

\mathcal{R}^4 counterterm and $E_{7(7)}$ symmetry in maximal supergravity

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Abstract

The coefficient of a potential \mathcal{R}^4 counterterm in $\mathcal{N} = 8$ supergravity has been shown previously to vanish in an explicit three-loop calculation. The \mathcal{R}^4 term respects $\mathcal{N} = 8$ supersymmetry; hence this result poses the question of whether another symmetry could be responsible for the cancellation of the three-loop divergence. In this article we investigate possible restrictions from the coset symmetry $E_{7(7)}/SU(8)$, using a double-soft scalar limit relation derived recently by Arkani-Hamed *et al.* In order to implement the relation, we make use of the fact that the \mathcal{R}^4 term occurs in the low-energy expansion of closed-string tree-level amplitudes. We find that the matrix elements of \mathcal{R}^4 that we investigated all obey the double-soft scalar limit relation, suggesting that $E_{7(7)}$ is also respected by the \mathcal{R}^4 term.

Contents

1	Introduction	2
2	Coset structure, hidden symmetry and double-soft limit	5
3	String theory corrections to field theory amplitudes	8
3.1	Tree-level amplitudes in $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ Supergravity	8
3.2	Amplitudes in open and closed string theory	10
4	Setting up the calculation	13
4.1	KLT relations	13
4.2	Choosing a suitable amplitude	15
4.3	Supersymmetric Ward identities	17
4.4	$\mathcal{N} = 1$ supersymmetric Ward identities	18
4.5	The second $\mathcal{N} = 1$ SUSY diamond	20
5	$E_{7(7)}$ symmetry for α'-corrected amplitudes?	22
6	Conclusion	25

1 Introduction

Divergences of four-dimensional gravity theories have been under investigation practically since the advent of quantum field theory. While pure gravity can be shown to be free of ultraviolet divergences at one loop, the addition of scalars or other particles renders the theory nonrenormalizable [1]. At the two-loop level, the counterterm

$$R^3 \equiv R_{\mu\nu}^{\lambda\rho} R_{\lambda\rho}^{\sigma\tau} R_{\sigma\tau}^{\mu\nu} \quad (1.1)$$

has been shown to respect all symmetries, to exist on-shell [2, 3] and to have a nonzero coefficient for pure gravity [4, 5].

Supersymmetry is known to improve the ultraviolet behavior of many quantum field theories. In fact, supersymmetry forbids the R^3 counterterm in any supersymmetric version of four-dimensional gravity, provided that all particles are in the same multiplet as the graviton. That is because the operator R^3 generates a scattering amplitude [6, 7] that can be shown to vanish by supersymmetric Ward identities (SWI) [8, 9, 10, 11]. However, the next possible counterterm [12, 13, 14] is

$$R^4 \equiv t_8^{\mu_1\nu_1\dots\mu_4\nu_4} t_8^{\rho_1\sigma_1\dots\rho_4\sigma_4} R_{\mu_1\nu_1\rho_1\sigma_1} R_{\mu_2\nu_2\rho_2\sigma_2} R_{\mu_3\nu_3\rho_3\sigma_3} R_{\mu_4\nu_4\rho_4\sigma_4}, \quad (1.2)$$

where t_8 is defined in eq. (4.A.21) of ref. [15]. This operator, also known as the square of the Bel-Robinson tensor [16], on dimensional grounds can appear as a counterterm at three loops. It is compatible with supersymmetry, not just $\mathcal{N} = 1$ but all the way up to maximal $\mathcal{N} = 8$ supersymmetry. This property follows from the appearance of R^4 in the low-energy effective action of the $\mathcal{N} = 8$ supersymmetric closed superstring [17]; indeed, it represents the first correction term beyond the limit of $\mathcal{N} = 8$ supergravity [18], appearing at order α'^3 . We denote by \mathcal{R}^4 the $\mathcal{N} = 8$ supersymmetric multiplet of operators containing R^4 .

The issue of possible counterterms in maximal $\mathcal{N} = 8$ supergravity [19, 20] is under perpetual investigation. Many of the current arguments rely on (linearized) superspace formulations and nonrenormalization theorems [21, 22], which in turn depend on the existence of an off-shell superspace formulation. It was a common belief for some time that a superspace formulation of maximally-extended supersymmetric theories could be achieved employing off-shell formulations with at most half of the supersymmetry realized. On the other hand, an off-shell harmonic superspace with $\mathcal{N} = 3$ supersymmetry for $\mathcal{N} = 4$ super-Yang-Mills (SYM) theory was constructed a while ago [23]. Assuming the existence of a similar description realizing six of the eight supersymmetries of $\mathcal{N} = 8$ supergravity would postpone the onset of possible counterterms at least to the five-loop level, while realizing seven of eight would postpone it to the six-loop level [22]. However, an explicit construction of such superspace formalisms has not yet been achieved in the gravitational case.

Another way to explore the divergence structure of $\mathcal{N} = 8$ supergravity is through direct computation of on-shell multi-loop graviton scattering amplitudes. The two-loop four-graviton scattering amplitude [24] provided the first hints that the \mathcal{R}^4 counterterm might have a vanishing coefficient at three loops. The full three-loop computation then demonstrated this vanishing explicitly [25, 26]. A similar cancellation has been confirmed at four loops recently [27]. The latter cancellation in four dimensions is not so surprising for the four-point amplitude because operators of the form $\partial^2 R^4$ can be eliminated in favor of R^5 using equations of motion [28], and it has been shown that there is no $\mathcal{N} = 8$ supersymmetric completion of R^5 [29, 30]. (This is consistent with the absence of R^5 terms from the closed-superstring effective action [31].) On the other hand, the explicit multi-loop amplitudes show an even-better-than-finite ultraviolet behavior, as good as that for $\mathcal{N} = 4$ super-Yang-Mills theory, which strengthens the evidence for an yet unexplored underlying symmetry structure.

There are also string- and M-theoretic arguments for the excellent ultraviolet behavior observed to date. Using a nonrenormalization theorem developed in the pure spinor formalism for the closed superstring [32], Green, Russo and Vanhove argued [33] that the first divergence in $\mathcal{N} = 8$ supergravity might be delayed until nine loops. Arguments based on M-theory dualities suggest the possibility of finiteness to all loop orders [34]. However, the applicability of arguments based on string and M theory to $\mathcal{N} = 8$ supergravity is subject to issues related to the decoupling of massive states [35].

There have also been a variety of attempts to understand the ultraviolet behavior of $\mathcal{N} = 8$ supergravity more directly at the amplitude level. The “no triangle” hypothesis [36, 37], now a theorem [38, 39], states in essence that the ultraviolet behavior of $\mathcal{N} = 8$ supergravity at one loop is as good as that of $\mathcal{N} = 4$ super-Yang-Mills theory. It also implies many, though not all, of the cancellations seen at higher loops [40]. Some of the one-loop cancellations are not just due to supersymmetry, but to other properties of gravitational theories [41], including their non-color-ordered nature [42].

These one-loop considerations, and the work of ref. [22], suggest that conventional $\mathcal{N} = 8$ supersymmetry alone may not be enough to dictate the finiteness of $\mathcal{N} = 8$ supergravity. However, since the construction of $\mathcal{N} = 8$ supergravity [43, 19, 20] it has been realized that another symmetry plays a key role — the exceptional, noncompact symmetry $E_{7(7)}$. Could this symmetry contribute somehow to an explanation of the (conjectured) finiteness of the theory?

The general role of the $E_{7(7)}$ symmetry, regarding the finiteness of maximal supergravity, has been a topic of constant discussion. (Aspects of its action on the Lagrangian in light-cone gauge [44], and covariantly [45], have also been considered recently.) While a manifestly $E_{7(7)}$ -invariant counterterm was presented long ago at eight loops [46], newer results using the light-cone formalism cast a different light on the question [47].

In this article we investigate whether restrictions on the appearance of the \mathcal{R}^4 term could

originate directly from the exceptional symmetry. One way to test whether \mathcal{R}^4 is invariant under $E_{7(7)}$ is to utilize properties of the on-shell amplitudes that \mathcal{R}^4 produces. This method is convenient because it turns out that the amplitudes can be computed, using string theory, even when a full nonlinear expression for \mathcal{R}^4 in four dimensions is unavailable. (The nonlinear expression for \mathcal{R}^4 in ten-dimensional type IIB supergravity is known [48], but not to our knowledge its reduction to four dimensions.)

Arkani-Hamed, Cachazo and Kaplan (ACK) [39] provided a very useful tool for an amplitude-based approach. Working in pure $\mathcal{N} = 8$ supergravity, they considered the emission of two additional soft scalar particles from a hard scattering amplitude, and thereby derived a relation between amplitudes differing by two in the number of legs. The relation should hold for any theory with $E_{7(7)}$ symmetry. If one could show agreement of the ACK relation for all amplitudes derived from a modified $\mathcal{N} = 8$ supergravity action, in this case perturbing it by the \mathcal{R}^4 term, then this action should obtain no restrictions from $E_{7(7)}$.

Actually, for this conclusion to hold, $E_{7(7)}$ should remain a good symmetry at the quantum level. Although there is evidence in favor of this, we know of no all-orders proof. At one loop, the cancellation of anomalies for currents from the $SU(8)$ subgroup of $E_{7(7)}$ was demonstrated quite a while ago [49]. The analysis was subtle because a Lagrangian for the vector particles cannot be written in a manifestly $SU(8)$ -covariant fashion. Thus the vectors contribute to anomalies, cancelling the more-standard contributions from the fermions. More recently, the question of whether the full $E_{7(7)}$ is a good quantum symmetry has been re-examined using the methods of ACK. He and Zhu recently showed that the infrared-finite part of single-soft scalar emission vanishes at one loop [50] as it does at tree level. A similar argument by Kaplan [51] shows that the double-soft scalar limit relation in $\mathcal{N} = 8$ supergravity can also be extended to one loop. These results support the conjecture that the full $E_{7(7)}$ is a good quantum symmetry of the theory, at least at the one-loop level.

The purpose of this article is to test the $E_{7(7)}$ invariance of eq. (1.2), by exploring the validity of the ACK relation for the four-dimensional $\mathcal{N} = 8$ supergravity action modified by adding the supersymmetric extension of the R^4 term. Amplitudes corresponding to the modified action will be derived from closed superstring theory, whose four-dimensional field-theory limit receives string corrections originating in a supersymmetric extension of eq. (1.2). As we will see, we need to go to six-point NMHV amplitudes to get the first nontrivial result. The strategy for obtaining information about higher-order α' -terms in closed-string scattering is the same as used in a recent article by Stieberger [31]: We will fall back to open-string calculations [52] and derive the corresponding closed-string results by employing the Kawai-Lewellen-Tye (KLT) [53] relations.

The remainder of this article is organized as follows. Sections 2 and 3 collect the background information on symmetries of $\mathcal{N} = 8$ supergravity, including the double-soft scalar limit of amplitudes, and they illuminate the state and availability of open-string amplitude calculations. In

section 4 the calculation is set up. We start by introducing the KLT relations connecting open- and closed-string amplitudes in subsection 4.1. A suitable amplitude for probing the double-soft scalar limit relation is singled out in subsection 4.2. The $\mathcal{N} = 1$ supersymmetric Ward identities needed to make use of the available open-string amplitudes are described in detail in subsections 4.3, 4.4 and 4.5. The main result of this article, the testing of possible restrictions originating from $E_{7(7)}$ symmetry, by employing the double-soft scalar limit relation on amplitudes produced by the \mathcal{R}^4 term, is presented in section 5. In section 6 we draw our conclusions.

2 Coset structure, hidden symmetry and double-soft limit

The physical field content of the maximal supersymmetric gravitational theory in four dimensions, $\mathcal{N} = 8$ supergravity [19, 20], consists of a vierbein (or graviton), 8 gravitini, 28 abelian gauge fields, 56 Majorana gauginos of either helicity, and 70 real (or 35 complex) scalars, which can be collected together in a single massless $\mathcal{N} = 8$ (on-shell) supermultiplet.

Starting from the fact that the vector bosons form an antisymmetric tensor representation of $SO(8)$ in the ungauged theory, Bianchi identities and equations of motion can be considered in order to realize a much larger symmetry, which leads to the notion of generalized electric-magnetic duality transformations. Investigating these transformations more closely and enlarging the corresponding duality group maximally by adding further scalars, not all of which turn out to be physical. After gauging a resulting local $SU(8)$ symmetry in order to reduce the degrees of freedom of the generalized duality group, 70 physical scalars remain. These scalars parameterize the coset $\frac{E_{7(7)}}{SU(8)}$ [20, 54], where $E_{7(7)}$ denotes a noncompact real form of E_7 , which has $SU(8)$ as its maximal compact subgroup. In other words, the scalars can be identified with the noncompact generators of $E_{7(7)}$. The resulting gauge is called *unitary*.

More explicitly, in unitary gauge the 63 compact generators T_I^J of $SU(8)$ can be joined with 70 generators $X_{I_1 \dots I_4}$ to form the adjoint representation of $E_{7(7)}$. Here $X_{I_1 \dots I_4}$ transforms under $SU(8)$ in the four-index antisymmetric tensor representation ($I, J = 1, \dots, 8$). The commutation relations between those generators are given schematically by

$$[T, T] \sim T, \quad [X, T] \sim X, \quad \text{and} \quad [X, X] \sim T. \quad (2.1)$$

The first commutator is just the usual $SU(8)$ Lie algebra, and the second one follows straightforwardly from the identification of X with the **70** of $SU(8)$. The more nontrivial statement about $E_{7(7)}$ invariance resides in the third commutator in eq. (2.1). Assuming the two scalars to be represented as $X_1^{I_1 \dots I_4}$ and $X_{2 I_5 \dots I_8}$, where the upper-index version can be obtained by employing the $SU(8)$ -invariant tensor,

$$X^{I_1 \dots I_4} = \frac{1}{24} \varepsilon^{I_1 I_2 I_3 I_4 I_5 I_6 I_7 I_8} X_{I_5 \dots I_8}, \quad (2.2)$$

the third relation reads explicitly (see *e.g.* ref. [39]),

$$-i [X_1^{I_1 \dots I_4}, X_{2 I_5 \dots I_8}] = \varepsilon_{I_5 I_6 I_7 I_8}^{J I_2 I_3 I_4} T_J^{I_1} + \varepsilon_{I_5 I_6 I_7 I_8}^{I_1 J I_3 I_4} T_J^{I_1} + \dots + \varepsilon_{I_5 I_6 I_7 J}^{I_1 I_2 I_3 I_4} T_{I_8}^J. \quad (2.3)$$

Here $\varepsilon_{I_5 I_6 I_7 I_8}^{I_1 I_2 I_3 I_4} = 1, -1, 0$ if the upper index set is an even, odd or no permutation of the lower set, respectively. (For a more general discussion of the properties of $E_{7(7)}$, see appendix B of ref. [20].)

Amplitudes in $\mathcal{N} = 8$ supergravity are invariant under $SU(8)$ rotations by construction. On the other hand, the action of the coset symmetry $\frac{E_{7(7)}}{SU(8)}$ on amplitudes is not obvious. One can understand the connection by recalling that the vacuum state of the theory is specified by the expectation values of the physical scalars. Because the scalars are Goldstone bosons, the soft emission of scalars in an amplitude changes the expectation value and moves the theory to another point in the vacuum manifold.

Arkani-Hamed, Cachazo and Kaplan [39] used the BCFW recursion relations [55, 56] to investigate how the noncompact part of $E_{7(7)}$ symmetry controls the soft emission of scalars in $\mathcal{N} = 8$ supergravity. Consider first the emission of a single soft scalar (which was also studied in refs. [57, 58]). The corresponding amplitudes can be traced back via the BCFW recursion relations to the three-particle amplitude, whose vanishing in the soft limit can be shown explicitly. Hence the emission of a single scalar from any amplitude vanishes in $\mathcal{N} = 8$ supergravity.

Moving on to double-soft emission, several different situations have to be distinguished, which are labelled by the number of common indices between the sets $\{I_1, I_2, I_3, I_4\}$ and $\{I_5, I_6, I_7, I_8\}$ in eq. (2.3). Four common indices allow the creation of an $SU(8)$ singlet, corresponding to the emission of a single soft graviton. This case is not interesting because $[X, X]$ vanishes. Similarly, if the scalars share one or two indices, the situation corresponds to a single soft limit in one of the subamplitudes generated by the BCFW recursion relations; thus this limit vanishes, and does not probe the commutator in eq. (2.3). Another way to see the vanishing is to reconsider eq. (2.3) explicitly: there are simply not enough indices to saturate the right-hand side. The only interesting configuration occurs if the two scalars X_1 and X_2 agree on exactly three of their indices. This result is in accordance with the commutation relation eq. (2.3), where three equal indices are necessary for the commutator of two noncompact generators to yield a result proportional to an $SU(8)$ generator.

Performing an explicit calculation of an $(n + 2)$ -point supergravity tree amplitude M_{n+2} containing two scalars sharing three indices and considering the double-soft limit on X_1 and X_2 results in the double-soft limit [39]

$$M_{n+2}(1, 2, \dots) \xrightarrow{p_1, p_2 \rightarrow 0} \frac{1}{2} \sum_{i=3}^{n+2} \frac{p_i \cdot (p_2 - p_1)}{p_i \cdot (p_1 + p_2)} T(\eta_i) M_n(3, 4, \dots), \quad (2.4)$$

where

$$T(\eta_i)_K^J = T([X^{I_1 \dots I_4}, X_{I_5 \dots I_8}])_K^J = \varepsilon_{I_5 I_6 I_7 I_8}^{I_1 I_2 I_3 I_4 K} \times \eta_{iK} \partial_{\eta_{iJ}} \quad (2.5)$$

acts on $(M_n)_J^K$; the n -point amplitude M_n has open $SU(8)$ indices due to the particular choice of indices of the scalars. Again, $\varepsilon_{I_5 I_6 I_7 I_8}^{I_1 I_2 I_3 I_4 K} = 1, -1, 0$ if the upper index set is an even, odd or no permutation of the lower set.

The Grassmann variables η_{iA} in the argument of eq. (2.5) refer to the description of an amplitude in the so-called on-shell superspace formalism [59]. They are a set of $8n$ anticommuting objects, where the index $i = 1, \dots, n$ numbers the particles and A is an $SU(8)$ index. Using these variables, one can write down a generating functional for MHV amplitudes in supergravity [57],

$$\Omega_n = \frac{1}{256} \frac{M_n(B_1^-, B_2^-, B_3^+, B_4^+, \dots, B_n^+)}{\langle 12 \rangle^8} \prod_{A=1}^8 \sum_{i,j=1}^n \langle ij \rangle \eta_{iA} \eta_{jA}, \quad (2.6)$$

where B^\pm are positive and negative helicity gravitons. Particle states of the $\mathcal{N} = 8$ multiplet can be identified with derivatives with respect to the anticommuting variables

$$\begin{aligned} 1 \leftrightarrow B_i^+ & \quad \frac{\partial}{\partial \eta_{iA}} \leftrightarrow F_i^{A+} \quad \dots \quad \frac{\partial^4}{\partial \eta_{iA} \partial \eta_{iB} \partial \eta_{iC} \partial \eta_{iD}} \leftrightarrow X^{ABCD} \quad \dots \\ \dots & \quad - \frac{1}{7!} \varepsilon_{ABCDEFGH} \frac{\partial^7}{\partial \eta_{iB} \partial \eta_{iC} \dots \partial \eta_{iH}} \leftrightarrow F_{iA}^- \quad \dots \quad \frac{1}{8!} \varepsilon_{ABCDEFGH} \frac{\partial^8}{\partial \eta_{iA} \partial \eta_{iB} \dots \partial \eta_{iH}} \leftrightarrow B_i^-, \end{aligned} \quad (2.7)$$

where the number of η 's is connected to the helicity of the state, and F^\pm denote gravitini of either helicity. Acting with these operators on the generating functional (2.6), one obtains the correct expressions for the corresponding component amplitudes, which automatically obey the MHV supersymmetry Ward identities. For example a two-gravitino two-graviton amplitude will read:

$$\begin{aligned} \langle F_5^{5+} F_5^- B^+ B^- \rangle & \equiv M_4(F_1^{5+}, F_{2,5}^-, B_3^+, B_4^-) \\ & = - \left(\frac{\partial}{\partial \eta_{15}} \right) \left(\frac{1}{7!} \varepsilon_{12345678} \frac{\partial^7}{\partial \eta_{21} \dots \partial \eta_{24} \partial \eta_{26} \dots \partial \eta_{28}} \right) \\ & \quad \times \left(\frac{1}{8!} \varepsilon_{12345678} \frac{\partial^8}{\partial \eta_{41} \dots \partial \eta_{48}} \right) \Omega_4. \end{aligned} \quad (2.8)$$

As we will see below, the $SU(8)$ generator (2.5) will act consistently on the remnant of the six-point amplitude represented in the above formalism.

In the double-soft limit (2.4), the amplitude with two soft scalars sharing three indices becomes a sum of amplitudes with only hard momenta; in each summand one leg gets $SU(8)$ rotated by an amount depending on its momentum. This relation has been proven by ACK at tree-level for pure $\mathcal{N} = 8$ supergravity. Here we will construct a suitable α' -corrected amplitude, derived from an action containing the supersymmetrized version of the R^4 term, and then take the double-soft limit numerically in order to test the $E_{7(7)}$ invariance of this term.

In order to do so, we will first have a look at string theory corrections to field theory amplitudes in the next section, before we set up the actual calculation in section 4.

3 String theory corrections to field theory amplitudes

Tree amplitudes for Type I open and Type II closed string theory have been computed and expanded in α' for various collections of external states. The leading terms in the low-energy effective action are $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ supergravity, respectively [18]. Indeed, in the zero Regge slope limit ($\alpha' \rightarrow 0$), the string amplitudes agree with the corresponding field theory results.

Expanding the string theory amplitude further in α' yields corrections to the field-theoretical expressions, which can be summarized by a series of local operators in the effective field theory. Terms which have to be added to the $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ supergravity actions in order to reproduce the α' corrections have been identified for low orders in α' . In particular, the first nonzero string correction to the action of $\mathcal{N} = 8$ supergravity is the supersymmetrized version of the possible R^4 counterterm eq. (1.2) discussed above [17].

The next subsection reviews properties of amplitudes in maximally supersymmetric field theories. Some recent computations of string theory amplitudes and their low-energy expansions are discussed in the following subsection.

3.1 Tree-level amplitudes in $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ Supergravity

A general amplitude in $\mathcal{N} = 4$ SYM can be color-decomposed as

$$\mathcal{A}_n^{\text{SYM}}(1, 2, \dots, n) = g_{YM}^{n-2} \sum_{\sigma \in S_n / \mathbb{Z}_n} \text{Tr}(T^{a_{\sigma(1)}} \dots T^{a_{\sigma(n)}}) A_n^{\text{SYM}}(\sigma(1), \sigma(2), \dots, \sigma(n)), \quad (3.1)$$

where the summation is over all $(n-1)!$ non-cyclic permutations of $i = 1, 2, \dots, n$. The number i is understood as a collective label for the momentum p_i and helicity h_i of particle i , *e.g.* $1 \equiv (p_1, h_1)$, and the T^{a_i} are matrices in the fundamental representation of the Yang-Mills gauge group $SU(N_c)$, normalized to $\text{Tr}(T^a T^b) = \delta^{ab}$.

The gauge-invariant subamplitudes A_n^{SYM} are independent of the color structure and can be shown to exhibit the following properties [60]:

- invariance under cyclic permutations: $A_n^{\text{SYM}}(1, 2, \dots, n) = A_n^{\text{SYM}}(2, 3, \dots, n, 1)$
- reflection identity: $A_n^{\text{SYM}}(1, 2, \dots, n) = (-1)^n A_n^{\text{SYM}}(n, n-1, \dots, 2, 1)$
- photon decoupling (or dual Ward) identity:

$$A_n^{\text{SYM}}(1, 2, 3, \dots, n) + A_n^{\text{SYM}}(2, 1, 3, \dots, n) + A_n^{\text{SYM}}(2, 3, 1, \dots, n) + \dots + A_n^{\text{SYM}}(2, 3, \dots, 1, n) = 0. \quad (3.2)$$

In addition, amplitudes in maximally supersymmetric theories are classified by their helicity structure. Employing supersymmetric Ward identities (see section 4.3), pure-gluon amplitudes with helicity structure $(\pm + \cdots +)$ can be shown to vanish [8, 9]. The simplest nonvanishing configurations $(- - + \cdots +)$ are called maximally helicity violating (MHV) amplitudes. In the case that all external legs are gluons g^\pm , they are given by [61]:

$$A_n^{\text{SYM}}(g_1^-, g_2^-, g_3^+, \dots, g_n^+) = i \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}, \quad (3.3)$$

where $|k_i^\pm\rangle$ are massless Weyl spinors, normalized by $\langle ij \rangle [ji] = 2k_i \cdot k_j$, and $\langle k_i^- | k_j^+ \rangle = \langle ij \rangle$, $\langle k_i^+ | k_j^- \rangle = [ij]$. The simplicity of the MHV sector is also expressed in the relations between different MHV amplitudes: any MHV amplitude is related directly to the pure-gluon one by supersymmetric Ward identities (see section 4.3), so that the knowledge of eq. (3.3) determines the complete set of MHV amplitudes.

While in the four- and five-point case the only nonvanishing configurations are MHV (or anti-MHV), the advent of a sixth leg introduces a new class of helicity structures, the so-called next-to-MHV (NMHV) amplitudes. Here it is necessary to distinguish three different helicity orderings

$$X : (- - - + + +) \quad Y : (- - + - + +) \quad Z : (- + - + - +). \quad (3.4)$$

Expressions for the amplitudes are distinct for the different orderings X , Y and Z . However, there is no procedural difference in deriving the expressions, so we will generally illustrate the amplitudes and supersymmetry relations for the helicity configuration X . Explicit results for all six-point pure-gluon NMHV amplitudes can be found in ref. [60], for example. More compact expressions result from use of the BCFW recursion relations. Using these relations, a prescription for determining all tree-level amplitudes in $\mathcal{N} = 4$ SYM from superconformal invariants has been derived [62].

We note that the supersymmetric Ward identities, reflection symmetry and cyclic invariance — as well as parity, or spinor conjugation — relate amplitudes within a certain MNHV helicity ordering only (X , Y or Z). On the other hand, the photon decoupling identity is an example of a relation among amplitudes featuring different helicity orderings.

Next we turn to amplitudes in $\mathcal{N} = 8$ supergravity. In this case, the color trace, which forces particles in gauge-theory subamplitudes to remain in a certain cyclic order, does not exist. Instead, supergravity amplitudes are symmetric under exchange of particles with the same helicity. We write the full amplitude $\mathcal{M}_n^{\text{SUGRA}}(1, 2, \dots, n)$ as

$$\mathcal{M}_n^{\text{SUGRA}}(1, 2, \dots, n) = \left(\frac{\kappa}{2}\right)^{(n-2)} M_n^{\text{SUGRA}}(1, 2, \dots, n), \quad (3.5)$$

where only the gravitational coupling constant $\kappa = \sqrt{32\pi G_N}$ has been removed from M_n^{SUGRA} . The four- and five-point MHV amplitudes for gravitons B^\pm are given by [63]

$$\begin{aligned} M_4^{\text{SUGRA}}(B_1^-, B_2^-, B_3^+, B_4^+) &= i \langle 12 \rangle^8 \frac{[12]}{\langle 34 \rangle N(4)}, \\ M_5^{\text{SUGRA}}(B_1^-, B_2^-, B_3^+, B_4^+, B_5^+) &= i \langle 12 \rangle^8 \frac{\varepsilon(1, 2, 3, 4)}{N(5)}, \end{aligned} \quad (3.6)$$

where

$$\varepsilon(i, j, m, n) = 4i\varepsilon_{\mu\nu\rho\sigma} k_i^\mu k_j^\nu k_m^\rho k_n^\sigma = [ij]\langle jm \rangle [mn]\langle ni \rangle - \langle ij \rangle [jm]\langle mn \rangle [ni] \quad (3.7)$$

and

$$N(n) \equiv \prod_{i=1}^{n-1} \prod_{j=i+1}^n \langle ij \rangle. \quad (3.8)$$

The higher-point MHV graviton amplitudes were first written down in ref. [63]. Explicit expressions for other helicity configurations are rare. However, in ref. [64] a prescription is given how to calculate any $\mathcal{N} = 8$ supergravity tree-level amplitude by employing ‘‘gravity subamplitudes’’, BCFW recursion relations, and superconformal invariants [65, 62].

In the on-shell superspace formalism introduced above, the determination of the type of amplitude away from those containing gluons (gravitons) exclusively can be done by counting derivatives acting on the appropriate generating functional. While 8 (16) derivatives are necessary for MHV amplitudes in $\mathcal{N} = 4$ SYM ($\mathcal{N} = 8$ supergravity), there are 12 (24) derivatives in the NMHV sector.

3.2 Amplitudes in open and closed string theory

Open-string tree amplitudes \mathcal{A}_n have the same color decomposition (3.1), with A_n^{SYM} replaced by the color-ordered string subamplitude A_n . At the four-point level, the two subamplitudes are related by the Veneziano formula,

$$\begin{aligned} A_4(1^-, 2^-, 3^+, 4^+) &= V^{(4)}(s_1, s_2) A_4^{\text{SYM}}(1^-, 2^-, 3^+, 4^+) \\ &= \frac{\Gamma(1+s_1)\Gamma(1+s_2)}{\Gamma(1+s_1+s_2)} A_4^{\text{SYM}}(1^-, 2^-, 3^+, 4^+). \end{aligned} \quad (3.9)$$

The above expression is given in terms of kinematical invariants defined via

$$\llbracket i \rrbracket_n = \alpha' (k_i + k_{i+1} + \cdots + k_{i+n})^2, \quad s_j = s_{jj+1} = \llbracket j \rrbracket_1, \quad t_j = \llbracket j \rrbracket_2, \quad (3.10)$$

which are $s_1 = \llbracket 1 \rrbracket_1 = s_{12} = 2\alpha' k_1 \cdot k_2$ and $s_2 = \llbracket 2 \rrbracket_1 = s_{23} = 2\alpha' k_2 \cdot k_3$ on-shell. Expanding the form-factor $V^{(4)}$ in powers of α' one finds

$$V^{(4)}(s_1, s_2) = 1 - \zeta(2)s_1s_2 + \zeta(3)s_1s_2(s_1+s_2) + \mathcal{O}(\alpha'^4), \quad (3.11)$$

where the leading correction to the pure Yang-Mills amplitude arises from the interaction term of four gauge field-strength tensors [66, 15, 67].

The full open string amplitude is quite simple in the four-point case (3.9). On the other hand, its generalizations to more external legs turn out to involve generalized hypergeometric functions [68]. Any n -point open string amplitude can be expressed in terms of $(n-3)!$ hypergeometric basis integrals. Expanding those functions in powers of α' yields expressions for the string-corrected five- and six-point MHV amplitudes

$$\begin{aligned} A_5 &= \left[V^{(5)}(s_j) - \frac{i\alpha'^2}{2} \varepsilon(1, 2, 3, 4) P^{(5)}(s_j) \right] A_5^{\text{SYM}} \\ A_6 &= \left[V_6^{\text{open}}(s_j, t_j) - \frac{i\alpha'^2}{2} \sum_{k=1}^5 \varepsilon_k P_k^{(6)}(s_j, t_j) \right] A_6^{\text{SYM}}, \end{aligned} \quad (3.12)$$

where

$$\varepsilon_1 = \varepsilon(2, 3, 4, 5), \quad \varepsilon_2 = \varepsilon(1, 3, 4, 5), \quad \varepsilon_3 = \varepsilon(1, 2, 4, 5), \quad \varepsilon_4 = \varepsilon(1, 2, 3, 5), \quad \varepsilon_5 = \varepsilon(1, 2, 3, 4). \quad (3.13)$$

Expansions in α' are given by [69]

$$\begin{aligned} V^{(5)}(s_i) &= 1 - \frac{\zeta(2)}{2} (s_1 s_2 + s_2 s_3 + s_3 s_4 + s_4 s_5 + s_5 s_1) \\ &\quad + \frac{\zeta(3)}{2} (s_1^2 s_2 + s_2^2 s_3 + s_3^2 s_4 + s_4^2 s_5 + s_5^2 s_1 + s_1 s_2^2 + s_2 s_3^2 + s_3 s_4^2 + s_4 s_5^2 + s_5 s_1^2 \\ &\quad \quad \quad + s_1 s_3 s_5 + s_2 s_4 s_1 + s_3 s_5 s_2 + s_4 s_1 s_3 + s_5 s_2 s_4) + \mathcal{O}(\alpha'^4), \end{aligned} \quad (3.14)$$

$$P^{(5)}(s_i) = \zeta(2) - \zeta(3)(s_1 + s_2 + s_3 + s_4 + s_5) + \mathcal{O}(\alpha'^2), \quad (3.15)$$

and explicit expressions for $V^{(6)}$ and $P_k^{(6)}$ can be found in the same reference.

Stieberger and Taylor have pushed the calculations even further [52]. In the process of determining all pure-gluon NMHV six-point amplitudes, they computed the following additional auxiliary amplitudes for the helicity configuration X defined in eq. (3.4):

$$\langle \phi^- \phi^- \phi^- \phi^+ \phi^+ \phi^+ \rangle, \quad \langle \phi^- \phi^- \lambda^- \lambda^+ \phi^+ \phi^+ \rangle, \quad \text{and} \quad \langle \phi^- \phi^- g^- g^+ \phi^+ \phi^+ \rangle, \quad (3.16)$$

as well the analogous quantities for Y and Z . Here λ denotes a gluino and ϕ a scalar. In order to get an impression of the complexity of the result, we provide the pure-gluon NMHV six-point amplitude in helicity configuration X [52], which will be expressed employing the following kinematic variables:

$$\alpha_X = -[12]\langle 34 \rangle [6|X|5], \quad \beta_X = [12]\langle 45 \rangle [6|X|3], \quad \gamma_X = [61]\langle 34 \rangle [2|X|5], \quad (3.17)$$

where $X \equiv p_6 + p_1 + p_2$. The subamplitude reads¹

$$A_6(g_1^+, g_2^+, g_3^-, g_4^-, g_5^-, g_6^+) = \frac{\alpha'^5}{s_5} \left(N_1^X \frac{\alpha_X^2}{s_1^2 s_3^2} + N_2^X \frac{\beta_X^2}{s_1^2} + N_3^X \frac{\gamma_X^2}{s_3^2} + N_4^X \frac{\alpha_X \beta_X}{s_1^2 s_3} + N_5^X \frac{\alpha_X \gamma_X}{s_1 s_3^2} + N_6^X \frac{\beta_X \gamma_X}{s_1 s_3} \right), \quad (3.18)$$

where the expansion of the functions N^X to $\mathcal{O}(\alpha'^2)$ is:

$$\begin{aligned} N_1^X &= -\zeta(2) s_1 s_3 + \dots, \\ N_2^X &= \frac{s_1}{s_2 s_4 t_1} - \zeta(2) \left(\frac{s_1 s_6}{s_2 s_4} + \frac{s_1^2}{s_4 t_1} + \frac{s_1 s_5}{s_2 t_1} \right) + \dots, \\ N_3^X &= \frac{s_3}{s_2 s_6 t_2} - \zeta(2) \left(\frac{s_3 s_4}{s_2 s_6} + \frac{s_3 s_5}{s_2 t_2} + \frac{s_3^2}{s_6 t_2} \right) + \dots, \\ N_4^X &= \zeta(2) \left(\frac{s_1 t_2}{s_2} + \frac{s_1 t_3}{s_4} \right) + \dots, \\ N_5^X &= \zeta(2) \left(\frac{s_3 t_1}{s_2} + \frac{s_3 t_3}{s_6} \right) + \dots, \\ N_6^X &= \frac{t_3}{s_2 s_4 s_6} + \zeta(2) \left(\frac{s_1 + s_3 - s_5}{s_2} - \frac{t_1 t_3}{s_2 s_4} - \frac{t_2 t_3}{s_2 s_6} - \frac{t_3^2}{s_4 s_6} \right) + \dots. \end{aligned} \quad (3.19)$$

The low-energy limit of closed Type II string theory in four dimensions is $\mathcal{N} = 8$ supergravity. The first correction to the low-energy effective action can be determined from the expression for the closed string four-point amplitude, or Virasoro-Shapiro amplitude,

$$\begin{aligned} M_4(1^-, 2^-, 3^+, 4^+) &= V_{\text{closed}}^{(4)}(s_1, s_2) M_4^{\text{SUGRA}}(1^-, 2^-, 3^+, 4^+) \\ &= \frac{\Gamma(1+s_1)\Gamma(1+s_2)\Gamma(1-s_1-s_2)}{\Gamma(1-s_1)\Gamma(1-s_2)\Gamma(1+s_1+s_2)} M_4^{\text{SUGRA}}(1^-, 2^-, 3^+, 4^+). \end{aligned} \quad (3.20)$$

The expansion of $V_{\text{closed}}^{(4)}$ has the first nonvanishing correction at $\mathcal{O}(\alpha'^3)$,

$$V_{\text{closed}}^{(4)}(s_1, s_2) = 1 + 2\zeta(3) s_1 s_2 (s_1 + s_2) + \mathcal{O}(\alpha'^4), \quad (3.21)$$

which corresponds to a supersymmetrized version of eq. (1.2) in the low energy effective action [17]. In other words, keeping terms up to order $\mathcal{O}(\alpha'^3)$ in the closed-string amplitudes is equivalent to working with a theory whose effective action is given by

$$S_{\text{corr}} = \int d^4x \sqrt{-g} (\mathcal{R} + \alpha'^3 \mathcal{R}^4) + \mathcal{O}(\alpha'^4). \quad (3.22)$$

While α' -corrected six-point amplitudes in open string theory ($\mathcal{N} = 4$ SYM) are already very cumbersome to calculate, the situation is even worse for closed string theory ($\mathcal{N} = 8$ supergravity). For higher-point tree amplitudes it is therefore more convenient to rely on the KLT

¹Note the shifted ordering of helicities compared to eq. (3.4). A cyclic shift $(1, 2, 3, 4, 5, 6) \rightarrow (3, 4, 5, 6, 1, 2)$ has to be performed in order to match the results analytically with ref. [52].

relations, which express closed string amplitudes as simple quadratic combinations of open string amplitudes.

Several different cyclic orderings of the open string amplitudes are required as input to the KLT relations. Fortunately, there are several open string amplitudes available. In particular, a couple of six-point NMHV amplitudes have been computed [52]², which will serve below as input to the calculation of a suitable α' -corrected $\mathcal{N} = 8$ supergravity amplitude.

4 Setting up the calculation

Arkani Hamed, Cachazo and Kaplan have proven eq. (2.4) analytically, by employing BCFW recursion relations for $\mathcal{N} = 8$ supergravity with $E_{7(7)}$ realized on-shell. Because invariance under $E_{7(7)}$ is a necessary condition for the relation to be valid, eq. (2.4) provides a useful tool for testing other theories, or operators, for their symmetry properties under $E_{7(7)}$. In particular, if the double-soft limit of all $(n + 2)$ -point amplitudes derived from eq. (3.22) coincides with the $SU(8)$ rotated sum of the corresponding n -point amplitudes, that would be strong evidence that $E_{7(7)}$ symmetry does not restrict the appearance of \mathcal{R}^4 as a counterterm in $\mathcal{N} = 8$ supergravity.

The analytical approach that ACK used to prove eq. (2.4) does not hold for the α' -corrected $\mathcal{N} = 8$ amplitudes. Higher-dimension operators lead to poorer large-momentum behavior, so that amplitudes shifted by large complex momenta will not fall off fast enough for the BCFW recursion relations to be valid. Instead we will find explicit (if lengthy) expressions for suitable and available string theory amplitudes, from which the α' -corrected amplitudes corresponding to eq. (3.22) can be deduced, and their double-soft limits inspected (numerically).

After we give a short introduction to the KLT relations in subsection 4.1, we will explore the constraints on the α' -corrected $\mathcal{N} = 8$ supergravity amplitude originating from the double-soft limit relation eq. (2.4) in subsection 4.2. Appropriate $\mathcal{N} = 8$ amplitudes will be identified and decomposed into $\mathcal{N} = 4$ SYM matrix elements using the KLT relations. The required (α' -corrected) $\mathcal{N} = 4$ SYM matrix elements can be related to the available open string amplitudes by carefully examining the NMHV supersymmetric Ward identities. In subsections 4.3 and 4.4, the $\mathcal{N} = 1$ supersymmetric Ward identities will be reviewed in detail and used to finally obtain expressions for the $\mathcal{N} = 4$ amplitudes, which serve as input to the KLT relations, in section 5.

4.1 KLT relations

Tree-level amplitudes in closed and open string theories are linked by the KLT relations [53], which arise from the fact that any closed-string vertex operator can be represented as a product

²We are grateful to Stephan Stieberger and Tomasz Taylor for providing us with expressions for the amplitudes from ref. [52] through order α'^3 .

of two open-string vertex operators,

$$V^{\text{closed}}(z_i, \bar{z}_i) = V_{\text{left}}^{\text{open}}(z_i) \bar{V}_{\text{right}}^{\text{open}}(\bar{z}_i). \quad (4.1)$$

While in the closed-string amplitude the insertion points z_i, \bar{z}_i of vertex operators are integrated over a two-sphere, in the open-string case the real z_i are integrated over the boundary of a disk. Thus the closed-string integrand equals the product of two open-string integrands. KLT related the two sets of string amplitudes by evaluating the closed-string integrals via a contour deformation in terms of the open-string integrals.

The KLT relations for four-, five- and six-point amplitudes are

$$M_4(1, 2, 3, 4) = \frac{-i}{\alpha' \pi} \sin(\pi s_{12}) A_4(1, 2, 3, 4) A_4(1, 2, 4, 3), \quad (4.2)$$

$$M_5(1, 2, 3, 4, 5) = \frac{i}{\alpha'^2 \pi^2} \sin(\pi s_{12}) \sin(\pi s_{34}) A_5(1, 2, 3, 4, 5) A_5(2, 1, 4, 3, 5) \\ + \mathcal{P}(2, 3), \quad (4.3)$$

$$M_6(1, 2, 3, 4, 5, 6) = \frac{-i}{\alpha'^3 \pi^3} \sin(\pi s_{12}) \sin(\pi s_{45}) A_6(1, 2, 3, 4, 5, 6) \\ \times [\sin(\pi s_{35}) A_6(2, 1, 5, 3, 4, 6) + \sin(\pi(s_{34} + s_{35})) A_6(2, 1, 5, 4, 3, 6)] \\ + \mathcal{P}(2, 3, 4). \quad (4.4)$$

Formulae for higher-point amplitudes can be derived straightforwardly [53]. In the field-theory ($\alpha' \rightarrow 0$) limit, a closed form has been obtained for all n [36].

The above equalities are exact relations between string theory amplitudes, and so they are valid order by order in α' . In order to calculate the string correction to an $\mathcal{N} = 8$ supergravity amplitude at a certain order in α' from known α' -corrected expressions in $\mathcal{N} = 4$ SYM, one has to determine all combinations of terms from the expansions of the amplitudes and the sine functions, whose multiplication results in the correct power of α' . For instance the second-order correction to the five-point amplitude in supergravity corresponds to terms of $\mathcal{O}(\alpha'^4)$, due to the prefactor of $\frac{1}{\alpha'^2}$. Taking the absence of first-order corrections to $\mathcal{N} = 4$ SYM amplitudes into account, four combinations have to be considered in eq. (4.3), according to the following table:

$\sin(\pi s_{12})$	$\sin(\pi s_{34})$	$A_5(1, 2, 3, 4, 5)$	$A_5(2, 1, 4, 3, 5)$
$\mathcal{O}(\alpha'^1)$	$\mathcal{O}(\alpha'^1)$	$\mathcal{O}(\alpha'^0)$	$\mathcal{O}(\alpha'^2)$
$\mathcal{O}(\alpha'^1)$	$\mathcal{O}(\alpha'^1)$	$\mathcal{O}(\alpha'^2)$	$\mathcal{O}(\alpha'^0)$
$\mathcal{O}(\alpha'^3)$	$\mathcal{O}(\alpha'^1)$	$\mathcal{O}(\alpha'^0)$	$\mathcal{O}(\alpha'^0)$
$\mathcal{O}(\alpha'^1)$	$\mathcal{O}(\alpha'^3)$	$\mathcal{O}(\alpha'^0)$	$\mathcal{O}(\alpha'^0)$

yielding

$$M_5^{\mathcal{O}(\alpha'^2)} = \frac{is_{12}s_{34}}{\alpha'^2} \left[A_5^{\text{SYM}}(1, 2, 3, 4, 5) A_5^{\mathcal{O}(\alpha'^2)}(2, 1, 4, 3, 5) + A_5^{\mathcal{O}(\alpha'^2)}(1, 2, 3, 4, 5) A_5^{\text{SYM}}(2, 1, 4, 3, 5) - \frac{\pi^2}{6}(s_{12}^2 + s_{34}^2) A_5^{\text{SYM}}(1, 2, 3, 4, 5) A_5^{\text{SYM}}(2, 1, 4, 3, 5) \right] + \mathcal{P}(2, 3). \quad (4.5)$$

The above expression can be shown to vanish analytically, in accordance with the higher-point generalization of eq. (3.21), or alternatively eq. (3.22), the statement that the first correction to the closed-string effective action is at $\mathcal{O}(\alpha'^3)$.

Although the KLT relations are often applied to pure-graviton and pure-gluon amplitudes, their use is not limited to these scenarios. Any pair of consistent open-string amplitudes is related to an amplitude in closed string theory and vice versa. Considering the combination of two open-string vertex operators into a closed one in eq. (4.1), one can immediately determine which type of particle has to appear at a certain position on the supergravity side by adding the helicities and combining the indices, according to the tensor-product decomposition of the Fock space,

$$[\mathcal{N} = 8] \leftrightarrow [\mathcal{N} = 4]_L \otimes [\mathcal{N} = 4]_R. \quad (4.6)$$

Somewhat remarkably, the opposite statement is true as well: given a certain operator, corresponding to a particular state in $\mathcal{N} = 8$ supergravity, the helicity, global symmetry properties, and the consistent action of supercharges in either of the theories are sufficient to unambiguously determine the decomposition into $\mathcal{N} = 4$ SYM states [57]. The decompositions relevant for the calculation to follow are

$$\begin{aligned} B^+ &= g^+ \tilde{g}^+, & F^{a+} &= \lambda^{a+} \tilde{g}^+, & F^{r+} &= g^+ \tilde{\lambda}^{r+}, \\ B^- &= g^- \tilde{g}^-, & F_a^- &= \lambda_a^- \tilde{g}^-, & F_r^- &= g^- \tilde{\lambda}_r^-, \\ X^{abcd} &= \varepsilon^{abcd} g^- \tilde{g}^+, & X^{abcr} &= \varepsilon^{abcd} \lambda_d^- \tilde{\lambda}^{r+}, & X^{abrs} &= \phi^{ab} \tilde{\phi}^{rs}, \\ X_{abcd} &= \varepsilon_{abcd} g^+ \tilde{g}^-, & X_{abcr} &= \varepsilon_{abcd} \lambda^{d+} \tilde{\lambda}_r^-, & X_{abrs} &= \phi_{ab} \tilde{\phi}_{rs}, \end{aligned} \quad (4.7)$$

where capital letters B, F, X denote the graviton, gravitino and scalar particle in $\mathcal{N} = 8$ supergravity and g, λ, ϕ the gluon, gluino and scalar in $\mathcal{N} = 4$ SYM. Quantities with indices a, b, \dots correspond to the first $SU(4)$, while quantities with a tilde and indices r, s, \dots are in the second $SU(4)$. (In particular, \tilde{g} does *not* denote a gluino!) Finally, the superscripts $+$ and $-$ mark the helicity signature.

4.2 Choosing a suitable amplitude

The simplest scenario one might think of, in order to test the double-soft scalar limit relation (2.4), would be to start with a five-point amplitude, which in turn would lead to a sum of three-point

amplitudes on the right-hand side of the relation. Three-point amplitudes are special as they require a setup with complex momenta in order to be non-trivial. However, here we have to take another constraint into account: we want to test amplitudes that receive nonvanishing corrections from the \mathcal{R}^4 term. Because the interactions originating in this counterterm candidate start at the four-point level, it is not sufficient to consider three-point amplitudes.

Therefore we will have to consider a six-point amplitude, which should reduce to a sum of four-point amplitudes in the double-soft limit. We again require that the four-point amplitudes on the right-hand side of eq. (2.4) are nonvanishing, which implies that they are MHV (or equivalently anti-MHV). Fortunately, corrections to all MHV-amplitudes with four legs are known up to $\mathcal{O}(\alpha'^3)$, indeed to arbitrary orders in α' , using eq. (3.20) and the MHV supersymmetry Ward identities.

On the left-hand side of eq. (2.4) the situation is more intricate. The four particles that appear already on the right-hand side are now accompanied by two additional scalars. According to eq. (2.7), the number of η derivatives acting on the generating functional is increased by eight, four for each scalar, so that the resulting amplitude resides in the NMHV sector. In addition, the two scalars have to share three $SU(8)$ indices, as elaborated on in section 2. Sorting out the distribution of the scalars' indices into two $SU(4)$ subgroups, there are finally five possible distinct choices³ satisfying the constraints. They are listed here, together with their respective KLT decompositions according to equation (4.7):

$$\langle X^{abrs} X_{abrt} \dots \rangle \rightarrow \langle \phi^{ab} \phi_{ab} \dots \rangle_L \times \langle \phi^{rs} \phi_{rt} \dots \rangle_R, \quad (4.8)$$

$$\langle X^{abrc} X_{abrs} \dots \rangle \rightarrow \langle \varepsilon^{abcd} \lambda_d^- \phi_{ab} \dots \rangle_L \times \langle \lambda^{r+} \phi_{rs} \dots \rangle_R, \quad (4.9)$$

$$\langle X^{abrc} X_{abrd} \dots \rangle \rightarrow \langle \lambda_d^- \lambda^{c+} \dots \rangle_L \times \langle \lambda^{r+} \lambda_r^- \dots \rangle_R, \quad (4.10)$$

$$\langle X^{abcr} X_{abcs} \dots \rangle \rightarrow \langle \lambda_d^- \lambda^{d+} \dots \rangle_L \times \langle \lambda^{r+} \lambda_s^- \dots \rangle_R, \quad (4.11)$$

$$\langle X^{abcd} X_{abcr} \dots \rangle \rightarrow \langle g^- \lambda^{d+} \dots \rangle_L \times \langle g^+ \lambda_r^- \dots \rangle_R. \quad (4.12)$$

Here the ellipses are understood to be filled with four particles such that the L - and R -amplitudes on the right-hand side of the KLT relation each transform as an $SU(4)$ singlet. In each of equations (4.10) to (4.12) we have left out a factor of $\varepsilon^{abcd} \varepsilon_{abcd}$. Because these indices are not summed over, this factor is equal to unity. Note that $\langle X^{abcd} X_{abce} \dots \rangle$ is absent because the five $SU(4)$ indices a, b, c, d, e cannot be made all distinct.

In order to proceed, we need to use supersymmetric Ward identities to relate one of the five decompositions (4.8)–(4.12) to the available open-string six-point results (see eq. (3.16) in section 3):

$$\begin{aligned} & \langle g^- g^- g^- g^+ g^+ g^+ \rangle, \quad \langle \phi^- \phi^- \phi^- \phi^+ \phi^+ \phi^+ \rangle, \\ & \langle \phi^- \phi^- \lambda^- \lambda^+ \phi^+ \phi^+ \rangle \quad \text{and} \quad \langle \phi^- \phi^- g^- g^+ \phi^+ \phi^+ \rangle. \end{aligned} \quad (4.13)$$

³Another five combinations can be obtained by switching the left and right $SU(4)$.

Supersymmetric Ward identities can be classified by the amount of supersymmetry employed (*e.g.*, $\mathcal{N} = 1, 2, 4$), as well as the number of legs and the sector (MHV, NMHV, *etc.*) characterizing the amplitudes. We deal with six-point NMHV amplitudes exclusively here. The notation $\mathcal{N} = 4$ SWI will refer to the set of supersymmetric Ward identities relating six-point NMHV amplitudes built from the full $\mathcal{N} = 4$ multiplet $(g^\pm, \lambda_m^\pm, \phi_n^\pm)$, where $m = 1, 2, 3, 4$ and $n = 1, 2, 3$. (Note that a superscript \pm on ϕ implies a complex field with a different index labelling from the real ϕ_{ab} used above.) In the original article [52], $\mathcal{N} = 2$ supersymmetric Ward identities have been served to relate the latter three amplitudes in eq. (4.13) to the pure-gluon one. So the obvious idea would be to search in the decompositions (4.8)–(4.12) for one in which the amplitudes contain particles from a single $\mathcal{N} = 2$ multiplet (plus its CPT conjugate), $(g^\pm, \lambda_m^\pm, \phi_1^\pm)$ with $m = 1, 2$.

However, the third amplitude in eq. (4.13) contains only one type of fermion, which points into the direction of a $\mathcal{N} = 1$ multiplet. Setting up the calculation employing $\mathcal{N} = 1$ SWI exclusively is a bit simpler than using $\mathcal{N} = 2$ SWI: For six-point NMHV amplitudes an explicit and simple solution to the $\mathcal{N} = 1$ SWI is known [57, 9]. (We note that very recently the supersymmetric Ward identities in maximally supersymmetric $\mathcal{N} = 4$ super-Yang-Mills theory and $\mathcal{N} = 8$ super-gravity were solved, quite remarkably, for arbitrary n -point N^k MHV amplitudes [70] in terms of basis amplitudes, in a manifestly supersymmetric form. These results may prove very useful in extending the considerations of this paper to greater numbers of legs.)

Now the decompositions (4.8) to (4.12) are not all equally suited to the use of an $\mathcal{N} = 1$ SWI. For example, the left $SU(4)$ amplitude of eq. (4.9) contains three distinct $SU(4)$ indices, a, b, d , thus requiring a full $\mathcal{N} = 4$ multiplet. The other four decompositions contain amplitudes which can be constructed from SWI with less supersymmetry. Indeed, the decomposition (4.12) contains only one index for the left $SU(4)$ amplitude, and one for the right one; this decomposition is the one we will use in this paper. As will be explained below, it is possible to obtain everything we need for testing the double-soft limit through eq. (4.12), by using a two-step procedure employing two different sets of $\mathcal{N} = 1$ SWI based on the multiplets (g^\pm, λ^\pm) and (ϕ^\pm, λ^\pm) .

The next three subsections introduce the SWI in general, elaborate on the $\mathcal{N} = 1$ SWI for (g^\pm, λ^\pm) in particular, and then describe the analogous set of $\mathcal{N} = 1$ SWI for the multiplet (ϕ^\pm, λ^\pm) . Then, in section 5, we will assemble these ingredients in order to test the $E_{7(7)}$ symmetry.

4.3 Supersymmetric Ward identities

Supersymmetric Ward identities can be derived using the fact that supercharges annihilate the vacuum of the theory, $Q|0\rangle = 0$, so that

$$0 = \langle [Q, \beta_1 \beta_2 \cdots \beta_n] \rangle = \sum_{i=1}^n \langle \beta_1 \beta_2 \cdots [Q, \beta_i] \cdots \beta_n \rangle. \quad (4.14)$$

Here the β_i are arbitrary states from the multiplet under consideration, $Q = Q(\eta) = \langle Q\eta \rangle$ is a supersymmetry operator, which has been bosonized by contraction with the Grassmann variable η , and $\langle \beta_1 \beta_2 \cdots \beta_n \rangle$ will be called the *source term* for the SWI. Source terms need to have an odd number of fermions, because amplitudes derived by acting on terms with an even number of fermions will vanish trivially. An immediate and standard result implied by eq. (4.14) is the disappearance of all amplitudes with helicity structure $\langle + + + \cdots + \rangle$ and $\langle + - - \cdots - \rangle$. With only little more effort one can show that maximally helicity violating amplitudes (MHV) are related pairwise by SWI, which in turn means that knowing one amplitude determines the whole MHV sector for a particular number of legs [71]. In the NMHV sector this is no longer true; here each supersymmetric Ward identity relates three amplitudes, which requires two known amplitudes in order to determine a third one.

Stieberger and Taylor have explicitly proven for open string theory on the disk that the forms of the supersymmetric Ward identities to all orders in α' are identical to those in the corresponding four-dimensional field-theoretical limit [72]. So the exploration in the next two subsections will be valid as well for the α' -corrected amplitudes under investigation.

4.4 $\mathcal{N} = 1$ supersymmetric Ward identities

As an example, let us investigate the set of amplitudes involving gluons (g^+, g^-) and a single pair of gluinos (λ^+, λ^-) (from which we drop the $SU(4)$ index for simplicity). The states are related by $\mathcal{N} = 1$ supersymmetry via

$$\begin{aligned}
[Q(\eta), g^+(p)] &= [p\eta] \lambda^+(p), \\
[Q(\eta), \lambda^+(p)] &= -\langle p\eta \rangle g^+(p), \\
[Q(\eta), g^-(p)] &= \langle p\eta \rangle \lambda^-(p), \\
[Q(\eta), \lambda^-(p)] &= -[p\eta] g^-(p),
\end{aligned} \tag{4.15}$$

where $Q(\eta) = \langle Q\eta \rangle$.

For each NMHV helicity sector, there are 20 distinct amplitudes related by $\mathcal{N} = 1$ SWI: a pure-gluon amplitude, a pure-gluino amplitude, nine two-gluino four-gluon amplitudes, and nine four-gluino two-gluon amplitudes, as shown in figure 1. In the following, we assume that amplitudes are drawn from helicity configuration X in eq. (3.4). For the two other configurations Y and Z , the relations are completely analogous.

Amplitudes in adjacent rows of figure 1 are related by the $\mathcal{N} = 1$ SWI. Acting for example with the supersymmetry operator $Q(\eta)$ on the source term $\langle g^- g^- g^- \lambda^+ g^+ g^+ \rangle$ yields

$$\begin{aligned}
&\langle 4\eta \rangle \langle g^- g^- g^- g^+ g^+ g^+ \rangle - \langle 1\eta \rangle \langle \lambda^- g^- g^- \lambda^+ g^+ g^+ \rangle \\
&- \langle 2\eta \rangle \langle g^- \lambda^- g^- \lambda^+ g^+ g^+ \rangle - \langle 3\eta \rangle \langle g^- g^- \lambda^- \lambda^+ g^+ g^+ \rangle = 0,
\end{aligned} \tag{4.16}$$

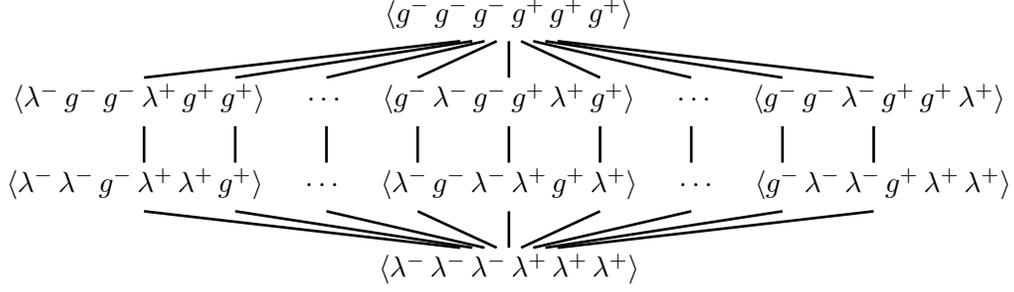


Figure 1: Amplitudes related by $\mathcal{N} = 1$ supersymmetric Ward identities.

which relates the pure-gluon amplitude to the two-gluino four-gluon ones from the second row in figure 1. Due to the freedom in choosing the two-component supersymmetry parameter η , the result is a system of equations which has rank 2. In order to find all relations between the pure gluon amplitude (first row) and the amplitudes in the second row, the action of $Q(\eta)$ on all possible source terms featuring one gluino and five gluons,

$$\begin{aligned}
&\langle \lambda^- g^- g^- g^+ g^+ g^+ \rangle, \quad \langle g^- \lambda^- g^- g^+ g^+ g^+ \rangle, \quad \langle g^- g^- \lambda^- g^+ g^+ g^+ \rangle, \\
&\langle g^- g^- g^- \lambda^+ g^+ g^+ \rangle, \quad \langle g^- g^- g^- g^+ \lambda^+ g^+ \rangle, \quad \langle g^- g^- g^- g^+ g^+ \lambda^+ \rangle,
\end{aligned} \tag{4.17}$$

has to be considered. The resulting system, linking ten amplitudes from the first and second rows, turns out to have rank eight, thus requiring two known amplitudes in order to derive all the others.

Repeating the analysis for the second and third rows, there are notably more identities to consider. They are generated by acting with $Q(\eta)$ on any of the 18 different source terms built from three gluinos and the same number of gluons, *e.g.* $\langle \lambda^- \lambda^- g^- g^+ \lambda^+ g^+ \rangle$. Interestingly this system connecting 18 unknown amplitudes is of rank 16, meaning that again two amplitudes have to be known in order to fix all the others.

Finally, the relations between the third row and the pure-gluino amplitude (fourth row) mirror the situation found for the top of the diagram and are also of rank eight.

Combining all of the above into one large system of equations, the total rank of the supersymmetric Ward identities pictured in figure 1 turns out to be 18. So, given any two of the 20 distinct amplitudes, one can calculate any other from this set employing the complete collection of $\mathcal{N} = 1$ SWI. The corresponding result has already been found by Grisaru and Pendleton in the context of $\mathcal{N} = 1$ supergravity [9], and recast recently in modern spinor-helicity form [57].

More explicitly, any two-gluino four-gluon amplitude $F_{i,I}$, with the gluinos situated at positions i and I , can be expressed in terms of the pure-gluon and pure-gluino amplitude as

$$F_{i,I} = \frac{4\langle Ij \rangle [ij] \langle g^- g^- g^- g^+ g^+ g^+ \rangle - \varepsilon_{ijk} \langle jk \rangle \varepsilon_{IJK} [JK] \langle \lambda^- \lambda^- \lambda^- \lambda^+ \lambda^+ \lambda^+ \rangle}{-2 \sum_{m,n \in \{i,j,k\}} \langle mn \rangle [nm]}, \tag{4.18}$$

where i, j, k and I, J, K mark the set of negative and positive helicity particles respectively, and the numerator contains implicit sums over j, k, J, K . For example,

$$F_{3,4} = \langle g^- g^- \lambda^- \lambda^+ g^+ g^+ \rangle = \frac{\langle 4|(1+2)|3\rangle \langle g^- g^- g^- g^+ g^+ g^+ \rangle + \langle 12\rangle [56] \langle \lambda^- \lambda^- \lambda^- \lambda^+ \lambda^+ \lambda^+ \rangle}{(k_1 + k_2 + k_3)^2}. \quad (4.19)$$

A similar formula for all four-gluino two-gluon amplitudes can be found in the appendix of ref. [57].

4.5 The second $\mathcal{N} = 1$ SUSY diamond

Recall [52] that the pure-gluon amplitude can be calculated from the latter three amplitudes in eq. (4.13), namely

$$\langle \phi^- \phi^- \phi^- \phi^+ \phi^+ \phi^+ \rangle, \quad \langle \phi^- \phi^- \lambda^- \lambda^+ \phi^+ \phi^+ \rangle \quad \text{and} \quad \langle \phi^- \phi^- g^- g^+ \phi^+ \phi^+ \rangle. \quad (4.20)$$

The question that immediately arises is whether this set forms a basis for the complete set of all six-point NMHV $\mathcal{N} = 2$ amplitudes⁴ in helicity configuration X ? We were not aware of a direct answer to that question, so we took the following approach. As mentioned already in subsection 4.2, we will consider a second set of six-point NMHV $\mathcal{N} = 1$ supersymmetric Ward identities, in addition to the $\mathcal{N} = 1$ SWI for (g^\pm, λ^\pm) described in the previous subsection.

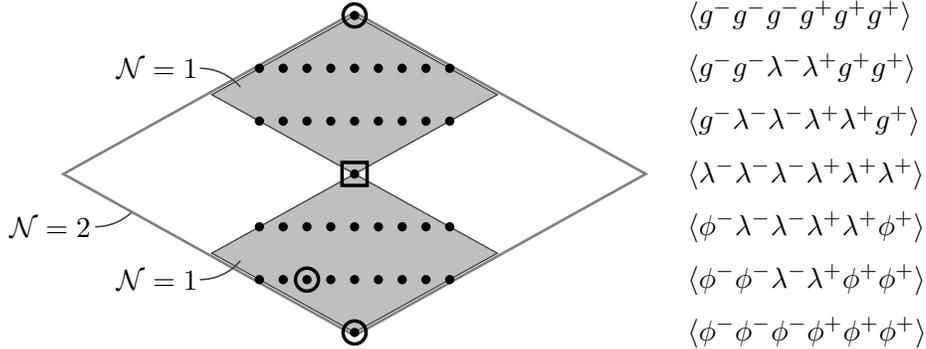


Figure 2: Amplitudes involving particles from a single $\mathcal{N} = 2$ multiplet containing two $\mathcal{N} = 1$ subsets.

In figure 2 the collection of six-point NMHV $\mathcal{N} = 2$ amplitudes is depicted in helicity configuration X . Every black dot denotes a particular amplitude. The top point represents the pure-gluon amplitude $\langle g^- g^- g^- g^+ g^+ g^+ \rangle$, the lowest point refers to the pure-scalar amplitude $\langle \phi^- \phi^- \phi^- \phi^+ \phi^+ \phi^+ \rangle$, and the central point denotes the pure-gluino amplitude $\langle \lambda^- \lambda^- \lambda^- \lambda^+ \lambda^+ \lambda^+ \rangle$. Supersymmetric Ward identities relate certain amplitudes from adjacent rows and the elements of eq. (4.13) are encircled. The upper diamond-shaped region corresponds precisely to figure 1: it is the subset of six-point NMHV $\mathcal{N} = 1$ amplitudes built from the multiplet (g^\pm, λ^\pm) within

⁴The term $\mathcal{N} = 2$ amplitudes refers to all possible amplitudes that can be constructed exclusively from particles from a single $\mathcal{N} = 2$ multiplet and its CPT conjugate, $(g^\pm, \lambda_m^\pm, \phi^\pm)$ with $m = 1, 2$ [73].

the $\mathcal{N} = 2$ amplitudes. (There are additional states in the full $\mathcal{N} = 2$ diamond in figure 2, of course, even in the second row.)

However, the upper diamond-shaped region is not the only subset of six-point NMHV $\mathcal{N} = 2$ amplitudes which can be related by $\mathcal{N} = 1$ supersymmetric Ward identities. Stretching between the pure-gluino and the pure-scalar amplitude there is a second region (referred to as the lower diamond in the following), which satisfies relations similar to those in the upper $\mathcal{N} = 1$ diamond. The modified supersymmetry operator \tilde{Q} will now act on a multiplet consisting of scalars (ϕ^+, ϕ^-) and gluinos (λ^+, λ^-) via

$$\begin{aligned} [\tilde{Q}(\eta), \phi^+(p)] &= \langle p\eta \rangle \lambda^+(p), \\ [\tilde{Q}(\eta), \lambda^+(p)] &= -[p\eta] \phi^+(p), \\ [\tilde{Q}(\eta), \phi^-(p)] &= [p\eta] \lambda^-(p), \\ [\tilde{Q}(\eta), \lambda^-(p)] &= -\langle p\eta \rangle \phi^-(p), \end{aligned} \tag{4.21}$$

which can be easily derived by identifying the supercharges of $\mathcal{N} = 2$ supersymmetry, Q_1 and Q_2 , with Q and \tilde{Q} respectively.

Writing down the set of supersymmetric Ward identities generated by acting with a supersymmetry generator \tilde{Q} on the source term $\langle \phi^- \phi^- \phi^- \lambda^+ \phi^+ \phi^+ \rangle$, one encounters the same structure derived in eq. (4.16):

$$\begin{aligned} [4\eta] \langle \phi^- \phi^- \phi^- \phi^+ \phi^+ \phi^+ \rangle - [1\eta] \langle \lambda^- \phi^- \phi^- \lambda^+ \phi^+ \phi^+ \rangle \\ - [2\eta] \langle \phi^- \lambda^- \phi^- \lambda^+ \phi^+ \phi^+ \rangle - [3\eta] \langle \phi^- \phi^- \lambda^- \lambda^+ \phi^+ \phi^+ \rangle = 0. \end{aligned} \tag{4.22}$$

In fact, one can show that the complete system of supersymmetric Ward identities and amplitudes for the lower diamond, ranging from the pure-gluino to the pure-scalar amplitude, can be obtained from the original $\mathcal{N} = 1$ system considered in figure 1 by exchanging

$$\begin{aligned} Q &\leftrightarrow \tilde{Q} \\ [\] &\leftrightarrow \langle \ \rangle \\ g^+ &\leftrightarrow \phi^+ \\ g^- &\leftrightarrow \phi^-. \end{aligned} \tag{4.23}$$

This symmetry corresponds geometrically to reflecting figure 2 about a horizontal line passing through the central point $\langle \lambda^- \lambda^- \lambda^- \lambda^+ \lambda^+ \lambda^+ \rangle$.

The second system of supersymmetric Ward identities in the lower diamond is obviously of the same rank as the original system. However, in contrast to the upper diamond it contains two of the known amplitudes from ref. [52],

$$\langle \phi^- \phi^- \phi^- \phi^+ \phi^+ \phi^+ \rangle \quad \text{and} \quad \langle \phi^- \phi^- \lambda^- \lambda^+ \phi^+ \phi^+ \rangle, \tag{4.24}$$

which allows the calculation of any other amplitude in the lower $\mathcal{N} = 1$ set. In particular, the pure-gluino amplitude $\langle \lambda^- \lambda^- \lambda^- \lambda^+ \lambda^+ \lambda^+ \rangle$ (\blacksquare in figure 2), which is the element connecting the upper and lower set of equations, can be determined. Having done so, there are now two known amplitudes from the upper $\mathcal{N} = 1$ diamond, the pure-gluino and the pure-gluon amplitude [52], which in turn is the precondition for determining any amplitude from the upper $\mathcal{N} = 1$ region. In other words: any six-point NMHV amplitude in the two shaded regions in figure 2 can be calculated from eq. (4.13).

In the next section, we will complete the ellipses on the left-hand side of the decomposition (4.12) by two gravitini and two gravitons, and KLT-factorize the result in such a way that the desired six-point closed-string ($\mathcal{N} = 8$ supergravity) amplitude can be related to a set of two-gluino four-gluon $\mathcal{N} = 4$ SYM amplitudes. The SYM amplitudes are available in turn by the two-step procedure described above.

5 $E_{7(7)}$ symmetry for α' -corrected amplitudes?

As explained in the last section, the most accessible way of testing the double-soft scalar limit relation is to calculate the $\mathcal{N} = 8$ supergravity amplitude,

$$\langle X^{1234} X_{1235} F^{5+} F_4^- B^+ B^- \rangle = \text{KLT} \left[\langle g^- \lambda^{4+} g^+ \lambda_4^- g^+ g^- \rangle_L \times \langle g^+ \lambda_5^- \lambda^{5+} g^- g^+ g^- \rangle_R \right], \quad (5.1)$$

a particular version of eq. (4.12). The determination of the right-hand side of eq. (5.1) will be done by employing the two-step procedure described in the last subsection.

How should we obtain the pure-gluino amplitude $\langle \lambda^- \lambda^- \lambda^- \lambda^+ \lambda^+ \lambda^+ \rangle$ from the amplitudes in eq. (4.24) in the first step? An expression relating any six-point NMHV two-fermion four-boson amplitude to the pure-fermion and pure-boson one has been given in eq. (4.18). We start from eq. (4.19), employ the correspondence eq. (4.23) which transforms the pure-gluon amplitude into the pure-scalar one, and solve the resulting equation for the pure-gluino amplitude:

$$\langle \lambda^- \lambda^- \lambda^- \lambda^+ \lambda^+ \lambda^+ \rangle = \frac{(k_1 + k_2 + k_3)^2 \langle \phi^- \phi^- \lambda^- \lambda^+ \phi^+ \phi^+ \rangle - \langle 3|(1+2)|4 \rangle \langle \phi^- \phi^- \phi^- \phi^+ \phi^+ \phi^+ \rangle}{\langle 56 \rangle [12]}. \quad (5.2)$$

In the second step, we employ eq. (4.18) to obtain analytical expressions for all two-gluino four-gluon amplitudes, allowing us to assemble finally the $\mathcal{N} = 8$ amplitude.

In the same manner as explained in subsection 4.1 for the expansion to $\mathcal{O}(\alpha'^2)$ of a five-point gravity amplitude, appropriate combinations of orders in α' have to be added and permuted on the right-hand side of eq. (5.1) in order to obtain the result including the \mathcal{R}^4 perturbation.

Explicitly, the third order in α' can be obtained by evaluating

$$\begin{aligned}
M_6^{\mathcal{O}(\alpha'^3)} &= \frac{-i}{\alpha'^3} s_{12} s_{45} \left(A_6^{\text{SYM}}(1, 2, 3, 4, 5, 6) \right. \\
&\quad \times \left[s_{35} A_6^{\mathcal{O}(\alpha'^3)}(2, 1, 5, 3, 4, 6) + (s_{34} + s_{35}) A_6^{\mathcal{O}(\alpha'^3)}(2, 1, 5, 4, 3, 6) \right] \\
&\quad + A_6^{\mathcal{O}(\alpha'^3)}(1, 2, 3, 4, 5, 6) \\
&\quad \times \left[s_{35} A_6^{\text{SYM}}(2, 1, 5, 3, 4, 6) + (s_{34} + s_{35}) A_6^{\text{SYM}}(2, 1, 5, 4, 3, 6) \right] \Big) \\
&\quad + \mathcal{P}(2, 3, 4). \tag{5.3}
\end{aligned}$$

All amplitudes needed on the right-hand side of eq. (5.3) are two-gluino four-gluon amplitudes for the helicity configurations X , Y or Z , which we have related by supersymmetry to the amplitudes considered in ref. [52].

Now we turn to the right-hand side of the double-soft limit relation (2.4). Given the particular choice of amplitude (5.1), it is straightforward to find an expression for the right-hand side. The operator

$$T_5^4 = \varepsilon_{12354}^{12345} \eta_{i5} \partial_{\eta_{i4}} = -\eta_{i5} \partial_{\eta_{i4}} \tag{5.4}$$

will act on the remnant of the six-point amplitude as

$$\begin{aligned}
& - \sum_{i=3}^6 \eta_{i5} \partial_{\eta_{i4}} \langle F^{5+} F_4^- B^+ B^- \rangle \\
&= \sum_{i=3}^6 \eta_{i5} \partial_{\eta_{i4}} \left(\frac{\partial}{\partial \eta_{35}} \right) \left(\frac{1}{7!} \varepsilon_{12345678} \frac{\partial^7}{\partial \eta_{41} \dots \partial \eta_{43} \partial \eta_{45} \dots \partial \eta_{48}} \right) \left(\frac{1}{8!} \varepsilon_{12345678} \frac{\partial^8}{\partial \eta_{61} \dots \partial \eta_{68}} \right) \Omega_4 \\
&= \langle F^{4+} F_4^- B^+ B^- \rangle - \langle F^{5+} F_5^- B^+ B^- \rangle. \tag{5.5}
\end{aligned}$$

Acting on particle 3, the operator changes the derivative with respect to η_{35} into a derivative with respect to η_{34} , thus effectively transforming the positive helicity gravitino F^{5+} into F^{4+} . Correspondingly, by acting on particle 4, again a derivative with respect to η_{45} will be changed into one with respect to η_{44} , this time transforming F_4^- into F_5^- .

Restoring the kinematical weight factors in eq. (2.4), the final comparison will be made according to the following formula:

$$\begin{aligned}
& \langle X^{1234} X_{1235} F^{5+} F_4^- B^+ B^- \rangle \Big|_{\mathcal{O}(\alpha'^3)} \rightarrow \\
& \frac{1}{2} \left[\frac{p_3 \cdot (p_2 - p_1)}{p_3 \cdot (p_1 + p_2)} \langle F^{4+} F_4^- B^+ B^- \rangle \Big|_{\mathcal{O}(\alpha'^3)} - \frac{p_4 \cdot (p_2 - p_1)}{p_4 \cdot (p_1 + p_2)} \langle F^{5+} F_5^- B^+ B^- \rangle \Big|_{\mathcal{O}(\alpha'^3)} \right]. \tag{5.6}
\end{aligned}$$

Given the complexity of the higher-order α' corrections in the available amplitudes (see *e.g.* eq. (3.18) at only $\mathcal{O}(\alpha'^2)$), the analytical computation of the left-hand side of eq. (5.6) would be

very cumbersome. Instead the computation and comparison have been performed numerically for a sufficient number of kinematical points.

For reference, we give numerical values at one sample double-soft kinematical point, with all outgoing momenta fulfilling $k_i^2 = 0$ and $\sum_{i=1}^6 k_i^\mu = 0$:

$$\begin{aligned}
k_1 &= (-0.853702542142, +0.696134406758, -0.306157335124, +0.387907984368) \times 10^{-4}, \\
k_2 &= (+0.711159367201, -0.099704627834, -0.295472686856, +0.639142021830) \times 10^{-4}, \\
k_3 &= (+0.818866370407, +0.408234512914, -0.661447772542, -0.257630664418), \\
k_4 &= (-1.098195656456, -0.551965696904, -0.598319787466, +0.737143813124), \\
k_5 &= (-0.618073260483, +0.143671541012, +0.362410922160, -0.479615853707), \\
k_6 &= (+0.897416800850, +0.000000000000, +0.897416800850, +0.000000000000). \tag{5.7}
\end{aligned}$$

At this point, with a particular external-state phase convention, the left- and right-hand sides of the supergravity ($\mathcal{O}(\alpha'^0)$) version of eq. (5.6) are given respectively by

$$-0.30572232 - i 0.89270274 \approx -0.30615989 - i 0.89271337, \tag{5.8}$$

while the desired $\mathcal{O}(\alpha'^3)$ terms in eq. (5.6) are,

$$3.08397954 + i 9.00278816 \approx 3.08775134 + i 9.00339016. \tag{5.9}$$

The difference between the left- and right-hand sides is due merely to the finite separation of the point (5.7) from the double-soft limit. It can be made as small as desired by working closer to the limit, using higher precision kinematics to avoid roundoff error.

The result is surprising: for any double-soft kinematical configuration considered, the left- and the right-hand side of eq. (5.6) show complete agreement within numerical errors.

Given the available amplitudes from the two shaded regions in figure 2, one can perform further tests for other $\mathcal{N} = 8$ amplitudes. In addition to eq. (5.6), we have tested the double-soft scalar limit for the following amplitudes

$$\begin{aligned}
&\langle X^{1234} X_{1235} F^{5+} F_4^- F^{4+} F_4^- \rangle \Big|_{\mathcal{O}(\alpha'^3)} \rightarrow \\
&\frac{1}{2} \left[+ \frac{p_3 \cdot (p_2 - p_1)}{p_3 \cdot (p_1 + p_2)} \langle F^{4+} F_4^- F^{4+} F_4^- \rangle \Big|_{\mathcal{O}(\alpha'^3)} \right. \\
&\quad - \frac{p_4 \cdot (p_2 - p_1)}{p_4 \cdot (p_1 + p_2)} \langle F^{5+} F_5^- F^{4+} F_4^- \rangle \Big|_{\mathcal{O}(\alpha'^3)} \\
&\quad \left. - \frac{p_6 \cdot (p_2 - p_1)}{p_6 \cdot (p_1 + p_2)} \langle F^{5+} F_4^- F^{4+} F_5^- \rangle \Big|_{\mathcal{O}(\alpha'^3)} \right] \tag{5.10}
\end{aligned}$$

and

$$\begin{aligned}
& \langle X^{1234} X_{1235} X^{1235} X_{1235} X^{1235} X_{1234} \rangle \Big|_{\mathcal{O}(\alpha'^3)} \rightarrow \\
& \frac{1}{2} \left[+ \frac{p_3 \cdot (p_2 - p_1)}{p_3 \cdot (p_1 + p_2)} \langle X^{1234} X_{1235} X^{1235} X_{1234} \rangle \Big|_{\mathcal{O}(\alpha'^3)} \right. \\
& \quad + \frac{p_5 \cdot (p_2 - p_1)}{p_5 \cdot (p_1 + p_2)} \langle X^{1235} X_{1235} X^{1234} X_{1234} \rangle \Big|_{\mathcal{O}(\alpha'^3)} \\
& \quad \left. - \frac{p_6 \cdot (p_2 - p_1)}{p_6 \cdot (p_1 + p_2)} \langle X^{1235} X_{1235} X^{1235} X_{1235} \rangle \Big|_{\mathcal{O}(\alpha'^3)} \right]. \tag{5.11}
\end{aligned}$$

Each limit shows complete agreement for any double-soft kinematical point.

6 Conclusion

Our computation shows that the double-soft limit of three distinct six-point $\mathcal{O}(\alpha'^3)$ -corrected $\mathcal{N} = 8$ matrix elements yields the corresponding weighted sum of four-point amplitudes, precisely as dictated by $E_{7(7)}$ invariance [39]. Based on the amplitudes we considered, there are no restrictions from $E_{7(7)}$ on the appearance of the \mathcal{R}^4 counterterm in a renormalized action of $\mathcal{N} = 8$ supergravity. However, the specific calculations still leave open the question for other matrix elements.

The hope of explaining the three-loop cancellations [25, 26] by a simple symmetry argument that originates in the $\frac{E_{7(7)}}{SU(8)}$ coset symmetry of $\mathcal{N} = 8$ supergravity is therefore unfulfilled at present. It is still possible that $E_{7(7)}$ plays a more subtle role in the excellent ultraviolet behavior of the theory, perhaps by relating somehow the coefficients of certain loop integrals making up the full three-loop amplitude.

One might wonder whether the methods of this paper could be used to study the $E_{7(7)}$ invariance properties of $\mathcal{N} = 8$ supermultiplets of operators with dimensions greater than that of \mathcal{R}^4 . In principle, the answer is yes, by going to higher orders in the α' expansion. However, a complication arises, once there is more than one candidate $\mathcal{N} = 8$ supermultiplet of operators with a given dimension. In this case, closed string theory will provide, at the appropriate order in the α' expansion, the matrix elements corresponding to a particular effective action, containing a unique linear combination of operator multiplets. On the other hand, for investigating all candidate supergravity counterterms one would like to have arbitrary linear combinations available. This complication may appear as early as order α'^5 .

Completely understanding the role of $E_{7(7)}$ will very likely be part of a fundamental explanation of the conjectured finiteness of $\mathcal{N} = 8$ supergravity. However, whether supersymmetry and the coset symmetry alone are sufficient ingredients remains to be shown.

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