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Wilson Loop in $N = 4$ Super Yang-Mills Theory

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Abstract

The Wilson loop in $N = 4$ super Yang-Mills theory admits a dual description as a macroscopic string configuration in the adS/CFT correspondence. We discuss the correction to the quark anti-quark potential arising from the fluctuations of the superstring.

One important ingredient of the string dualities is the twofold description of D-branes as solitonic supergravity solutions and as submanifolds of spacetime where open strings may end. The second description leads to a gauge field theory on the world-volume of D-branes. This general idea is at the heart of Maldacena's conjecture[1] that $\mathcal{N} = 4$ supersymmetric $SU(N)$ Yang Mills theory in four dimensions is dual to the type IIB string theory on $adS_5 \times S^5$ space. The radii of the S^5 and the adS_5 are equal and related to the Yang-Mills coupling via $R^2/\alpha' = \sqrt{4\pi g_{\mathrm{YM}}^2 N}$. (The string coupling g is $g = g_{\mathrm{YM}}^2$.) For type II supergravity to be good effective description, we need the geometry to be large and also the string coupling to be small. This means that we need g_s small but $g_s N$ large. The latter is however the effective coupling constant in the large N field theory which is thus strongly coupled. Maldacena's conjecture provides a new possibility to gain insight into strongly coupled gauge theory by studying weakly coupled string theory. As an application Wilson loops have been computed in [2]. The string configuration for a quark-antiquark separated by a distance L is a long string on $adS_5 \times S^5$, the ends of which are restricted to the (four dimensional) boundary of adS_5, where they are a distance L apart. The expectation value of the Wilson loop is then given by the effective energy of the string. This was computed in saddle-point approximation in [2] and found to be

$$E = -\frac{4\pi\sqrt{g_{\mathrm{YM}}^2 N}}{\Gamma\,(1/4)^4\, L}. \tag{1}$$

This strong coupling result differs from the perturbative field theory computation ($g_{\mathrm{YM}}^2 N$ small), which predicts a Coulomb law with $E \sim g_{\mathrm{YM}}^2 N/L$. In general the numerator is a function of $g_{\mathrm{YM}}^2 N$ which interpolates between these two forms, and hence there ought to be corrections to the saddle point approximation (1). Since $adS_5 \times S^5$ is an exact string background [3, 4, 5], the first correction comes from the fluctuations of the superstring (R^2/α' correction). In this talk we will report on work in this direction[6].

Recall the string background in the computation of the Wilson loop. The $adS_5 \times S^5$ space has the metric

$$ds^2 = R^2 \left[U^2 \left(-\left(dx^0\right)^2 + \left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2 \right) + \frac{dU^2}{U^2} + d\Omega_5^2 \right]. \tag{2}$$

As compared to [2] we have rescaled $U \to R^2 U$, and set $\alpha' = 1$. In addition there is a constant dilaton and N units of RR-4-form flux through S^5. The quantum theory of type IIB string in this background is described by the action in [4]. This is a Green-Schwarz type sigma model action on a target supercoset. The usual sigma model expansion in R^2/α' results in a power series in $1/\sqrt{N g_{\mathrm{YM}}^2}$. Since conformal invariance prevents the appearance of a new scale these corrections are not expected to change the $1/L$ dependence of E on dimensional grounds. However they can modify the 'Coulomb charge'. The classical solution with the boundary condition that the ends of the string sweep out a rectangular loop with spatial distance L and (infinite) time duration T, is given by the static configuration [2]

$$x^0 = \tau \quad , \quad x^1 = \sigma, \qquad \text{and} \qquad \partial_\sigma U = \pm \frac{U^2}{U_0^2} \sqrt{U^4 - U_0^4}. \tag{3}$$

The rest of the coordinates are constant. Here U_0 is an integration constant which is related to L by $U_0 = (2\pi)^{3/2} / \Gamma(1/4)^2 L$. The fermionic coordinates of the supercoset are zero. Plugging this solution into the action in [4] (or equivalently the Nambu-Goto action), one obtains the result (1). (However one needs to regularize the expression by subtracting an infinite contribution due to the self-energy of the heavy quarks.)

The leading order correction is obtained by expanding around the classical configuration (3) to second order in fluctuations. We parameterize the fluctuations by the normal coordinates[7]: ξ^a, ($a = 0, \cdots, 4; 5, \cdots, 9$ are local (flat) Lorentz indices) on $adS_5 \times S^5$. At second order, the bosonic fluctuations in adS_5, S^5 directions and the fermionic fluctuations decouple. Before writing the action, we define the combinations

$$\xi^{\parallel} = \frac{U_0^2}{U^2} \xi^1 + \frac{\sqrt{U^4 - U_0^4}}{U^2} \xi^4, \qquad \xi^{\perp} = -\frac{\sqrt{U^4 - U_0^4}}{U^2} \xi^1 + \frac{U_0^2}{U^2} \xi^4, \tag{4}$$

which parameterize fluctuations along the longitudinal and perpendicular directions of the background string in the one-four plane. Now the adS_5 part of the action becomes

$$S^{(2)}_{adS} = \frac{1}{4\pi} \int d^2\sigma \sqrt{-h} \left(h^{ij} \left(\sum_{a=2,3,\perp} \partial_i \xi^a \partial_j \xi^a \right) + 2\left(\xi^2\right)^2 + 2\left(\xi^3\right)^2 + 2 \frac{U^4 - U_0^4}{U^4} \left(\xi^{\perp}\right)^2 \right), \tag{5}$$

where h_{ij} is (up to a factor of R^2) the classical induced world-sheet metric[1])

$$ds^2 = -U^2 \left(d\sigma^0\right)^2 + \frac{U^6}{U_0^4} \left(d\sigma^1\right)^2. \tag{6}$$

Observe that ξ^0 and $\xi^{\parallel}$ do not appear in (5) (total derivatives have been dropped). Hence a natural choice to fix world-sheet diffeomorphisms is

$$\xi^0 = \xi^{\parallel} = 0. \tag{7}$$

The action quadratic in fluctuations in S^5 directions comes out to be

$$S^{(2)}_{S^5} = \frac{1}{4\pi} \int d^2\sigma \sqrt{-h} h^{ij} \sum_{a=5}^{9} \partial_i \xi^a \partial_j \xi^a. \tag{8}$$

[1]) For our purpose it is more convenient to work with the induced metric rather than the standard (conformally) flat one on the world-sheet.

The fermionic part of the action is given by plugging in the background (3) into the action of [4] and keeping terms quadratic in fermions. This action has local fermionic $\varkappa$-symmetry which has to be fixed for the one-loop calculation. There is a proposal in the literature[8] to this end. For our purpose, however, it turns out that the following choice is most convenient. We will set (in the notation of [4])

$$\left(\gamma^{-}\right)^{\alpha}_{\beta}\vartheta^{1\beta\beta'} = 0 \quad , \quad \left(\gamma^{+}\right)^{\alpha}_{\beta}\vartheta^{2\beta\beta'} = 0 \tag{9}$$

where $\gamma^{\pm} = \gamma^0 \pm \gamma^{\parallel}$ with $\gamma^{\parallel} = \frac{U_0^2}{U^2}\gamma^1 + \frac{\sqrt{U^4 - U_0^4}}{U^2}\gamma^4$ (Cf. (4)). With this choice the target space spinors 'metamorphose' into world-sheet spinors, and the action is found to simplify substantially. The corresponding equations of motion are most compactly written for the 'two-component' world-sheet spinors $\begin{pmatrix}\vartheta^1\\ \vartheta^2\end{pmatrix}$:

$$\left[i\left(\varrho^m \nabla_m\right)^{\alpha}_{\beta} - \delta^{\alpha}_{\beta}\varrho^3\right]\begin{pmatrix}\vartheta^{1\beta\alpha'}\\ \vartheta^{2\beta\alpha'}\end{pmatrix} = 0. \tag{10}$$

The notation needs explanation. Firstly, the covariant derivatives act as

$$\left(\nabla_{\pm}\vartheta^I\right)^{\alpha\alpha'} = \left[\delta^{\alpha}_{\beta}\left(\partial_{\pm} \pm \frac{\omega_{\pm}}{2}\right) + \left(A_{\pm}\right)^{\alpha}_{\beta}\right]\vartheta^{I\beta\alpha'}, \tag{11}$$

where the tangent space derivatives $\partial_{\pm} = \frac{1}{U}\partial_{\tau} \pm \frac{U_0^2}{U^3}\partial_{\sigma}$ are defined with the help of a set of (inverse) zweibein ε_m of the metric (6), $\omega^{01}_m = \varepsilon^{\tau}_m\omega^{01}_{\tau}$ being the corresponding spin connection. There is an additional gauge field $A_{\pm} = \pm\frac{U_0^2}{U^2}\gamma^{14}$ for local rotations in the tangent one-four-plane. Finally, the matrices

$$\varrho^{+} = \begin{pmatrix} 0 & 0 \\ \gamma^0 & 0 \end{pmatrix} \quad , \quad \varrho^{-} = \begin{pmatrix} 0 & \gamma^0 \\ 0 & 0 \end{pmatrix} \tag{12}$$

satisfy a two dimensional Clifford algebra, and $\varrho^3 = [\varrho^+, \varrho^-]$. The fermionic action is easily inferred from the equations of motion (10).

Collecting our results (5), (8) and (10) one can write a formal expression for the 1-loop contribution to the effective action as a ratio of determinants of two-dimensional generalized Laplace operators[6]. These determinants suffer from divergences and can be regularized by, say, the heat kernel technique[9]. The quadratic divergences cancel, but a logarithmic divergence remains, which can be absorbed in the (infinite) mass of the external quarks[6]. A closed form expression for the regularized effective action is still lacking.

Finally we remark that after this talk was given, ref.[10] appeared which also discusses corrections to the Wilson loop due to stringy fluctuations. This work is however complementary to ours in the sense that their approximation seems to work only in the finite temperature case which we have not considered. Our techniques can also be applied to other cases where the *adS*/CFT duality conjecture works, *e.g.*[11].

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