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Future non-linear stability of the Einstein-Vlasov system with reflection Bianchi II and VI$_0$ symmetry

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Abstract. Assuming that the space-time is close to special solutions which will play the role of the ω-limit and that the maximal velocity of the particles is small, we have been able to show that for reflection symmetric Bianchi II and reflection symmetric Bianchi VI$_0$ spacetimes with collisionless matter the asymptotic behaviour at late times is close to the special case of dust. The key was a bootstrap argument.

Keywords: Vlasov, Bianchi II, Bianchi VI$_0$, stability, reflection symmetry, kinetic theory, cosmology
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MOTIVATION

A starting point to understand general models are the homogeneous models. There are a lot of results concerning this subject and in particular two books [1], [2] which are an excellent introduction and a great summary of many of the results obtained.

In general the focus has been on the fluid model since it appears (theoretically) relatively natural when dealing with isotropic universes and from observations we also know that the Universe is almost isotropic. However to have a deeper understanding of the dynamics one should go beyond the study of isotropic universes. General statements may vary then depending on the choice of the matter model. It is also important to note as is pointed out in [3] that a quasi-isotropic epoch is compatible with all Bianchi models and thus it is interesting to study the dynamics of all the different types.

We will deal with the *future asymptotics* of some homogeneous cosmological models within the so called Bianchi class A and the matter is described via an ensemble of free falling particles also called collisionless matter. For all the models treated here the fundamental questions are on a firm ground, i.e. future geodesic completeness has been shown for these models [4], [5].

Concerning the late time behaviour of the Universe one believes in the cosmic no hair conjecture [6], [7]. This conjecture states roughly speaking that all expanding cosmological models with a positive cosmological constant approach asymptotically the de Sitter solution (empty and flat universe with a positive cosmological constant).

In absence of a cosmological constant there are also different results concerning the future. For the Einstein-non-linear scalar field system future non-linear stability has been

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shown for a variety of scalar fields. For recent work on this subject we refer to [8], [9], [10] and [11]. The late-time behaviour of Bianchi spacetimes with a non-tilted fluid is well understood [12], [13]. In particular all non-tilted perfect fluid orthogonal Bianchi models except IX with a linear equation of state where $0 < \gamma < \frac{2}{3}$ are future asymptotic to the flat FL model [14]. Note the restriction on γ here. One cannot expect isotropization for most of the Bianchi models. However there are two important characteristics of the future asymptotics of some of the Bianchi models which have been the 'conjectures' of the present work:

1. The spacetimes considered tend to special (self-similar) solutions
2. For expanding models the dispersion of the velocities of the particles decays

This second 'conjecture' means that asymptotically there is a dust-like behaviour for collisionless matter which is the matter model we will use. This has been achieved already for locally rotationally symmetric (LRS) models in the cases of Bianchi I, II, III and IX [15],[16],[17],[18] and for reflection symmetric models in the case of Bianchi I [19]. These results have been obtained using dynamical systems theory.

The Einstein-Vlasov system remains a system of partial differential equations (PDE's) even if one assumes spatial homogeneity. The reason is that although the distribution function written in a suitable frame will not depend on the spatial point, the dependence with respect to the momenta remains (since the Vlasov equation is defined on the mass shell). However in the results mentioned a reduction to a system of ordinary differential equations was possible due to the additional symmetry assumptions. This is no longer possible if one drops some of these additional symmetries (see [20]-[21] for the reasons). Thus if one wants to generalize these results the theory of finite dimensional dynamical systems is not enough.

Most of the results obtained until now rely on the theory of dynamical systems. Thus one might be tempted to use techniques coming from the theory of infinite-dimensional dynamical systems. The first important difficulty would be to choose the suitable (weighted) norm. Another one is that important theorems which have been used for the finite-dimensional case cannot be used here. All this may work, but this is not the approach taken.

Here the main tool used was a bootstrap argument which is often used in non-linear PDE's. We will present results concerning the late-time behavior of some expanding Bianchi A spacetimes with collisionless matter where we have assumed small data. This assumption will be specified later, but roughly it means that the universe is close to the special self-similar solution mentioned earlier and that the velocity dispersion of the particles is bounded.

The results obtained are as follows. For reflection symmetric Bianchi II and reflection symmetric Bianchi VI_0 we have been able to show that their late-time behaviour remains the same if the LRS condition is dropped. We will show that these spacetimes, reflection symmetric Bianchi II and reflection symmetric Bianchi VI_0, will tend to solutions which are even more symmetric. In the case of Bianchi II we will show that it will become LRS, a Bianchi model whose isometry group of the spatial metric is four-dimensional. In this case there exists a one-dimensional isotropy group and one can show that a spacetime of Bianchi class A admits a four-dimensional isometry group, if and only if two structure constants are equal and if the corresponding metric components are equal as well. Bianchi VI_0 cannot be LRS, however it is compatible with an additional discrete symmetry (Appendix B.1 of [22]). The analysis of the asymptotics shows that the Bianchi VI_0 spacetimes tend to this special class. Note that for VI_0 there is no corresponding LRS/previous result.

All the results show that the dust model usually assumed in observational cosmology in the 'matter-dominated' Era is robust. Another way of saying the same is that asymptotically collisionless matter is well approximated by the dust system.

MAIN RESULTS

Using the 3+1 formulation our initial data are $(g_{ij}(t_0), k_{ij}(t_0), f(t_0))$, i.e. a Riemannian metric, a second fundamental form and the distribution function of the Vlasov equation, respectively, on a three-dimensional manifold $S(t_0)$. This is the initial data set at $t = t_0$ for the Einstein-Vlasov system. We can decompose the second fundamental form introducing σ_{ab} as the trace-free part: $k_{ab} = \sigma_{ab} - Hg_{ab}$, where $H = -\frac{1}{3}k$ is the Hubble parameter. We define $\Sigma_a^b = \frac{\sigma_a^b}{H}$, $\Sigma_+ = -\frac{1}{2}(\Sigma_2^2 + \Sigma_3^3)$ and $\Sigma_- = -\frac{1}{2\sqrt{3}}(\Sigma_2^2 - \Sigma_3^3)$. Now we will introduce formulas which are valid in the diagonal case. There (ijk) denotes a cyclic permutation of (123) and the Einstein summation convention is suspended for the next two formulas. Let us define:

$$N_i = \frac{\nu_i}{H}\sqrt{\frac{g_{ii}}{g_{jj}g_{kk}}}$$

where for Bianchi II the only non-trivial ν_i is $\nu_1 = 1$ and for Bianchi VI_0 $\nu_2 = 1$ and $\nu_3 = -1$. We have a number (different from zero) of particles at possibly different momenta and we define P as the supremum of the absolute value of these momenta at a given time t:

$$P(t) = \sup\{|p| = (g^{ab}p_a p_b)^{\frac{1}{2}} | f(t,p) \neq 0\}$$

The main results are the following:

Theorem 1 *Consider any C^∞ solution of the Einstein-Vlasov system with reflection and Bianchi II symmetry and with C^∞ initial data. Assume that $|\Sigma_+(t_0) - \frac{1}{8}|$, $|\Sigma_-(t_0)|$, $|N_1(t_0) - \frac{3}{4}|$ and $P(t_0)$ are sufficiently small. Then at late times the following estimates hold:*

$$\begin{aligned}
H(t) &= \frac{2}{3}t^{-1}(1 + O(t^{-\frac{1}{2}})) \\
\Sigma_+ - \frac{1}{8} &= O(t^{-\frac{1}{2}}) \\
\Sigma_- &= O(t^{-1}) \\
N_1 - \frac{3}{4} &= O(t^{-\frac{1}{2}}) \\
P(t) &= O(t^{-\frac{1}{2}})
\end{aligned}$$

Theorem 2 *Consider any C^∞ solution of the Einstein-Vlasov system with reflection Bianchi VI_0 symmetry and with C^∞ initial data. Assume that $|\Sigma_+(t_0)+\frac{1}{4}|$, $|\Sigma_-(t_0)|$, $|N_2(t_0)-\frac{3}{4}|$, $|N_3(t_0)+\frac{3}{4}|$ and $P(t_0)$ are sufficiently small. Then at late times the following estimates hold:*

$$\begin{aligned} H(t) &= \frac{2}{3}t^{-1}(1+O(t^{-\frac{1}{2}})) \\ \Sigma_+ + \frac{1}{4} &= O(t^{-\frac{1}{2}}) \\ \Sigma_- &= O(t^{-\frac{1}{2}}) \\ N_2 - \frac{3}{4} &= O(t^{-\frac{1}{2}}) \\ N_3 + \frac{3}{4} &= O(t^{-\frac{1}{2}}) \\ P(t) &= O(t^{-\frac{1}{2}}) \end{aligned}$$

We will present the proof in a different paper.

REFERENCES

1. M. P. Ryan, and L. C. Shepley, *Homogeneous relativistic cosmologies*, Princeton University Press, Princeton, 1975.
2. J. Wainwright, and G. Ellis, *Dynamical Systems in Cosmology*, Cambridge University Press, Cambridge, 1997.
3. G. F. R. Ellis, and J. Wainwright, "Cosmological observations," in *Dynamical Systems in Cosmology*, edited by J. Wainwright, and G. F. R. Ellis, Cambridge University Press, Cambridge, 1997.
4. A. D. Rendall, *Ann. Phys.* **233**, 82–96 (1994).
5. A. D. Rendall, *Math. Proc. Camb. Phil. Soc.* **118**, 511–526 (1995).
6. G. W. Gibbons, and S. W. Hawking, *Phys. Rev. D* **15**, 2738–2751 (1977).
7. S. W. Hawking, and I. G. Moss, *Phys. Lett.* **110B**, 35–38 (1982).
8. H. Ringström, *Invent. Math.* **173**, 123–208 (2008).
9. H. Ringström, *Commun. Math. Phys.* **290**, 155–218 (2009).
10. A. Alho, F. P. Mena, and J. A. Valiente Kroon, *arXiv:1006.3778v2 [gr-qc]* (2011).
11. J. L. Costa, A. Alho, and J. Natário, *arXiv:1107.0802v1 [gr-qc]* (2011).
12. J. Wainwright, "Bianchi cosmologies: non-tilted class A models," in *Dynamical Systems in Cosmology*, edited by J. Wainwright, and G. F. R. Ellis, Cambridge University Press, Cambridge, 1997.
13. C. G. Hewitt, and J. Wainwright, "Bianchi cosmologies: non-tilted class B models," in *Dynamical Systems in Cosmology*, edited by J. Wainwright, and G. F. R. Ellis, Cambridge University Press, Cambridge, 1997.
14. C. G. Hewitt, C. Uggla, and J. Wainwright, "Bianchi cosmologies: extending the scope," in *Dynamical Systems in Cosmology*, edited by J. Wainwright, and G. F. R. Ellis, Cambridge University Press, Cambridge, 1997.
15. A. D. Rendall, and K. P. Tod, *Class. Quant. Grav.* **16**, 1705–1726 (1998).
16. A. D. Rendall, and C. Uggla, *Class. Quant. Grav.* **17**, 4697–4713 (2000).
17. A. D. Rendall, *Gen. Rel. Grav.* **34**, 1277–1294 (2002).
18. S. Calogero, and J. M. Heinzle, *SIAM J. Appl. Dynam. Syst.* **9**, 1244–1262 (2010).
19. A. D. Rendall, *J. Math. Phys.* **37**, 438–451 (1996).
20. R. Maartens, and S. D. Maharaj, *J. Math. Phys.* **26**, 2869–2880 (1985).
21. R. Maartens, and S. D. Maharaj, *Gen. Rel. Grav.* **22**, 595–607 (1990).
22. S. Calogero, and J. M. Heinzle, *Physica* **D240**, 636–669 (2011).