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From weak coupling to spinning strings

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Abstract

We identify the gauge theory dual of a spinning string of minimal energy with spins S_1, S_2 on AdS_5 and charge J on S^5. For this purpose we focus on a certain set of local operators with two different types of covariant derivatives acting on complex scalar fields. We analyse the corresponding nested Bethe equations for the ground states in the limit of large spins. The auxiliary Bethe roots form certain string configurations in the complex plane, which enable us to derive integral equations for the leading and sub-leading contribution to the anomalous dimension. The results can be expressed through the observables of the $\mathfrak{sl}(2)$ sub-sector, i.e. the cusp anomaly $f(g)$ and the virtual scaling function $B_L(g)$, rendering the strong-coupling analysis straightforward. Furthermore, we also study a particular sub-class of these operators specialising to a scaling limit with finite values of the second spin at weak and strong coupling.

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1 Introduction and Summary

Twist operators have so far played a major role in dynamical tests of the AdS/CFT correspondence in the planar limit. The main reason is their special scaling property at large values of the Lorentz spin. In particular, the anomalous dimension grows logarithmically with the spin, *cf.* [1]. This scaling behaviour is not a unique feature of the maximally supersymmetric Yang-Mills theory in four dimensions, but rather a special case of the so-called Sudakov scaling [2] common to many gauge theories.

A typical representative of these operators in the $\mathfrak{sl}(2)$ sector of $\mathcal{N}=4$ SYM is built from L complex scalar fields, $\mathcal{Z}$, and M covariant light-cone derivatives, $\mathcal{D}$, acting on the scalar background fields within the trace

$$\operatorname{tr}\mathcal{D}^M\mathcal{Z}^L+\dots\,. \tag{1.1}$$

The anomalous dimension of these operators occupy a band [3]. A distinguished sub-set among these is formed by the ground states, i.e. the lowest operators in the band, which enjoy several additional symmetry properties. For the two lowest possible values of the length, $L=2$ and for the ground state of $L=3$, analytic expressions for the anomalous dimensions can be found at high orders in perturbation theory [4, 5].

For twist-two operators these coincide with the maximal transcendental terms [6] of the known QCD results up to three-loop order, see [7] and references therein. At four-loop order the splitting functions of QCD are unknown, nevertheless the anomalous dimension of these operators may be determined [8, 9]. The result agrees with constraints from the BFKL equation [6]. Moreover, the result for $M=2$ coincides with the explicit Feynman diagram computations of [10].

The ground states for $L=3$ enjoy a similar solvability and the leading wrapping correction may be explicitly found [11]. In the special case of two excitations, the result is confirmed by the super-graph computation of [12].

For $L>3$ it is unknown whether closed expressions for the anomalous dimensions can be found, even for the ground states. Nevertheless, the anomalous dimensions of the latter enjoy very interesting scaling properties in the limit $M\to\infty$,

$$\gamma_{\mathrm{sl}(2)}(L,M)=f(g)(\log M+\gamma_{\mathrm{E}}-(L-2)\log 2)+B_L(g)+\mathcal{O}\big(\tfrac{1}{\log M}\big)\,. \tag{1.2}$$

The function $f(g)$ is also referred to as the cusp anomalous dimension. It is conjectured to be independent of the length L and consequently not influenced by wrapping interactions. Thus, one can use the asymptotic Bethe equations to derive an integral equation [13], which allows to compute $f(g)$ to arbitrary loop order. The scaling function resulting from the weak-coupling solution of this equation coincides up to four-loop order with the explicit perturbative computations of [14]. At strong coupling, the solution to this integral equation [15, 16] leads to a remarkable agreement with the corresponding string theory results, see [17, 18, 19]. Hence $f(g)$ embodies the first known interpolating function of AdS/CFT. A special phenomenological interest in this object is due to its appearance in multi-loop gluon scattering amplitudes as well as in expectation values of certain Wilson lines. That is, the scaling function $f(g)$ determines the leading $1/\epsilon^2$ pole structure of the logarithm of gluon amplitudes [14] as well as the logarithmic growth of the anomalous dimension of light-like Wilson loops with a cusp [20], as first noted in the strong coupling limit [21].

The virtual scaling function, $B_L(g)$, appearing in (1.2) explicitly depends on the twist L and it is less obvious that it remains unaffected by wrapping effects. The integral equation corresponding to $B_L(g)$ has been derived in [22], see also [23]. Interestingly, the solution to this equation may be related to the solution of the integral equation determining $f(g)$. This intertwines the strong coupling analysis of both functions and the methods developed for the cusp anomalous dimension may be directly applied also to the case of the virtual scaling function. The resulting strong-coupling expansion [22] is in perfect agreement with the string theory predictions at leading and next-to-leading order in λ, see [24], suggesting that the wrapping interactions can be neglected also for the first finite-spin corrections. Also this quantity appears in the context of gluon amplitudes and Wilson loops. It enters the sub-leading $1/\epsilon$ poles as part of the collinear anomalous dimension, see [25].

In this paper we will go beyond the $\mathfrak{sl}(2)$ sector. We introduce and investigate a gauge theory dual of a minimal energy spinning string configuration with two spins, S_1 and S_2, in AdS_5 and charge J in S^5. The field content of these operators can be schematically represented by

$$\operatorname{tr} \mathcal{D}^{n+m} \dot{\mathcal{D}}^m \mathcal{Z}^L \,. \tag{1.3}$$

The charges of the string are related to m and n through the identification

$$S_1 = n + m - \tfrac{1}{2} \,, \qquad S_2 = m - \tfrac{1}{2} \,, \qquad J = L \,. \tag{1.4}$$

At weak coupling we extensively examine the limit $m, n \to \infty$ with $n/m = \alpha$ fixed, in which $S_1, S_2 \to \infty$ with $S_1/S_2 = 1 + \alpha$ fixed. We start with the analysis of the corresponding nested one-loop Bethe equations. Surprisingly, the states with minimal length, $L = 3$, are again solvable and the respective Baxter functions may be found. While we extensively use this analytic solution for a numerical study of the behaviour of the one-loop Bethe roots in the large m, n limit, we defer its derivation to Appendix A. We find that the auxiliary roots may be decoupled at the first two orders in $m, n \gg 1$, leaving a remainder in the main equation. Upon introducing the density of roots, this effective equation may be turned into a solvable integral equation. Subsequently, we use its leading one-loop solution as the starting point for the derivation of the all-loop integral equation for the density. The solution to this all-loop equation allows to determine the corresponding anomalous dimension. For the first two orders in m, n, we find

$$\gamma(L, m, \alpha) = \frac{3}{2}\, \gamma_{\mathrm{sl}(2)}(\tfrac{2}{3}\, L, m) + \frac{f(g)}{2} \log\Big(\frac{1}{2}(1+\alpha)(2+\alpha)\Big) + \mathcal{O}\Big(\frac{1}{\log m}\Big) \,. \tag{1.5}$$

This result is very surprising, since it suggests a deep relation between the one-spin and two-spin solutions at the first two leading orders. This unexpected link calls for further investigation on the string theory side.

At leading order in spin the result $\sim \frac{3}{2} f(g)$ agrees with the energy scaling of spiky-strings [26, 27] that consist of three arcs, each of them contributing a factor of $\frac{1}{2}$, see [28, 29]. At finite order the results differ [24, 30].

Very recently it was shown by A. Tirziu and A. Tseytlin that in the case of $\alpha = 0$ the string theory calculation agrees with the formula (1.5) at leading order in λ, see [31]. This gives strong evidence for the absence of wrapping effects in our gauge theory calculation and justifies the use of the asymptotic Bethe equations.

We furthermore examine the large n limit of (1.3) for arbitrary finite values of $m \geq 2$. The strong coupling solution cannot be expressed in terms of known observables. However, the same methodology as in [22] can be used, to solve the corresponding integral equation at strong coupling. At leading order in n this solution resembles the folded string configuration.

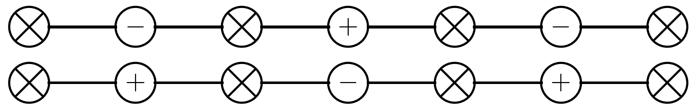

Figure 1: Dynkin diagrams of $\mathfrak{su}(2,2|4)$ with different gradings [32].

2 Definitions and the string theory dual

The highest-weight states corresponding to operators (1.3) have the following Dynkin labels with respect to the upper Dynkin diagram in figure 1, *cf.* [32],

$$(\Delta_0, s_1, s_2, q_1, p, q_2, B, L) = (L + 2m + n - \tfrac{3}{2}, 2m + n - 1, n, 1, L - 3, 2, \tfrac{1}{2}, L)\,. \tag{2.1}$$

This suggests that the string theory dual of these operators is a spinning string with two spins S_1, S_2 on AdS_5 and charge L on the S^5. Upon the usual $SO(4)$ rotation, we can read of the spins directly from (2.1)

$$S_1 = m + n - \tfrac{1}{2}\,, \qquad S_2 = m - \tfrac{1}{2}\,. \tag{2.2}$$

In view of the comparison with the string results, we will consider large values of m and n while fixing their ratio to $\frac{n}{m} = \alpha$. For finite values of α both spins become large and

$$\frac{S_1}{S_2} = 1 + \alpha\,. \tag{2.3}$$

In particular, the case of $\alpha = 0$ corresponds to the symmetric case of the spinning string with equal spins. The well-studied case of the folded string with one large spin, on the other hand, corresponds to $\alpha = \infty$. In this publication, we will only study the gauge theory states with minimal anomalous dimension, which correspond to minimal energy states on the string theory side.

In contradistinction to the $\mathfrak{sl}(2)$ sub-sector, the Bethe equations corresponding to the operators (1.3) are nested. The number of nesting levels depends on the choice of representation. The minimal number of levels is equal to three. However, for the sake of convenience, we will mostly use a non-minimal representation corresponding to the lower Dynkin diagram in figure 1.

A subset of (1.3) with $L = 3$ has already been studied in the literature, *cf.* [33]. It was found experimentally that the corresponding one-loop anomalous dimension is given by the following closed formula

$$\gamma_{n,m} = 4\,H_1\left(\tfrac{m}{2} - \tfrac{1}{2}\right) + 4\,H_1\left(m + \tfrac{n}{2}\right) + 4\,H_1\left(\tfrac{m}{2} + \tfrac{n}{2}\right) - 4\,H_1\left(-\tfrac{1}{2}\right)\,. \tag{2.4}$$

Here, $H_1(N)$ denotes N-th harmonic number. We will prove this formula in Appendix A, by constructing an explicit one-loop solution for $L = 3$. This is a counterpart of the $L = 2$ solution found in the $\mathfrak{sl}(2)$ sector [34]. The corresponding Baxter functions (see (A.28), (A.29) and (A.34)) become quite complicated and are not hypergeometric orthogonal polynomials anymore. Nevertheless, the explicit one-loop solution enormously facilitates numerical studies of the Bethe solutions at large values of m and n.

3 Large spins solution at one-loop order

In this section we will analyse the operators (1.3) in the limit $m, n \to \infty$ and $n/m = \alpha =$ fixed at leading order in perturbation theory. As in the case of the $\mathfrak{sl}(2)$ operators [34], the leading solution *does not* depend on the length L. The minimal set of equations at one-loop consists of two coupled nesting levels

$$\left(\frac{u_{4,k}+\frac{i}{2}}{u_{4,k}-\frac{i}{2}}\right)^L = \prod_{\substack{j=1\\ j\neq k}}^{n+2m} \frac{u_{4,k}-u_{4,j}-i}{u_{4,k}-u_{4,j}+i} \prod_{j=1}^{m} \frac{u_{4,k}-u_{5,j}+\frac{i}{2}}{u_{4,k}-u_{5,j}-\frac{i}{2}} \tag{3.1}$$

$$1 = \prod_{\substack{j=1\\ j\neq k}}^{m} \frac{u_{5,k}-u_{5,j}-i}{u_{5,k}-u_{5,j}+i} \prod_{j=1}^{n+2m} \frac{u_{5,k}-u_{4,j}+\frac{i}{2}}{u_{5,k}-u_{4,j}-\frac{i}{2}}\,. \tag{3.2}$$

From the numerical studies of the analytic solution for $L = 3$ presented in Appendix A we infer that the roots u_5 form strings

$$\begin{aligned} &u_5 \to \pm i/2, \pm 3i/2, \pm \ldots \quad m \text{ even},\\ &u_5 \to 0, \pm i, \pm 2i, \pm \ldots \quad m \text{ odd}. \end{aligned} \tag{3.3}$$

However, it is *incorrect* to assume, even at the leading order, that all u_5 will exhibit this behaviour. We thus introduce an effective cut-off $c(\alpha)$ such that

$$\begin{aligned} &u_5 \to \pm i/2, \pm 3i/2, \ldots, \pm i(m/c(\alpha) - 1/2) \quad m \text{ even},\\ &u_5 \to 0, \pm i, \pm 2i, \ldots, \pm i(m/c(\alpha) - 1/2) \quad m \text{ odd}. \end{aligned} \tag{3.4}$$

In the following we will explicitly fix $c(\alpha)$ in the limits $m, n \to \infty$. We have also checked numerically that the remaining roots, i.e. not belonging to the effective strings (3.4), scale as $\sim m^2$ at large values of m and thus are not relevant for the leading and the sub-leading order. In the large n limit and for finite values of m the strings (3.3), as we will show in what follows, become exact and $c(\infty) = 2$ as expected. In either case, one can completely decouple the u_5 roots. The effective equation for the middle-node roots thus takes the following form

$$\left(\frac{u_{4,k}+\frac{i}{2}}{u_{4,k}-\frac{i}{2}}\right)^L = \left(\frac{u_{4,k}+\frac{i\,m}{c(\alpha)}}{u_{4,k}-\frac{i\,m}{c(\alpha)}}\right) \prod_{\substack{j=1\\ j\neq k}}^{m(\alpha+2)} \frac{u_{4,k}-u_{4,j}-i}{u_{4,k}-u_{4,j}+i}\,. \tag{3.5}$$

In the large m and/or n limit this equation may be turned into an integral equation along the lines presented in [34]. Explicitly, one obtains

$$\begin{aligned} \frac{2\,L}{m(\alpha+2)} \arctan(2u_k) = &\frac{2\,\pi}{m(\alpha+2)} \widetilde{n}(u) - 2\int_{-b(\alpha)}^{b(\alpha)} du' \rho(u') \arctan(u-u') \\ &+ \frac{2}{m(\alpha+2)} \arctan(\tfrac{c(\alpha)\,u_k}{m})\,. \end{aligned} \tag{3.6}$$

Here, $\widetilde{n}(u)$ is the function for the mode numbers. It is related to the density $\rho_0(u)$ through

$$\widetilde{n}(u) = \frac{L-3}{2}\text{sgn}(u) - \frac{m(\alpha+2)}{2} + \int_{-b(\alpha)}^{u} du' \rho(u')\,. \tag{3.7}$$

3.1 The $\alpha = 0$ solution

Upon rescaling the momentum-carrying roots $u_4 = 2m\,\bar{u}_4$ and the corresponding density $\bar{\rho}_0(\bar{u}) = 2m\,\rho_0(u)$, the leading part of the equation (3.6) is given by

$$3\pi\,\mathrm{sgn}(\bar{u}) - 2\,\arctan(2\,c\,\bar{u}) + 2 \fint_{-\bar{b}(0)}^{\bar{b}(0)} d\bar{u}'\,\frac{\bar{\rho}_0(\bar{u}')}{(\bar{u}-\bar{u}')} = 0\,. \tag{3.8}$$

The solution to this equation is given by

$$\bar{\rho}_0(\bar{u}) = \frac{3}{2}\,\bar{\rho}_{\mathrm{K}}\Big(\frac{\bar{u}}{2\,\bar{b}(0)}\Big) - \frac{1}{2\pi}\log\left(\frac{\sqrt{1-\frac{\bar{u}^2}{\bar{b}(0)^2}} + \frac{\sqrt{4\bar{b}(0)^2\,c^2+1}}{2\bar{b}(0)c}}{\frac{\sqrt{4\bar{b}(0)^2c^2+1}}{2\bar{b}(0)c} - \sqrt{1-\frac{\bar{u}^2}{\bar{b}(0)^2}}}\right)\,. \tag{3.9}$$

Here, with $\rho_{\mathrm{K}}(u)$ we have denoted the one-loop density of the Bethe roots corresponding to the ground state of twist-L operators in $\mathfrak{sl}(2)$. The normalisation condition

$$1 = \int_{-\bar{b}(0)}^{\bar{b}(0)} d\bar{u}\,\bar{\rho}_0(\bar{u})\,, \tag{3.10}$$

relates the boundary of the root distribution to the dimension of the effective strings c. Explicitly, one finds

$$\bar{b}(0) = \frac{1}{16\,c}(\sqrt{9-4\,c+4\,c^2}+6\,c-3)\,. \tag{3.11}$$

To determine the constant c we proceed as follows. The density $\rho_0(u)$ has the following large m expansion

$$\rho_0(u) = \frac{3}{4m}\left(\frac{2}{\pi}\log m + C - \frac{2}{\pi^2}\log(u^2)\right) + \dots\,, \tag{3.12}$$

with the constant term C given by

$$C = \frac{2}{\pi}\log\big(4\,\bar{b}(0)\big) - \frac{2}{3\pi}\log\left(2\,\bar{b}(0)\,c + \sqrt{4\,\bar{b}(0)^2\,c^2+1}\right)\,. \tag{3.13}$$

One may thus split the density in (3.6) as follows

$$\rho(u) = \rho_0(u) + r(u)\,, \tag{3.14}$$

and use the expansion (3.12) to obtain a leading integral equation for $r(u)$. Upon Fourier transformation one finds

$$\hat{r}(t) = \frac{1}{2m}\left(3\frac{e^{-|t|/2}}{1-e^{-|t|}} - \frac{3}{|t|} - \frac{L}{1+e^{-|t|/2}}\right)\,. \tag{3.15}$$

One can now use (3.14) to calculate the one-loop anomalous dimension. After straightforward integration one finds

$$\gamma_0 = 12\log m + 12\gamma_{\mathrm{E}} - 8(L-3)\log 2 + 6\,\pi\,C. \tag{3.16}$$

This should be compared with the large m expansion of the $L=3$ analytic result (2.4)

$$\gamma_0^{L=3} = 12\log m + 12\gamma_{\mathrm{E}}\,. \tag{3.17}$$

Thus, the constant c is determined by the condition $C = 0$ in conjunction with (3.11). Numerically one may determine $c = 2.83181(\dots)$, which means that approximately $\frac{7}{10}$ of the u_5 roots form effective strings. We have checked numerically that $\bar{b}(0)$ obtained by inserting this value of c is consistent with the scaling properties of the largest u_4 roots.

3.2 The solution for general $\alpha > 0$

The limit $m \to \infty, n \to \infty$ with $\frac{n}{m} = \alpha =$ fixed is a simple generalisation of the $\alpha = 0$ case discussed above. The roots u_5 form again strings, and one expects the cut-off parameter to depend on α, i.e. $c = c(\alpha)$. Upon rescaling the roots by $n + 2m = m(\alpha + 2)$, one derives the following integral equation

$$3\pi\,\mathrm{sgn}(\bar{u}) - 2\,\arctan((2+\alpha)\,c(\alpha)\,\bar{u}) + 2 \fint_{-\bar{b}(\alpha)}^{\bar{b}(\alpha)} d\bar{u}' \frac{\bar{\rho}_0(\bar{u}', \alpha)}{(\bar{u} - \bar{u}')} = 0\,. \tag{3.18}$$

The solution to this equation is given by

$$\bar{\rho}_0(\bar{u}, \alpha) = \frac{3}{2}\,\bar{\rho}_{\mathrm{K}}\Big(\frac{\bar{u}}{2\,\bar{b}(\alpha)}\Big) - \frac{1}{2\pi} \log \left(\frac{\sqrt{1 - \frac{\bar{u}^2}{\bar{b}(\alpha)^2}} + \frac{\sqrt{\bar{b}(\alpha)^2(\alpha+2)^2 c(\alpha)^2 + 1}}{\bar{b}(\alpha)(\alpha+2)c(\alpha)}}{\frac{\sqrt{\bar{b}(\alpha)^2(\alpha+2)^2 c(\alpha)^2 + 1}}{\bar{b}(\alpha)(\alpha+2)c(\alpha)} - \sqrt{1 - \frac{\bar{u}^2}{\bar{b}(\alpha)^2}}} \right)\,. \tag{3.19}$$

The normalisation condition yields the relation

$$\bar{b}(\alpha) = \frac{1}{8\,(2+\alpha)\,c(\alpha)} \left(-3 + 3\,(2+\alpha)\,c(\alpha) + \sqrt{9 - 2\,(2+\alpha)\,c(\alpha) + (2+\alpha)^2\,c(\alpha)^2} \right)\,. \tag{3.20}$$

The constant $c(\alpha)$ may be determined by the same procedure as in the $\alpha = 0$ case. The denisty $\rho_0(u)$ has the following large m expansion

$$\rho_0(u, \alpha) = \frac{3}{2m(2+\alpha)} \left(\frac{2}{\pi} \log m + C(\alpha) - \frac{2}{\pi^2} \log(u^2) \right)\,,$$

with

$$C(\alpha) = \frac{2}{\pi} \log\big(2\,\bar{b}(\alpha)(\alpha+2)\big) - \frac{2}{3\pi} \log\left(\bar{b}(\alpha)(\alpha+2)c(\alpha) + \sqrt{\bar{b}(\alpha)^2(\alpha+2)^2 c(\alpha)^2 + 1} \right)\,. \tag{3.21}$$

Splitting again the leading density as

$$\rho(u, \alpha) = \rho_0(u, \alpha) + r(u, \alpha) + \ldots\,, \tag{3.22}$$

one determines $\hat{r}(t, \alpha)$ to be

$$\hat{r}(t, \alpha) = \frac{1}{m(2+\alpha)} \left(3 \frac{e^{-|t|/2}}{1 - e^{-|t|}} - \frac{3}{|t|} - \frac{L}{1 + e^{-|t|/2}} \right)\,. \tag{3.23}$$

This immediately leads to

$$\gamma_0 = 12 \log m + 12\gamma_{\mathrm{E}} - 8(L-3)\log 2 + 6\,\pi\,C(\alpha). \tag{3.24}$$

Expanding (2.4) for general α, one finds

$$\gamma_0 = 12 \log m + 12\gamma_{\mathrm{E}} + 4 \log\left(\tfrac{1}{2}(1+\alpha)(2+\alpha)\right)\,. \tag{3.25}$$

One thus concludes that

$$3\,\pi\,C(\alpha) = 2 \log\left(\tfrac{1}{2}(1+\alpha)(2+\alpha)\right)\,. \tag{3.26}$$

The above formula in conjunction with (3.21) and (3.20) determines $c(\alpha)$ uniquely. The behaviour of $c(\alpha)$ as a function of α is shown in figure 2. Clearly, the effective strings become exact in the large α limit

$$\lim_{\alpha \to \infty} c(\alpha) \to 2\,. \tag{3.27}$$

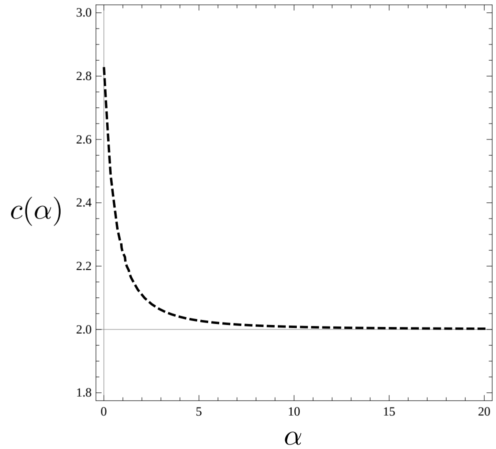

Figure 2: The plot of $c(\alpha)$ as function of α.

3.3 The $\alpha = \infty$ solution

In view of (3.27), one concludes that for $\alpha \to \infty$ all roots u_5 form strings as in (3.3). Rescaling the main roots as $u_4 = n\,\bar{u}_4$ and taking the large n limit we find

$$0 = 2\pi\,\mathrm{sgn}(\bar{u}) - 2 \fint_{-b/n}^{b/n} d\bar{u}' \frac{\bar{\rho}_0(\bar{u})}{(\bar{u} - \bar{u}')}. \tag{3.28}$$

The above equation, together with the momentum constraint which fixes b to $n/2$, is solved by the known $\mathfrak{sl}(2)$ density, $\bar{\rho}_{\mathrm{K}}(\bar{u})$, found in [35]. This formally corresponds to taking $\alpha \to \infty$, although the latter is not unique due to the fact that any dependence on finite values of the spin S_2 is lost.

4 The all-loop equations

Beyond the one-loop order the spectral equations for the operators (1.3) can be derived from the full system of the asymptotic Bethe equations conjectured in [32]. In what follows, we will work with a set of four coupled Bethe equations

$$\begin{aligned}
\left(\frac{x^+_{4,k}}{x^-_{4,k}}\right)^L &= \prod_{j\neq k}^{n+2m} \frac{x^-_{4,k} - x^+_{4,j}}{x^+_{4,k} - x^-_{4,j}} \frac{1 - g^2/x^+_{4,k}x^-_{4,j}}{1 - g^2/x^-_{4,k}x^+_{4,j}}\, \sigma^2(u_{4,k}, u_{4,j}) \\
&\times \prod_{j=1}^{m} \frac{x^+_{4,k} - x_{5,j}}{x^-_{4,k} - x_{5,j}} \prod_{j=1}^{m-2} \frac{1 - g^2/x^+_{4,k}x_{7,j}}{1 - g^2/x^-_{4,k}x_{7,j}} \\
1 &= \prod_{j=1}^{m-1} \frac{u_{5,k} - u_{6,j} - i/2}{u_{5,k} - u_{6,j} + i/2} \prod_{j=1}^{n+2m} \frac{x_{5,k} - x^-_{4,j}}{x_{5,k} - x^+_{4,j}} \\
1 &= \prod_{j=1}^{m-1} \frac{u_{6,k} - u_{6,j} + i}{u_{6,k} - u_{6,j} - i} \prod_{j=1}^{m} \frac{u_{6,k} - u_{5,j} - i/2}{u_{6,k} - u_{5,j} + i/2} \prod_{j=1}^{m-2} \frac{u_{6,k} - u_{7,j} - i/2}{u_{6,k} - u_{7,j} + i/2} \\
1 &= \prod_{j=1}^{m-1} \frac{u_{7,k} - u_{6,j} - i/2}{u_{7,k} - u_{6,j} + i/2} \prod_{j=1}^{n+2m} \frac{1 - g^2/x_{7,k}x^-_{4,j}}{1 - g^2/x_{7,k}x^+_{4,j}}.
\end{aligned} \tag{4.1}$$

Our convention for the coupling constant is $g^2 = g^2_{\rm YM} N_c/(16\pi^2)$. The deformation of the spectral parameter reads $x(u) = \frac{1}{2}(u + \sqrt{u^2 - 4g^2})$, with the conventional notation $x^\pm = x(u \pm \frac{i}{2})$. The form of the dressing factor σ^2 is given in [13]. Note that at one-loop order, the generating functions of the u_6 and u_7 roots can be immediately obtained from the u_5 roots. Consequently, these roots inherit the behaviour of the u_5 roots and also form effective strings. We will denote the effective cut-off parameter for the u_7 roots by $d(\alpha)$.

In the large m, n limit, the leading positions of the inner u_5 roots are again given by (3.4). The quantum corrections to these roots vanish as $m, n \to \infty$. The outer roots, on the other hand, grow very rapidly. The same is true for the auxiliary roots u_7. Thus, effectively, the system (4.1) reduces to

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{j\neq k}^{n+2m} \frac{u_k - u_j - i}{u_k - u_j + i} \left(\frac{1 - g^2/x_k^+ x_j^-}{1 - g^2/x_k^- x_j^+}\right)^2$$
$$\times \frac{u_k + i\frac{m}{c(\alpha)}}{u_k - i\frac{m}{c(\alpha)}} \, \frac{1 - g^2/x_k^- x(i\frac{m}{c(\alpha)})}{1 + g^2/x_k^+ x(i\frac{m}{c(\alpha)})} \, \frac{1 + g^2/x_k^- x(i\frac{m}{d(\alpha)})}{1 - g^2/x_k^+ x(i\frac{m}{d(\alpha)})} \sigma^2(u_k, u_j). \tag{4.2}$$

From this set of effective Bethe equations we will derive an integral for the fluctuation density, $\sigma(g, u, \alpha)$,

$$\rho(g, u, \alpha) = \rho(u, \alpha) - \frac{8\, g^2}{m(\alpha + 2)}\, \sigma(g, u, \alpha)\,. \tag{4.3}$$

In what follows, we will treat the cases $\alpha = 0$, $\alpha > 0$ and $\alpha = \infty$ separately, even though the first two directly interpolate between each other. The case of $\alpha = \infty$ requires a more careful analysis.

4.1 The equal spin case $\alpha = 0$

Knowing (3.12) and (3.15), one can easily derive the integral equation for the Fourier-Laplace transform[1] of the fluctuation density $\hat\sigma(g, t, 0)$

$$\hat\sigma(g,t,0) = \frac{t}{e^t - 1}\Big[\big(\tfrac{3}{2}\log m + \tfrac{3}{2}\gamma_{\rm E} - (L-3)\log 2\big)\, K(2gt, 0) - \frac{L}{8g^2 t}\,(J_0(2gt) - 1)$$
$$+ \frac{1}{2}\int_0^\infty dt' \Big(\frac{3}{e^{t'} - 1} - \frac{L-3}{e^{t'/2} + 1}\Big)\,\big(K(2gt, 2gt') - K(2gt, 0)\big)$$
$$-4g^2 \int_0^\infty dt' K(2gt, 2gt')\hat\sigma(g, t', 0)\Big]\,. \tag{4.4}$$

It is written in terms of the usual integral kernel $K(t, t') = K_0(t, t') + K_1(t, t') + K_d(t, t')$, with the parity even and odd components given respectively by [13]

$$K_0(t,t') = \frac{tJ_1(t)J_0(t') - t'J_0(t)J_1(t')}{t^2 - t'^2} = \frac{2}{tt'}\sum_{n=1}^\infty (2n-1) J_{2n-1}(t) J_{2n-1}(t')\,,$$
$$K_1(t,t') = \frac{t'J_1(t)J_0(t') - tJ_0(t)J_1(t')}{t^2 - t'^2} = \frac{2}{tt'}\sum_{n=1}^\infty (2n) J_{2n}(t) J_{2n}(t')\,. \tag{4.5}$$

[1] We define the Fourier-Laplace transform of the fluctuation density by $\hat\sigma(t) = e^{-\frac{t}{2}} \int_{-\infty}^{\infty} du\, e^{-itu} \sigma(u)$.

The dressing kernel $K_d(t,t')$ is a convolution of the even and the odd part

$$K_d(t,t) = 8g^2 \int_0^\infty dt'' K_1(t, 2gt'') \frac{t''}{e^{t''}-1} K_0(2gt'', t') \,. \tag{4.6}$$

The anomalous dimension corresponds to the value of $\hat{\sigma}(g,t,0)$ at the origin

$$\gamma(L,m) = 16\, g^2\, \hat{\sigma}(g,0,0) \,. \tag{4.7}$$

By comparing (4.4) with the corresponding equation for the $\mathfrak{sl}(2)$ sector [22], one easily infers that terms proportional to $K(2gt,0)$ in the first line of (4.4) give rise to the cusp anomalous dimension $f(g)$, while the remaining terms yield the virtual scaling function $B_2(g)$ obtained in [22], and the first generalised scaling function $\epsilon_1(g)$ of [36, 37]. Hence, the anomalous dimension is given by

$$\begin{aligned}\gamma(L,m) &= \frac{3}{2} f(g)\,(\log m + \gamma_{\mathrm{E}}) + (L-3)\epsilon_1(g) + \frac{3}{2} B_2(g) + \mathcal{O}\Big(\frac{1}{\log m}\Big) \\ &= \frac{3}{2}\, \gamma_{\mathrm{sl}(2)}(\tfrac{2}{3}L, m) + \mathcal{O}\Big(\frac{1}{\log m}\Big) \,.\end{aligned} \tag{4.8}$$

Remarkably, the large m anomalous dimension is up to the order $\mathcal{O}(\frac{1}{\log m})$ proportional to the anomalous dimension of the twist operators. This can be traced back to the "screening properties" of the u_5 roots, *cf.* (3.4). It should be noted that although we have assumed $\alpha = 0$, one does not need to put $n = 0$. In contrary, (4.8) is valid also in the latter case and the first finite n corrections are sub-leading. We expect, in similarity to the case of $\mathfrak{sl}(2)$ operators, that the wrapping corrections will not affect the first two orders in the large spin, $S = 2S_1$, expansion. The findings of [31] confirm this hypothesis.

4.2 The case of $\alpha > 0$

It is straightforward to repeat the computation of the preceding paragraph for $\alpha > 0$, the only difference being an additional one-loop term in the energy, *cf.* (3.25). The resulting equation for $\hat{\sigma}(g,t,\alpha)$ takes the following form

$$\begin{aligned}\hat{\sigma}(g,t,\alpha) = \frac{t}{e^t-1}\Big[&\left(\tfrac{3}{2}\log m + \tfrac{3}{2}\gamma_{\mathrm{E}} - (L-3)\log 2 + \tfrac{1}{2}\log\left(\tfrac{1}{2}(1+\alpha)(2+\alpha)\right)\right) K(2gt,0) \\ &- \frac{L}{8g^2 t}\,(J_0(2gt)-1) + \frac{1}{2}\int_0^\infty dt' \Big(\frac{3}{e^{t'}-1} - \frac{L-3}{e^{t'/2}+1}\Big)\,(K(2gt,2gt') - K(2gt,0)) \\ &- 4g^2 \int_0^\infty dt' K(2gt,2gt')\hat{\sigma}(g,t',\alpha)\Big] \,.\end{aligned} \tag{4.9}$$

The anomalous dimension can be easily found

$$\gamma_L(g,m) = \frac{3}{2}\gamma_{\mathrm{sl}(2)}(\tfrac{2}{3}L,m) + \frac{f(g)}{2}\log\Big(\frac{1}{2}(1+\alpha)(2+\alpha)\Big) + \mathcal{O}\Big(\frac{1}{\log m}\Big) \,. \tag{4.10}$$

It is noteworthy that the dependence on α is logarithmic and that the corresponding prefactor is again proportional to $f(g)$. It would be interesting to understand from a string theory perspective why the energy of the general two-spin solution so closely resembles the one-spin solution.

4.3 One large spin limit

In this section we will discuss the case of $n \to \infty$ and *finite* values of m. It turns out that, contrary to the $\alpha = 0$ case, the sub-leading correction exhibits an interesting dependence on m.

The one-loop leading solution has been discussed in section 3.3 and is fully equivalent to the one-loop problem for the ground states in the $\mathfrak{sl}(2)$ sub-sector, *cf.* [34]. Proceeding in the same spirit as in the previous sections, we split off the leading density

$$\rho(u) = \rho_0(u) + r(u) \tag{4.11}$$

and derive a leading equation for $r(u)$, which subsequently may be solved by Fourier transformation

$$\hat{r}(t) = \frac{1}{n}\Big(- \frac{L-3}{1+e^{-|t|/2}} + 3\frac{e^{-|t|/2}}{1-e^{-|t|}} - \frac{e^{-m|t|/2}}{1-e^{-|t|}} - \frac{2}{|t|}\Big). \tag{4.12}$$

Using this expression, it is straightforward to calculate the one-loop anomalous dimension for the first two orders in n

$$E_0 = 4\log n + 6\gamma_{\rm E} + 2\psi_0(\tfrac{m+1}{2}) - 4(L-3)\log 2\,. \tag{4.13}$$

The derivation of higher-loop corrections goes along similar lines as in the preceding sections. Upon defining the fluctuation density by

$$\rho(g,u,m) = \rho(u,m) - \frac{8\,g^2}{n}\,\sigma(g,u,m)\,, \tag{4.14}$$

one derives the following closed integral equation for $m \geq 2$

$$\begin{aligned}\hat{\sigma}(g,t,m) = \frac{t}{e^t-1}\Big[&\big(\log n + \tfrac{3}{2}\gamma_{\rm E} + \tfrac{1}{2}\psi_0(\tfrac{m+1}{2}) - (L-3)\log 2\big)\,K(2gt,0) - \frac{L}{8g^2t}\,(J_0(2gt)-1)\\ &+\frac{1}{2}\int_0^\infty dt'\Big(\frac{3}{e^{t'}-1} - \frac{e^{-(m+1)t'/2}}{1-e^{-t'}} - \frac{L-3}{e^{t'/2}+1}\Big)\big(K(2gt,2gt') - K(2gt,0)\big)\\ &-4g^2\int_0^\infty dt' K(2gt,2gt')\hat{\sigma}(g,t') - \frac{1}{2}\int_0^\infty K_1(2gt,2gt')e^{-t'\frac{m-1}{2}}\,.\end{aligned} \tag{4.15}$$

The resulting anomalous dimension scales logarithmically with n

$$\gamma(L,n,m) = 16g^2\sigma(g,0,m) = f(g)\log n + \dots\,, \tag{4.16}$$

while the finite-spin corrections depend explicitly on m, as can be directly inferred from (4.15). At weak-coupling, one can easily determine the perturbative expansion at the first few orders

$$\begin{aligned}\gamma(L,n,m) = &f(g)(\log n + \tfrac{3}{2}\gamma_{\rm E} + \tfrac{1}{2}\psi_0(\tfrac{m+1}{2}) - (L-3)\log 2) - 2g^4(\psi_2(\tfrac{m+1}{2}) + 2(21-4L)\zeta(3))\\ &+\tfrac{1}{3}g^6\Big(\psi_4(\tfrac{m+1}{2}) + 2\pi^2\psi_2(\tfrac{m+1}{2}) + 24\psi_1(\tfrac{m+1}{2})\zeta(3) + \tfrac{96}{(m-1)^2}\zeta(3)\\ &+8(6-L)\pi^2\zeta(3) + 72(31-7L)\zeta(5)\Big) + \dots\,,\end{aligned} \tag{4.17}$$

For the choice of parameters $m = 2$ and $L = 3$ we find perfect agreement with the large n expansion of the anomalous dimension up to four-loop order which is available in [38].

However, since the g^8 contribution to (4.17) is quite lengthy, we merely give the first three loop orders.

Interestingly, equation (4.15) can also be solved at large values of the coupling by making use of the strong coupling expansion for twist operators in the $\mathfrak{sl}(2)$ sector [15]. We defer the strong-coupling analysis to Appendix B and only present the final result

$$\gamma(L,n,m) = \Big(4g - \frac{3\log 2}{\pi}\Big)\log\frac{n}{g} + 6\,g\,(\log 2 - 1) + (1-L) + \frac{2}{m-1} + \frac{m}{2} \\ + \frac{9\log 2}{\pi} - \frac{9(\log 2)^2}{2\pi} + \mathcal{O}\Big(\frac{1}{g}\Big)\,. \qquad (4.18)$$

5 Outlook

The appearance of the cusp anomalous dimension $f(g)$ and the virtual scaling function $B_L(g)$ beyond the $\mathfrak{sl}(2)$ sector in (1.5) is quite remarkable and certainly promotes their universality. For further sub-leading corrections in spin, the endpoints of the effective condensate may not be enough to completely determine the corresponding contribution. In this case the remaining roots should also be taken into account and it is questionable if the scaling still resembles the behaviour of twist operators.

It will be quite interesting to investigate if the generalised scaling function $f(g,j)$ of the $\mathfrak{sl}(2)$ sector [39] also appears in the refined limit $S\to\infty$, $L\to\infty$ with $j=\frac{L}{\log S}$ fixed.

In similarity to the known solvable cases of twist-two and three operators, it would be very interesting to see whether it is possible to construct higher loop contributions to the anomalous dimension (2.4) for general values of m and n. A first step in this direction has been made in [38], where the case of $m=2$ was analysed.

Furthermore, the decoupling procedure described in detail in Appendix A is based on iteratively splitting one bosonic node of the corresponding Dynkin diagram into two fermionic ones. Although this is straightforward at the level of the equations, it remains obscure to us what the corresponding algebraic interpretation might be.

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A The analytic one-loop solution

In this section we will study the following class of length-three operators

$$\mathrm{tr}\,\mathcal{D}^{n+m}\dot{\mathcal{D}}^{m}\mathcal{Z}^{3}\,. \tag{A.1}$$

For $m = 0$ this set reduces to twist-three operators of the $\mathfrak{sl}(2)$ sub-sector. Also the case of $m = 1$ is redundant as (A.1) then corresponds to descendents of twist-two operators. Therefore in what follows we will assume $m > 1$. A subgroup of these operators for $m = 2$ has been studied in [38] up to four-loop order. At one-loop order, on the other hand, a closed expression for the anomalous dimension of the ground states has been conjectured in [33] for *any* value of m and n, see (2.4). It is rather straightforward to prove this formula[2] using the analytic solution provided below.

The excitation pattern for the higher-loop Dynkin diagram in the upper part of figure 1 reads

$$(K_1, K_2, K_3, K_4, K_5, K_6, K_7) = (0, 0, n + 2m - 1, n + 2m, n + 2(m - 1), m - 1, 0)\,, \tag{A.2}$$

where K_ν denotes the excitation number of the ν-th node of the Dynkin diagram. After a dualisation of the u_3 roots the corresponding one-loop system of equations describing this class of operators is given by

$$\left(\frac{u_{4,k}+\frac{i}{2}}{u_{4,k}-\frac{i}{2}}\right)^3 = \prod_{j=1}^{n+2m-2}\frac{u_{4,k}-u_{5,j}-\frac{i}{2}}{u_{4,k}-u_{5,j}+\frac{i}{2}} \tag{A.3}$$

$$1 = \prod_{j=1}^{m-1}\frac{u_{5,k}-u_{6,j}+\frac{i}{2}}{u_{5,k}-u_{6,j}-\frac{i}{2}}\prod_{j=1}^{n+2m}\frac{u_{5,k}-u_{4,j}-\frac{i}{2}}{u_{5,k}-u_{4,j}+\frac{i}{2}} \tag{A.4}$$

$$1 = \prod_{\substack{j=1\\ j\neq k}}^{m-1}\frac{u_{6,k}-u_{6,j}+i}{u_{6,k}-u_{6,j}-i}\prod_{j=1}^{n+2m-2}\frac{u_{6,k}-u_{5,j}-\frac{i}{2}}{u_{6,k}-u_{5,j}+\frac{i}{2}}\,. \tag{A.5}$$

This set of Bethe equations is valid for $m \geq 1$. In the following section we will solve it *exactly* thanks to a hidden recurrence relation between roots for different value of m.

A.1 The one-loop recurrence

We start by dualizing equation (A.3). The system of equations (A.3)-(A.5) reduces to

$$\left(\frac{u_{5,k}+i}{u_{5,k}-i}\right)^3 = \prod_{\substack{j=1\\ j\neq k}}^{n+2m-2}\frac{u_{5,k}-u_{5,j}-i}{u_{5,k}-u_{5,j}+i}\prod_{j=1}^{m-1}\frac{u_{5,k}-u_{6,j}+\frac{i}{2}}{u_{5,k}-u_{6,j}-\frac{i}{2}} \tag{A.6}$$

$$1 = \prod_{\substack{j=1\\ j\neq k}}^{m-1}\frac{u_{6,k}-u_{6,j}-i}{u_{6,k}-u_{6,j}+i}\prod_{j=1}^{n+2m-2}\frac{u_{6,k}-u_{5,j}+\frac{i}{2}}{u_{6,k}-u_{5,j}-\frac{i}{2}}\,. \tag{A.7}$$

[2] M. Beccaria, private communication.

The derivation of the hidden recurrence is based on the observation that the last equation is equivalent to a system of two coupled equations

$$1 = \prod_{j=1}^{m-2} \frac{u_{6,k} - u_{7,j} - \frac{i}{2}}{u_{6,k} - u_{7,j} + \frac{i}{2}} \prod_{j=1}^{n+2m-2} \frac{u_{6,k} - u_{5,j} + \frac{i}{2}}{u_{6,k} - u_{5,j} - \frac{i}{2}} \tag{A.8}$$

$$1 = \prod_{j=1}^{m-1} \frac{u_{7,k} - u_{6,j} + \frac{i}{2}}{u_{7,k} - u_{6,j} - \frac{i}{2}}\,, \tag{A.9}$$

where we have introduced a new set of auxiliary roots u_7. We now introduce the Baxter function for the u_6 roots and their dual counterpart $\widetilde{u}_6$

$$\begin{aligned} R(u) \equiv & \prod_{j=1}^{m-2} (u - u_{7,j} - \tfrac{i}{2}) \prod_{j=1}^{n+2m-2} (u - u_{5,j} + \tfrac{i}{2}) - \prod_{j=1}^{m-2} (u - u_{7,j} + \tfrac{i}{2}) \prod_{j=1}^{n+2m-2} (u - u_{5,j} - \tfrac{i}{2}) \\ = & \, c_6 \prod_{j=1}^{m-1} (u - u_{6,j}) \prod_{j=1}^{n+2m-4} (u - \widetilde{u}_{6,j})\,. \end{aligned} \tag{A.10}$$

It is straightforward to derive the following two relations

$$\frac{R(u_{5,k} + \frac{i}{2})}{R(u_{5,k} - \frac{i}{2})} = \prod_{\substack{j=1 \\ j\neq k}}^{n+2m-2} \frac{u_{5,k} - u_{5,j} + i}{u_{5,k} - u_{5,j} - i} = \prod_{j=1}^{m-1} \frac{u_{5,k} - u_{6,j} + \frac{i}{2}}{u_{5,k} - u_{6,j} - \frac{i}{2}} \prod_{j=1}^{n+2m-4} \frac{u_{5,k} - \widetilde{u}_{6,j} + \frac{i}{2}}{u_{5,k} - \widetilde{u}_{6,j} - \frac{i}{2}}\,, \tag{A.11}$$

$$\frac{R(u_{7,k} + \frac{i}{2})}{R(u_{7,k} - \frac{i}{2})} = \prod_{\substack{j=1 \\ j\neq k}}^{m-2} \frac{u_{7,k} - u_{7,j} + i}{u_{7,k} - u_{7,j} - i} = \prod_{j=1}^{m-1} \frac{u_{7,k} - u_{6,j} + \frac{i}{2}}{u_{5,k} - u_{7,j} - \frac{i}{2}} \prod_{j=1}^{n+2m-4} \frac{u_{7,k} - \widetilde{u}_{6,j} + \frac{i}{2}}{u_{7,k} - \widetilde{u}_{6,j} - \frac{i}{2}}\,. \tag{A.12}$$

With the help of (A.11) we now rewrite (A.6) as

$$\left(\frac{u_{5,k} + i}{u_{5,k} - i}\right)^3 = \prod_{j=1}^{n+2m-4} \frac{u_{5,k} - \widetilde{u}_{6,j} - \frac{i}{2}}{u_{5,k} - \widetilde{u}_{6,j} + \frac{i}{2}}\,, \tag{A.13}$$

and once again dualize this set of equations by defining the polynomial $Q_5(u)$ as

$$Q_5(u) \equiv (u+i)^3 \prod_{j=1}^{n+2m-4} (u - \widetilde{u}_{6,j} + \tfrac{i}{2}) - (u-i)^3 \prod_{j=1}^{n+2m-4} (u - \widetilde{u}_{6,j} - \tfrac{i}{2}) = c_5 \prod_{j=1}^{n+2m-2} (u - u_{5,j}). \tag{A.14}$$

The function $Q_5(u)$ obeys the relation

$$\frac{Q_5(\widetilde{u}_{6,k} + \frac{i}{2})}{Q_5(\widetilde{u}_{6,k} - \frac{i}{2})} = \left(\frac{\widetilde{u}_{6,k} + \frac{3}{2}i}{\widetilde{u}_{6,k} - \frac{3}{2}i}\right)^3 \prod_{\substack{j=1 \\ j\neq k}}^{n+2m-4} \frac{\widetilde{u}_{6,k} - \widetilde{u}_{6,j} + i}{\widetilde{u}_{6,k} - \widetilde{u}_{6,j} - i} = \prod_{j=1}^{n+2m-2} \frac{\widetilde{u}_{6,k} - u_{5,j} + \frac{i}{2}}{\widetilde{u}_{6,k} - u_{5,j} - \frac{i}{2}}\,. \tag{A.15}$$

Since the $\widetilde{u}_6$ roots also solve (A.8), we can decouple the u_5 roots. Likewise, we use (A.12) to rewrite (A.9) in terms of $\widetilde{u}_6$. The resulting set of equations thus reads

$$\left(\frac{\widetilde{u}_{6,k}+\frac{3}{2}i}{\widetilde{u}_{6,k}-\frac{3}{2}i}\right)^3 = \prod_{\substack{j=1\\ j\neq k}}^{n+2m-4} \frac{\widetilde{u}_{6,k}-\widetilde{u}_{6,j}-i}{\widetilde{u}_{6,k}-\widetilde{u}_{6,j}+i} \prod_{j=1}^{m-2} \frac{\widetilde{u}_{6,k}-u_{7,j}+\frac{i}{2}}{\widetilde{u}_{6,k}-u_{7,j}-\frac{i}{2}} \tag{A.16}$$

$$1 = \prod_{\substack{j=1\\ j\neq k}}^{m-2} \frac{u_{7,k}-u_{7,j}-i}{u_{7,k}-u_{7,j}+i} \prod_{j=1}^{n+2m-4} \frac{u_{7,k}-\widetilde{u}_{6,j}+\frac{i}{2}}{u_{7,k}-\widetilde{u}_{6,j}-\frac{i}{2}}\,. \tag{A.17}$$

Comparing the two systems of equations, namely (A.6)-(A.7) with (A.16)-(A.17), we note that the value of m has been lowered by one, while the spin of the representation increased by $\frac{1}{2}$. Clearly, this procedure can be applied recursively until all second-level roots vanish and get *absorbed* into the spin representation of the first-level roots. Thus, after $(m-1)$ steps one can decouple (A.7) from (A.6) and one is left with a single system of equations[3]. Before using this recursion to solve the system (A.6)-(A.7), we will investigate the Baxter functions appearing in the intermediate steps.

Suppose that one has repeated the aforementioned procedure ℓ times. The intermediate equations then read

$$\left(\frac{u_k^{(\ell)}+(1+\frac{\ell}{2})i}{u_k^{(\ell)}-(1+\frac{\ell}{2})i}\right)^3 = \prod_{\substack{j=1\\ j\neq k}}^{n+2(m-1-\ell)} \frac{u_k^{(\ell)}-u_j^{(\ell)}-i}{u_k^{(\ell)}-u_j^{(\ell)}+i} \prod_{j=1}^{m-1-\ell} \frac{u_k^{(\ell)}-v_j^{(\ell)}+\frac{i}{2}}{u_k^{(\ell)}-v_j^{(\ell)}-\frac{i}{2}} \tag{A.18}$$

$$1 = \prod_{\substack{j=1\\ j\neq k}}^{m-1-\ell} \frac{v_k^{(\ell)}-v_j^{(\ell)}-i}{v_k^{(\ell)}-v_j^{(\ell)}+i} \prod_{j=1}^{n+2(m-1-\ell)} \frac{v_k^{(\ell)}-u_j^{(\ell)}+\frac{i}{2}}{v_k^{(\ell)}-u_j^{(\ell)}-\frac{i}{2}}\,, \tag{A.19}$$

with the initial values

$$u_k^{(0)} \equiv u_{5,k}\,, \qquad u_k^{(1)} \equiv \widetilde{u}_{6,k}\,, \tag{A.20}$$

$$v_k^{(0)} \equiv u_{6,k}\,, \qquad v_k^{(1)} \equiv u_{7,k}\,. \tag{A.21}$$

A single step of the iteration relates the polynomials

$$P_\ell(u) \equiv \prod_{j=1}^{n+2(m-1-\ell)} (u-u_j^{(\ell)})\,, \tag{A.22}$$

with consecutive values of ℓ through

$$P_\ell(u) = \big(u+(1+\tfrac{\ell}{2})\,i\big)^3\, P_{\ell+1}(u+\tfrac{i}{2}) - \big(u-(1+\tfrac{\ell}{2})\,i\big)^3\, P_{\ell+1}(u-\tfrac{i}{2})\,. \tag{A.23}$$

The initial polynomials (A.10) and (A.14) are respectively the second and the third member of this family, i.e. $\widetilde{Q}_6(u) = P_1(u)$ and $Q_5(u) = P_0(u)$. It should be clear that the first element is given by

$$P_{-1}(u) = Q_4(u) = c_4 \prod_{j=1}^{n+2m} (u-u_{4,j})\,. \tag{A.24}$$

[3]Please note, however, that in the penultimate step one should not perform the splitting, but rather dualize the nested set of equations directly.

A general solution to the recurrence relation (A.23) is given by

$$P_\ell(u) = \sum_{k=0}^{n}(-1)^k \binom{n}{k} \prod_{j=1}^{k}\left(u - \frac{(2j+\ell)i}{2}\right)^3 \prod_{j=1}^{n-k}\left(u + \frac{(2j+\ell)i}{2}\right)^3 P_{\ell+n}\left(u + \left(\frac{n}{2} - k\right)i\right), \tag{A.25}$$

with n being an arbitrary positive integer.

As already mentioned before, for $\ell = m-1$ equations (A.18)-(A.19) decouple and one is left with a single equation

$$\left(\frac{u_k^{(m-1)} + \frac{m+1}{2}i}{u_k^{(m-1)} - \frac{m+1}{2}i}\right)^3 = \prod_{\substack{j=1\\ j\neq k}}^{n} \frac{u_k^{(m-1)} - u_j^{(m-1)} - i}{u_k^{(m-1)} - u_6^{(m-1)} + i}. \tag{A.26}$$

It is noteworthy that these are the Bethe equations of a non-compact $\mathfrak{sl}(2)$ magnet in the spin-$(-\frac{m+1}{2})$ representation. Thus, we have proven an equivalence between both systems noticed in [33]. The corresponding Baxter equation can be solved exactly for the ground state. Please refer to Appendix A.2 for further details. The solution is the Wilson polynomial

$$\begin{aligned} F_{n,m}(u) \equiv P_{m-1}(u) &= c_{m-1}\prod_{j=1}^{n}(u - u_j^{(m-1)}) \\ &= {}_4F_3\left(\left.\begin{matrix} -\frac{n}{2},\ \frac{n}{2}+1+\frac{3m}{2},\ \frac{1}{2}+iu,\ \frac{1}{2}-iu \\ 1+\frac{m}{2},\ 1+\frac{m}{2},\ 1+\frac{m}{2} \end{matrix}\right| 1\right). \end{aligned} \tag{A.27}$$

Plugging this into (A.25), one finds an explicit solution for the primary roots

$$Q_4(u) = \sum_{k=0}^{m}(-1)^k \binom{m}{k} \prod_{j=1}^{k}\left(u - \tfrac{(2j-1)i}{2}\right)^3 \prod_{j=1}^{m-k}\left(u + \tfrac{(2j-1)i}{2}\right)^3 F_{n,m}\left(u + \left(\tfrac{m}{2} - k\right)i\right), \tag{A.28}$$

while the u_5 roots are generated by

$$Q_5(u) = \sum_{k=0}^{m-1}(-1)^k \binom{m-1}{k} \prod_{j=1}^{k}(u - j\,i)^3 \prod_{j=1}^{m-1-k}(u + j\,i)^3\, F_{n,m}\left(u + \left(\tfrac{m-1}{2} - k\right)i\right). \tag{A.29}$$

The Baxter function for the u_6 roots is a part of a *different* recursive scheme. Defining

$$K_\ell(u) \equiv \prod_{j=1}^{m-1-\ell}(u - v_j^{(\ell)}), \tag{A.30}$$

we find that K_ℓ obeys the following functional relation

$$K_\ell(u) = f_\ell(u)\, K_{\ell+1}\left(u + \tfrac{i}{2}\right) - \widetilde{f}_\ell(u)\, K_{\ell+1}\left(u - \tfrac{i}{2}\right), \tag{A.31}$$

with

$$f_\ell(u) = -\frac{P_\ell\left(u - \frac{i}{2}\right)}{P_{\ell+1}(u)} \qquad \text{and} \qquad \widetilde{f}_\ell(u) = -\frac{P_\ell\left(u + \frac{i}{2}\right)}{P_{\ell+1}(u)}. \tag{A.32}$$

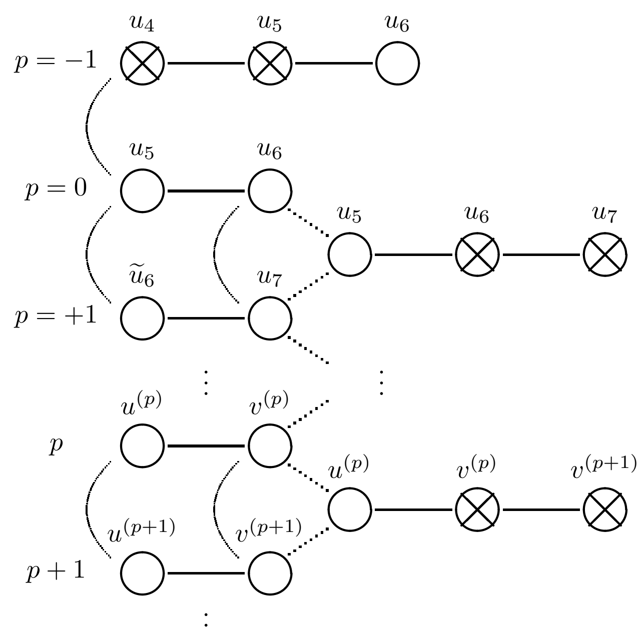

Figure 3: A schematic representation of the decoupling procedure. Dotted lines indicate the bosonic splitting and the dualization of the interjacent roots. The curved lines denote two kinds of recurrence relations P and K, respectively. The exact one-loop solutions Q_4 and Q_5 belong to P iteration, while Q_6 is part of the K recurrence.

A general solution to equation (A.31) is presented in Appendix A.3. We fix the boundary conditions by specifying

$$K_{m-1}=1 \qquad \text{and} \qquad K_0(u)\equiv Q_6(u)=\prod_{j=1}^{m-1}(u-u_{6,j})\,. \tag{A.33}$$

Thus, we find that Q_6 is given by the following expression

$$\begin{aligned} Q_6(u) = &\prod_{k=0}^{m-2} f_k\left(u+\frac{k}{2}\,i\right) \\ &+\sum_{r=1}^{m-1}(-1)^r\sum_{j_1=0}^{m-2}\sum_{j_2=0}^{j_1-1}\cdots\sum_{j_r=0}^{j_{r-1}-1}\prod_{s=1}^{r}\widetilde{f}_{j_s}\left(u+\frac{j_s-2(r-s)}{2}\,i\right)\prod_{k=0}^{j_r-1}f_k\left(u+\frac{k}{2}\,i\right) \\ &\times\prod_{s=2}^{r}\prod_{k=j_s+1}^{j_{s-1}-1}f_k\left(u+\frac{k-2(r-s+1)}{2}\,i\right)\prod_{k=j_1+1}^{m-2}f_k\left(u+\frac{k-2r}{2}\,i\right)\,, \end{aligned} \tag{A.34}$$

with $f_p(u)$ and $\widetilde{f}_p(u)$ given in (A.32) and

$$\begin{aligned} P_\ell(u) = &\sum_{k=0}^{m-1-\ell}(-1)^k\binom{m-1-\ell}{k}\prod_{j=1}^{k}\left(u-\frac{(2j+\ell)\,i}{2}\right)^3\prod_{j=1}^{m-1-\ell-k}\left(u+\frac{(2j+\ell)\,i}{2}\right)^3 \\ &\times F_{n,m}\left(u+\left(\frac{m-1-\ell}{2}-k\right)i\right)\,, \end{aligned} \tag{A.35}$$

as follows from (A.25) with $\ell + n = m - 1$.

The complete solution of the one-loop problem is hence given by (A.28), (A.29) and (A.34).

A.2 Solution to the higher spin magnet

The Bethe equations in (A.26) correspond to an integrable non-compact $XXX_{-\frac{m+1}{2}}$ magnet. The Baxter equation associated with XXX_{-s} magnets takes the following generic form

$$(u+is)^L Q(u+i) + (u-is)^L Q(u-i) = t_L(u) Q(u)\,. \tag{A.36}$$

For the special case of $L = 3$, the transfer matrix of the ground states can be determined explicitly

$$t_3(u) = 2u^3 + q_2 u + q_3\,, \quad q_2 = -(n^2 - n + 6\,n\,s + 6\,s^2)\,, \quad q_3 = 0\,. \tag{A.37}$$

The solution to (A.36) with (A.37) can be found by noting that the Wilson polynomials W_k,

$$\frac{W_k(u^2; a,b,c,d)}{(a+b)_k\,(a+c)_k\,(a+d)_k} \equiv {}_4F_3\left(\left. \begin{matrix} -k,\ k+a+b+c+d-1,\ a+i\,u,\ a-i\,u \\ a+b,\ a+c,\ a+d \end{matrix} \right| 1 \right)\,, \tag{A.38}$$

satisfy the following difference equation (see e.g. [41])

$$k(k+a+b+c+d-1)y(u) = B(u)y(u+i) - [B(u)+D(u)]y(u) + D(u)y(u-i)\,. \tag{A.39}$$

Here, $y(u) = W_k(u^2; a,b,c,d)$ and

$$B(u) = \frac{(a-iu)(b-iu)(c-iu)(d-iu)}{2iu(2iu-1)}\,,\ D(u) = \frac{(a+iu)(b+iu)(c+iu)(d+iu)}{2iu(2iu+1)}\,. \tag{A.40}$$

It is easy to check that (A.36) is embedded in (A.39) upon the identification

$$a = \frac{1}{2}\,, \quad b = c = d = s \quad \text{and} \quad k = \frac{n}{2}\,. \tag{A.41}$$

Hence, the solution is given by

$$Q(u) = {}_4F_3\left(\left. \begin{matrix} -\frac{n}{2},\ \frac{n}{2}+3s-\frac{1}{2},\ \frac{1}{2}+i\,u,\ \frac{1}{2}-i\,u \\ \frac{1}{2}+s,\ \frac{1}{2}+s,\ \frac{1}{2}+s \end{matrix} \right| 1 \right)\,. \tag{A.42}$$

For the special value of $s = \frac{m+1}{2}$ one finds (A.27).

A.3 Solution of the recurrence

The class of functional equations

$$A_p(u) = f_p(u)\,A_{p+1}\left(u+\frac{i}{2}\right) - \widetilde{f}_p(u)\,A_{p+1}\left(u-\frac{i}{2}\right) \tag{A.43}$$

is solved by

$$
\begin{aligned}
A_p(u) = &\prod_{k=0}^{n-1} f_{p+k}\left(u+\frac{k}{2}\,i\right) A_{p+n}\left(u+\frac{n}{2}\,i\right)\\
&+\sum_{r=1}^{n}(-1)^r \sum_{j_1=0}^{n-1}\sum_{j_2=0}^{j_1-1}\cdots\sum_{j_r=0}^{j_{r-1}-1}\prod_{s=1}^{r}\widetilde{f}_{p+j_s}\left(u+\frac{j_s-2(r-s)}{2}\,i\right)\prod_{k=0}^{j_r-1} f_{p+k}\left(u+\frac{k}{2}\,i\right)\\
&\times\prod_{s=2}^{r}\prod_{k=j_s+1}^{j_{s-1}-1} f_{p+k}\left(u+\frac{k-2(r-s+1)}{2}\,i\right)\prod_{k=j_1+1}^{n-1} f_{p+k}\left(u+\frac{k-2r}{2}\,i\right)\\
&\times A_{p+n}\left(u+\frac{n-2\,r}{2}\,i\right)\,.
\end{aligned}
\tag{A.44}
$$

The proof is by induction.

B The strong coupling limit

In order to analyse the sub-leading contribution to (4.16) we decompose the density $\hat{\sigma}(t)$ into its parity even and odd parts

$$
\hat{\sigma}(t)\frac{e^t-1}{t} = \frac{\gamma_+(2gt)}{2gt} + \frac{\gamma_-(2gt)}{2gt}\,. \tag{B.1}
$$

Since the kernels K_0 and K_1 in (4.5) are given by the sum over Bessel functions, the functions $\gamma_\pm$ take the form of a Neumann series

$$
\gamma_+(2gt) = 2\sum_{n=1}^{\infty} 2nJ_{2n}(2gt)\gamma_{2n}\,, \quad \gamma_-(2gt) = 2\sum_{n=1}^{\infty}(2n-1)J_{2n-1}(2gt)\gamma_{2n-1}\,. \tag{B.2}
$$

Using this decomposition of the fluctuation density it is possible to rewrite the integral equation (4.15) as an infinite system of equations with $n \geq 1$

$$
\begin{aligned}
&\int_0^\infty \frac{dt}{t}\left(\frac{\gamma_+(t)}{1-e^{-t/2g}} - \frac{\gamma_-(t)}{e^{t/2g}-1}\right) J_{2n}(t) = \frac{L}{8ng} + h_{2n} - \frac{1}{2}\int_0^\infty dt\frac{J_{2n}(2gt)}{2gt}e^{-t\frac{m-1}{2}}\,,\\
&\int_0^\infty \frac{dt}{t}\left(\frac{\gamma_-(t)}{1-e^{-t/2g}} + \frac{\gamma_+(t)}{e^{t/2g}-1}\right) J_{2n-1}(t) = h_{2n-1}\,,
\end{aligned}
\tag{B.3}
$$

where the term $h_n = h_n(g)$ is given by the expression

$$
h_n = \frac{1}{4}\int_0^\infty dt\left(\frac{3}{e^t-1} - \frac{e^{-(m+1)t/2}}{1-e^{-t}} - \frac{L-3}{e^{t/2}+1}\right)\left(\frac{J_n(2gt)}{gt} - \delta_{n,1}\right)\,. \tag{B.4}
$$

Since the left hand side of the equations (B.3) is the same as in the case of the BES equation, we expect to be able to express the solution in terms of the solution to the BES equation. For this purpose we introduce a new parameter j, which interpolates between the system corresponding to the BES equation and (B.3)

$$
\begin{aligned}
&\int_0^\infty \frac{dt}{t}\left(\frac{\gamma_+(t,j)}{1-e^{-t/2g}} - \frac{\gamma_-(t,j)}{e^{t/2g}-1}\right) J_{2n}(t) = \frac{jL}{8ng} + jh_{2n} - \frac{j}{2}\int_0^\infty dt\frac{J_{2n}(2gt)}{2gt}e^{-t\frac{m-1}{2}}\,,\\
&\int_0^\infty \frac{dt}{t}\left(\frac{\gamma_-(t,j)}{1-e^{-t/2g}} + \frac{\gamma_+(t,j)}{e^{t/2g}-1}\right) J_{2n-1}(t) = jh_{2n-1} + \frac{1}{2}(1-j)\delta_{n,1}\,.
\end{aligned}
\tag{B.5}
$$

Setting $j = 0$ gives back the BES equation while $j = 1$ corresponds to (B.3). Multiplying the first equation by $(2n)\gamma_{2n}(t, j')$ and the second by $(2n-1)\gamma_{2n-1}(t, j')$, summing over all n and finally subtracting the two equations leads to a left hand side that is symmetric under exchange of j and j', see [36] for details. Using this fact and setting $j = 0$ and $j' = 1$, we find

$$\gamma_1(g,1) = \frac{1}{4}\int_0^\infty dt \left(3 - e^{-t(m-1)/4g}\right)\left(\frac{\gamma_-(t,0)}{(e^{t/2g}-1)gt} + \frac{\gamma_+(t,0)}{(e^{-t/2g}-1)gt} - \frac{\gamma_1(g,0)}{(e^{t/2g}-1)g}\right)$$
$$-\frac{L-3}{4}\int_0^\infty dt \left(\frac{\gamma_-(t,0)}{(e^{t/4g}+1)gt} + \frac{\gamma_+(t,0)}{(e^{-t/4g}+1)gt} - \frac{\gamma_1(g,0)}{(e^{t/4g}+1)g}\right). \tag{B.6}$$

The finite order correction is then given by $16\, g^2\, \gamma_1(g,1)$. A change of variables as in [15],

$$2\gamma_\pm(t,0) = \left(1 - \mathrm{sech}(\tfrac{t}{2g})\right)\Gamma_\pm(t,0) \pm \tanh(\tfrac{t}{2g})\Gamma_\mp(t,0)\,, \tag{B.7}$$

leads to

$$\gamma_1(g,1) = \frac{1}{16g^2}(L-3)\epsilon_1(g) + \gamma_1(g,0)(L-3)\log 2 + \frac{3}{2}B_2(g)$$
$$+\frac{1}{2}\int_0^\infty dt e^{-t(m+1)/4g}\left(\frac{1}{4gt}\left(\Gamma_+(t,0) + \Gamma_-(t,0)\right) + \frac{\gamma_1(g,0)}{(e^{t/2g}-1)}\right). \tag{B.8}$$

At this stage, we make use of the solution of the BES equation obtained in [15],

$$\Gamma_+(t,0) = \sum_{k=0}^\infty (-1)^{k+1} J_{2k}(t)\Gamma_{2k}\,, \quad \Gamma_-(t,0) = \sum_{k=0}^\infty (-1)^{k+1} J_{2k-1}(t)\Gamma_{2k-1}\,,$$

where the coefficients Γ_k are given by

$$\Gamma_k = -\frac{1}{2}\Gamma_k^{(0)} + \sum_{p=1}^\infty \frac{1}{g^p}\left(c_p^- \Gamma_k^{(2p-1)} + c_p^+ \Gamma_k^{2p}\right)\,, \tag{B.9}$$

$$\Gamma_{2m}^{(p)} = \frac{\Gamma(m+p-\frac{1}{2})}{\Gamma(m+1)\Gamma(\frac{1}{2})}\,, \quad \Gamma_{2m-1}^{(p)} = \frac{(-1)^p\Gamma(m-\frac{1}{2})}{\Gamma(m+1-p)\Gamma(\frac{1}{2})}\,. \tag{B.10}$$

The prefactors $c_p^\pm$ explicitly depend on g and can be determined from the so-called all-loop quantization condition of [15]. Bearing that in mind, we find for the integral in (B.8)

$$I(g) = \frac{1}{2}\int_0^\infty dt\, e^{-t(m+1)/4g}\left(\frac{1}{4gt}\left(\Gamma_+(t,0) + \Gamma_-(t,0)\right) + \frac{\gamma_1(g,0)}{(e^{t/2g}-1)}\right)$$
$$= \frac{1}{8g}\sum_{k=1}^\infty (-1)^{k+1}\left(\left(\frac{ig}{x(i\frac{m-1}{2})}\right)^{2k}\frac{\Gamma_{2k}}{2k} + \left(\frac{ig}{x(i\frac{m-1}{2})}\right)^{2k-1}\frac{\Gamma_{2k-1}}{2k-1}\right)$$
$$+\frac{\gamma_1(g,0)}{4g}\int_0^\infty dt\,\frac{e^{-t(m-1)/4g}}{e^{t/2g}-1} - \frac{1}{8g}\int_0^\infty dt\, e^{-t(m-1)/4g}\frac{J_0(t)}{t}\Gamma_0$$
$$+\frac{1}{8g}\int_0^\infty dt\, e^{-t(m-1)/4g}\frac{J_1(t)}{t}\Gamma_{-1}\,. \tag{B.11}$$

Subsequently, using that $\Gamma_0 = 4\,g\,\gamma_1(g,0)$ and $\Gamma_{-1} = 1$, we can recast $I(g)$ as

$$I(g) = \frac{1}{8g}\sum_{k=1}^{\infty}(-1)^{k+1}\left(\left(\frac{ig}{x(i\frac{m-1}{2})}\right)^{2k}\frac{\Gamma_{2k}}{2k} + \left(\frac{ig}{x(i\frac{m-1}{2})}\right)^{2k-1}\frac{\Gamma_{2k-1}}{2k-1}\right)$$
$$+\frac{\gamma_1(g,0)}{4g}\int_0^{\infty} dt\, e^{-t(m-1)/4g}\left(\frac{1}{e^{t/2g}-1} - 2g\frac{J_0(t)}{t}\right) + \frac{1}{8g}\frac{ig}{x(i\frac{m-1}{2})}\,. \qquad \text{(B.12)}$$

Both sums and the integral in the above formula may be performed analytically leading to

$$I(g) = \frac{1}{8g^2(m-1)} + \frac{f(g)}{32g^2}\left(\log g - \psi_0\left(\tfrac{m+1}{2}\right) + \tfrac{m-1}{4g}\right)\,. \qquad \text{(B.13)}$$

Hence we obtain

$$\gamma_1(g,1) = \frac{1}{16g^2}(L-3)\epsilon_1(g) + \frac{f(g)}{16g^2}(L-3)\log 2 + \frac{3}{2}\frac{B_2(g)}{16g^2} + \frac{1}{8g^2(m-1)}$$
$$+\frac{f(g)}{32g^2}\left(\log g - \psi_0\left(\tfrac{m+1}{2}\right) + \tfrac{m-1}{4g}\right) + \ldots\,. \qquad \text{(B.14)}$$

The anomalous dimension for $m \geq 2$ is consequently given by

$$\gamma(L,n) = f(g)\left(\log n + \tfrac{3}{2}\gamma_{\text{E}} + \tfrac{1}{2}\psi_0\left(\tfrac{m+1}{2}\right) - (L-3)\log 2\right) + 16g^2\gamma_1(g,1)$$
$$= \left(4g - \frac{3\log 2}{\pi}\right)\log\frac{n}{g} + 6g(-1+\log 2) + (1-L) + \frac{2}{m-1} + \frac{m}{2}$$
$$+\frac{9\log 2}{\pi} - \frac{9(\log 2)^2}{2\pi} + \mathcal{O}\left(\tfrac{1}{g}\right)\,. \qquad \text{(B.15)}$$

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