}q, \quad U \rightarrow pU, \quad t_{N_f-1}^{-1} \rightarrow p^{(K+1)N_f-2N}t_{N_f-1}^{-1}, \quad y_{\tilde{N}} \rightarrow p^{\tilde{N}+NK-N_fK(K+1)/2}y_{\tilde{N}}$$

are very lengthy and require a lot of attention for reaching the needed statement. Consideration of the s -parameters certificates is equivalent to the t -case because of the symmetries of initial integrals' kernels.

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