

## Sugawara-type constraints in hyperbolic coset models

---

Thibault Damour<sup>1</sup>, Axel Kleinschmidt<sup>2</sup> and Hermann Nicolai<sup>3</sup>

<sup>1</sup>*Institut des Hautes Etudes Scientifiques  
35, Route de Chartres, FR-91440 Bures-sur-Yvette, France*

<sup>2</sup>*Physique Théorique et Mathématique  
Université Libre de Bruxelles & International Solvay Institutes  
ULB-Campus Plaine C.P. 231, BE-1050 Bruxelles, Belgium*

<sup>3</sup>*Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut  
Am Mühlenberg 1, DE-14476 Potsdam, Germany*

### Abstract

In the conjectured correspondence between supergravity and geodesic models on infinite-dimensional hyperbolic coset spaces, and  $E_{10}/K(E_{10})$  in particular, the constraints play a central role. We present a Sugawara-type construction in terms of the  $E_{10}$  Noether charges that extends these constraints infinitely into the hyperbolic algebra, in contrast to the truncated expressions obtained in [arXiv:0709.2691](#) that involved only finitely many generators. Our extended constraints are associated to an infinite set of roots which are all imaginary, and in fact fill the closed past light-cone of the Lorentzian root lattice. The construction makes crucial use of the  $E_{10}$  Weyl group and of the fact that the  $E_{10}$  model contains both  $D = 11$  supergravity and  $D = 10$  IIB supergravity. Our extended constraints appear to unite in a remarkable manner the different canonical constraints of these two theories. This construction may also shed new light on the issue of ‘open constraint algebras’ in traditional canonical approaches to gravity.

# 1 Introduction

In canonical formulations of gravity, the constraints are the essential ingredients for, and main obstacles to, carrying out a canonical quantization of gravity [1] (for an overview and bibliography see [2]). This applies in particular to the Hamiltonian (scalar) constraint determining evolution in ‘time’, and therefore the dynamics. The problem of properly setting up and defining the quantum constraints has been tackled in a variety of approaches but, arguably, the problem remains as open as in Bryce DeWitt’s seminal 1967 paper [1]. A further cause of difficulties, shared by all approaches so far, can be traced to the fact that the constraints form an *open algebra*, that is, the structure ‘constants’ are not constants, but field dependent.

At the level of classical maximal supergravity, progress has been made in the last years towards establishing a correspondence between the equations of  $D = 11$  supergravity on the one hand and a geodesic coset model based on the hyperbolic Kac–Moody structure  $E_{10}$  [3] on the other (similar correspondences exist for other supergravity models). The supergravity equations are treated canonically and therefore comprise *dynamical* (evolution) equations and *constraint* equations. There is a precise correspondence between a truncation of the *dynamical equations* and a truncation of the *geodesic equation* on the coset  $E_{10}/K(E_{10})$  [3]. The  $D = 11$  supergravity *constraint* equations can similarly be mapped to *constraints* that can be imposed consistently on the geodesic motion [4]. For instance, imposition of the Hamiltonian constraint implies that the geodesic is null. According to [4] the weakly conserved constraints of  $D = 11$  supergravity can be translated into weakly conserved coset model constraints, which in turn allow for a reformulation as bilinear expressions in terms of conserved charges, that is, as *strongly* conserved constraints.<sup>1</sup> As noted there, this construction is very reminiscent of the well-known Sugawara construction [5] for affine Lie algebras [6, 7]. It is the purpose of the present paper to follow up on this observation, making it more precise and giving the beginning of a generalized Sugawara construction for hyperbolic Kac–Moody algebras which makes the analogy with the affine construction much more compelling.

---

<sup>1</sup>As usual, the term ‘weakly conserved constraints’ here refers to a set of constraints  $\mathcal{C}$  satisfying (modulo the coset equations of motion)  $d\mathcal{C}/dt = f(\mathcal{C}) \approx 0$ , where  $f(\mathcal{C})$  is a function vanishing on the *constraint surface* defined by  $\mathcal{C} = 0$ , while ‘strongly conserved’ constraints satisfy  $d\mathcal{C}/dt = 0$  (upon use of the equations of motion).

Understanding and reformulating supergravity in these algebraic terms could prove very useful for the transition to the quantum theory (see [8] for first steps towards the quantization of the  $E_{10}/K(E_{10})$  model and [9] for pure gravity). An analogy to be kept in mind in this discussion is that of (bosonic) string theory. There, the dynamical equation for the embedding (target space) coordinates can be written as a free wave equation if one adopts a conformal gauge. This free wave equation admits an infinite set of conserved charges  $\alpha_n^\mu$ . The price to pay for the simple dynamical equation is that one has to impose the (Fubini-Veneziano-)Virasoro constraints,  $L \sim \alpha \alpha$ , on the solutions. In the quantum version, the Virasoro constraints and the existence of a proper Hilbert space imply the critical dimension [10]. Assuming the validity of the Kac–Moody/supergravity correspondence, the dynamical equations of supergravity also become simple, yielding geodesics on a symmetric space as their solutions. This system is fully integrable. It admits an infinite set of conserved charges,  $J$ , that do not (Poisson) commute among themselves, and one can formally write down the general solution in terms of  $J$  and some initial data. The complications and interesting structures are then again to be found in the constraints and their algebra. The fact that all constraints found so far admit a Sugawara-like structure, *i.e.*,  $\mathfrak{L} \sim J J$ , is tantalizing in this analogy, and may turn out to be crucial for the quantisation of the theory. The gauge symmetries encoded in the coset constraints are directly linked to the space-time and gauge symmetries that are known from the geometrical formulation of supergravity.

The replacement of the supergravity constraints by coset model constraints with an underlying algebraic structure may also shed new light on the old problem of open constraint algebras alluded to above, circumventing some of the seemingly insurmountable difficulties of the usual canonical formulation. The main new feature here is that the ‘structure constants’, while still dependent on the dynamical degrees of freedom (fields), become *constants of motion* in the present formulation. More explicitly, suppose the classical constraints  $\mathcal{C}^A(\phi)$  satisfy the first-class canonical (Poisson) algebra

$$\{\mathcal{C}^A(\phi), \mathcal{C}^B(\phi)\} = f^{AB}{}_C(\phi)\mathcal{C}^C(\phi), \quad (1.1)$$

where  $\phi$  denotes the canonical variables. In the standard formulation of canonical gravity and supergravity, the  $\phi$ -dependent structure ‘constants’  $f^{AB}{}_C(\phi)$  do not (Poisson) commute with the Hamiltonian and thus vary in time. By contrast, the structure constants obtained with the Sugawara-like form of the constraints do commute with the Hamiltonian constraint, and

are thus preserved in time, even though they still depend on the canonical variables  $\phi$ . Because the correspondence between the space-time based field theory and the one-dimensional  $E_{10}/K(E_{10})$  model is only very incompletely understood, it is, however, not clear how to translate the coset model constraints back into more conventional field theory language. At the very least, one can say that the relation between the field variables of the geometric theory and the  $E_{10}$  variables must be extremely non-local.

Obtaining a universal algebraic description of the constraints and their algebra is also desirable from an M-theory point of view. In the same way that the unique dynamical geodesic equation on  $E_{10}/K(E_{10})$  allows for maps to *different* maximal supergravity theories, depending on the level decomposition chosen to describe the infinite-dimensional Lie algebra [11, 12, 13, 14], the constraints should also exhibit this ‘versatility’. Our construction below has this property, albeit in a novel way. More precisely, we will define a ‘universal scaffold’ of hyperbolic Sugawara constraints by using null root vectors  $\alpha$  of the hyperbolic algebra, decomposed into a sum of two real roots  $\beta_1 + \beta_2 = \alpha$ , and the hyperbolic Weyl group. This will define an infinite number of constraints  $\mathfrak{L}_\alpha$  associated with a ‘skeleton’ of roots  $\alpha$  on the light-cone in terms of current bilinears. (The notions of skeleton and scaffold are depicted in figures 2 and 3 below.) Extending (away from the real  $\beta_i$  case) the set of current-bilinear contributions  $\mathfrak{L}_\alpha \sim J_{\beta_1} J_{\beta_2}$  to a given null-root constraint ( $\alpha^2 = 0$ ), or extending the skeleton of supporting roots  $\alpha$  constraints *into* the light-cone ( $\alpha^2 < 0$ ), however, seems to require the choice of a subalgebra of the hyperbolic algebra that is kept manifest. In analogy with affine algebras, this procedure is very suggestive of a choice of ‘spectral parameters’ for the hyperbolic algebra, even though we do not know whether such a realization of the hyperbolic algebra exists. However, the picture that emerges from the present work is that if such realizations exist, they do so only in combination with suitable constraints. Furthermore, such realizations cannot be unique, giving the algebra a ‘chameleon-like’ aspect. This feature would be in line with the conjectured emergence of a space-time structure from the Lie algebra, where the dimension of the emergent space would depend on the decomposition and the chosen form of the constraints, such that the ‘spectral parameters’ would become associated to spatial coordinates<sup>2</sup>. These points

---

<sup>2</sup>However, this association is likely to be more subtle than just a simple equality, as can already be seen for the affine spectral parameter in  $D = 2$  supergravities, cf. Eqn. (2.1) of [15] with  $\rho = t$  (time) and  $\tilde{\rho} = x^1$  (space).

will be elaborated on and explained below by means of the constraints of  $D = 11$  supergravity and of type IIB supergravity, respectively, but similar results are expected to hold for other decompositions, such as massive IIA theory, as well as for maximal supergravities in lower dimensions. Importantly, though the set of roots ‘supporting’ the constraints is clearly related to the weight diagram of particular highest-weight representations of  $E_{10}$ , the constraints themselves do *not* form (under Poisson commutation) a highest or lowest weight representation of the hyperbolic  $E_{10}$ , as already observed in [4], and explained in much more detail here. Rather, they indicate the existence of new unexplored algebraic structures inside the hyperbolic algebra and its envelopping algebra.

We emphasize that our approach is canonical and crucially relies on a split of space and time, as well as certain gauge choices required for matching the supergravity and coset model degrees of freedom. An earlier and conceptually different M-theory proposal based on the indefinite, but non-hyperbolic, ‘very extended’ Kac–Moody algebra  $E_{11}$  has been developed by Peter West and collaborators [16, 17]. In contradistinction to the present work, their approach is ‘covariant’ in the sense that neither a split of space-time nor gauge choices for the supergravity fields are required, and the issue of writing down canonical constraints thus does not arise in the same way. Instead, one needs to introduce extra gauge invariances encompassing the gauge transformations of supergravity, and the problem becomes one of ‘fitting’ such gauge symmetries into the  $E_{11}$  framework [18]. However, despite many similarities at the kinematical level, especially with regard to embedding the bosonic sectors of maximal supergravities [19, 20, 21, 22], it appears doubtful whether a gauge-fixed version of that approach matches with the structures presented here.

From the mathematical point of view, it would also be desirable to associate a Sugawara-type construction to a hyperbolic algebra. In the affine case, the existence of this construction is directly linked to the realization of affine algebras as loop algebras via the so-called spectral parameter. A similar description and understanding is lacking for hyperbolic algebras, the only known description is in terms of generators and relations in the Chevalley–Serre basis. Any construction hinting at an alternative description could shed light on the deeper and to date elusive structure of hyperbolic Kac–Moody algebras. After all, even not knowing about the current algebra realization of affine algebras, the existence of a preferred set of bilinear Virasoro operators in the envelopping algebra would almost inevitably lead to this realization.

Here, we are searching for a similarly distinguished structure in the enveloping algebra of the hyperbolic algebra.

The remainder of the paper is structured as follows. In section 2 we first review the affine Sugawara construction and rephrase it in a slightly unconventional form. We use this form to propose a (partly schematic) trial expression for Sugawara generators for hyperbolic algebras. In section 3 we then explore this trial expression in more detail in the case of  $E_{10}$  and show that our trial expression does not only serve to reproduce the  $D = 11$  constraints but also those of type IIB supergravity. This also allows for a more precise definition of the Sugawara constraints and an exploration of their structure in terms of a skeleton of constraints associated with null roots and terms induced by covariantization. In appendices, we collect some known results on level decomposition in order to render the presentation self-contained, as well as some more detailed computations.

## 2 Sugawara construction

Before proceeding to the discussion of the hyperbolic Sugawara construction we first review briefly the definition of Sugawara operators for affine Lie algebras, see [7] (as well as [5, 6] for earlier work).

### 2.1 Affine Sugawara construction

A non-twisted affine Lie algebra can be defined for any finite-dimensional Lie algebra. Let the finite-dimensional Lie algebra  $\mathfrak{g}$  be simple and generated by  $T^A$  ( $A = 1, \dots, \dim \mathfrak{g}$ ) with commutation relations  $[T^A, T^B] = f^{AB}{}_C T^C$  and non-degenerate invariant form  $\langle T^A | T^B \rangle = \kappa^{AB}$ . Then the corresponding affine Lie algebra  $\hat{\mathfrak{g}}$  has generators  $T_m^A$  (for  $m \in \mathbb{Z}$ ),  $c$  and  $d$  with non-trivial commutation relations

$$[T_m^A, T_n^B] = f^{AB}{}_C T_{m+n}^C + \kappa^{AB} m \delta_{m,-n} c, \quad [d, T_m^A] = -m T_m^A. \quad (2.1)$$

The generator  $c$  commutes with all Lie algebra generators and is called the *central element*,<sup>3</sup> while the generator  $d$  is called the *derivation*.<sup>4</sup>

---

<sup>3</sup>The central element of the affine Lie algebra, here denoted  $c$ , is often denoted  $K$ ; it should not be confused with the central element of the Virasoro algebra associated to the affine algebra.

<sup>4</sup>This terminology follows from the presentation of affine algebras as loop algebras where  $d$  is the derivative with respect to the spectral parameter [7].

In any irreducible highest weight representation, the central element  $c$  acts as a scalar; its eigenvalue  $k$  on that representation is called the *level* of the representation. For such a level  $k$  representation, the Sugawara generators are defined (within the enveloping algebra of the  $T_m^A$ 's) by [7] (for  $n \in \mathbb{Z}$ )

$$L_n = \frac{1}{2(k + h^\vee)} \sum_{m \in \mathbb{Z}} : T_{n-m}^A T_m^B : \kappa_{AB}, \quad (2.2)$$

where the colons denote normal ordering as appropriate for the highest weight representation and  $\kappa_{AB}$  is the inverse of  $\kappa^{AB}$ ;  $h^\vee$  is the dual Coxeter number defined by  $f^{AC}_D f^{BD}_C = 2h^\vee \kappa^{AB}$ . We note that there are two separate contributions to the normalization of the Sugawara generators (2.2): The first one is  $k$ , related to the central extension, the second one  $h^\vee$  comes from normal ordering. Both contributions are *quantum effects*. Below, we will treat these two contributions differently. In the hyperbolic extension, the central generator ceases to be central and is on par with all the other Lie algebra generators. Normal ordering, on the other hand, will be mostly ignored, as our discussion deals with the classical constraints only. Normal ordering ensures that the generators  $L_m$  are well defined on any element of the representation. The operators (2.2) obey a Virasoro algebra

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{k \dim \mathfrak{g}}{12(k + h^\vee)} m(m^2 - 1) \delta_{m,-n} \quad (2.3)$$

Their commutators with the affine generators are

$$[L_m, T_n^A] = -n T_{m+n}^A. \quad (2.4)$$

Here, we would like to take a more formal point of view and rewrite (2.2) as a quadratic expression in the generators without resorting to an integrable representation. The reason is that the normalization in (2.2) involves the *inverse* of the (shifted) eigenvalue of the central generator  $c$ . However, in the full hyperbolic algebra the element  $c$  is no longer central (in fact, the hyperbolic algebra does not possess any central elements), and a direct generalization of (2.2) would thus necessarily involve the inverse of an operator, which furthermore is no longer singled out in the full algebra. For this reason, we formally multiply (2.2) by the central element and drop the normalization constant. We also recall that affine Lie algebras have two different kinds of roots: *real* roots and *null* roots. In particular, there is a primitive null root

$\delta$  which can be used to describe all roots of the affine algebra via an affine ladder diagram: Let  $\Delta^{\text{fin}} \equiv \Delta(\mathfrak{g})$  be the set of roots of the finite-dimensional algebra  $\mathfrak{g}$  (where we include  $\alpha = 0$  for simplicity), then the root system of the affine extension  $\hat{\mathfrak{g}}$  is

$$\Delta^{\text{aff}} \equiv \Delta(\hat{\mathfrak{g}}) = \{ \alpha + n\delta : \alpha \in \Delta^{\text{fin}} \text{ and } n \in \mathbb{Z} \}, \quad (2.5)$$

that is, there are  $\mathbb{Z}$  copies of the finite root system. The roots  $n\delta$  are null roots and the associated root space  $\hat{\mathfrak{g}}_{n\delta}$  has dimension given by the rank:  $\text{mult}(n\delta) = \dim \hat{\mathfrak{g}}_{n\delta} = \text{rank}(\mathfrak{g})$  for  $n \neq 0$ . For  $n = 0$  the dimension is equal to that of the Cartan subalgebra and takes the value  $\text{rank}(\mathfrak{g}) + 2$  (the two extra elements are  $c$  and  $d$ ). All other roots are real and the corresponding root spaces are one-dimensional.

Using the structure of the affine root system we can rewrite the commutation relations (2.1) as

$$[T_{\alpha_1}, T_{\alpha_2}] = f_{\alpha_1 \alpha_2}^{\alpha_1 + \alpha_2} T_{\alpha_1 + \alpha_2} + \kappa_{\alpha_1, \alpha_2} c, \quad (2.6)$$

where we have suppressed the multiplicity index for null roots. The values of  $f_{\alpha_1 \alpha_2}^{\alpha_1 + \alpha_2}$  and  $\kappa_{\alpha_1, \alpha_2}$  can be obtained by comparison with (2.1). We furthermore define quadratic generators in the enveloping algebra  $U(\hat{\mathfrak{g}})$  by

$$L_{n\delta} := \sum_{\beta \in \Delta^{\text{aff}}} T_{n\delta - \beta} T_{\beta}, \quad (2.7)$$

where  $T_{\beta}$  is a canonically normalized element in the root space  $\hat{\mathfrak{g}}_{\beta}$ . If the  $\beta$  root space is degenerate, we choose an orthonormal basis and contract with the canonically conjugate basis. Then the definition (2.7) is unambiguous except when the root spaces of  $n\delta - \beta$  and  $\beta$  have different dimensions. This happens *only* when one of  $n\delta - \beta$  or  $\beta$  is equal to zero, *i.e.*, when one of the generators belongs to the Cartan subalgebra. In that case the generators are to be contracted according to the definition (2.2), *i.e.*, we omit any terms involving a contraction with  $c$  or  $d$ , but contract only with elements of the Cartan subalgebra of the horizontal  $\mathfrak{g}$ . Except for this point and the lack of normal ordering, the expression (2.7) is a reformulation of (2.2). Note that although we could have defined quadratic generators of the form (2.7) for any point on the root lattice, we do this only for *null roots*. To get a Virasoro algebra it is furthermore essential that the space of null roots has an additive structure since all null roots lie on a  $\mathbb{Z}$ -graded line.

The affine Weyl group is the semi-direct product of the finite Weyl group with a translation group [23]. After the standard embedding of the affine algebra into a hyperbolic algebra of over-extended type [24], the affine Weyl group can also be described as the subgroup of the hyperbolic Weyl group stabilizing an affine null root [25]; the so-called affine translations are then realized as Lorentz boosts along this null direction.<sup>5</sup> Since null roots  $n\delta$  are stabilized by the affine Weyl group  $\mathcal{W}^{\text{aff}}$ , the l.h.s. of the definition (2.7) is invariant under the action of the Weyl group. One can check that the r.h.s. is also invariant.

Besides the convention for null root spaces, the definition (2.7) differs from the standard one (2.2) by its lack of normal ordering. However, as is well known, this affects only the generator  $L_0$  for affine algebras. In addition, normal ordering is only required for the quantum theory, whereas we are here mainly concerned with the structure of the *classical* constraints. In the classical theory, one associates to each symmetry generator  $T_\alpha$  a corresponding conserved charge, say  $J_\alpha$ . Accordingly, we will below consider expressions such as (2.7) (with the replacement  $T_\alpha \rightarrow J_\alpha$ ) as functions on phase space and leave open the quantum definition of the constraints. We also remark that the generator  $L_0$  as defined in (2.2) differs from the Hamiltonian (quadratic Casimir) by a term proportional to  $cd$ . Omission of this term is admissible in the affine case, but not in the hyperbolic algebra. [In other words, our hyperbolic-algebra generalization of (2.2) will contain terms of the type  $cd$ , which do not enter the affine version of (2.2).] Correlatively, while the affine Hamiltonian is bounded below, *i.e.*,  $L_0 \geq 0$ , the full Hamiltonian is not because the Cartan-Killing metric on the Cartan subalgebra is indefinite for hyperbolic algebras (with  $\langle c|c \rangle = \langle d|d \rangle = 0$  and  $\langle c|d \rangle = 1$ ).

We now proceed to compute the algebra of the constraints as defined by (2.7). In the course of the following computations we manipulate infinite sums formally, well aware that they are not well-defined and normally would require a normal ordered evaluation on a representation space. With this in mind one computes in the universal enveloping algebra

$$[L_{m\delta}, T_\alpha] = -2\kappa_{\alpha, -\alpha} c T_{m\delta + \alpha}. \quad (2.8)$$

---

<sup>5</sup>We also note that the Weyl orbit of the ‘cusp’  $\delta$  is dense on the boundary of the hyperbolic space obtained by projecting the interior of the forward lightcone onto the unit hyperboloid. Equivalently, the rays through all the hyperbolic null roots cover the boundary of the lightcone densely.

which is the same as (2.4), but now expressed in terms of affine roots. The important point we wish to emphasize here is that the r.h.s is *bilinear* in affine generators since we multiplied the Sugawara generators by the central element. Continuing now to the commutator of two Sugawara generators (2.7) leads to

$$[L_{m\delta}, L_{n\delta}] = 2(m-n)cL_{(m+n)\delta}, \quad (2.9)$$

so that in this formulation the algebra closes with a pre-factor ( $=c$ ) that is itself an algebra generator. Due to the lack of normal ordering one does not obtain the central term as in (2.3). Neither is the shift by the dual Coxeter number visible in this formal computation in the enveloping algebra.

## 2.2 Hyperbolic Sugawara construction

The expression (2.7) can be formally generalized to hyperbolic Lie algebras of the over-extended type [24].<sup>6</sup> In the hyperbolic case the root system  $\Delta^{\text{hyp}}$  is much more complicated than (2.5): Besides the real and null roots there are now time-like (purely imaginary) roots  $\alpha$  with  $\alpha^2 < 0$ . The multiplicities of these roots grows exponentially and no closed formula for their multiplicities is known although these can be computed algorithmically, for example via the Peterson recursion formula. For each  $\alpha$  root space  $\mathfrak{g}_\alpha \subset \mathfrak{g}$ , we choose a basis

$$T_\alpha^{(s)} \quad \text{for } s = 1, \dots, \text{mult}(\alpha). \quad (2.10)$$

which is ‘null orthonormal’ (when using the standard bilinear form) with respect to the corresponding dual basis in the  $\mathfrak{g}_{-\alpha}$  root space:

$$\langle T_\alpha^{(s)} | T_\beta^{(s')} \rangle = \delta_{s,s'} \delta_{\alpha+\beta,0}. \quad (2.11)$$

The commutation relations are then

$$[T_{\alpha_1}^{(s_1)}, T_{\alpha_2}^{(s_2)}] = f_{\alpha_1 \alpha_2}^{(s_1)(s_2)} T_{\alpha_1 + \alpha_2}^{(s_{12})}. \quad (2.12)$$

Our hyperbolic generalization of the affine Sugawara construction (2.7) then consists of two elements:

---

<sup>6</sup>By ‘over-extension’ we mean the canonical extension via the non-twisted affine extension, whereby two nodes are added to the Dynkin diagram; adding a third node would yield ‘very-extended’ algebras [26].

- (i) the choice of a special set of ‘constraint’ generators, labelled by a subset, say  $\mathcal{C}$ , of the set of pairs  $(\alpha, \bar{s})$  labelling the roots (including their degeneracy); and
- (ii) a general expression for the hyperbolic Sugawara generator  $\mathfrak{L}_{\alpha, \bar{s}}$  (or ‘generalized Virasoro constraint’) associated to a particular pair<sup>7</sup>  $(\alpha, \bar{s}) \in \mathcal{C}$  of the form

$$\mathfrak{L}_{\alpha, \bar{s}} = \sum_{\substack{\beta_1, \beta_2 \in \Delta^{\text{hyp}} \\ \beta_1 + \beta_2 = \alpha}} \sum_{s_1, s_2} M_{s_1, s_2}(\beta_1, \beta_2) T_{\beta_1}^{(s_1)} T_{\beta_2}^{(s_2)}. \quad (2.13)$$

Here  $M_{s_1, s_2}(\beta_1, \beta_2)$  denote some numerical coefficients that we expect to be simply  $\pm 1$  or 0 (or possibly other rational numbers) for an appropriate choice of the dual bases  $T_{\pm\alpha}^{(s)}$  in the  $\pm\alpha$  root spaces.

We do not have yet a full understanding of the precise set  $\mathcal{C}$  of ‘constraint’ generators<sup>8</sup>, nor of the numerical coefficients  $M_{s_1, s_2}(\beta_1, \beta_2)$  entering the definition of our generalized Virasoro constraints  $\mathfrak{L}_{\alpha, \bar{s}}$ . We will argue that a distinguished role is played by the ‘null subset’ of  $\mathcal{C}$ , *i.e.*, by the case where  $\alpha$  is a null root. In that case, the corresponding constraint degeneracy index takes only one value (while the degeneracy of a null root within the hyperbolic algebra is equal to the rank). Moreover, still in the case where  $\alpha$  is a null root, we will be able to verify that the coefficients  $M_{s_1, s_2}(\beta_1, \beta_2)$  in (2.13) are indeed simply equal to  $\pm 1$  when both of  $\beta_1$  and  $\beta_2$  (such that  $\alpha = \beta_1 + \beta_2$ ) are *real* roots. In the following, we shall refer to the better understood ‘null’ subset of  $\mathcal{C}$  as being the *skeleton* of  $\mathcal{C}$ ; and we shall refer to the better understood set of special configurations  $(\alpha, \beta_1, \beta_2)$ , with  $\alpha$  null,  $\beta_1$ , and  $\beta_2$  real, and  $\alpha = \beta_1 + \beta_2$ , as being the universal *scaffold* at the basis of our construction.

As the name ‘skeleton’ suggests, there are more constraints than those associated to null roots. Below, we shall give explicit examples of (‘fleshy’) constraints associated with strictly imaginary roots  $\alpha^2 < 0$ . However, constraints associated to null roots play a distinguished role in our construction. The special role of light-like  $\alpha$  is already suggested by the affine Sugawara

---

<sup>7</sup>Note that while  $\alpha$  runs over a subset of  $\Delta$ ,  $\bar{s}$  correspondingly runs over a subset of the full degeneracy of the root  $\alpha \in \Delta$ .

<sup>8</sup>The letter  $\mathcal{C}$  is used here to evoke both the word ‘constraint’, and the fact that the set  $\mathcal{C}$  appears to have the structure of a convex cone.

construction (2.7) where constraints were *only* defined for null roots. In addition, the special configurations where both  $\beta_1$  and  $\beta_2$  are real introduce a significant simplification in our construction. Indeed, in that case the root spaces associated to  $\beta_1$  and  $\beta_2$  are one-dimensional, so that there exists a unique (up to sign) contraction between the associated step operators. By contrast, when not both of  $\beta_1$  and  $\beta_2$  are real, the root spaces that are paired are multidimensional, and moreover not necessarily of equal dimension. This leaves open many possibilities for ‘contracting’  $T_{\beta_1}^{(s_1)}$  with  $T_{\beta_2}^{(s_2)}$  in forming  $\mathfrak{L}_{\alpha, \bar{s}}$ . The information on how to contract the elements of different root spaces is then encoded in the choice of the coefficients  $M_{s_1, s_2}(\beta_1, \beta_2)$ . Let us note, however, that, given a certain pair  $(\alpha, \bar{s}) \in \mathcal{C}$ , *i.e.*, given a certain Lie algebra generator  $T_{\alpha}^{(\bar{s})}$ , there exists (when  $\alpha = \beta_1 + \beta_2$ ) a distinguished way of contracting (a part of) the  $\beta_1$  root space  $\mathfrak{g}_{\beta_1}$  with the  $\beta_2$  one  $\mathfrak{g}_{\beta_2}$ . Indeed, if we denote  $\beta_1 = \alpha - \beta$ , so that  $\beta_2 = +\beta$ , the adjoint action of  $T_{\alpha}^{(\bar{s})}$ ,  $\text{ad}_{T_{\alpha}^{(\bar{s})}} x \equiv [T_{\alpha}^{(\bar{s})}, x]$  maps  $\mathfrak{g}_{-\beta}$  onto (a part of)  $\mathfrak{g}_{\beta_1} = \mathfrak{g}_{\alpha-\beta}$ . We can then use the natural ‘dual’ pairing between  $\mathfrak{g}_{-\beta}$  and  $\mathfrak{g}_{+\beta}$  (*i.e.*, between  $\mathfrak{g}_{-\beta_2}$  and  $\mathfrak{g}_{+\beta_2}$ ) to write putative constraints of the form<sup>9</sup>

$$\mathfrak{L}_{\alpha, \bar{s}} = \sum_{\beta \in \Delta^{\text{hyp}}} \sum_s N(\alpha, \beta) [T_{\alpha}^{(\bar{s})}, T_{-\beta}^{(s)}] T_{\beta}^{(s)}. \quad (2.14)$$

Here the coefficients  $N(\alpha, \beta)$  no longer depend on the degeneracy index  $s$  within the dual spaces  $\mathfrak{g}_{\pm\beta}$ , and the sum over  $s$  is easily seen to be *independent* of the choice of (dual) bases  $T_{\pm\beta}^{(s)}$  (as long as the orthonormalization condition (2.11) is satisfied). We leave to future work further study of the usefulness of the special construction (2.14).

One advantage of expressing the constraints as in (2.13) is that, contrary to the expressions derived in [4] (which were formulated in terms of the  $GL(10)$  level decomposition of  $E_{10}$ ), such a definition a priori appears not to be tied to any particular level decomposition of the hyperbolic algebra. Therefore, this opens up the possibility of writing a ‘universal’ set of coset constraints, whose further (particular) level decompositions could give rise to the apparently *different* canonical constraints arising in different maximal supergravities (mIIA, IIB, ...). However, we shall give evidence below that this hope of a universal constraint construction is not fulfilled in this simple

---

<sup>9</sup>To see that expression (2.14) is indeed well-defined, one can invoke the invariance of the bilinear form, see Lemma 2.4 in [23].

way. Rather, we will encounter a more refined construction, where only the scaffold is universal. The reason appears to lie in the existence of various ways of contracting (multi-dimensional) root spaces, *i.e.*, in the possibility of various consistent choices for the coefficients  $M_{s_1, s_2}(\beta_1, \beta_2)$ . Each particular level decomposition might be tied to a particular corresponding choice for these coefficients. Even if this turns out to be the case, it seems that our construction still involves a universal part, namely the part of (2.13) involving the *skeleton* of ‘null’ constraints, and its associated *scaffold* of special configurations where a null root  $\alpha$  is decomposed into two real roots  $\beta_1$  and  $\beta_2$ . As we shall emphasize below, this universal part is invariant under the Weyl group of the hyperbolic algebra and already yields an infinite number of constraints (associated to the intersection of the light-cone with the root lattice). This ‘universal part’ is, however, *not invariant* under the hyperbolic algebra itself. As we shall see below, one can associate to each choice of a finite-dimensional subalgebra (used as a way of ‘slicing’ the hyperbolic algebra by means of a corresponding level decomposition) a way of generating additional constraints by covariantizing under that subalgebra. Each such covariantization procedure allows one to ‘flesh out’ the skeleton by adding new constraints inside the light cone and also terms with  $\beta_1$  and  $\beta_2$  not both real. The prescription will be made more precise in section 3 when we discuss the example of  $E_{10}$ .

A further general issue regarding (2.13) is the operator ordering. Below we will work with similar expressions involving functions on classical phase space which are commuting. [Note that they commute as functions, but do not ‘Poisson commute’.] For those the issue of ordering becomes relevant only after the transition to the quantum theory, which we will not consider here. Finally, as written, (2.13) is meant to define only one constraint per root even though null roots have multiplicity greater than one.

The structure of null roots in hyperbolic over-extended algebras is known to be given by Weyl orbits through

$$\Delta^{\text{null}} = \bigcup_{n \in \mathbb{Z} \setminus \{0\}} \mathcal{W} \cdot (n \delta), \quad (2.15)$$

where  $\mathcal{W}$  is the hyperbolic Weyl group and  $\delta$  the primitive null root of the affine algebra embedded in the hyperbolic extension. Restricting the construction (3.5) to affine generators reduces all the Weyl orbits to points since  $\delta$  is invariant under the affine Weyl group. Hence the construction gives

constraints only for the roots  $\alpha = n\delta$  in agreement with the affine Sugawara construction (2.7).

At this point, we stress a possible qualitative difference between the usual affine Sugawara construction (2.7) and the corresponding hyperbolic construction (2.13) at the present stage of our understanding of the construction. The affine Virasoro constraints  $L_{n\delta}$  form a *two-sided* tower, where  $n$  runs over the set of integers  $\mathbb{Z}$ , while it seems consistent that the hyperbolic constraints  $\mathfrak{L}_{\alpha,\bar{s}}$  run over a set  $\mathcal{C}$  which is a *one-sided* convex cone, contained within the *past light-cone* of the Lorentzian root lattice. This one-sided structure of the constraints was clearly apparent in [4], where only constraints  $\mathfrak{L}_\alpha$  corresponding to negative imaginary  $\alpha$  were found, as will be shown in section 3 below.<sup>10</sup>

This asymmetry between the two-sidedness of the usual affine (Virasoro) constraints, and the one-sidedness of the hyperbolic ones, seems to be deeply rooted in the different physics (and mathematics) associated to the origin of these constraints. In the usual affine case, the origin of the constraints is a gauge invariance under reparametrizations of (two) periodic (world-sheet light-cone) variables  $\sigma_\pm = \tau \pm \sigma$ . The periodic nature of these variables, and the real (or hermitian) character of the worldsheet embedding functions, e.g.  $\partial_\pm X^\mu(\tau, \sigma)$ , implies the existence of two-sided Fourier expansions involving, for each choice of sign in  $\sigma_\pm$  the two complex-conjugated basis functions  $\exp(+in\sigma_\pm)$  and  $\exp(-in\sigma_\pm)$ . By contrast, the hyperbolic coset models should describe the gravitational physics taking place near a space-like singularity, *i.e.*, in a time-asymmetric situation of the type  $t \rightarrow 0^+$ , say. Moreover, the hyperbolic coset model is itself parametrized asymmetrically in terms of *positive* roots only. The analysis of the dynamics of supergravity in [3] found evidence for relating the supergravity fields to *one-sided* towers of coset variables. This tower consists of the so-called ‘gradient generators’ that are conjectured to correspond to multiple spatial gradients, roughly in terms of a spatial Taylor expansion. It is then natural to conjecture that the usual space-dependent supergravity constraints will also give rise to one-sided-only towers of ‘gradient cousins’ of the (already one-sided) low-level constraints discussed in [4].

---

<sup>10</sup>There was a further one-sidedness in [4] related to the fact that we were working in a *truncated* coset whence only a Borel subalgebra of the hyperbolic algebra played a role. This effect is an artefact of the truncation and irrelevant to the present construction.

Another (related) argument for expecting that the tower of coset constraints be one-sided only, is the idea proposed in [4] that the set of constraints be just large enough to reduce the exponentially infinite number of variables entering the hyperbolic coset models to a much smaller number of degrees of freedom involving only a rather small vicinity of the future light-cone in root space (*i.e.*, essentially the gradient generators, plus a relatively manageable set of extra M-theoretic degrees of freedom). To achieve such a strong reduction in the number of degrees of freedom, without killing them all, it is natural to have a set of constraints  $\mathcal{C}$  which fills, like the coset variables, a one-sided cone and whose degeneracies do not grow faster than the ones of the roots. Note, however, that our intuitive argument cannot exclude the possibility that the constraints fill a double-sided cone, if the degeneracies of the constraints are such that the sum of the positive-sided and negative-sided ones does not grow faster than the positive-root degeneracies.

Whatever be the ultimate definition of the physically correct set of coset constraints,  $\mathfrak{L}_{\alpha, \bar{s}}$ , one would expect it to satisfy some commutation relations (of the general type  $[\mathfrak{L}, \mathfrak{L}] = O(\mathfrak{L})$ ) reflecting some aspects of the (currently unknown) underlying gauge symmetry of the hyperbolic models, in the same way that the Virasoro algebra (2.9) is a gauge-fixed remnant of the world-sheet diffeomorphism symmetry of the underlying (Nambu-Goto-type) string action. Given the trial expression (2.13) one can wonder what algebra these expressions satisfy, *i.e.*, whether there is a generalization of the Virasoro algebra (2.9) associated with our construction. While a conclusive answer to this question would require a knowledge of the  $E_{10}$  algebra which is presently not available, we can at least formulate the following expectation. Under the Poisson (or Dirac) bracket the grading of the algebra implies that the simplest type of commutation relation one might have is of the form

$$\{\mathfrak{L}_\alpha, \mathfrak{L}_\beta\} = \sum_\gamma J_{\alpha+\beta-\gamma} \mathfrak{L}_\gamma. \quad (2.16)$$

As we shall discuss in the next section below, relations of the type (2.16) *do hold* if we consider only the (truncated, low-level) constraints of [4]. However, the vast generalization of the definition of the constraints introduced in the present paper makes the validity of a result of the type (2.16) highly non-trivial and dependent upon delicate structures that we do not currently understand in detail. Indeed, there are two non-trivial assertions contained

in the expected result (2.16). The first one is that the trilinear<sup>11</sup> expression in current components on the r.h.s. organizes itself into products between constraints and certain current components, much in the same way as for the affine Virasoro algebra (cf. (2.9) where the r.h.s. is a product of a constraint  $L_{m\delta}$  by a (conserved) algebra generator  $c$ ). The second claim relates to the roots  $\gamma$  contributing on the r.h.s. and the question whether these only cover constraints that had been defined previously. Both points are important for ascertaining the closure of the constraint algebra. The fact that only strongly conserved coefficients appear in the algebra of constraints is important for the discussion of open algebras, as mentioned in the introduction. We note one point concerning (2.16) in comparison to the affine Virasoro algebra (2.9). There it was important that an additive structure existed on the set of all roots for which generators  $L_{m\delta}$  were defined. Here, we expect that this additive structure will be replaced by a certain convexity-related structure of the cone  $\mathcal{C}$ , akin to the structure of integrable highest-weight representations [23]. Though we do not yet fully comprehend this structure, we shall see below that our proposed ‘fleshing out’ of the skeleton ensures (when starting from a past-light-cone-only skeleton) the convex structure of a solid cone, *i.e.*, all  $\alpha$ ’s generated by our construction lie on or inside the light-cone.

### 3 Universality and relation to supergravity

In this section we specialize to the case of  $E_{10}$  whose Dynkin diagram is given in figure 1. The relation to supergravity will help to make the construction of the preceding section more concrete. An important role will be seen to be played by the relation between  $D = 11$  supergravity (or type IIA in  $D = 10$ ), and type IIB in  $D = 10$ .

#### 3.1 Consistency with supergravity constraints: $D = 11$

The Sugawara constraints (2.13) can be interpreted as constraints to be imposed on geodesics on the infinite-dimensional coset space  $E_{10}/K(E_{10})$  as follows [4]. The global  $E_{10}$  symmetry gives rise to conserved Noether charges  $\mathcal{J} \in \text{Lie}(E_{10})$  that can be expanded in the orthonormal basis  $\{T_\alpha^{(s)} \mid \alpha \in$

---

<sup>11</sup>The hyperbolic Lie algebra structure  $\{J, J\} = J$  guarantees that the commutator of two  $J$ -bilinear constraints  $\mathcal{L}$  is only trilinear in the  $J$ ’s.

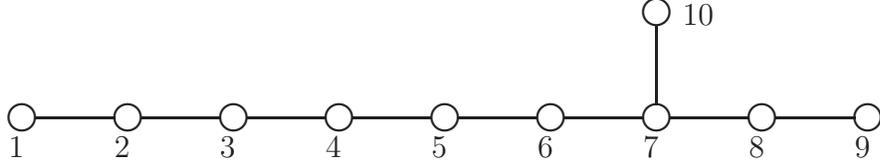


Figure 1: Dynkin diagram of  $E_{10}$  with numbering of nodes.

$\Delta^{\text{hyp}}, s = 1, \dots, \text{mult } \alpha\}$  as

$$\mathcal{J} = \sum_{\alpha \in \Delta^{\text{hyp}}} \sum_{s=1}^{\text{mult } \alpha} J_{\alpha}^{(s)} T_{\alpha}^{(s)}. \quad (3.1)$$

The pairing between charges and generators is as in [4]:

$$\begin{aligned} \mathcal{J} = & \dots + \frac{1}{3!} J^{(-1)}_{m_1 m_2 m_3} F_{m_1 m_2 m_3} + J^{(0)}_m K^n_m \\ & + \frac{1}{3!} J^{(1)}_{m_1 m_2 m_3} E^{m_1 m_2 m_3} + \dots, \end{aligned} \quad (3.2)$$

where we have for definiteness chosen the  $\mathfrak{gl}(10)$  level decomposition of  $E_{10}$  that is reviewed in appendix A.1. An important point to note here is that tensor generators and coefficients transform contragrediently. For instance, for the Chevalley-Serre generators this translates into the following identification

$$\begin{aligned} T_{\alpha_1} &= K^1_2 & [\sim e_1] \\ J_{\alpha_1} &= J^2_1 & [\sim f_1 = -\omega(e_1)] \end{aligned} \quad (3.3)$$

and so on, where  $\omega$  is the Chevalley involution on  $E_{10}$ . With this identification of algebra generators and current components we can work either in the universal enveloping algebra, generated by the  $T_{\alpha}$ , or in the Poisson algebra, generated by the current components  $J_{\alpha}$ . Namely, when considered as elements of a Poisson algebra on phase space, the components  $J_{\alpha}^{(s)}$  close into the same hyperbolic algebra under Poisson commutation, as follows directly from the Hamiltonian formulation of the coset space dynamics. That is, we have the canonical brackets

$$\{J_{\alpha_1}^{(s_1)}, J_{\alpha_2}^{(s_2)}\} = f_{\alpha_1 \alpha_2}^{(s_1)(s_2)} J_{\alpha_1 + \alpha_2}^{(s_{12})}, \quad (3.4)$$

identical (including the sign) to the commutation relations of the hyperbolic algebra (2.11). The classically conserved charges of the  $E_{10}/K(E_{10})$  model are commuting functions on phase space in terms of which we write the classical constraints as

$$\mathfrak{L}_\alpha = \sum_{\beta \in \Delta^{\text{hyp}}} \sum_{s,s'} M_{s,s'}(\alpha, \beta) J_{\alpha-\beta}^{(s)} J_\beta^{(s')}. \quad (3.5)$$

without specifying the summation over the ‘internal’ degrees of freedom at this point (that is, the matrix  $M_{s,s'}(\alpha, \beta)$ ). The Hamiltonian (scalar) constraint entering the coset model of [3] can be represented as the special member of the hierarchy of constraints (3.5) corresponding to  $\alpha = 0$

$$\mathfrak{L}_0 \equiv \mathcal{H} = \sum_{\beta \geq 0} \sum_{s=1}^{\text{mult } \beta} J_{-\beta}^{(s)} J_\beta^{(s)}. \quad (3.6)$$

In this way one confirms that all Noether charges  $J_\alpha^{(s)}$  are indeed classically conserved because they Poisson commute with  $\mathcal{H}$ :

$$\{\mathcal{H}, J_\alpha^{(s)}\} = 0. \quad (3.7)$$

This is a direct consequence of the fact that  $\mathcal{H}$  is just the quadratic Casimir operator for the hyperbolic algebra (see chapter 2 of [23] for a proof and the explicit computation). We note also that for the Hamiltonian constraint (3.6) the issues of contracting generators from root spaces of different dimensions are absent since the root spaces of  $\alpha$  and  $-\alpha$  always have the same dimension. Since all components of  $\mathcal{J}$  are conserved, *any* expression of the type (3.5) is strictly conserved for any geodesic. We can therefore consistently constrain the geodesic motion on the coset space by demanding that the initial conditions satisfy  $\mathfrak{L}_\alpha = 0$ .

In [4] we have shown (with the same truncation of higher order spatial gradients as in [3]) that the canonical constraints of  $D = 11$  supergravity can be successively rewritten in two different (but related) forms. Our analysis used an  $A_9 = \mathfrak{sl}(10)$  level decomposition of the  $E_{10}$  algebra, corresponding to the removal of node 10 in fig. 1. The results of this level decomposition of [3, 27] are reproduced in appendix A.1. The explicit computation involved the determination of various numerical coefficients in the  $E_{10}$  expressions that were originally fixed by requiring weak conservation of the constraint surface under the coset model equations of motion. Comparison with the

canonical  $D = 11$  supergravity constraints and use of the dictionary then showed precise agreement of these numerical coefficients, thus extending the correspondence between the  $E_{10}/K(E_{10})$  coset model and the (truncated)  $D = 11$  supergravity equations of motion to the full canonical formulation. In section 3.1.2, we shall show that, remarkably, these specific numerical coefficients found for the supergravity constraints in [4] *coincide* with our proposed sum over canonically normalized current components (3.5) when both  $\beta$  and  $\alpha - \beta$  are real and for *unit* coefficients  $M_{s,s'}(\alpha - \beta, \beta)$ . In addition to this unearthing of a hidden simplicity in the definition of the constraints, another advantage of writing the constraints in the form (3.5) is that this will allow us to evaluate them also for other level decompositions, and in this way to verify agreement with the canonical constraints of massive IIA and IIB supergravity as well. The agreement between the dynamical (evolution) equations of these theories with the coset model equations in appropriate truncations had already been established in [11, 13, 14]. Moreover, the form (3.5) is directly amenable to an affine reduction, and brings out more clearly the analogy with the affine Sugawara construction.

### 3.1.1 On the roots associated to the supergravity constraints

Let us first turn to the detailed consideration of the set of roots, including their multiplicities, that are associated to supergravity constraints. In the case of  $D = 11$  supergravity, these constraints are, respectively, the diffeomorphism and Gauss constraints, and the Bianchi identities for the 4-form field strength and the Riemann tensor.<sup>12</sup> The analysis of [4] was based on a  $\mathfrak{gl}(10)$  level decomposition truncated at level  $\ell = 3$ , such that, when expressed in terms of the conserved  $E_{10}$  Noether current in this decomposition, the constraints take the form

$$\mathfrak{L}^{(-3)}{}^{n_1\dots n_9} = 28 \mathfrak{J}^{(-1)}{}_{[n_1 n_2 n_3} \mathfrak{J}^{(-2)}{}_{n_4\dots n_9]} + 3 \mathfrak{J}^{(-3)}{}_{p|[n_1\dots n_8} \mathfrak{J}^{(0)}{}_{n_9]p}, \quad (3.8a)$$

$$\begin{aligned} \mathfrak{L}^{(-4)}{}_{m_1\dots m_{10}|n_1 n_2} &= \frac{21}{10} \mathfrak{J}^{(-2)}{}_{n_1[m_1\dots m_5} \mathfrak{J}^{(-2)}{}_{m_6\dots m_{10}]n_2} + \frac{3}{2} \mathfrak{J}^{(-3)}{}_{n_2|[m_1\dots m_8} \mathfrak{J}^{(-1)}{}_{m_9 m_{10}]n_1} \\ &\quad - (n_1 \leftrightarrow n_2), \end{aligned} \quad (3.8b)$$

---

<sup>12</sup>In a more conventional canonical analysis, one would not interpret the Bianchi identities as proper constraints, as they are not directly associated to gauge transformations, unlike the diffeomorphism and Gauss constraints. In the present setting, however, they would correspond to generators of gauge transformations on the *dual* fields, *i.e.*, on the 7-form field and the ‘dual graviton’.

for the diffeomorphism and Gauss constraints and

$$\mathfrak{L}^{(-5)}_{m_1 \dots m_{10} | n_1 \dots n_5} = 3 \mathfrak{J}^{(-2)}_{m_1 m_2 [n_1 \dots n_4} \mathfrak{J}^{(-3)}_{n_5] m_3 \dots m_{10}}, \quad (3.8c)$$

$$\mathfrak{L}^{(-6)}_{m_1 \dots m_{10} | n_0 | n_1 \dots n_7} = 9 \mathfrak{J}^{(-3)}_{n_0 | m_1 \dots m_8} \mathfrak{J}^{(-3)}_{m_9 | m_{10} n_1 \dots n_7}. \quad (3.8d)$$

for the Bianchi identities. Here, we have changed the normalization of the charge  $J^{(-3)}$  compared to [4, 12] so that all highest weight states are uniformly normalized to unity (the usefulness of this re-definition was already pointed out in footnote 19 of [4]). Explicitly, the normalizations of the  $E_{10}$  generators, in their  $A_9$  decomposition, are

$$\begin{aligned} \langle J^a_b | J^c_d \rangle &= \delta^a_d \delta^c_b - \delta^a_b \delta^c_d, & \langle J^{(-1) a_1 a_2 a_3} | J_{b_1 b_2 b_3} \rangle &= 3! \delta^{a_1 a_2 a_3}_{b_1 b_2 b_3}, \\ \langle J^{(-2) a_1 \dots a_6} | J_{b_1 \dots b_6} \rangle &= 6! \delta^{a_1 \dots a_6}_{b_1 \dots b_6}, \end{aligned} \quad (3.9)$$

By contrast, for the mixed symmetry field on level  $|\ell| = 3$  we shall take here a normalization that differs from the one given in Eq. (2.30) of [12] by a factor  $1/9$ , *viz.*

$$\langle J^{(-3) a_0 | a_1 \dots a_8} | J_{b_0 | b_1 \dots b_8} \rangle = \frac{8 \cdot 8!}{9} \left( \delta^{a_0}_{b_0} \delta^{a_1 \dots a_8}_{b_1 \dots b_8} - \delta^{a_0}_{[b_1} \delta^{a_1 \dots a_7 a_8}_{b_2 \dots b_8] b_0} \right). \quad (3.10)$$

This normalization is chosen so that operators associated to real roots (two indices identical) have unit norm, like the highest weight

$$\langle J^{(-3)_{10} | 345678910} | J_{10 | 345678910} \rangle = 1 \quad (3.11)$$

whereas for operators associated to null roots (all indices different)

$$\langle J^{(-3)_{2 | 345678910} | J_{2 | 345678910} \rangle = \frac{8}{9}. \quad (3.12)$$

In addition to these normalizations, we have used in (3.8) the same implicit antisymmetrization conventions as in [4]. For instance, the expression in (3.8c), corresponding to a Bianchi constraint on the four-form field strength, is understood to be antisymmetrized (with weight one) over  $m_1 \dots m_{10}$ ; furthermore the last relation (3.8d) is to be projected onto a  $(7, 1)$  hook for the indices  $n_1 \dots n_7$  and  $n_0$ . We note that for the constraints listed in (3.8) there are no ordering ambiguities in a possible transition to

operator expressions in a quantum theory, except for  $\mathfrak{L}^{(-6)}$  in (3.8d), since all commutator terms vanish by Jacobi or Serre relations; for instance

$$\left[ J^{(-1)}_{[m_1 m_2 m_3]}, J^{(-2)}_{[m_4 \dots m_9]} \right] \propto J^{(-3)}_{[m_1 | m_2 \dots m_9]} = 0 \quad (3.13)$$

Let us now exhibit the roots underlying the diffeomorphism constraint (3.8a). For this, we first consider its highest component, corresponding to the indices 2345678910. To identify the root  $\alpha$  to which it belongs we must find the eigenvalues under the ten Cartan generators of  $E_{10}$ . (Indeed, the ‘covariant’ components,  $\alpha_i \equiv \alpha(h_i)$  of a root precisely encode the eigenvalues in  $[h_i, e_\alpha] = \alpha(h_i)e_\alpha$ .) Since we are working with the current components  $J$  we display the Cartan elements in this description. In the  $\mathfrak{gl}(10)$  basis the Cartan elements are

$$\begin{aligned} h_i &= J^i_{\phantom{i}i} - J^{i+1}_{\phantom{i+1}i+1} \quad (i = 1, \dots, 9), \\ h_{10} &= -\frac{1}{3} (J^1_{\phantom{1}1} + \dots + J^7_{\phantom{7}7}) + \frac{2}{3} (J^8_{\phantom{8}8} + J^9_{\phantom{9}9} + J^{10}_{\phantom{10}10}). \end{aligned} \quad (3.14)$$

Alternatively, one can do the calculation with Lie algebra elements, using the more familiar expressions of the Cartan generators  $h_i$  in terms of Lie algebra generators recalled in appendix A.1. (In that case, one notes that the constraint  $\mathfrak{L}^{(-3)2345678910}$  is associated with the contragredient Lie-algebra basis element  $F_{2345678910}$ .) An easy calculation shows that the only non-zero eigenvalue corresponds to  $h_1$  (first node in figure 1), and is equal to +1. Hence, the list of ‘covariant’ components  $\alpha_i \equiv \alpha(h_i)$ , also known as ‘Dynkin labels’, is  $[+1, 0, 0, 0, 0, 0, 0, 0, 0, 0]$ . This is equivalent to saying that the root associated to the highest component of the diffeomorphism constraint is equal to the fundamental weight  $\Lambda_1$  associated to the simple root  $\alpha_1$ .<sup>13</sup> To explicitly write the root  $\alpha = \Lambda_1$  associated to the highest diffeomorphism constraint in terms of the simple roots, we must convert its Dynkin labels to root labels, *i.e.*, pass from covariant indices to contravariant ones by using the inverse of the Cartan matrix  $A_{ij} = \langle h_i | h_j \rangle$ . This leads to the corresponding root

---

<sup>13</sup> The fundamental weights  $\Lambda_i$  are defined as dual to the simple roots  $\alpha_j$  w.r.t. the Cartan inner product:  $\langle \Lambda_i | \alpha_j \rangle = +\delta_{ij}$ . The fact that  $\Lambda_1$ , and the integrable highest-weight representation  $L(\Lambda_1)$  built from it, is related to the tower of constraints was already discussed at some length in [4]. This relation does not mean, however, that  $\mathfrak{L}^{(-3)2345678910}$  is a highest weight vector for the action of all the  $E_{10}$  generators. Actually, as was already shown in [4], and will be further discussed in section 3.3, it fails to be one.

$\alpha = -(\alpha_2 + 2\alpha_3 + 3\alpha_4 + 4\alpha_5 + 5\alpha_6 + 6\alpha_7 + 4\alpha_8 + 2\alpha_9 + 3\alpha_{10}) \equiv -\delta$ , where the (positive) root  $\delta$  denotes the primitive null root of  $E_9 \subset E_{10}$ . In particular, this shows that the root  $\alpha = \Lambda_1$  associated to the highest component of the diffeomorphism constraint is a *negative null* root.<sup>14</sup> We can therefore write for this particular component

$$\mathfrak{L}^{(-3)2345678910} \equiv \mathfrak{L}_\alpha \quad \text{with } \alpha = \Lambda_1 = -\delta \equiv -\delta^{(3)}. \quad (3.15)$$

Let us now consider consider the roots associated to the other components of the diffeomorphism constraint (3.8a). They are obtained by the action of the permutation group  $\mathcal{S}_{10}$  on the indices. Since the permutation group is the Weyl group of  $\mathfrak{sl}(10)$ , we conclude that *all* components of the diffeomorphism constraint are associated with (negative) null roots, forming a single orbit of the Weyl group  $W(\mathfrak{sl}(10))$ . These null roots can be obtained by acting with the corresponding Weyl transformation on  $\delta$ , such that

$$w(\mathfrak{L}_\alpha) = \mathfrak{L}_{w(\alpha)} \quad (3.16)$$

where  $w$  on the left hand side acts on the indices of the constraint  $\mathfrak{L}$  by permuting them.

Let us now proceed to considering the roots associated to the higher-level (or rather ‘lower-level’, as the levels are negative) constraints. To find the roots for the level  $\ell = -4$  and  $\ell = -5$  constraints in (3.8b) and (3.8c), we consider their highest weight components. These are  $\mathfrak{L}^{(-4)12345678910||910}$  and  $\mathfrak{L}^{(-5)12345678910||678910}$ , respectively. A straightforward calculation gives the eigenvalues  $[0, 0, 0, 0, 0, 0, 0, 1, 0, -1]$  and  $[0, 0, 0, 0, 1, 0, 0, 0, 0, -1]$ , respectively. The corresponding roots are again found to be *null* and negative. In view of the fact, recalled in (2.15), that all null roots are Weyl images of the basic one-dimensional string of affine null roots  $n\delta$ , we can look for the specific affine root  $n\delta$  from which they descend. We find that it is  $-\delta$ , *i.e.*,  $n = -1$ . In other words, in addition to being null, the roots associated to

---

<sup>14</sup>We note that the association of the ‘null’ ( or ‘cusp’) fundamental weight  $\Lambda_1$  to the diffeomorphism constraint is valid not only for maximal supergravity and  $E_{10}$ , but also for other (super)gravity theories. For instance, for pure gravity in any spatial dimension  $d$  the basic (diffeomorphism) constraint is always associated to roots of the form  $-\mu_a$ , where  $\mu_a$  (with  $a = 1, \dots, d$ ) denotes the null roots that are contained within the  $GL(d)$  multiplet of the ‘gravity root’. The notation  $\mu_a = -\beta^a + \sum_c \beta^c$  is the notation used in [28]. Note that the null root  $-\mu_1$  is indeed the fundamental weight associated with the ‘hyperbolic’ node of  $AE_d$  (as explicitly displayed in equation (3.14) of [29]).

the level  $\ell = -4$  and  $\ell = -5$  constraints can be obtained from the ‘basic’  $\ell = -3$  ‘diffeomorphism-constraint’ root  $\alpha = \Lambda_1 = -\delta \equiv -\delta^{(3)}$  by applying some  $E_{10}$  Weyl reflection:  $w_\alpha(\beta) = \beta - (\alpha \cdot \beta)\alpha$  (here simplified by taking into account the fact that  $\alpha \cdot \alpha = 2$  for the roots of a simply laced algebra). More explicitly, we have:

$$\delta^{(4)} = w_\theta(\delta^{(3)}), \quad \theta := \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_{10} \quad (3.17)$$

and

$$\delta^{(5)} = w_{\theta'}(\delta^{(4)}), \quad \theta' := \alpha_6 + 2\alpha_7 + 2\alpha_8 + \alpha_9 + \alpha_{10} \quad (3.18)$$

where we have given the explicit Weyl reflections in  $\mathcal{W}(E_{10})$  that move between the different levels. Note that  $\theta$  is the highest root of the embedded  $A_8$  algebra associated with the IIB theory, and  $\theta'$  is the highest weight of an embedded  $D_5$  algebra. Finally, similarly to the case of the roots associated to  $\mathfrak{L}^{(-3)}$ , the fact that the Young tableaux describing the  $GL(10)$  index structure of  $\mathfrak{L}^{(-4)}$  and  $\mathfrak{L}^{(-5)}$  are totally antisymmetric guarantees that *all* the roots associated to the other components of these constraints are obtained from the basic ones (3.17) and (3.18) by  $GL(10)$  permutations, *i.e.*, by further Weyl reflections. In particular, all of them are *null*.

So far all the roots associated to the first three levels of constraints have been found to be light-like (and negative). The constraint  $\mathfrak{L}^{(-6)}$  differs from the lower level ones in that it is the first in the hierarchy of constraints to involve a non-trivial Young tableau. As a consequence, we are going to see that it contains a mixture of null ( $\alpha^2 = 0$ ) and time-like ( $\alpha^2 = -2$ ) roots. More precisely, the highest weight component  $\mathfrak{L}^{(-6)}_{1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10||10|4\ 5\ 6\ 7\ 8\ 9\ 10}$  is easily checked to be associated to a null root, which can be obtained from  $-\delta^{(5)}$  by the following Weyl transformation

$$\delta^{(6)} = w_{\theta''}(\delta^{(5)}), \quad \theta'' = \alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8 + \alpha_{10} \quad (3.19)$$

Here,  $\theta''$  is the highest root of an embedded  $D_6$  algebra. Covariantizing this component under the action of the  $\mathfrak{sl}(10) = A_9$  subalgebra gives a representation of  $(7, 1)$  hook type which is not a pure antisymmetric tensor unlike the constraints on levels  $-3$ ,  $-4$  and  $-5$ . From the point of view of the permutation group  $\mathcal{S}_{10} = \mathcal{W}(\mathfrak{sl}(10))$  this means that there are two separate orbits under  $\mathcal{W}(\mathfrak{sl}(10))$ . The ‘outer’ orbit consists of permutations of the lowest weight indices and corresponds to null roots of  $E_{10}$ . The inner orbit

corresponds to imaginary  $E_{10}$  roots with  $\alpha^2 = -2$ . In terms of the supergravity constraint (3.8d) these two orbits correspond to cases when there are two identical indices on the (7, 1) hook part or when they are all different, respectively. The ‘skeleton’ of null roots  $-\delta^{(3)}, -\delta^{(4)}, -\delta^{(5)}, \dots$ , together with their multiples (discussed below) and their time-like descendants, is sketched in figure 2.

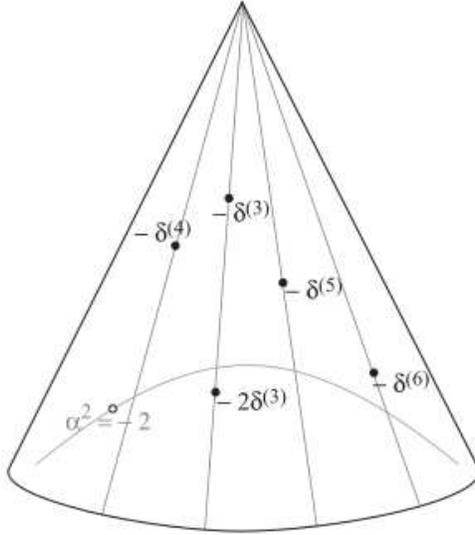


Figure 2: Sketch of the set  $\mathcal{C}$  of roots (and notably its ‘skeleton’ of null roots on the past light-cone) labelling the extended set of constraints constructed in this paper.

Let us finally note that all null roots  $\alpha$  appearing in these constraints appear with multiplicity one, although the same roots, considered as  $E_{10}$  roots have the non-trivial root multiplicity eight. That the null roots appear with multiplicity one in the Sugawara construction should be so by consistency with the affine case. By contrast, the purely imaginary roots belonging to the inner orbit of  $\mathfrak{L}^{(-6)}$  have multiplicity seven as constraints compared to multiplicity 44 as roots of  $E_{10}$ .

### 3.1.2 Supergravity constraints and canonical normalization

So far we have analyzed the roots  $\alpha$  labelling the l.h.s. of our basic Sugawara-like expression (2.13). Next we analyze the roots  $\beta_1, \beta_2$  contributing to the

*right hand side* of (2.13). Our principal aim here will be to see what are the values of the numerical coefficients  $M_{s_1, s_2}(\beta_1, \beta_2)$  that enter the Sugawara-like sum. We start here from the explicit  $GL(10)$ -decomposed form (3.8). To this aim let us consider the components of the currents  $J$  on the r.h.s. where the indices are distributed in a specific way. For example, we can pick out two representative terms where only operators for *real roots* appear and obtain

$$\begin{aligned} \mathfrak{L}_{2345678910}^{(-3)} &\ni 28 \cdot \frac{3! \cdot 6!}{9!} \frac{(-1)_{234}}{J} \frac{(-2)_{5678910}}{J} + 3 \cdot \frac{8!}{9!} \frac{(-3)_{2|23456789}}{J} \frac{(0)_{10}}{J} \\ &= \frac{1}{3} \left( \frac{(-1)_{234}}{J} \frac{(-2)_{5678910}}{J} + \frac{(-3)_{2|23456789}}{J} \frac{(0)_{10}}{J} \right). \end{aligned} \quad (3.20)$$

Hence, we find the remarkable fact that the combinatorial factors appearing in (3.8a) are precisely such as to imply, in the root basis, a relative normalization equal to *unity*. As the overall prefactor  $1/3$  (as well as the corresponding  $1/60$  in the formulas below) is merely chosen to agree with the normalisations in [4], it might eventually be traded for a more convenient one. Thus, all terms in the bracket belong to real roots and are canonically normalized, justifying *in retrospect* the relative factor in (3.8a) by (3.5).

For the Gauss constraint (3.8b) one similarly finds

$$\begin{aligned} \mathfrak{L}_{12345678910||910}^{(-4)} &\ni \frac{21}{10} \cdot \frac{2 \cdot 5! \cdot 5!}{10!} \frac{(-2)_{9123104}}{J} \frac{(-2)_{5678910}}{J} \\ &+ \frac{3}{2} \cdot \frac{1}{2} \cdot \frac{2 \cdot 8!}{10!} \frac{(-3)_{9|910123456}}{J} \frac{(-1)_{7810}}{J} \\ &= \frac{1}{60} \left( \frac{(-2)_{9123104}}{J} \frac{(-2)_{5678910}}{J} + \frac{(-3)_{9|910123456}}{J} \frac{(-1)_{7810}}{J} \right). \end{aligned} \quad (3.21)$$

Again, the terms appear with the same relative coefficient and confirm the expression (3.5) for real roots.

For the constraints (3.8c) and (3.8d) on levels  $\ell = -5$  and  $\ell = -6$  there is nothing to check since there is only one type of term. The basic ‘scaffold’ of Sugawara constraints exhibiting a decomposition  $\alpha = \beta_1 + \beta_2$  with  $\alpha$  null and  $\beta_1, \beta_2$  real is illustrated in figure 3. Note that the relations  $\alpha^2 = 0$  and  $\beta_1^2 = \beta_2^2 = 2$  imply that  $\alpha \cdot \beta_1 = 0 = \alpha \cdot \beta_2$ , *i.e.*, that  $\beta_1$  and  $\beta_2$  are orthogonal to  $\alpha$ , so that they belong to the hyperplane tangent to the light-cone along the considered null root (see figure 3, where one has chosen  $\alpha = -\delta$ ). One has to imagine the infinite ‘scaffold’ made by the tangent hyperplanes associated to the infinite skeleton of Weyl images of  $-\delta$ .

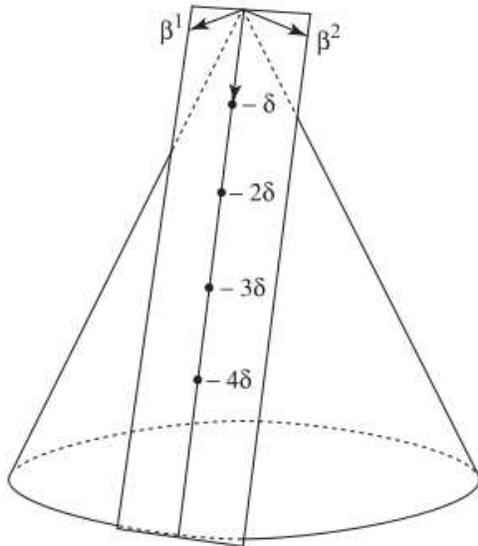


Figure 3: Sketch of one of the basic elements of the infinite ‘scaffold’ of special Sugawara configurations  $\alpha = \beta_1 + \beta_2$  with  $\alpha$  null and  $\beta_1, \beta_2$  real. The real roots  $\beta_1, \beta_2$  lie within the hyperplane tangent to the light-cone along the null root (here chosen to be  $\alpha = -\delta$ ). One must imagine completing the structure shown here by all its Weyl images.

### 3.1.3 General structure of constraints

We note that there are also terms contributing to (3.5) where not both of  $J_{\alpha-\beta}$  and  $J_\beta$  are real. For example, (3.8a) contains a term

$$\mathfrak{L}^{(-3)}_{2345678910} \ni \frac{1}{3} J^{(-3)}_{1|23456789} J^{(0)}_{10}_1, \quad (3.22)$$

where an imaginary level three root is contracted with a real level zero root (albeit positive). Similar contractions appear also for the other constraints. Note that, though, after removing the same prefactor (1/3) as above, we have again a simple coefficient unity, the time-like-root generator associated to the level  $-3$  root is such that its normalization involves the fraction 8/9, see (3.12).

At this stage, we start seeing several patterns appearing within the structure of the constraints, and notably in the set  $\mathcal{C}$  labelling the roots (together with their multiplicity) associated to the constraints. A first pattern is that, so far, all the constraints can be labelled by the members of the integrable

highest-weight representation descending from the fundamental weight  $\Lambda_1$ , which is dual to the first (‘hyperbolic’) node of the Dynkin diagram, figure 1. A second, closely related, pattern is that the pattern of roots comprise many null roots, and that the null constraint-roots studied so far all belong to the Weyl orbit of  $\Lambda_1 = -\delta$ . A third pattern is the simple (unit) relative normalization of the contributions  $(\alpha, \beta_1, \beta_2)$  to the Sugawara expression (2.13) involving the decomposition of a null root  $\alpha$  into two real roots  $(\beta_1, \beta_2)$ . A fourth pattern is that the null roots associated to non purely antisymmetric Young tableaux give rise, upon covariantization under  $GL(10)$ , to a set of roots which ‘penetrate’ within the past light-cone, *i.e.*, which are time-like (and past-directed) rather than light-like.

It is tantalizing to generalize these patterns to the infinite tower of coset constraints that we are trying to construct. We can first assume that the set  $\mathcal{C}$  of ‘constraint roots’ contains the full weight diagram, say  $P(\Lambda_1)$ , of the fundamental representation  $L(\Lambda_1)$  based on  $\Lambda_1 = -\delta$ . By proposition 10.1 of [23] we know that  $P(\Lambda_1)$  (including its multiplicities) is invariant under the full  $E_{10}$  Weyl group,  $\mathcal{W}(E_{10})$ . In particular, this would imply, in view of (2.15), that there is an *infinite sequence of null constraints* related to the orbit of minus the primitive null element  $-\delta$ . Upon covariantization of the resulting highest weight vectors under  $\mathfrak{sl}(10)$  we obtain a series of constraints related to  $\delta$  as indicated in the first row of the following table

|            | $\ell = -3$                       | $\ell = -4$                       | $\ell = -5$                       | $\ell = -6$                        | $\ell = -7$                       | $\ell = -8$                        | ... |
|------------|-----------------------------------|-----------------------------------|-----------------------------------|------------------------------------|-----------------------------------|------------------------------------|-----|
| $-\delta$  | $\mathfrak{L}_{(-\delta)}^{(-3)}$ | $\mathfrak{L}_{(-\delta)}^{(-4)}$ | $\mathfrak{L}_{(-\delta)}^{(-5)}$ | $\mathfrak{L}_{(-\delta)}^{(-6)}$  | $\mathfrak{L}_{(-\delta)}^{(-7)}$ | $\mathfrak{L}_{(-\delta)}^{(-8)}$  | ... |
| $-2\delta$ |                                   |                                   |                                   | $\mathfrak{L}_{(-2\delta)}^{(-6)}$ |                                   | $\mathfrak{L}_{(-2\delta)}^{(-8)}$ | ... |
| $\vdots$   |                                   |                                   |                                   |                                    |                                   |                                    |     |

Here, we added a subscript  $(-\delta)$  to all the constraints in the  $\mathcal{W}(E_{10})$  orbit of  $-\delta$  and suppressed the labels for the  $\mathcal{W}(\mathfrak{sl}(10))$  suborbits in the columns. Let us also recall the existence of the Hamiltonian constraint,  $\mathfrak{L}_0$ , which could be thought of as being associated to the 0th multiple of  $\delta$ .

In addition to the ‘skeleton’ of null roots constituting the Weyl orbit of  $\Lambda_1 = -\delta$ , the weight diagram  $P(\Lambda_1)$  of  $L(\Lambda_1)$  contains *all (past-directed) time-like roots*. This follows from proposition 11.2a of [23]. To apply this proposition, we need, for each putative weight  $\mu$  within the Weyl chamber ( $\mu \in P_+$ ; *i.e.*  $\mu = \sum_{i=1}^{10} p_i \Lambda_i$ , with  $p_i \geq 0$ ), to control the ‘support’ of the root  $\Lambda_1 - \mu$ , *i.e.* the non-zero coefficients  $m_j$  in its simple-root decomposition:

$\Lambda_1 - \mu = \sum_{j=1}^{10} m_j \alpha_j$ . Using  $\langle \Lambda_1 | \alpha_j \rangle = \delta_{1j}$ , and  $\langle \alpha_i | \alpha_j \rangle = A_{ij}$ , the root-basis integers  $m_j$  are easily seen to be related to the weight-basis integers  $p_i$  via the knowledge of the *inverse* of the  $E_{10}$  Cartan matrix  $A_{ij}$ . Now, by explicit inspection of this inverse Cartan matrix (see, e.g. [30]), one finds that the only place in it where there is a zero in the first column is in the first row. This shows that any element of the Weyl chamber  $\mu = \sum_{i=1}^{10} p_i \Lambda_i$  such that  $p_i \neq 0$  for at least one  $i$  among  $2, \dots, 10$ , the vector  $\Lambda_1 - \mu$  has non-vanishing ‘support’  $m_1$  on the first node and hence is ‘non-degenerate w.r.t  $\Lambda_1$ ’ (in the sense defined in section 11.2 of [23]). Hence, by Kac’s proposition 11.2a such  $\mu$ ’s are indeed weights (together with their Weyl images). The only exceptional case is when  $p_j = 0$  for  $j = 2, \dots, 10$ , which corresponds to  $\mu = p_1 \Lambda_1$ . In other words, we have found that all the negative time-like weights belong to  $P(\Lambda_1)$ , but that the multiples of  $\Lambda_1 = -\delta$  are not part of the weight diagram  $P(\Lambda_1)$ .<sup>15</sup>

Though the set  $P(\Lambda_1)$  is already quite large, it only corresponds to the  $GL(10)$  covariantization of the first row in the table above. In view of the structure of the usual affine Virasoro-Sugawara constraints  $L_{n\delta}$  recalled above, together with the known structure of  $E_{10}$  null roots (2.15), it is now quite natural to conjecture that the ‘null skeleton’ of  $\mathcal{C}$  contains, in addition to the orbit of  $-\delta$  (first row in the table) the Weyl orbits of (negative) multiples of  $\delta$ :  $-n\delta$ . This amounts to conjecturing that, besides the weight diagram  $P(\Lambda_1)$  of the fundamental representation  $L(\Lambda_1)$ , we must add the weight diagrams  $P(n\Lambda_1)$  (with  $n = 2, 3, \dots$ ) corresponding to the multiple tensor product of  $L(\Lambda_1)$  with itself:  $L(\Lambda_1) \otimes L(\Lambda_1)$ ,  $L(\Lambda_1) \otimes L(\Lambda_1) \otimes L(\Lambda_1)$ , etc..

Besides this mathematical argument for conjecturing an extension of the set of constraints beyond the ones related to the Weyl orbit of  $-\delta$  (and its covariantization), there is a physical argument suggesting the necessity of this extension. Indeed, all the constraints discussed so far correspond, in view of the ‘dictionary’ of [3], to the values *at one spatial point*, of some space-dependent supergravity constraints. For instance,  $\mathfrak{L}^{(-3)n_1 \dots n_9}$  is the

---

<sup>15</sup>Another way of seeing this is by using proposition 11.3 of [23] where  $P(\Lambda_1)$  is described as the convex hull of the Weyl orbit of  $\Lambda_1$ . The infinitely many Weyl images of  $\Lambda_1$  all lie on the light-cone (and densely approximate any null direction) and one might think that the convex hull covers all points on the light-cone. This is not true since one is constructing the convex hull as an infinite union of closed sets but this is not necessarily closed. In the present case it is open and misses exactly the multiples of  $\Lambda_1$  and their Weyl images but the convex hull covers all points inside the light-cone.

spatial  $\epsilon^{n_1 \dots n_{10}}$  dual of the diffeomorphism constraint  $\mathcal{H}_m(\mathbf{x}_0)$ , taken at the specific spatial point  $\mathbf{x}_0$  around which one analyzes the asymptotic behaviour of the supergravity fields as  $t \rightarrow 0$ . However, the full supergravity diffeomorphism constraint consists of imposing the vanishing of  $\mathcal{H}_m(\mathbf{x})$  at all spatial points. When expanding the diffeomorphism constraint  $\mathcal{H}_m(\mathbf{x})$  in a (ten-dimensional) spatial Taylor expansion around the base point  $\mathbf{x}_0$ , we see that we should replace the unique constraint  $\mathcal{H}_m(\mathbf{x}_0) \sim \mathfrak{L}^{(-3)n_1 \dots n_9}$  by an *infinite gradient tower* of spatial derivatives of the form  $\partial_{m_1 \dots m_k} \mathcal{H}_m(\mathbf{x}_0)$ . For instance, at the first spatial-gradient level  $m = 1$ , we should be considering the two irreducible  $GL(10)$  tensors contained in  $\partial_m \mathcal{H}_n(\mathbf{x}_0)$ , *i.e.*, its symmetric and antisymmetric parts. Dualizing back these first-gradient constraints by means of  $\epsilon^{n_1 \dots n_{10}}$ , we are led to expecting that the ‘first-gradient descendants’ of  $\mathfrak{L}^{(-3)n_1 \dots n_9}$  will comprise two  $GL(10)$  tensors bearing 18 contravariant indices, and belonging to two different Young tableaux: one with [9,9] boxes (corresponding to the symmetric combination) and one with [10,8] boxes (corresponding to the antisymmetric combination). The former corresponds to the null root  $-2\delta = 2\Lambda_1$ , whereas the latter corresponds to the imaginary  $\Lambda_2$  and so lies inside the past light-cone.

The extension of this gradient construction to the other supergravity constraints (Gauss, etc.) then naturally leads us to conjecture the existence of the second row of the table. Then, when considering higher spatial gradients we are led to conjecturing the existence of further rows ‘stemming’ from  $-3\delta$ ,  $-4\delta$ , etc. One finds that the putative constraints associated with the Weyl orbit of  $-n\delta$  start on level  $\ell = -3n$  and are spaced by  $n$ . Finally, it seems that the full table is describing all possible weights on or inside the (past) light-cone.

The notation in the table is condensed and does not display the  $\mathfrak{sl}(10)$  representation structure of the various constraints. For example, the set of constraints labelled by  $\mathfrak{L}_{(-\delta)}^{(-6)}$  and  $\mathfrak{L}_{(-2\delta)}^{(-6)}$  transform in different  $\mathfrak{sl}(10)$  representations. The former one is in the hook representation of (3.8d), whereas the latter has two sets of antisymmetric 9-tuples. Explicitly, one has the following two index structures

$$\mathfrak{L}_{(-\delta)}^{(-6)}{}_{m_1 \dots m_{10} | n_0 | n_1 \dots n_7} \text{ and } \mathfrak{L}_{(-2\delta)}^{(-6)}{}_{m_1 \dots m_9 | n_1 \dots n_9} . \quad (3.23)$$

In the affine truncation to  $E_9$  only one member in each infinite sequence (row) for a given  $-n\delta$  is non-trivial because of the presence of 10-tuples of antisymmetrized indices in the higher components. In the example (3.23)

above, the first tensor vanishes in the affine truncation, whereas the second one is non-zero. In addition, all the surviving constraints from the beginning of each sequence reduce to singlets under  $\mathfrak{sl}(9)$ . These are the  $\mathfrak{L}_{(-n\delta)}^{(-3n)}$ . This (one-sided) sequence of constraints naturally correspond to the generators  $L_{-n\delta}$  (for  $n > 0$ ) of the affine Sugawara construction that we had introduced in (2.7). (We will return below to specific issues concerning the contractions of the null roots and Cartan subalgebra generators).

We do not present an explicit expression for the second rung of constraints, like the second term in (3.23), but note that on the contractions of real root spaces it is given by the same general formula (3.5) as the other constraints we have considered so far. Among the other constraints in  $\mathfrak{L}_{(-n\delta)}^{(-n\ell)}$ , some have an index structure similar to the elementary  $\mathfrak{L}_{(-\delta)}^{(-\ell)}$ , but with all tuples replicated  $n$  times. For example, the index structure of  $\mathfrak{L}_{(-2\delta)}^{(-8)}$  contains a tensor with two 10-tuples and two 2-tuples

$$\mathfrak{L}_{(-2\delta)}^{(-8)} \quad m_1 \dots m_{10} || p_1 \dots p_{10} || n_1 n_2 | q_1 q_2 . \quad (3.24)$$

To complete this discussion, let us point out the following ‘experimental’ relation between the constraints and the level decomposition of the adjoint of  $E_{10}$  under  $A_9$  [27]. ‘Admissible’  $A_9$  representations in the level decomposition rarely appear with outer multiplicity zero. Here, ‘admissible’ refers to solving necessary diophantine conditions on the lowest weight vectors of a possible  $A_9$  representation occurring in the adjoint representation of  $E_{10}$ , see eqns. (6) and (7) in [3]. The only cases up to  $\ell \leq 28$  for which the outer multiplicity of an admissible representation is zero are those when the associated lowest root in the representation is null.<sup>16</sup> More precisely, the only entries with vanishing outer multiplicities in the tables of [27] occur at<sup>17</sup>

---

<sup>16</sup>This is no longer necessarily true when considering Kac–Moody algebras different from  $E_{10}$  [21] or decompositions other than that under  $A_9$ .

<sup>17</sup>We use two different notations for describing elements  $\alpha$  of the (self-dual)  $E_{10}$  root lattice, namely in terms of either the basis of simple roots  $\alpha_i$  or of the basis of fundamental weights  $\Lambda_i$ :  $\alpha = \sum_i m_i \alpha_i = \sum_i p_i \Lambda_i$ . In the former we write the ten-tuple of coefficients with round parentheses  $(m_1, \dots, m_{10})$  and in the latter with square brackets  $[p_1, \dots, p_{10}]$ . The  $p_i$  are often referred to as Dynkin labels. The  $A_9$  weight is obtained from  $[p_1, \dots, p_{10}]$  by dropping the last entry  $p_{10}$  since this corresponds to the node that is deleted in the  $A_9$  level decomposition.

| Level       | $E_{10}$ root                     | $A_9$ weight                  |
|-------------|-----------------------------------|-------------------------------|
| $\ell = 3n$ | $n(0, 1, 2, 3, 4, 5, 6, 4, 2, 3)$ | $[n, 0, 0, 0, 0, 0, 0, 0, 0]$ |
| $\ell = 4n$ | $n(1, 2, 3, 4, 5, 6, 7, 4, 2, 4)$ | $[0, 0, 0, 0, 0, 0, 0, n, 0]$ |
| $\ell = 5n$ | $n(1, 2, 3, 4, 5, 7, 9, 6, 3, 5)$ | $[0, 0, 0, 0, n, 0, 0, 0, 0]$ |

The first line corresponds to the root  $n\delta$  and both the second and the third line can be obtained from the first line by the Weyl transformations given explicitly in eqs. (3.17) and (3.18). These entries have vanishing outer multiplicities since the corresponding  $E_{10}$  generators are already contained in the gradient representations on the relevant level. One potentially important implication of the vanishing outer multiplicities is that there are no ordering ambiguities because the relevant commutators always vanish, as in (3.13), whereas ordering ambiguities will occur in general for higher level constraints like  $\mathfrak{L}_{(-\delta)}^{(-6)}$ .

### 3.1.4 Algebra of constraints

Let us now return to the question of the constraint algebra (2.16) raised at the end of section 2. We discuss this issue by using the explicit expressions for the constraints (3.8). As discussed above, one would like the constraint algebra to close with structure constants given by current components. From the results of [4] it follows that one can generate higher level constraints from lower level constraints by the action of the *negative level* current operator,  $J^{(-1)}$ , *i.e.*, that schematically

$$\mathfrak{L}^{(-(\ell+1))} = \left\{ \begin{matrix} (-1) \\ J \end{matrix}, \begin{matrix} (-\ell) \\ \mathfrak{L} \end{matrix} \right\}, \quad (3.25)$$

is valid for  $\ell = -3, -4, -5$ . This property is equivalent to the result of [4] that the constraints are ‘covariant’ under the upper Borel group  $E_{10}^+$  (*i.e.*, that they form a representation of  $E_{10}^+$ ; even if they do not form a representation of the full group  $E_{10}$ ). In addition, the *level-three truncated* constraints (3.8) have the property that their Sugawara expression contains *only negative level* currents, *i.e.*, schematically

$$\mathfrak{L}^{(-\ell)} = \sum_{p+q=\ell} \begin{matrix} (-p) \\ J \end{matrix} \cdot \begin{matrix} (-q) \\ J \end{matrix}, \quad (3.26)$$

It is now easy to see that the two properties (3.25) and (3.26) imply that the Poisson bracket of two constraints closes in the desired manner of (2.16).

This is certainly an encouraging result, which suggests that the structure of the constraints incorporates special features allowing for the existence of a closed algebra of the type of a generalized Virasoro algebra (2.16).

However, it is not clear whether the two special properties (3.25) and (3.26) continue to hold for the generalized infinite tower of  $E_{10}$  constraints whose construction was sketched above. We shall see that the property (3.26) is likely to be violated when implementing a certain ‘see-saw’ construction defined below. As for the property (3.25) (which says that the constraints form a representation of  $E_{10}^+$ ), one reason for believing that it might not be universally valid comes from the example of the affine Sugawara construction. There the constraints do *not* transform in a representation of the affine algebra:  $\mathfrak{L}^{(-\ell-n)} \neq \{J^{(-n)}, \mathfrak{L}^{(-\ell)}\}$ . Rather one finds that it is the algebra which transforms under the constraints, *i.e.*,  $J^{(-\ell-n)} = \{J^{(-n)}, \mathfrak{L}^{(-\ell)}\}$ . We leave to future work further discussion of this important issue.

### 3.2 Universality: $D = 11$ , IIB and massive IIA

The full  $E_{10}$  Lie algebra can be obtained from the closure (via commutators) of two of its finite-dimensional sub-algebras: (i) its  $A_9$  subalgebra (relevant for  $D = 11$  supergravity), and (ii) its  $A_8 \oplus A_1$  subalgebra (relevant for type IIB supergravity). The  $A_9$  subalgebra corresponds to nodes 1, 2, 3, 4, 5, 6, 7, 8, 9 of the Dynkin diagram in fig. 1; the  $A_8 \oplus A_1$  algebra corresponds to the nodes 1, 2, 3, 4, 5, 6, 7, 10 and 9 in fig. 1. The two subalgebras  $A_9$  and  $A_8 \oplus A_1$  together cover all ten nodes of the  $E_{10}$  diagram, and therefore their closure is all of  $E_{10}$ . For the  $A_8 \oplus A_1$  decomposition, the term ‘level’ refers to node 8. For low levels, the decomposition under this  $A_8 \oplus A_1$  subalgebra, originally performed in [21, 13], is reproduced in appendix A.2.

The two decompositions under  $A_9$  and  $A_8 \oplus A_1$  provide two different bases for the same Lie algebra  $E_{10}$ . In order to distinguish them we use the letter  $J$  for the current components in the  $A_9$  decomposition, as already done for example in (3.8), and the letter  $I$  for current components in the  $A_8 \oplus A_1$  decomposition. Since the real root spaces are one dimensional it is usually straightforward to explicitly work out the ‘change of basis’ between the current components expressed in the  $J$  basis or the  $I$  basis. For example, the root space of the real root  $\alpha = -\alpha_{10}$  contains the current component  $J^{8910}$  in the  $A_9$  decomposition. In the  $A_8 \oplus A_1$  decomposition this root space is part of the  $A_8$  ‘gravity line’ and therefore one obtains the following relation

between the vectors of the two bases in the  $\alpha = -\alpha_{10}$  root space

$$J^{(-1)8910} = I^{(0)8}_9 \quad \text{corresponding to} \quad E_{-\alpha_{10}} \equiv f_{10}. \quad (3.27)$$

That the two generators are not on the same level with regard to the two decompositions of the  $E_{10}$  algebra will be of crucial importance for the construction we shall discuss next. In appendix A, we also recall the association of the level decompositions with low-lying generators in an explicit tensor basis for the two decompositions.

The fact that  $A_9$  and  $A_8 \oplus A_1$  together generate the whole  $E_{10}$  algebra allows in principle to extend the lowest level supergravity constraints to arbitrarily high levels by the following mechanism (which for obvious reasons we will refer to as a ‘*see-saw mechanism*’). Among the root components contributing to a given known constraint in one level decomposition, there are some that correspond to ‘unknown’ levels in a different decomposition. Covariantizing the resulting expression with regard to the  $\mathfrak{gl}(n, \mathbb{R})$  subalgebra relevant for that new decomposition we generate new components, which in turn can be analyzed in terms of the first decomposition. Covariantizing again, but now with respect to the first decomposition, we again generate new components. It is easy to see that this procedure never stops, and so continues *ad infinitum*.

To see how this construction works in a concrete example consider the following terms in the  $D = 11$  diffeomorphism constraint (3.8a), see also (3.20),

$$\begin{aligned} \mathfrak{L}^{(-3)2345678910} \ni & \frac{1}{3} \left( J^{(-1)234} J^{(-2)5678910} + J^{(-3)9|234567910} J^{(0)8}_9 \right. \\ & \left. + J^{(-3)9|23456789} J^{(0)10}_9 - J^{(-3)8|234568910} J^{(0)7}_8 \right). \end{aligned} \quad (3.28)$$

All the terms in the bracket correspond to canonically normalized real root components of the current. In analogy with (3.27) one can now convert these terms into the alternative basis provided by the  $A_8 \oplus A_1$  decomposition. In

this way we obtain (see appendix A for the notation)

$$\begin{aligned}
J_{234}^{(-1)} &= I_{2349}^{(-2)}, & J_{5678910}^{(-2)} &= I_{5678}^{(-2)}, \\
J_{9|234567910}^{(-3)} &= I_{234567,i}^{(-3)}, & J_8^{(0)} &= I_{89,\dot{2}}^{(-1)}, \\
J_{9|23456789}^{(-3)} &= I_{23456789,ii}^{(-4)}, & J_{10}^{(0)} &= I_{\dot{1}}^{(0)}, \\
J_{8|234568910}^{(-3)} &= I_{8|2345689}^{(-4)}, & J_7^{(0)} &= I_7^{(0)}.
\end{aligned} \tag{3.29}$$

where dotted indices refer to the  $\mathfrak{sl}(2, \mathbb{R})$  algebra associated with node 9. Putting this back into (3.28) one can see that this is part of a  $GL(9, \mathbb{R}) \times SL(2, \mathbb{R})$  covariant expression of the form<sup>18</sup>

$$\begin{aligned}
\mathfrak{C}^{(-4)}_{n_1 \dots n_8} &= \frac{35}{3} I_{[n_1 \dots n_4]}^{(-2)} I_{n_5 \dots n_8]}^{(-2)} - \frac{28}{3} I_{[n_1 n_2, \alpha]}^{(-1)} I_{n_3 \dots n_8], \beta}^{(-3)} \epsilon_{\alpha \beta} \\
&\quad - \frac{1}{3} I_{n_1 \dots n_8, \alpha \gamma}^{(-4)} I_{\beta \gamma}^{(0)} \epsilon_{\alpha \beta} - \frac{8}{3} I_{p|[n_1 \dots n_7}^{(-4)} I_{n_8]}^{(0)}_p + \dots \tag{3.30}
\end{aligned}$$

where for clarity of notation we use the symbol  $\mathfrak{C}$  to denote the IIB constraints. Remarkably, this expression is *exactly the diffeomorphism constraint of IIB supergravity* when the correspondence with  $E_{10}$  of [13] is used. This is explained in more detail in appendix B.<sup>19</sup> Indeed, using the expressions (A.5) and (A.6) for the Cartan generators expressed in IIB variables, one finds that the component 23456789 of the IIB diffeomorphism constraint is associated with the root space of  $-\delta$ , just as is the component 2345678910 of the  $D = 11$  diffeomorphism constraint, see appendix B. This suggests that, possibly the two expressions agree completely. Inspecting all the different root components and  $E_{10}$  generators one verifies

$$\mathfrak{L}^{(-3)}_{2345678910} \Big|_{\text{real roots}} = \mathfrak{C}^{(-4)}_{23456789} \Big|_{\text{real roots}} = \frac{1}{3} \mathcal{L}_{-\delta} \Big|_{\text{real roots}}, \tag{3.31}$$

*i.e.*, the expressions agree on the bilinear expressions involving two real root generators — as was, in fact, guaranteed by our use of the Weyl group in

<sup>18</sup>The 8-index tensor on the l.h.s. is fully antisymmetric. We use the convention that  $\epsilon^{i\dot{2}} = +1 = -\epsilon_{i\dot{2}}$ .

<sup>19</sup>We take this opportunity to point out a typo in the Einstein equation (67) in [13]: The terms involving the (self-dual) five-form field strength should be multiplied by 1/2. This does not affect the dictionary derived in that paper.

the covariantization procedure. We find it remarkable that there is such an agreement between the constraints of two different physical theories expressed in the simple algebraic fashion (3.5).

However, considering the bilinear terms contributing to the two expressions, one finds that there are terms that differ, an explicit example can be found in appendix C. One way to interpret this difference is the following: The full set of constraints can be divided in two parts: (i) a universal part, based on the ‘skeleton’ of null roots, and comprising the ‘scaffold’ of special configurations

$$\mathfrak{L}_\alpha = \sum_{\substack{\beta_1 + \beta_2 = \alpha \\ \beta_1, \beta_2 \text{ real}}} J_{\beta_1} J_{\beta_2} \quad \text{for } \alpha \text{ null} \quad (3.32)$$

and, (ii) a non-universal part (the ‘flesh’) that depends on the choice of subgroup under which one covariantizes the ‘scaffold’ part (3.32).

The universal part of the construction (3.32) has the property of being preserved by the action of the discrete Weyl group  $W(E_{10})$  and its subgroups  $W(A_9)$  and  $W(A_8 \oplus A_1)$ . By contrast, the covariantization of the skeleton under the corresponding continuous groups  $GL(10, \mathbb{R})$  (for  $D = 11$ ) and  $GL(9, \mathbb{R}) \times SL(2, \mathbb{R})$  (for type IIB) leads to different results on the additional new terms inside the light-cone that are generated by the covariantization. That different new terms are possible is due to the fact that in those terms one has to specify the coefficients  $M_{s_1, s_2}(\beta_1, \beta_2)$  for the contraction of root spaces of different dimensions.<sup>20</sup> These are fixed by covariance under a chosen level decomposition subgroup.<sup>21</sup>

We believe there is some evidence that hyperbolic algebras may admit a realisation akin to the realisation of affine algebras in terms of a spectral parameter<sup>22</sup>, but our results here strongly suggest that, if there is such a realisation, it will not be unique. Thinking of Sugawara constructions as being associated with spectral parameters, this can be interpreted by saying that the  $E_{10}$  algebra may not possess a single or unique set of spectral parameters. Rather, one can and has to choose a set of spectral parameters by covariantizing under a subalgebra of one’s choice. If the spectral parameters are related

---

<sup>20</sup>A similar difference was already noted for the  $E_9$  contraction in [4].

<sup>21</sup>See, however, the suggestion above, Eq. (2.14) that one might use the Lie algebra generator associated to the considered constraint-root  $\alpha$  to define a universal way of pairing the two different root spaces  $\mathfrak{g}_{\beta_1}, \mathfrak{g}_{\beta_2}$ .

<sup>22</sup>Some evidence from the structure of the compact subgroup  $K(E_{10})$  was given in [31].

to space variables (as is the case for the affine algebras appearing in  $D = 2$  supergravities), then this would be in good agreement with the anticipation that one can make space-times of different dimensions emergent from  $E_{10}$ , depending on the choice of level decomposition [32]. From this point of view the hyperbolic Sugawara construction considered here is less unique than in the affine case since it depends on the choice of level decomposition. At the same time it nicely incorporates the expected possibility of having different spaces emerging from an U-duality (Weyl group) invariant scaffold.

On the other hand, restricting only to real root generators, we can now use the agreement between the two expressions to construct new terms involving higher level generators, showing the full power of the approach. The crucial point is that the IIB diffeomorphism constraint (3.30) also contains other components that are not contained in the previous expressions (3.8) corresponding to the  $A_9$  level decomposition with the level truncation appropriate to  $D = 11$  supergravity, for example the real root combination

$$\mathfrak{e}^{(-4)23456789} \ni \frac{1}{3} I^{(-4)8|2345678} I^{(0)9}_8. \quad (3.33)$$

Translating again between the two different bases of  $E_{10}$  using

$$I^{(0)9}_8 = J^{(+1)8910}, \quad I^{(-4)8|2345678} = J^{(-4)8910|2345678910}. \quad (3.34)$$

we infer that this is part of an extended  $\mathfrak{sl}(10)$  covariant expression, namely

$$\mathfrak{L}^{(-3)m_1\dots m_9} \rightarrow (3.8a) + \frac{1}{3 \cdot 3!} J^{(+1)p_1p_2p_3} J^{(-4)p_1p_2p_3|m_1\dots m_9}. \quad (3.35)$$

The normalization is fixed by the term in the IIB expansion. This is also the only possible contraction between  $A_9$  level  $+1$  and  $-4$  contributing to the diffeomorphism constraint in  $D = 11$ . (The mass deformation generator on  $\ell = 4$  does not contribute to the diffeomorphism constraint [14].) We note that the generator appearing in this new piece of the  $D = 11$  diffeomorphism constraint is a gradient generator in the language of [3]. The new term in the  $D = 11$  constraint now has components on IIB level  $\ell = -5$  and  $\ell = -6$  that can be covariantized now under  $\mathfrak{sl}(9) \oplus \mathfrak{sl}(2)$  generating new terms. We have carried out this procedure one step farther and found the following

expressions for the ‘diffeomorphism constraints’ in  $A_9$  decomposition

$$\begin{aligned}
\mathfrak{L}^{(-3)}_{m_1\dots m_9} &= 28 \frac{(-1)}{J}{}_{m_1 m_2 m_3} \frac{(-2)}{J}{}_{m_4\dots m_6} + 3 \frac{(-3)}{J}{}_{p|m_1\dots m_8} \frac{(0)}{J}{}_{m_9}{}_p \quad (3.36) \\
&+ \frac{1}{3 \cdot 3!} \frac{(-4)}{J}{}_{p_1 p_2 p_3 | m_1\dots m_9} \frac{(+1)}{J}{}_{p_1 p_2 p_3} + \frac{1}{3 \cdot 6!} \frac{(-5)}{J}{}_{p_1\dots p_6 | m_1\dots m_9} \frac{(+2)}{J}{}_{p_1\dots p_6} \\
&+ \dots
\end{aligned}$$

with implicit antisymmetrization over  $[m_1 \dots m_9]$ , and a corresponding expression in  $A_8 \oplus A_1$  decomposition

$$\begin{aligned}
\mathfrak{C}^{(-4)}_{m_1\dots m_8} &= \frac{35}{3} \frac{(-2)}{I}{}_{m_1\dots m_4} \frac{(-2)}{I}{}_{m_5\dots m_8} - \frac{28}{3} \frac{(-1)}{I}{}_{m_1 m_2, \alpha} \frac{(-3)}{I}{}_{m_3\dots m_8, \beta} \epsilon_{\alpha\beta} \\
&- \frac{1}{3} \frac{(-4)}{I}{}_{m_1\dots m_8, \alpha\gamma} \frac{(0)}{I}{}_{\beta}{}^{\gamma} \epsilon_{\alpha\beta} - \frac{8}{3} \frac{(-4)}{I}{}_{p|m_1\dots m_7} \frac{(0)}{I}{}_{m_8}{}_p \quad (3.37) \\
&+ \frac{1}{3 \cdot 2!} \frac{(-5)}{I}{}_{p_1 p_2 | m_1\dots m_8, \alpha} \frac{(+1)}{I}{}_{p_1 p_2, \alpha} + \frac{1}{3 \cdot 4!} \frac{(-6)}{I}{}_{p_1\dots p_4 | m_1\dots m_8} \frac{(+2)}{I}{}_{p_1\dots p_4} \\
&+ \dots
\end{aligned}$$

with implicit antisymmetrization over  $[m_1 \dots m_8]$ . Note that the index range of the world indices is different in the two decompositions: In the  $D = 11$  case, corresponding to  $A_9$ , the index range is  $m = 1, \dots, 10$  and in the type IIB case, corresponding to  $A_8 \oplus A_1$ , the index range is  $m = 1, \dots, 9$ . By construction, these two expressions have the property that they agree on the real roots. In this way one produces an expression for a Sugawara constraint  $\mathcal{L}_{-\delta}$  which extends to arbitrarily positive and negative step operators. We also note the appearance of *gradient generators* precisely in accord with (2.2), as these generators are the ones that reduce to the higher level affine generators in the truncation of  $E_{10}$  to  $E_9$  [31]. The gradient generators are those generators related to real roots of the affine  $E_9$  [3, 27]. It is straightforward to see that the infinite prolongation of our procedure will give rise to all the terms needed to match with the full sum in (2.2), for negative values of  $n$ .

Our see-saw mechanism not only demands the extension of the constraints  $\mathfrak{L}_\alpha$  (for a given  $\alpha$ ) to infinite strings of bilinears of Noether charges in agreement with the affine Sugawara construction, but *also allows to switch between constraints that are distinct as supergravity constraints*. For instance, certain components of the IIB diffeomorphism constraint metamorphose into components of the  $D = 11$  Gauss constraint when viewed in a different level decomposition! To see this more explicitly, consider the following component

of the IIB diffeomorphism constraint (3.37)

$$\mathfrak{E}^{(-4)}_{12345678} \ni \frac{1}{3} \left( I^{(-2)}_{1234} I^{(-2)}_{5678} + \dots \right) \quad (3.38)$$

where we only picked out one real root combination for simplicity. Translating this to the  $A_9$  basis via

$$I^{(-2)}_{1234} = J^{(-2)}_{1234910}, \quad I^{(-2)}_{5678} = J^{(-2)}_{5678910}, \quad (3.39)$$

we find that it is part of a covariant expression

$$\mathfrak{L}^{(-4)}_{m_1 \dots m_{10} | n_1 n_2} = 42 \left( J^{(-2)}_{n_1 [m_1 \dots m_5} J^{(-2)}_{m_6 \dots m_{10}] n_2} - (n_1 \leftrightarrow n_2) \right) \quad (3.40)$$

where the overall normalization differs by a factor of 20 from (3.8b), see also (3.20) in comparison to (3.21). This is exactly the combination that appears in the Gauss constraint of  $D = 11$  supergravity, a result not too surprising from the point of view of U-duality. Evidently, this process could now be continued *ad libitum*.

One can similarly generate new terms for the Gauss constraint in  $D = 11$ , given up to  $\ell = 3$  in (3.8b). Starting from the following components of the IIB diffeomorphism constraint

$$\mathfrak{E}^{(-4)}_{12345678} \ni \frac{1}{3} \left( I^{(-4)}_{12345678,ii} I^{(0)}_{i^2} + I^{(-4)}_{8|2345678} I^{(0)}_{1^8} \right) + \dots \quad (3.41)$$

They can be mapped to  $A_9$  quantities using the two distinct  $A_9$  level  $\ell = 4$  representations

$$I^{(-4)}_{12345678,ii} = J^{(-4)}_{9|9|12345678910}, \quad I^{(-4)}_{8|2345678} = J^{(-4)}_{8910|2345678910} \quad (3.42)$$

to give  $\mathfrak{sl}(10)$  covariant additions to (3.8b) via

$$\begin{aligned} \mathfrak{L}^{(-4)}_{m_1 \dots m_{10} | n_1 n_2} &\rightarrow (3.8b) + \frac{2}{3} J^{(-4)}_{m_1 \dots m_{10} | p [n_1} J^{(0)}_{n_2] p} \\ &+ \frac{10}{3} J^{(-4)}_{m_1 \dots m_9 | n_1 n_2 p} J^{(0)}_{m_{10} p} + \dots \quad (3.43) \end{aligned}$$

We note that here both the gradient *and* non-gradient generator on  $A_9$  level  $\ell = 4$  contribute. Since the mass deformation parameter of massive type IIA is contained in the non-gradient generator, this is in agreement with the fact that the Gauss constraint of massive IIA gets modified by the Romans mass [33, 14].

### 3.3 General remarks on the construction

Let us summarize the construction and comment on some open questions concerning this procedure. Starting from a single constraint  $\mathcal{L}_{-\delta}$ , associated to the primitive null root, we construct the ‘scaffold’ as in (3.32), based on the decomposition  $\alpha = \beta_1 + \beta_2$  for  $\alpha$  null and  $\beta_1, \beta_2$  real.<sup>23</sup> Since all root spaces involved are real, they are one-dimensional and there is no ambiguity in the contraction. There are also no ordering ambiguities at this level. We can then act on the expression (3.32) with  $W(E_{10})$  to generate similar expressions for all null roots. This constitutes the full scaffold of the hyperbolic Sugawara constraints which is invariant (only) under  $W(E_{10})$ . The constraints of the type in (3.32) are both infinite in number and each consists of an infinite number of bilinears in the current components.

In order to construct constraints for the full  $E_{10}$  one then needs to choose a level decomposition under a regular, finite-dimensional subalgebra. Covariance under this subalgebra induces additional terms on top of those already contained in the skeleton. The precise form of these additional terms depends on the subalgebra one chose in a systematic way, as is apparent from the explicit expressions in appendix C. From that point of view it is clear that our construction is not covariant with respect to the full  $E_{10}$  Lie algebra, but involves only the Weyl group  $\mathcal{W}(E_{10})$  in a canonical way. Everything beyond that depends on the chosen subalgebra for the level decomposition.<sup>24</sup>

We can also bring out the lack of ‘ $E_{10}$  covariance’ by relating our construction to the question of an  $E_{10}$  representation structure in the bilinear expression in  $E_{10}$  generators. As already pointed out in footnote 13 one might have liked to identify the constraint  $\mathcal{L}_{-\delta}$  with a highest weight vector of an integrable  $E_{10}$  representation with highest weight  $\Lambda_1 = -\delta$ . If this were the case the constraint should be annihilated by all raising operators. Here, we recall that we express the step operators in terms of current components (rather than in terms of the ‘contragredient’  $E_{10}$  Lie algebra generators  $T_\alpha^{(s)}$ ), so that for example  $e_1 = J^2_1$ . Using the explicit expression for  $\mathcal{L}_{-\delta}$  in  $A_9$  decomposition we find that

$$\left[ e_i, \mathfrak{L}^{(-3)2345678910} \right] = \left[ J^{(0)_{i+1}}_i, \mathfrak{L}^{(-3)2345678910} \right] = 0 \quad \text{for } i = 1, \dots, 9 \quad (3.44)$$

---

<sup>23</sup>We note that decompositions of imaginary roots into real roots have been considered in a different context in [34].

<sup>24</sup>In some sense this is also true for the affine  $E_9$  Sugawara construction which uses as choice of subalgebra for the level decomposition  $E_8$ .

where all commutators should be read as Poisson (or Dirac) brackets in the canonical setting. However, for the  $e$  generator corresponding to the omitted node we get

$$\left[ e_{10}, \mathfrak{L}^{(-3)2345678910} \right] = \left[ J_{8910}, \mathfrak{L}^{(-3)2345678910} \right] \neq 0, \quad (3.45)$$

showing that this component of the constraint generator *is only a highest weight state with respect to the  $A_9$  subalgebra, but not the full  $E_{10}$  algebra*. Since the  $A_9$  expression agrees with the  $A_8 \oplus A_1$  expression on the real roots, we can repeat the calculation in IIB variables to find

$$\left[ e_i, \mathfrak{L}^{(-4)23456789} \right] = 0 \quad \text{for } i = 1, \dots, 7, 9, 10 \quad (3.46)$$

and

$$\left[ e_8, \mathfrak{L}^{(-4)23456789} \right] \neq 0, \quad (3.47)$$

Being a highest weight vector now with respect to  $A_8 \oplus A_1$ , this is different from the result for the  $A_9$  decomposition, but again illustrates the lack of full  $E_{10}$  covariance.

Similar conclusions hold for the  $D_9 \equiv SO(9,9)$  decomposition of [11]. Without going into the details of the calculation, the lowest order constraint for the  $D_9$  decomposition is

$$\mathfrak{L}^{(-2)I} = \frac{1}{2} J^{KL} J^{(-2)IKL} + \frac{1}{2} J^{(-1)A} (\mathcal{C}\Gamma^I)_{AB} J^{(-1)B} + \dots \quad (3.48)$$

where  $I, K, L = 1, \dots, 18$  and  $A, B = 1, \dots, 256$  are  $SO(9,9)$  vector and spinor indices, respectively, and we use again the symbol  $J$  to denote the components of the conserved  $E_{10}$  current, but now in the  $D_9$  decomposition. The  $18=9+9$  constraints in (3.48) correspond to the diffeomorphism constraint and the Gauss constraint for the Neveu-Schwarz 2-form field of IIA theory; alternatively, they might be interpreted as a doubled set of diffeomorphism constraints w.r.t. the nine spatial target space coordinates  $X^i$  and their (world-sheet) ‘duals’  $\tilde{X}^i$  [11]. As before it is the omitted node (*i.e.*, node 9 for the massive IIA theory) which causes failure of the construction: by  $SO(9,9)$  covariance, the dilaton field associated with this node cannot

appear in (3.48). Accordingly, it is now the generator  $e_9$  which does not annihilate the relevant component of  $\mathfrak{L}^{(-2)}$ .

To summarize: The failure of the constraint to be a highest weight vector w.r.t. the full  $E_{10}$  algebra is invariably associated with the node that has been deleted for the given level decomposition. In Appendix C we show that a related statement applies to the dependence of the constraints on the Cartan subalgebra generators.

One further interesting aspect of our construction is that, to start with, it associates a constraint with every Weyl image of the fundamental null root  $-\delta$ . In the same way one can associate constraints to the Weyl images of  $-n\delta$  and in this way obtains a constraint for every  $E_{10}$  root on the (past) light-cone. After choosing a level decomposition subalgebra one generates additional constraints *inside* the light-cone by covariantization under this subalgebra.

It is possible that, as indicated in subsection 3.1.3, the set of roots  $\mathcal{C}$  ‘supporting’ the full set of constraints be *universally* given by all the weights inside the (past) light-cone. This set can also be described as the union of the weight diagrams of the representations  $L(\Lambda_1)$ ,  $L(\Lambda_1) \otimes L(\Lambda_1)$ ,  $L(\Lambda_1) \otimes L(\Lambda_1) \otimes L(\Lambda_1)$ , etc.. On the other hand, the precise Sugawara-like expression defining the constraint  $\mathfrak{L}_\alpha$  associated to some  $\alpha \in \mathcal{C}$  seems to depend on the choice of a level decomposition.

Finally, note that since we are defining an infinity of constraints associated with all null roots of the hyperbolic algebra  $E_{10}$ , one might worry whether there are any solutions that satisfy the geodesic equation and all the Sugawara constraints. It is reassuring to note that there are such solutions, namely for example the Kasner cosmologies. These correspond to only non-vanishing Cartan subalgebra components of the current and hence all constraints except the Hamiltonian constraint (3.6) are trivially satisfied. Other solutions correspond to specific cases of Bianchi cosmologies. The exact count of the remaining number of degrees of freedom is quite involved and beyond the scope of this paper.

**Acknowledgements.** We would like to thank Ofer Gabber and Victor Kac for informative discussions. AK is a Research Associate of the Fonds de la Recherche–FNRS, Belgium, and would like to thank IHES and AEI for hospitality. This work has been supported in part by IISN-Belgium (conventions 4.4511.06, 4.4505.86 and 4.4514.08) and by the Belgian Federal Science Policy Office through the Interuniversity Attraction Pole P6/11.

## A Level decompositions

For the reader's convenience, we collect in this appendix some results on the level decompositions of  $E_{10}$  appropriate for  $D = 11$  supergravity and for type IIB in  $D = 10$ . These appeared originally in [3, 27] and [21, 13], respectively.

### A.1 Level decomposition under $A_9$

The  $A_9 \cong \mathfrak{sl}(10)$  subalgebra relevant for  $D = 11$  supergravity is obtained by removing node 10 from the Dynkin diagram of fig. 1.

| $\ell$ | $A_9$ Dynkin labels | $E_{10}$ root for lowest weight | $\mu$ | $\alpha^2$ |
|--------|---------------------|---------------------------------|-------|------------|
| 0      | [1,0,0,0,0,0,0,0,1] | (-1,-1,-1,-1,-1,-1,-1,-1,0)     | 1     | 2          |
| 0      | [0,0,0,0,0,0,0,0,0] | (0,0,0,0,0,0,0,0,0)             | 1     | 0          |
| 1      | [0,0,0,0,0,0,1,0,0] | (0,0,0,0,0,0,0,0,1)             | 1     | 2          |
| 2      | [0,0,0,1,0,0,0,0,0] | (0,0,0,0,1,2,3,2,1,2)           | 1     | 2          |
| 3      | [0,1,0,0,0,0,0,0,1] | (0,0,1,2,3,4,5,3,1,3)           | 1     | 2          |
| 3      | [1,0,0,0,0,0,0,0,0] | (0,1,2,3,4,5,6,4,2,3)           | 0     | 0          |
| 4      | [0,0,0,0,0,0,0,0,2] | (1,2,3,4,5,6,7,4,1,4)           | 1     | 2          |
| 4      | [0,0,0,0,0,0,0,1,0] | (1,2,3,4,5,6,7,4,2,4)           | 0     | 0          |
| 4      | [1,0,0,0,0,0,1,0,0] | (0,1,2,3,4,5,6,4,2,4)           | 1     | 2          |
| 5      | [0,0,0,0,0,1,0,0,1] | (1,2,3,4,5,6,8,5,2,5)           | 1     | 2          |
| 5      | [0,0,0,0,1,0,0,0,0] | (1,2,3,4,5,7,9,6,3,5)           | 0     | 0          |
| 5      | [1,0,0,1,0,0,0,0,0] | (0,1,2,3,5,7,9,6,3,5)           | 1     | 2          |
| 6      | [0,0,0,1,0,0,0,1,0] | (1,2,3,4,6,8,10,6,3,6)          | 1     | 2          |
| 6      | [0,0,1,0,0,0,0,0,1] | (1,2,3,5,7,9,11,7,3,6)          | 1     | 0          |
| 6      | [0,1,0,0,0,0,0,0,0] | (1,2,4,6,8,10,12,8,4,6)         | 1     | -2         |
| 6      | [1,1,0,0,0,0,0,0,1] | (0,1,3,5,7,9,11,7,3,6)          | 1     | 2          |
| 6      | [2,0,0,0,0,0,0,0,0] | (0,2,4,6,8,10,12,8,4,6)         | 0     | 0          |
| 7      | [0,0,0,0,0,0,0,0,1] | (2,4,6,8,10,12,14,9,4,7)        | 1     | -4         |
| 7      | [0,0,1,0,0,1,0,0,0] | (1,2,3,5,7,9,12,8,4,7)          | 1     | 2          |
| 7      | [0,1,0,0,0,0,0,1,1] | (1,2,4,6,8,10,12,7,3,7)         | 1     | 2          |
| 7      | [0,1,0,0,0,0,1,0,0] | (1,2,4,6,8,10,12,8,4,7)         | 1     | 0          |
| 7      | [1,0,0,0,0,0,0,0,2] | (1,3,5,7,9,11,13,8,3,7)         | 1     | 0          |
| 7      | [1,0,0,0,0,0,0,1,0] | (1,3,5,7,9,11,13,8,4,7)         | 2     | -2         |
| 7      | [2,0,0,0,0,0,1,0,0] | (0,2,4,6,8,10,12,8,4,7)         | 1     | 2          |
| 8      | [0,0,0,0,0,0,0,1,2] | (2,4,6,8,10,12,14,8,3,8)        | 1     | 2          |
| 8      | [0,0,0,0,0,0,0,2,0] | (2,4,6,8,10,12,14,8,4,8)        | 0     | 0          |

| $\ell$ | $A_9$ Dynkin labels | $E_{10}$ root for lowest weight | $\mu$ | $\alpha^2$ |
|--------|---------------------|---------------------------------|-------|------------|
| 8      | [0,0,0,0,0,0,1,0,1] | (2,4,6,8,10,12,14,9,4,8)        | 2     | -2         |
| 8      | [0,0,0,0,0,1,0,0,0] | (2,4,6,8,10,12,15,10,5,8)       | 2     | -4         |
| 8      | [0,0,2,0,0,0,0,0,0] | (1,2,3,6,9,12,15,10,5,8)        | 1     | 2          |
| 8      | [0,1,0,0,1,0,0,0,1] | (1,2,4,6,8,11,14,9,4,8)         | 1     | 2          |
| 8      | [0,1,0,1,0,0,0,0,0] | (1,2,4,6,9,12,15,10,5,8)        | 1     | 0          |
| 8      | [1,0,0,0,0,0,1,1,0] | (1,3,5,7,9,11,13,8,4,8)         | 1     | 2          |
| 8      | [1,0,0,0,0,1,0,0,1] | (1,3,5,7,9,11,14,9,4,8)         | 2     | 0          |
| 8      | [1,0,0,0,1,0,0,0,0] | (1,3,5,7,9,12,15,10,5,8)        | 2     | -2         |
| 8      | [2,0,0,1,0,0,0,0,0] | (0,2,4,6,9,12,15,10,5,8)        | 1     | 2          |

The low-lying generators are denoted by  $(a, b, \dots = 1, \dots, 10)$

$$\begin{aligned}
\ell = 0 & : K^a_b \\
\ell = 1 & : E^{abc} = E^{[abc]} \\
\ell = 2 & : E^{a_1 \dots a_6} = E^{[a_1 \dots a_6]} \\
\ell = 3 & : E^{a_0 | a_1 \dots a_8} = E^{a_0 | [a_1 \dots a_8]} \\
\ell = 4 & : E^{a_1 a_2 a_3 | b_1 \dots b_9} = E^{[a_1 a_2 a_3] | [b_1 \dots b_9]} \quad \text{and} \quad E^{a|b|c_1 \dots c_{10}} = E^{(a|b)|[c_1 \dots c_{10}]}
\end{aligned} \tag{A.1}$$

(with the usual irreducibility conditions  $E^{[a_0 | a_1 \dots a_8]} = 0$ , etc.). They are related to the Chevalley–Serre generators by

$$e_i = K^i_{i+1}, \quad f_i = K^{i+1}_i, \quad h_i = K^i_i - K^{i+1}_{i+1} \quad (i = 1, \dots, 9) \tag{A.2}$$

and

$$\begin{aligned}
e_{10} & = E^{8910}, \quad f_{10} = F_{8910}, \\
h_{10} & = -\frac{1}{3}K + K^8_8 + K^9_9 + K^{10}_{10},
\end{aligned} \tag{A.3}$$

where  $K = \sum_{a=1}^{10} K^a_a$ . Commutation relations for these generators can be found in [12, 14] but note that we have rescaled all generators such that their lowest weight elements (e.g.  $E^{10|345678910}$ ) have norm 1.

## A.2 Level decomposition under $A_8 \oplus A_1$

The  $A_8 \oplus A_1 \cong \mathfrak{sl}(9) \oplus \mathfrak{sl}(2)$  subalgebra relevant for type IIB supergravity is obtained by removing node 8 from the Dynkin diagram of fig. 1.

| $\ell$ | $A_8 \oplus A_1$ Dynkin labels | $E_{10}$ root for lowest weight | $\mu$ | $\alpha^2$ |
|--------|--------------------------------|---------------------------------|-------|------------|
| 0      | [1,0,0,0,0,0,0,1][0]           | (-1,-1,-1,-1,-1,-1,-1,0,0,-1)   | 1     | 2          |
| 0      | [0,0,0,0,0,0,0,0][0]           | (0,0,0,0,0,0,0,0,0,0)           | 1     | 0          |
| 0      | [0,0,0,0,0,0,0,0][2]           | (0,0,0,0,0,0,0,0,-1,0)          | 1     | 2          |
| 1      | [0,0,0,0,0,0,1,0][1]           | (0,0,0,0,0,0,0,1,0,0)           | 1     | 2          |
| 2      | [0,0,0,0,1,0,0,0][0]           | (0,0,0,0,0,1,2,2,1,1)           | 1     | 2          |
| 3      | [0,0,1,0,0,0,0,0][1]           | (0,0,0,1,2,3,4,3,1,2)           | 1     | 2          |
| 4      | [0,1,0,0,0,0,0,1][0]           | (0,0,1,2,3,4,5,4,2,2)           | 1     | 2          |
| 4      | [1,0,0,0,0,0,0,0][0]           | (0,1,2,3,4,5,6,4,2,3)           | 0     | 0          |
| 4      | [1,0,0,0,0,0,0,0][2]           | (0,1,2,3,4,5,6,4,1,3)           | 1     | 2          |
| 5      | [0,0,0,0,0,0,0,1][1]           | (1,2,3,4,5,6,7,5,2,3)           | 1     | 0          |
| 5      | [1,0,0,0,0,0,1,0][1]           | (0,1,2,3,4,5,6,5,2,3)           | 1     | 2          |
| 6      | [0,0,0,0,0,0,1,1][0]           | (1,2,3,4,5,6,7,6,3,3)           | 1     | 2          |
| 6      | [0,0,0,0,0,1,0,0][0]           | (1,2,3,4,5,6,8,6,3,4)           | 0     | 0          |
| 6      | [0,0,0,0,0,1,0,0][2]           | (1,2,3,4,5,6,8,6,2,4)           | 1     | 2          |
| 6      | [1,0,0,0,1,0,0,0][0]           | (0,1,2,3,4,6,8,6,3,4)           | 1     | 2          |
| 7      | [0,0,0,0,1,0,0,1][1]           | (1,2,3,4,5,7,9,7,3,4)           | 1     | 2          |
| 7      | [0,0,0,1,0,0,0,0][1]           | (1,2,3,4,6,8,10,7,3,5)          | 1     | 0          |
| 7      | [1,0,1,0,0,0,0,0][1]           | (0,1,2,4,6,8,10,7,3,5)          | 1     | 2          |
| 8      | [0,0,0,1,0,0,1,0][0]           | (1,2,3,4,6,8,10,8,4,5)          | 1     | 2          |
| 8      | [0,0,1,0,0,0,0,1][0]           | (1,2,3,5,7,9,11,8,4,5)          | 1     | 0          |
| 8      | [0,0,1,0,0,0,0,1][2]           | (1,2,3,5,7,9,11,8,3,5)          | 1     | 2          |
| 8      | [0,1,0,0,0,0,0,0][0]           | (1,2,4,6,8,10,12,8,4,6)         | 2     | -2         |
| 8      | [0,1,0,0,0,0,0,0][2]           | (1,2,4,6,8,10,12,8,3,6)         | 1     | 0          |
| 8      | [1,1,0,0,0,0,0,1][0]           | (0,1,3,5,7,9,11,8,4,5)          | 1     | 2          |
| 8      | [2,0,0,0,0,0,0,0][0]           | (0,2,4,6,8,10,12,8,4,6)         | 0     | 0          |
| 8      | [2,0,0,0,0,0,0,0][2]           | (0,2,4,6,8,10,12,8,3,6)         | 1     | 2          |

The low-lying generators are (now  $a, b, \dots = 1, \dots, 9$  are  $\mathfrak{sl}(9)$  vector indices and  $\alpha, \beta = \dot{1}, \dot{2}$  are  $\mathfrak{sl}(2)$  vector indices)

$$\begin{aligned}
\ell = 0 & : K^a_b \quad \text{and} \quad K^\alpha_\beta \quad (\text{with } \delta^\beta_\alpha K^\alpha_\beta = 0) \\
\ell = 1 & : E^{ab,\alpha} = E^{[ab],\alpha} \\
\ell = 2 & : E^{a_1 a_2 a_3 a_4} = E^{[a_1 a_2 a_3 a_4]} \\
\ell = 3 & : E^{a_1 \dots a_6, \alpha} = E^{[a_1 \dots a_6], \alpha} \\
\ell = 4 & : E^{a_0 | a_1 \dots a_7} = E^{a_0 | [a_1 \dots a_7]} \quad \text{and} \quad E^{a_1 \dots a_8, \alpha \beta} = E^{[a_1 \dots a_8], (\alpha \beta)}
\end{aligned} \tag{A.4}$$

The relation to the Chevalley–Serre generators is now given by

$$\begin{aligned}
e_i &= K^i_{i+1}, & f_i &= K^{i+1}_i, & h_i &= K^i_i - K^{i+1}_{i+1} \quad (i = 1, \dots, 7) \\
e_{10} &= K^8_9, & f_{10} &= K^9_8, & h_{10} &= K^8_8 - K^9_9, \\
e_9 &= K^1_{\dot{2}}, & f_9 &= K^{\dot{2}}_1, & h_9 &= K^1_1 - K^{\dot{2}}_{\dot{2}}.
\end{aligned} \tag{A.5}$$

The explicit dots on the indices indicate numerical values for  $\mathfrak{sl}(2)$  vector indices. For the deleted node 8 one has

$$\begin{aligned}
e_8 &= E^{89,\dot{2}}, & f_8 &= F_{89,\dot{2}}, \\
h_8 &= -\frac{1}{4}K + K^8_8 + K^9_9 - \frac{1}{2} \left( K^1_1 - K^{\dot{2}}_{\dot{2}} \right),
\end{aligned} \tag{A.6}$$

where now  $K = \sum_{a=1}^9 K^a_a$  is the trace in  $\mathfrak{gl}(9)$ . Commutation relations for these generators can be found in [13], where we used an  $\mathfrak{so}(1, 2)$  spinor and vector notation instead of  $\mathfrak{sl}(2)$  tensors as above.

## B Constraints of type IIB supergravity and universality

The Einstein equation of motion of IIB supergravity can be written as

$$\begin{aligned}
R_{AB} &= -\frac{1}{4}S_A^\alpha S_{B,\alpha} + \frac{1}{96}F_A^{C_1\dots C_4}F_{BC_1\dots C_4} \\
&\quad + \frac{1}{4}H_A^{C_1C_2,\alpha}H_{BC_1C_2,\alpha} - \frac{1}{48}\eta_{AB}H^{C_1\dots C_3,\alpha}H_{C_1C_2C_3\alpha}
\end{aligned} \tag{B.1}$$

in flat indices, where we corrected a factor of two compared to [13].

The diffeomorphism constraint is obtained as the  $0a$  component of this equation. Using self-duality of  $F$  and the dictionary of [13] one finds, up to overall normalization, the expression

$$\begin{aligned}
\mathfrak{C}^{(-4)m_1\dots m_8} &= \frac{35}{3} I^{(-2)m_1\dots m_4} I^{(-2)m_5\dots m_8} + \frac{28}{3} I^{(-1)m_1m_2,\alpha} I^{(-3)m_3\dots m_8,\beta} \epsilon_{\alpha\beta} \\
&\quad + \frac{1}{3} I^{(-4)m_1\dots m_8,\alpha\gamma} I^{(0)\beta}_{\gamma} \epsilon_{\alpha\beta} + \frac{8}{3} I^{(-4)p|m_1\dots m_7} I^{(0)m_8}_p
\end{aligned} \tag{B.2}$$

in terms of the  $E_{10}$  current components in  $A_8 \oplus A_1$  decomposition.

## C Explicit expressions involving Cartan generators

In this appendix, we give explicit expressions for the contractions between the Cartan subalgebra and the  $\delta$  root space to show that the  $A_9$  and  $A_8 \oplus A_1$  covariant expressions (3.36) and (3.37) differ, thereby illustrating that (3.31) is indeed only valid on contractions of real root spaces.

Consider the contributions from the Cartan subalgebra to the highest component of (3.36). They come exclusively from the  $J^{(-3)}J^{(0)}$  contraction and are

$$\begin{aligned} 3 \mathfrak{L}^{(-3)2345678910} &\ni J^{(-3)2|345678910} J^{(0)2}_2 + J^{(-3)3|456789102} J^{(0)3}_3 \\ &\quad + \dots + J^{(-3)10|23456789} J^{(0)10}_{10} \\ &= J^{(-3)2|345678910} (h_2 + h_3 + \dots + h_8 + h_9) + \dots + J^{(-3)9|102345678} h_9 \end{aligned} \quad (\text{C.1})$$

where the hook symmetry

$$J^{(-3)[2|345678910]} = 0 \quad (\text{C.2})$$

of the level three element was used and we identified for simplicity the current component with the corresponding Cartan generators using (A.2) and (A.3). We see that only the Cartan generators of the  $A_9$  ‘gravity line’ appear in this contraction. The only ‘missing’ ones are the one from the deleted node 10 and the hyperbolic node 1. The latter is related to our choice of (highest) component.

Repeating the same calculation for the  $A_8 \oplus A_1$  decomposition and (3.37) one finds similarly

$$\begin{aligned} 3 \mathfrak{L}^{(-4)23456789} &\ni I^{(-4)2|3456789} I^{(0)2}_2 + \dots + I^{(-4)9|2345678} I^{(0)9}_9 \\ &\quad + I^{(-4)23456789,i\dot{2}} \left( I^{(0)i}_i - I^{(0)\dot{2}}_{\dot{2}} \right) \\ &= I^{(-4)2|3456789} (h_2 + \dots + h_7 + h_{10}) + I^{(-4)8|9234567} h_{10} \\ &\quad + I^{(-4)23456789,i\dot{2}} h_9 \end{aligned} \quad (\text{C.3})$$

The Cartan generators that appear in this expression are those from the  $A_8 \oplus A_1$  gravity line, so the ‘missing’ generators are that of the deleted node

8 and of the hyperbolic node 1. The latter is again related to our choice of component of the diffeomorphism constraint so that the real discrepancy between the two expressions can be traced again to the different deleted nodes. This is related to the failure of this constraint to be a highest weight vector, see the expressions (3.45) and (3.47).

Finally, it is clear that the Cartan generator missing in (3.48) is  $h_9$ , as the diagonal generators among the  $SO(9, 9)$  generators  $J^{(0)KL}$  are identified with  $h_1, \dots, h_8, h_{10}$ , while  $h_9$  is associated with the dilaton, again confirming our general conclusion.

## References

- [1] B. S. DeWitt, *Quantum Theory of Gravity. 1. The Canonical Theory*, Phys. Rev. **160** (1967) 1113.
- [2] C. Kiefer, *Quantum gravity*, Int. Ser. Monogr. Phys. **124** (Oxford University Press, 2004).
- [3] T. Damour, M. Henneaux and H. Nicolai,  *$E_{10}$  and a 'small tension expansion' of M Theory*, Phys. Rev. Lett. **89** (2002) 221601 [arXiv:hep-th/0207267].
- [4] T. Damour, A. Kleinschmidt and H. Nicolai, *Constraints and the  $E_{10}$  Coset Model*, Class. Quant. Grav. **24** (2007) 6097 [arXiv:0709.2691 [hep-th]].
- [5] H. Sugawara, *A Field theory of currents*, Phys. Rev. **170** (1968) 1659.
- [6] K. Bardakçi and M. B. Halpern, *New dual quark models*, Phys. Rev. D **3** (1971) 2493.
- [7] P. Goddard and D. I. Olive, *Kac-Moody And Virasoro Algebras In Relation To Quantum Physics*, Int. J. Mod. Phys. A **1** (1986) 303.
- [8] A. Kleinschmidt, M. Koehn and H. Nicolai, *Supersymmetric quantum cosmological billiards*, Phys. Rev. D **80** (2009) 061701 [arXiv:0907.3048[gr-qc]]
- [9] L. A. Forte, *Arithmetical Chaos and Quantum Cosmology*, Class. Quant. Grav. **26** (2009) 045001 [arXiv:0812.4382 [gr-qc]].

- [10] P. Goddard and C. B. Thorn, *Compatibility of the Dual Pomeron with Unitarity and the Absence of Ghosts in the Dual Resonance Model*, Phys. Lett. B **40** (1972) 235.
- [11] A. Kleinschmidt and H. Nicolai,  *$E_{10}$  and  $SO(9,9)$  invariant supergravity* JHEP **0407** (2004) 041 [arXiv:hep-th/0407101].
- [12] T. Damour and H. Nicolai, *Eleven dimensional supergravity and the  $E_{10}/K(E_{10})$  sigma-model at low  $A_9$  levels*, arXiv:hep-th/0410245.
- [13] A. Kleinschmidt and H. Nicolai, *IIB supergravity and  $E_{10}$* , Phys. Lett. B **606** (2005) 391 [arXiv:hep-th/0411225].
- [14] M. Henneaux, E. Jamsin, A. Kleinschmidt and D. Persson, *On the  $E_{10}$ /Massive Type IIA Supergravity Correspondence*, Phys. Rev. D **79** (2009) 045008 [arXiv:0811.4358 [hep-th]].
- [15] H. Nicolai and H.A.J. Samtleben, *On  $K(E_9)$* , Q.J. Pure Appl. Math **1** (2005) 180, [hep-th/0407055]
- [16] P. C. West,  *$E_{11}$  and  $M$  theory*, Class. Quant. Grav. **18** (2001) 4443 [arXiv:hep-th/0104081].
- [17] P. C. West,  *$E(11)$ ,  $SL(32)$  and central charges*, Phys. Lett. **B575** (2003) 333 [arXiv: hep-th/0307098]
- [18] F. Riccioni and P. West, *Local  $E_{11}$* , JHEP **0904** (2009) 051 [arXiv:0902.4678 [hep-th]].
- [19] I. Schnakenburg and P. C. West, *Kac-Moody symmetries of IIB supergravity*, Phys. Lett. B **517** (2001) 421 [arXiv:hep-th/0107181].
- [20] I. Schnakenburg and P. C. West, *Massive IIA supergravity as a non-linear realisation*, Phys. Lett. B **540** (2002) 137 [arXiv:hep-th/0204207].
- [21] A. Kleinschmidt, I. Schnakenburg and P. C. West, *Very-extended Kac-Moody algebras and their interpretation at low levels*, Class. Quant. Grav. **21** (2004) 2493 [arXiv:hep-th/0309198].
- [22] P. C. West, *The IIA, IIB and eleven-dimensional theories and their common  $E(11)$  origin*, Nucl. Phys. **B693** (2004) 76 [arXiv: hep-th/0402140]

- [23] V. G. Kac, *Infinite dimensional Lie algebras* (Cambridge University Press, 1995).
- [24] T. Damour, S. de Buyl, M. Henneaux and C. Schomblond, *Einstein billiards and overextensions of finite-dimensional simple Lie algebras*, JHEP **0208** (2002) 030 [arXiv:hep-th/0206125].
- [25] R. W. Gebert and H. Nicolai, *An affine string vertex operator construction at arbitrary level*, J. Math. Phys. **38** (1997) 4435 [arXiv:hep-th/9608014].
- [26] M. R. Gaberdiel, D. I. Olive and P. C. West, *A class of Lorentzian Kac-Moody algebras*, Nucl. Phys. B **645** (2002) 403 [arXiv:hep-th/0205068].
- [27] H. Nicolai and T. Fischbacher, *Low level representations for  $E_{10}$  and  $E_{11}$* , Cont. Math. **343** 191 (American Mathematical Society, 2004) [arXiv:hep-th/0301017].
- [28] T. Damour, M. Henneaux and H. Nicolai, *Cosmological billiards* Class. Quant. Grav. **20**, R145 (2003) [arXiv:hep-th/0212256].
- [29] T. Damour, M. Henneaux, B. Julia and H. Nicolai, *Hyperbolic Kac-Moody algebras and chaos in Kaluza-Klein models*, Phys. Lett. B **509**, 323 (2001) [arXiv:hep-th/0103094].
- [30] V. Kac, R.V. Moody and M. Wakimoto, *On  $E_{10}$* , in: “Differential geometrical methods in theoretical physics”, eds. K. Bleuler and M. Werner, (Kluwer, 1988).
- [31] A. Kleinschmidt, H. Nicolai and J. Palmkvist,  *$K(E_9)$  from  $K(E_{10})$* , JHEP **0706** (2007) 051 [arXiv:hep-th/0611314].
- [32] T. Damour and H. Nicolai, *Symmetries, Singularities and the De-Emergence of Space*, Int. J. Mod. Phys. **D17** (2008) 525 [arXiv:0705.2643 [hep-th]].
- [33] L. J. Romans, *Massive  $N=2a$  Supergravity In Ten-Dimensions*, Phys. Lett. B **169** (1986) 374.
- [34] J. Brown, O. J. Ganor and C. Helfgott, *M-theory and  $E_{10}$ : Billiards, branes, and imaginary roots*, JHEP **0408** (2004) 063 [arXiv:hep-th/0401053].