

# Hamiltonian evolution and quantization for extremal black holes

Claus Kiefer<sup>1</sup>

Fakultät für Physik, Universität Freiburg, Hermann-Herder-Straße 3, D-79104 Freiburg, Germany

Jorma Louko<sup>2</sup>

Max-Planck-Institut für Gravitationsphysik, Schlaatzweg 1, D-14473 Potsdam, Germany

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**Abstract.** We present and contrast two distinct ways of including extremal black holes in a Lorentzian Hamiltonian quantization of spherically symmetric Einstein-Maxwell theory. First, we formulate the classical Hamiltonian dynamics with boundary conditions appropriate for extremal black holes only. The Hamiltonian contains no surface term at the internal infinity, for reasons related to the vanishing of the extremal hole surface gravity, and quantization yields a vanishing black hole entropy. Second, we give a Hamiltonian quantization that incorporates extremal black holes as a limiting case of nonextremal ones, and examine the classical limit in terms of wave packets. The spreading of the packets, even the ones centered about extremal black holes, is consistent with continuity of the entropy in the extremal limit, and thus with the Bekenstein-Hawking entropy even for the extremal holes. The discussion takes place throughout within Lorentz-signature spacetimes.

**Keywords:** Quantum black holes; Black-hole thermodynamics; Variational methods in gravitation

## 1 Introduction

Extremal black holes have a special and controversial status in black hole thermodynamics. On the one hand, one might expect macroscopic thermodynamical quantities to be continuous functions of the black hole parameters when one passes from nonextremal black holes to the extremal limit, and one would then be led to the conclusion that extremal black holes must possess the standard Bekenstein-Hawking entropy, equal to one quarter of the horizon area [1–5]. Support for this view is also provided by state-counting calculations of certain extremal and near-extremal black holes in string theory; for a review, see Ref. [6]. On the other hand, extremal and nonextremal black hole geometries have qualitative differences that raise doubts about limiting arguments. In particular, the absence of a bifurcate Killing horizon and the vanishing of the horizon surface gravity in the Lorentzian extremal black hole spacetime furnish the Riemannian section of

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<sup>1</sup> E-mail: Claus.Kiefer@physik.uni-freiburg.de

<sup>2</sup> E-mail: louko@aei-potsdam.mpg.de

the spacetime with certain properties that, within the path-integral approach to black-hole thermodynamics [7–9], have lead to the conclusion of a vanishing entropy [10, 11].

In this paper we make two observations that aim to clarify the distinctive status of extremal black hole geometries within Hamiltonian quantization. We consider the limited but concrete context of spherically symmetric geometries in four space-time dimensions, within the Einstein-Maxwell theory.

First, we formulate the classical Hamiltonian dynamics of Lorentzian spacetimes under boundary conditions appropriate for *only* extremal black holes, such that the spacelike hypersurfaces are asymptotic to hypersurfaces of constant Killing time at the internal infinity. We show that the action does not depend on how, or whether, the evolution of the spacelike hypersurfaces near the internal infinity is prescribed in the variational principle, and we trace the geometrical reason for this directly to the vanishing of the surface gravity. We also find explicitly the reduced Hamiltonian theory, with a two-dimensional reduced phase space. The action does not contain at the internal infinity a boundary term of the kind that gives rise to the Bekenstein-Hawking entropy in the Hamiltonian quantization of nonextremal black holes [12–19]. Quantizing this Hamiltonian theory on its own would therefore lead to the conclusion of a vanishing extremal black hole entropy.

Second, we formulate a Hamiltonian quantization that incorporates extremal black holes as a limiting case of nonextremal ones, and we explore in this quantization wave packets centered around a given black hole. The wave packets centered around extremal and non-extremal holes have some qualitative differences, in particular in how they spread as a function of time. However, the spreading behavior also suggests that in this approach both extremal and nonextremal holes should have the standard Bekenstein-Hawking entropy. Physically, for a wave packet centered about an extremal hole, the extremal configuration itself is a set of measure zero, and the entropy should then in essence be determined by the nonextremal configurations in a small but finite neighborhood of the packet center.

Neither of these results is surprising, and they are fully in line with the general expectation that treating extremal black holes on their own lead to a vanishing entropy while treating them as limiting cases of nonextremal ones lead to the usual Bekenstein-Hawking entropy. However, with their express emphasis on a Lorentz-signature spacetime and on a Hamiltonian framework of quantization, we view our results as filling a gap in the literature. Concurring observations were recently made, within a technically different approach that relies on Euclidean-signature spacetimes, in Ref. [20].

The rest of the paper is as follows. In section II we set up a classical Hamiltonian theory under boundary conditions appropriate for extremal black holes, and in section III we reduce this theory to its unconstrained, two-dimensional phase space. A Hamiltonian quantization including both extremal and nonextremal holes is analyzed in section IV. Section V presents brief concluding remarks. Some relevant properties of the family of spacetimes are collected in the Appendix. We use the geometrical units [21] in which  $c = G = 1$ , but we keep Planck's constant  $\hbar$ .

## 2 Canonical formalism for extremal holes

In this section we present a Hamiltonian formulation for spherically symmetric Einstein-Maxwell spacetimes under boundary conditions appropriate for the exterior region of an extremal black hole, such that the spacelike hypersurfaces are asymptotic to the hypersurfaces of constant Killing time near the internal infinity. For concreteness, we consider a negative cosmological constant; the cases of a positive or vanishing cosmological constant can be handled with straightforward modifications and similar conclusions. As our new results concern just the internal infinity, we leave the boundary conditions and boundary terms at the “outer, end of the spacelike hypersurfaces unspecified. Examples of how to handle the outer end can be found in Refs. [12–19].

The notation follows Ref. [5]. We write the cosmological constant as  $-3\ell^{-2}$ , where  $\ell > 0$ .

We consider the general spherically symmetric ADM metric

$$ds^2 = -N^2 dt^2 + A^2(dr + N^r dt)^2 + R^2 d\Omega^2, \quad (2.1)$$

where  $d\Omega^2$  is the metric on the unit two-sphere, and  $N$ ,  $N^r$ ,  $A$  and  $R$  are functions of  $t$  and  $r$ . We take the electromagnetic bundle to be trivial, and describe the electromagnetic field by the globally-defined spherically symmetric one-form

$$A = \Gamma dr + \Phi dt, \quad (2.2)$$

where  $\Gamma$  and  $\Phi$  are functions of  $t$  and  $r$ . The coordinate  $r$  takes the semi-infinite range  $[0, \infty)$ . We assume both the spatial metric and the spacetime metric to be nondegenerate, and  $A$ ,  $R$ , and  $N$  to be positive.

The bulk contribution to the Hamiltonian action of Einstein-Maxwell theory reads

$$S_\Sigma = \int dt \int_0^\infty dr (P_A \dot{A} + P_R \dot{R} + P_\Gamma \dot{\Gamma} - NH - N^r H_r - \tilde{\Phi} G), \quad (2.3)$$

where the super-Hamiltonian constraint  $H$ , the radial supermomentum constraint  $H_r$ , and the Gauss law constraint  $G$  are given by

$$H = -R^{-1} P_R P_A + \frac{1}{2} R^{-2} A (P_A^2 + P_\Gamma^2) + A^{-1} R R'' - A^{-2} R R' A' + \frac{1}{2} A^{-1} R'^2 - \frac{1}{2} A - \frac{3}{2} \ell^{-2} A R^2, \quad (2.4 a)$$

$$H_r = P_R R' - A P_A' - \Gamma P_\Gamma', \quad (2.4 b)$$

$$G = -P_\Gamma', \quad (2.4 c)$$

and we have defined

$$\tilde{\Phi} := \Phi - N^r \Gamma. \quad (2.5)$$

We regard  $N$ ,  $N^r$ , and  $\tilde{\Phi}$  as the independent Lagrange multipliers in the action (2.3). Local variation of (2.3) yields the three constraint equations  $H = H_r = G = 0$ , and the six dynamical equations that give the time derivatives of the coordinates and the momenta [15]. These equations correctly reproduce the spherically symmetric Einstein-Maxwell equations.

We now wish to adopt near  $r = 0$  boundary conditions that enforce every classical solution to be (part of) an exterior region of an extremal Reissner-Nordström-

anti-de Sitter (RNAdS) black hole (see the Appendix), and such that the constant  $t$  hypersurfaces are asymptotic to the constant Killing time hypersurfaces at the internal infinity as  $r \rightarrow 0$ . To this end, we take the variables to have the small  $r$  expansion

$$A(t, r) = A_{-1}(t) r^{-1} + O(1), \quad (2.6 \text{ a})$$

$$R(t, r) = R_0(t) + R_1(t) r + O(r^2), \quad (2.6 \text{ b})$$

$$P_A(t, r) = O(r^3), \quad (2.6 \text{ c})$$

$$P_R(t, r) = O(r), \quad (2.6 \text{ d})$$

$$N(t, r) = A^{-1} R' (\tilde{N}_0(t) + O(r)), \quad (2.6 \text{ e})$$

$$N'(t, r) = O(r), \quad (2.6 \text{ f})$$

$$\Gamma(t, r) = \Gamma_0(t) + O(r), \quad (2.6 \text{ g})$$

$$P_T(t, r) = Q_0(t) + O(r^2), \quad (2.6 \text{ h})$$

$$\tilde{\Phi}(t, r) = \tilde{\Phi}_0(t) + O(r), \quad (2.6 \text{ i})$$

where  $A_{-1}$ ,  $R_0$ , and  $R_1$  are positive. Here  $O(r^n)$  stands for a term whose magnitude at  $r \rightarrow 0$  is bounded by  $r^n$  times a constant, and whose derivatives with respect to  $r$  and  $t$  fall off accordingly. It is straightforward to verify that the falloff conditions (2.6) are consistent with the constraints, and that they are preserved by the time evolution equations when the constraints hold for the initial data.

To see that the falloff (2.6) accomplishes what we wish, consider a classical solution satisfying this falloff. First, as  $R_1$  is positive, the radius of the two-sphere is not constant on the spacetime, and the solution cannot belong to the Bertotti-Robinson-type family [22]. The solution is therefore part of a RNAdS spacetime. Second, recall that on a classical solution, the functions  $F$  and  $T$  defined in the Appendix can be written in terms of the canonical variables as [23, 15]

$$F = \left( \frac{R'}{A} \right)^2 - \left( \frac{P_A}{R} \right)^2, \quad (2.7 \text{ a})$$

$$-T' = R^{-1} F^{-1} A P_A, \quad (2.7 \text{ b})$$

for which equations (2.6) imply the falloff

$$F = \frac{R_1^2 r^2}{A_{-1}^2} + O(r^3), \quad (2.8 \text{ a})$$

$$T' = O(1). \quad (2.8 \text{ b})$$

Equation (2.8b) implies that the  $r \rightarrow 0$  end of a constant  $t$  hypersurface tends to a finite value of the Killing time. Third, equation (2.6a) shows that the proper distance on a constant  $t$  hypersurface from any positive value of  $r$  to  $r = 0$  is infinite, while equation (2.6b) shows that  $R$  must tend to a finite and nonzero value. These properties imply that the solution, if it exists, is an extremal RNAdS hole, and that the constant  $t$  hypersurfaces are asymptotic to the constant Killing time hypersurfaces at the internal infinity as  $r \rightarrow 0$ . Finally, it is easy to verify that the extremal RNAdS black hole *can* be written in coordinates satisfying (2.6); for example, a

static gauge satisfying (24) is obtained from the curvature coordinates of the Appendix via  $T = t$  and  $R = M(1 + r)$ . The falloff (2.6) thus has the desired properties.

Note that when the equations of motion hold,  $Q_0$  is the charge parameter and  $R_0$  is the horizon radius, and for an extremal black hole these are related by equation (A4) of the Appendix: this relation arises in the Hamiltonian formulation as the leading-order term in the small  $r$  expansion of the constraint equation  $H = 0$ . Note also from (2.6), (2.8a), and (A1a) that for a classical solution  $\tilde{N}_0$  is equal to  $dT/dt$  at  $r = 0$ .  $\tilde{\Phi}_0$  is related to the electromagnetic gauge at  $r = 0$  [15].

Consider now the variational principle. The variation of the bulk action (2.3) contains a volume term proportional to the equations of motion, boundary terms from the initial and final hypersurfaces, and boundary terms from  $r = 0$  and  $r = \infty$ . With the falloff (2.6), the boundary term from  $r = 0$  is  $-\int dt \tilde{\Phi}_0 \delta Q_0$ . Therefore, adding to (2.3) the boundary term

$$\int dt \tilde{\Phi}_0 Q_0 \tag{2.9}$$

(plus appropriate boundary terms at  $r = \infty$ ) yields a consistent action functional for prescribing  $\tilde{\Phi}_0(t)$  (and the appropriate quantities at  $r = \infty$ ).

We emphasize that one does *not* need to add to the action a boundary term that would refer to the small  $r$  behavior of  $N(t, r)$ . The action functional is consistent, without any change in the boundary terms, whether or not one chooses to restrict the variations of  $N(t, r)$  at  $r = 0$  in some fashion, such as by prescribing  $\tilde{N}_0(t)$ . This is a crucial difference between the extremal and nonextremal black hole variational principles. For a nonextremal hole, with boundary conditions that make the spacelike hypersurfaces at  $r = 0$  asymptotic to constant Killing time hypersurfaces at the bifurcation two-sphere [12–19], the variation of (2.3) contains at  $r = 0$  also the boundary term

$$-\frac{1}{2} \int dt [(N/A)' \delta(R^2)]_{r=0}. \tag{2.10}$$

Now, both the extremal and nonextremal classical black hole solutions satisfy  $[(N/A)']_{r=0} = \kappa (dT/dt)_{r=0}$ , where  $\kappa$  is the surface gravity of the hole with respect to the Killing field  $\partial_T$ ; one way to see this is to use the fact that, in terms of the function  $F(R)$  (A1b) given in the Appendix,  $\kappa = \frac{1}{2} [\partial F(R)/\partial R]_{R=R_0}$ . The geometrical reason why the surface term (2.10) is not present under the extremal hole falloff is therefore the vanishing surface gravity of the extremal hole.

We also emphasize that our falloff (2.6) is not a special case of the nondegenerate horizon falloff adopted in Refs. [12–19]. Rather, the conditions (2.6) imply at the very outset the distinctive horizon characteristics of the extremal hole, including the vanishing of the surface gravity. We shall return to this issue in sections IV and V.

### 3 Hamiltonian reduction

We wish to eliminate the constraints from the Hamiltonian theory formulated in section II and express the reduced theory in terms of an explicit canonical chart. As the dynamical content of the theory depends on the boundary conditions, we now make a concrete choice for the falloff at  $r \rightarrow \infty$ , taking the spatial hypersurfaces there to be asymptotic to hypersurfaces of constant Killing time as in Ref.

[15]. We follow closely the method of Ref. [15], which is an adaptation for the formalism developed for spherically symmetric vacuum geometries by Kuchař [23].<sup>1</sup> We shall not aim at a self-contained presentation, but we shall elaborate on the steps where the new boundary conditions bring about new features.

The total action consists of the bulk action (2.3) and the boundary action given by

$$S_{\partial\Sigma} := \int dt (\tilde{\Phi}_0 \dot{Q}_0 - \tilde{\Phi}_+ \dot{Q}_+ - \tilde{N}_+ \dot{M}_+). \quad (3.1)$$

The first term in (3.1) is the boundary term (2.9). In the second term,  $\tilde{\Phi}_+$  and  $\dot{Q}_+$  are the asymptotic values of respectively  $\tilde{\Phi}$  and  $P_\Gamma$  as  $r \rightarrow \infty$ . The quantity  $\tilde{N}_+$  in the third term characterizes the asymptotic evolution of the spacelike hypersurfaces at the infinity: on a classical solution, the value of  $dT/dt$  at  $r \rightarrow \infty$  is equal to  $\tilde{N}_+$ . Finally,  $\dot{M}_+$  is the asymptotic value of a phase space function whose value on the classical solution is just the mass parameter. This action is appropriate for a variational principle that fixes  $\tilde{\Phi}_0$  at  $r = 0$ , and  $\tilde{\Phi}_+$  and  $\tilde{N}_+$  at  $r \rightarrow \infty$  [15].

As a first step, we make a Kuchař-type [23] canonical transformation from the phase space chart  $\{\mathcal{A}, R, \Gamma, P_{\mathcal{A}}, P_R, P_\Gamma\}$  to the new chart  $\{M, R, Q, P_M, P_R, P_Q\}$  by equations (3.7) of Ref. [15]. It is easy to find the falloff of the new variables, and to verify that the transformation is canonical under our boundary conditions. Of the geometrical meaning of the new variables, it is here sufficient to recall that on a classical solution,  $M$  and  $Q$  are constants whose values are just the mass and charge parameters of the spacetime.

The bulk action in the new chart reads

$$S_\Sigma = \int dt \int_0^\infty dr (P_M \dot{M} + P_R \dot{R} + P_Q \dot{Q} - N^M M' - N^Q Q' - N^R P_R), \quad (3.2)$$

where  $N^M$ ,  $N^Q$ , and  $N^R$  are a set of new Lagrange multipliers, related to the old ones by equations (3.16) of Ref. [15]. We are interested in the boundary terms in the variation of (3.2). At  $r \rightarrow \infty$ , the situation is as in Ref. [15]: the asymptotic values of  $M$ ,  $Q$ ,  $N^M$ , and  $N^Q$  are respectively  $M_+$ ,  $Q_+$ ,  $-\tilde{N}_+$ , and  $-\tilde{\Phi}_+$ , and the boundary term in the variation of (3.2) at  $r \rightarrow \infty$  is  $\int dt (\tilde{N}_+ \delta M_+ + \tilde{\Phi}_+ \delta Q_+)$ . At  $r \rightarrow 0$ , on the other hand, we have

$$M = M_0 + M_1 r + O(r^2), \quad (3.3 \text{ a})$$

$$Q = Q_0 + O(r^2), \quad (3.3 \text{ b})$$

$$N^M = -\tilde{N}_0 + O(r), \quad (3.3 \text{ c})$$

$$N^Q = -\tilde{\Phi}_0 + (Q_0/R_0) \tilde{N}_0 + O(r), \quad (3.3 \text{ d})$$

where

$$M_0 = \frac{R_0}{2} \left( \frac{R_0^2}{\ell^2} + 1 + \frac{Q_0^2}{R_0^2} \right), \quad (3.4 \text{ a})$$

$$M_1 = \frac{R_1}{2} \left( \frac{3R_0^2}{\ell^2} + 1 - \frac{Q_0^2}{R_0^2} \right). \quad (3.4 \text{ b})$$

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<sup>1</sup> Related reduction methods have been previously and subsequently considered in a variety of contexts; in addition to Refs. [12–19], see in particular Refs. [24–31]. A more extensive list of references is given in Ref. [32].

The boundary term from  $r = 0$  in the variation of (3.2) therefore reads

$$- \int dt \left[ \frac{\tilde{N}_0}{2} \left( \frac{3R_0^2}{\ell^2} + 1 - \frac{Q_0^2}{R_0^2} \right) \delta R_0 + \tilde{\Phi}_0 \delta Q_0 \right]. \quad (3.5)$$

The first term under the integral in (3.5) is proportional to  $M_1$ , which vanishes when the bulk constraint equation  $M' = 0$  holds.<sup>2</sup> This first term in (3.5) therefore vanishes as a consequence of the bulk variational equations, and only the second term in (3.5) remains.

Collecting these observations, we see that when  $\tilde{N}_+$ ,  $\tilde{\Phi}_+$ , and  $\tilde{\Phi}_0$  are prescribed, the boundary action to be added to the bulk action (3.2) is again given by (3.1). We emphasize that, as in section II, this conclusion is independent of whether the variation of  $\tilde{N}_0$  might also be restricted in some way.

A Hamiltonian reduction in the new variables is straightforward. The constraints  $M' = 0$  and  $Q' = 0$  imply  $M(t, r) = \mathbf{m}(t)$  and  $Q(t, r) = \mathbf{q}(t)$ , but equations (3.4) shows also that  $\mathbf{m}$  and  $\mathbf{q}$  are not independent. A convenient independent parameter is  $\mathbf{r}(t) := R_0 > 0$ , in terms of which we have

$$\mathbf{m} = \mathbf{r} (1 + 2\mathbf{r}^2 \ell^{-2}), \quad (3.6 \text{ a})$$

$$\mathbf{q} = \epsilon \mathbf{r} (1 + 3\mathbf{r}^2 \ell^{-2})^{1/2}, \quad (3.6 \text{ b})$$

where  $\epsilon$  is a discrete parameter taking the values  $\pm 1$ . The reduced action reads

$$S = \int dt (\mathbf{p}_r \dot{\mathbf{r}} - \mathbf{h}), \quad (3.7)$$

where

$$\mathbf{p}_r = (1 + 6\mathbf{r}^2 \ell^{-2}) \left( \int_0^\infty dr P_M \right) + \frac{\epsilon (1 + 6\mathbf{r}^2 \ell^{-2})}{(1 + 3\mathbf{r}^2 \ell^{-2})^{1/2}} \left( \int_0^\infty dr P_Q \right), \quad (3.8)$$

and the reduced Hamiltonian  $\mathbf{h}$  is

$$\mathbf{h} = \epsilon (\tilde{\Phi}_+ - \tilde{\Phi}_0) \mathbf{r} (1 + 3\mathbf{r}^2 \ell^{-2})^{1/2} + \tilde{N}_+ \mathbf{r} (1 + 2\mathbf{r}^2 \ell^{-2}). \quad (3.9)$$

$\mathbf{p}_r$  can be interpreted geometrically in terms of the Killing time evolution and the electromagnetic gauge choice at the two ends of the constant  $t$  hypersurfaces as in Ref. [15], and the dynamics derived from  $\mathbf{h}$  can be verified to have the correct geometric content.

We note that it would be possible to do the reduction in two stages, imposing in the first stage all the constraints *except* what setting  $M_1$  (3.4b) to zero implies for the interdependence of  $\mathbf{m}$  and  $\mathbf{q}$ . After this first stage, one arrives at a four-dimensional phase space on which  $\mathbf{m}$  and  $\mathbf{q}$  and their conjugate momenta, found as in Ref. [15], provide a canonical chart. The single remaining constraint is an algebraic relation between  $\mathbf{m}$  and  $\mathbf{q}$ , and it clearly Poisson commutes with the Hamiltonian,

<sup>2</sup> Note that  $M_1 = 0$  is equivalent to the equation (A4) for the horizon radius of an extremal hole. The extremality condition thus emerges in the new variables as the leading-order term in the small  $r$  expansion of the constraint  $M' = 0$ , just as it did in the old variables as the leading-order term in the small  $r$  expansion of the constraint  $H = 0$ . In the next-to-leading order in  $r$ , each of these constraint equations can be verified to imply the relation  $(R_0/A_{-1})^2 = (Q_0/R_0)^2 + 3(R_0/\ell)^2$ .

which does not depend on the momenta. Elimination of the last constraint then duly leads to the fully reduced two-dimensional phase space found above.<sup>3</sup>

The reduced Hamiltonian  $\mathbf{h}$  (3,9) depends on  $\tilde{N}_+$ ,  $\tilde{\Phi}_+$ , and  $\tilde{\Phi}_0$ , but the action (3.7) depends in no way on  $\tilde{N}_0$ . In particular, the action (3.7) does not contain a horizon term of the kind that produces the Bekenstein-Hawking entropy upon quantizing the analogous Hamiltonian formulation for nonextremal black holes [12–19]. One is led to conclude that a Hamiltonian quantization of the theory (3.7) along the lines of the nonextremal Hamiltonian quantization in Refs. [12–16] would lead to a vanishing extremal black hole entropy.

#### 4 Wave packets in Hamiltonian quantization

We now wish to include extremal black holes as a limiting case in a quantum theory that is initially formulated for nonextremal black holes, and examine the classical limit of the theory, both for extremal and nonextremal holes, in terms of wave packets. For concreteness, in this section we set the cosmological constant to zero.

In the Hamiltonian theory for nonextremal holes, we take one end of the space-like hypersurfaces to be at the asymptotically flat infinity and the other end at the horizon bifurcation two-sphere, such that the hypersurfaces are at each end asymptotic to hypersurfaces of constant Killing time [15, 17, 18]. In the quantum theory, we then arrive at plane-wave-like wave functions of the form

$$\Psi_{mq}(\alpha, \tau, \lambda) = \exp \left[ \frac{i}{\hbar} \left( \frac{A(m, q) \alpha}{8\pi} - m\tau - q\lambda \right) \right], \quad (4.1)$$

where the parameters  $m$  and  $q$  labeling the plane waves have the interpretation as the mass and charge: they satisfy  $m > |q|$ , and we have  $A(m, q) = 4\pi R^2(m, q)$  and  $R(m, q) = m + \sqrt{m^2 - q^2}$ , so that  $A$  is the area and  $R$  the area-radius of the horizon. Of the three arguments  $(\alpha, \tau, \lambda)$  of the wave function,  $\tau$  has an interpretation as the Killing time at the infinity,  $\lambda$  is related to the electromagnetic gauge choice at the infinity and at the horizon, and  $\alpha$  is the rapidity parameter of the normal vector to the spacelike hypersurfaces at the bifurcation two-sphere. One way to arrive at the wave functions (4.1) is the leading-order semiclassical approximation to the Wheeler-DeWitt equation in the metric variables [17–19].<sup>4</sup> Another way is to introduce  $\tau$  and  $\alpha$  as reparametrization clocks and perform an exact quantization of the Hamiltonian theory along the lines of equations (191)–(192) of Ref. [23]. Note that all the three arguments  $(\alpha, \tau, \lambda)$  stand on an equal footing as “configuration, variables, and the wave function does not depend on an additional, external “time” variable.

It is useful to point out the parallels between the plane waves (4.1) and the plane wave states for the free nonrelativistic particle, proportional to

<sup>3</sup> We thank Bernard Whiting for discussions on this point.

<sup>4</sup> In this case, going to higher orders in the semiclassical approximation would change the wave function in ways that are important when fields with local degrees of freedom come into play; how the Hawking radiation can be obtained from the wave functional at the next order was shown in Ref. [33].



$\exp(ikx - i\omega t)$ . As  $A = A(m, q)$ , the number of parameters in the states  $\Psi_{mq}$  is one less than the number of arguments; the same holds for the particle, as there  $\omega = \omega(k) = k^2/2m$ . The phase of (4.1) is  $\hbar^{-1}$  times a particular solution to the Hamilton-Jacobi equation, labeled by  $m$  and  $q$ , and varying the phase with respect to the parameters yields the equations [18, 19]

$$\alpha = 8\pi \left( \frac{\partial A}{\partial m} \right)^{-1} \tau \equiv \kappa \tau, \quad (4.2a)$$

$$\lambda = \frac{\kappa}{8\pi} \frac{\partial A}{\partial q} \tau \equiv \phi \tau, \quad (4.2b)$$

where  $\kappa$  denotes the surface gravity and  $\phi$  the electrostatic potential difference between the infinity and the horizon: these are the equations for the family of classical spacetimes recovered from the particular solution to the Hamilton-Jacobi equation. In comparison, for the free particle the corresponding extremization yields the particular classical trajectory  $x = kt/m$ .

Now, if we were to perform a similar quantization for the extremal holes on their own, the first term in the exponent in (4.1) would not be present, as the analysis in section II shows. (Note that this is consistent with equation (4.2a), as the surface gravity for the extremal hole vanishes.) In the analogy with the free particle, this is as if a particular value for the momentum, say  $p_0$ , were special in the sense that no dynamical variables  $(x, p)$  existed for  $p = p_0$ . However, a classical correspondence for the free particle is not gained from the plane wave solutions itself, but from wave *packets* that are obtained by superposing different wave numbers  $k$ . Only such superpositions yield quantum states that are sufficiently concentrated near individual classical trajectories, such as  $x = k_0 t/m$ . We shall proceed similarly with the quantum state (4.1) and first build for nonextremal holes wave packets that are concentrated along the classical relations (4.2). After having built these packets, we then extend them, by hand, to the extremal limit, and let this limit *define* what we mean by extremal holes in the quantum theory. This procedure might be called “extremization after quantization, and it mirrors the spirit of the path integral approach in Ref. [34].

For the explicit construction of the wave packets, we integrate over  $A$  and  $q$  and express the mass  $m$  as a function of these variables,

$$m(A, q) = \frac{R}{2} \left( 1 + \frac{q^2}{R^2} \right) = \frac{A + 4\pi q^2}{4\sqrt{\pi A}}. \quad (4.3)$$

The integration range is  $A > 4\pi q^2$ . For the weight functions we choose Gaussians that are peaked around the values  $A = A_0$  and  $q = q_0$ :

$$\begin{aligned} \psi(\alpha, \tau, \lambda) = & \int_{A > 4\pi q^2} dA dq \exp \left( -\frac{(A - A_0)^2}{2(\Delta A)^2} - \frac{(q - q_0)^2}{2(\Delta q)^2} \right) \\ & \times \exp \left[ \frac{i}{\hbar} \left( \frac{A\alpha}{8\pi} - m(A, q) \tau - q\lambda \right) \right]. \end{aligned} \quad (4.4)$$

Provided  $A_0$  and  $q_0$  are not close to the extremal limit,  $A_0 = 4\pi q_0^2$ , and provided  $\Delta A$  and  $\Delta q$  are chosen suitably, it is a good approximation to expand  $m(A, q)$

around  $A_0$  and  $q_0$  to quadratic order and then take the integral over all real  $A$  and  $q$ . We denote the values of  $m$ ,  $\phi$ , and  $\kappa$  at  $(A_0, q_0)$  by  $m_0$ ,  $\phi_0$ , and  $\kappa_0$ , respectively. The corresponding horizon radius is called  $R_0$ . The calculation is lengthy but straightforward. Apart from overall normalization and phase factors, the result reads

$$\begin{aligned} \psi(\alpha, \tau, \lambda) = & \mathcal{N} \exp \left[ \frac{i}{\hbar} \left( \frac{A_0 \alpha}{8\pi} - m_0 \tau - q_0 \lambda \right) \right] \\ & \times \exp \left( - \frac{(\lambda - \phi_0 \tau)^2}{2\hbar^2} \frac{\mathcal{F}}{\mathcal{B}} - \frac{(\alpha - \kappa_0 \tau)^2}{2\hbar^2} \frac{\mathcal{G}}{\mathcal{B}} \right) \\ & \times \exp \left( \frac{(\lambda - \phi_0 \tau)(\alpha - \kappa_0 \tau)}{2\hbar^2} \frac{\mathcal{H}}{\mathcal{B}} \right) \times (\text{phase factors}), \end{aligned} \quad (4.5)$$

where

$$\mathcal{B} = \left( 1 + \frac{\tau^2 \kappa_0 (\Delta A)^2 (\Delta q)^2}{64 \hbar^2 \pi^2 R_0^3} \right)^2 + \frac{4\pi \tau^2}{\hbar^2 A_0} \left( (\Delta q)^2 + \frac{(\Delta A)^2 (1 - 3\kappa_0 R_0)}{16\pi A_0} \right)^2, \quad (4.6 a)$$

$$\mathcal{F} = (\Delta q)^2 + \frac{\tau^2 (\Delta A)^2 (\Delta q)^2}{8\hbar^2 A_0^2} \left( (\Delta q)^2 (1 - 2\kappa_0 R_0) + \frac{(\Delta A)^2 (1 - 3\kappa_0 R_0)}{8\pi A_0} \right) \quad (4.6 b)$$

$$\mathcal{G} = \frac{1}{64\pi^2} \left( (\Delta A)^2 + \frac{4\pi \tau^2 (\Delta A)^2 (\Delta q)^2}{\hbar^2 A_0} \left[ (\Delta q)^2 + \frac{(\Delta A)^2 (1 - 2\kappa_0 R_0)}{16\pi A_0} \right] \right) \quad (4.6 c)$$

$$\mathcal{H} = \frac{\tau^2 (\Delta A)^2 (\Delta q)^2 q_0}{2\hbar^2 A_0^2} \left( (\Delta q)^2 + \frac{(\Delta A)^2 (1 - 3\kappa_0 R_0)}{16\pi A_0} \right). \quad (4.6 d)$$

The packet is, as expected, concentrated around the classical values (4.2), but it has – analogously to the free particle – a width that “spreads” with increasing time  $\tau$ . We note that the term  $1 - 2\kappa_0 R_0$  occurring in these expressions becomes zero for vanishing charge (Schwarzschild case), while  $1 - 3\kappa_0 R_0$  becomes zero for  $q_0^2 = 3m_0^2/4$ , which is the thermodynamical stability boundary for charged black holes with fixed charge [4].

Let us pause to comment on the special case of Schwarzschild black hole. The charge terms are absent, and one is left with the wave packet

$$\psi(\alpha, \tau) = \mathcal{N} \exp \left( \frac{iA_0 \alpha}{8\pi \hbar} - \frac{im_0 \tau}{\hbar} - \frac{(\alpha - \kappa_0 \tau)^2}{2\hbar^2} \frac{\mathcal{G}'}{\mathcal{B}'} \right) \times (\text{phase factors}), \quad (4.7)$$

where

$$\mathcal{B}' = 1 + \frac{\tau^2 (\Delta A)^4}{256\pi \hbar^2 A_0^3}, \quad \mathcal{G}' = \frac{(\Delta A)^2}{64\pi^2}. \quad (4.8)$$

From the expression for  $\mathcal{B}'$  one can read off the time scale,  $\tau_*$ , of the spreading:

$$\tau_* = \frac{16\hbar \sqrt{\pi} A_0^{3/2}}{(\Delta A)^2}. \quad (4.9)$$

The minimal value for  $\Delta A$  should be of the order of the Planck length squared, i.e.,  $\Delta A \propto \hbar \approx 2.6 \times 10^{-66} \text{ cm}^2$ . This corresponds to a black hole as classical as possible. The corresponding dispersion time from (4.9) is

$$\tau_* = \frac{128\pi^2 R_0^3}{\hbar} \approx 10^{73} \left( \frac{m_0}{m_\odot} \right)^3 \text{ sec.} \quad (4.10)$$

Note that this is just of the order of the black hole evaporation time! The occurrence of this timescale is not very surprising, however, since the evaporation time gives also the timescale for the breakdown of the semiclassical approximation.

In the general case of nonvanishing charge, the dispersion time also depends on the charge uncertainty  $\Delta q$ . A direct comparison with (4.10) can be made if only the last term in (4.6a) – the term that only depends on  $\Delta A$  – is taken into account: For small charge, the dispersion time (4.10) increases according to

$$\tau_* \rightarrow \tau_* \left( 1 + \frac{3q_0^2}{R_0^2} \right). \quad (4.11)$$

This may be interpreted as being due to the fact that the Hawking temperature for charged holes is smaller than for uncharged ones. Taking into account also the  $\Delta q$ -terms in (4.6a), the dispersion time generally decreases. Consider now the extremal limit, in which the center of the packet is driven to  $A_0 = 4\pi q_0^2$ . As this center is now on the boundary of the integration domain in (4.4), the approximations made above in the evaluation of the integrals are no longer fully justified. However, as the integrand in (4.4) is a smooth function of  $A$  and  $q$  at and beyond the boundary, the expressions (4.5) and (4.6) should still remain qualitatively correct. Assuming this is the case, and recognizing that  $A_0 = 4\pi q_0^2$  implies  $\kappa_0 = 0$ , we see that the widths of the Gaussians in (4.5) are  $\tau$ -independent for large enough  $\tau$ . This is, again, not surprising, since the extremal hole has vanishing temperature and thus does not evaporate. For example, taking the large  $\tau$  limit and choosing the minimal widths  $\Delta A \propto \hbar$  and  $\Delta q \propto \sqrt{\hbar}$ , one finds that the  $\alpha$ -dependence of the wave packet is the Gaussian factor

$$\exp \left( -\frac{\alpha^2}{128\pi^2} \right). \quad (4.12)$$

This factor is independent of both  $\tau$  and  $\hbar$ .

It is evident that although our wave packet for  $A_0 = 4\pi q_0^2$  is peaked at vanishing  $\alpha$ , the packet has support also at  $\alpha \neq 0$ , and the packet does not seem to be qualitatively different from one for which  $A_0$  is close to but not exactly equal to  $4\pi q_0^2$ . In this approach, one would thus expect the extremal hole to have the usual Bekenstein-Hawking entropy.

## 5 Concluding remarks

We have discussed two complementary approaches to a Lorentzian Hamiltonian quantum theory that would encompass extremal black holes. In our classical Hamiltonian theory comprising only extremal black holes, the Hamiltonian does not

contain a horizon surface term, and it is difficult to see how quantization of such a theory could lead to a nonvanishing result for the black hole entropy. If, on the other hand, the extremal case is understood as a certain limit in a quantum theory that encompasses both the extremal and nonextremal cases, such a term will emerge, since the extremal case is only “of measure zero”. This becomes especially transparent from the analogy with the free particle, for which it would seem peculiar to separately quantize some lower-dimensional set of classical solutions. We emphasize again that the very notion of a black hole is a classical one, analogous to the notion of a classical trajectory in particle mechanics.

Although we discussed how the Bekenstein-Hawking entropy for an extremal black hole would in principle arise from the quantum theory of section IV, we did not attempt a direct computation of the entropy. One might expect that the entropy could be recovered as an “entanglement entropy” between the wave packets built in section IV. Examining this question lies, however, beyond the scope of this paper.

We have throughout the paper understood the extremal limit of nonextremal black holes so that the structure of the asymptotic infinity remains qualitatively unchanged. This meant that the limiting spacetime is indeed a black hole, and the horizon structure experiences a qualitative change in the limit. However, it is possible to take the extremal limit also in a way that preserves a bifurcate Killing horizon, at the cost of having the infinity undergo a qualitative change. The resulting spacetimes are of the Bertotti-Robinson type, in which the radius of the two-sphere is constant on the spacetime [22, 35–37]. These spacetimes are geodesically complete and cannot be interpreted as black holes, and the horizon is an acceleration horizon rather than an event horizon. Nevertheless, if one works under boundary conditions that do not require an infinity, for example by taking the “outer, end of the spacelike hypersurfaces to be at a finite “box”, it is possible to include the Bertotti-Robinson type spacetimes in a Hamiltonian formalism that handles the horizon bifurcation two-sphere as in Refs. [12–19]. As a result, one finds that the acceleration horizon is associated with an entropy equal to one quarter of the area [35, 36]. This is similar to the result for the acceleration horizon in Rindler spacetime [38].

We wish to add here some remarks on the situation in string theory (for a recent review, see [6]). The issue of black hole entropy is also there addressed in the framework of a semiclassical approximation: while the semiclassical approximation used in section IV can be found through an expansion with respect to the gravitational constant [19], the semiclassical approximation in string theory is accomplished through an expansion with respect to the string length. In addition, one can vary the string coupling constant at the level of the effective action and thereby connect the large-coupling regime of black holes with the small-coupling regime of D-branes in Minkowski spacetime. From a methodological point of view, string theory employs the “quantization before extremization”-method used in the wave packet construction of section IV. For the purpose of calculating the entropy, one looks in the quantum theory for states that are eigenstates of the Hamiltonian and some gauge generator with respective eigenvalues  $m$  (mass) and  $q$  (generalized charges); the so-called BPS states are then defined as the states in the “small representation, for which  $m = |q|$ , giving the condition of extremality. In principle

one should be able to perform superpositions and construct wave packets in the manner of section IV also in string theory, although this has, to our knowledge, not been done.

An interesting open problem is the possible occurrence of a naked singularity. The boundary conditions of section II clearly do not comprise a singular three-geometry. In the formulation of section IV, however, the wave packet (4.5) contains also parameter values that would correspond to such singular geometries. These geometries could be avoided if one imposed the boundary condition that the wave function in the momentum representation vanish for such values. Continuity would then also enforce that the wave function vanish at the boundary itself, i.e., for the extremal case. Consequently, quantum gravity would forbid the existence of extremal holes! Such a consequence would also follow in string theory. This is certainly an interesting aspect that should deserve further investigation.

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## Appendix: Extremal Reissner-Nordström-anti-de Sitter black holes

In this appendix we recall some relevant properties of the extremal Reissner-Nordström and Reissner-Nordström-anti-de Sitter metrics.

In the curvature coordinates  $(T, R)$ , the Reissner-Nordström-anti-de Sitter (RNAdS) metric reads

$$ds^2 = -F dT^2 + F^{-1} dR^2 + R^2 d\Omega^2, \quad (\text{A1 a})$$

where  $d\Omega^2$  is the metric on the unit two-sphere and

$$F := \frac{R^2}{\ell^2} + 1 - \frac{2M}{R} + \frac{Q^2}{R^2}. \quad (\text{A1 b})$$

We take  $M$  and  $Q$  to be real and  $\ell > 0$ . Together with the electromagnetic potential

$$A = \frac{Q}{R} dT, \quad (\text{A2})$$

the metric (A1) is a solution to the Einstein-Maxwell equations with the cosmological constant  $-3\ell^{-2}$  [22, 39]. The parameters  $M$  and  $Q$  are referred to respectively as the mass and the (electric) charge.

We are interested in the case where the quartic polynomial  $R^2F(R)$  has a positive double root,  $R = R_0$ , such that  $F$  is positive for  $R > R_0$ . The necessary and sufficient condition for this to happen is that  $Q \neq 0$  and  $M = M_{\text{crit}}(Q)$ , where

$$M_{\text{crit}}(Q) := \frac{\ell}{3\sqrt{6}} \left( \sqrt{1 + 12(Q/\ell)^2} + 2 \right) \left( \sqrt{1 + 12(Q/\ell)^2} - 1 \right)^{1/2}. \quad (\text{A3})$$

We then have  $R_0 = R_{\text{crit}}(Q)$ , where

$$R_{\text{crit}}(Q) := \frac{\ell}{\sqrt{6}} \left( \sqrt{1 + 12(Q/\ell)^2} - 1 \right)^{1/2}. \quad (\text{A4})$$

The metric is uniquely determined by the value of  $Q \neq 0$ , or alternatively by the value of  $R_0 > 0$  and the sign of  $Q$ . The region  $R_0 < R < \infty$  covers one exterior region of the extremal RNAdS hole. The Penrose diagram of the maximal analytic extension can be found in Refs. [40, 41].

The extremal Reissner-Nordström metric is obtained from the extremal RNAdS metric in the limit  $\ell \rightarrow \infty$ , in which case (A3) and (A4) reduce to  $M_{\text{crit}}(Q) = (Q) = |Q|$ . The region  $R_0 < R < \infty$  covers one exterior region, and the Penrose diagram of the maximal analytic extension can be found for example in Ref. [42].

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