HU-Mathematik-2012-10
HU-EP-12/33
AEI-2012-107

The Tetrahedron Zamolodchikov Algebra and the $AdS_5 \times S^5$ S-matrix

Vladimir Mitev [a], Matthias Staudacher [a,b], Zengo Tsuboi [a] ,

[a] *Institut für Mathematik und Institut für Physik, Humboldt-Universität zu Berlin IRIS Haus, Zum Großen Windkanal 6, 12489 Berlin, Germany*

[b] *Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut Am Mühlenberg 1, 14476 Potsdam, Germany*

mitev•math.hu-berlin.de
matthias•aei.mpg.de
ztsuboi•yahoo.co.jp

Abstract

The S-matrix of the $AdS_5 \times S^5$ string theory is a tensor product of two centrally extended $\mathrm{su}(2|2) \ltimes \mathbb{R}^2$ S-matrices, each of which is related to the R-matrix of the Hubbard model. The R-matrix of the Hubbard model was first found by Shastry, who ingeniously exploited the fact that, for zero coupling, the Hubbard model can be decomposed into two XX models. In this article, we review and clarify this construction from the AdS/CFT perspective and investigate the implications this has for the $AdS_5 \times S^5$ S-matrix.

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1 Introduction and Conclusion

The integrability of a 1-dimensional quantum system (or a 2-dimensional classical system) is directly linked to the Yang-Baxter equation for the R-matrix of the model. Currently many of the known R-matrices are derived from symmetry arguments, using quantum affine Lie (super)algebras. The most famous exception to this appears to be the R-matrix of the Hubbard model found by Shastry [1]. Interestingly, this R-matrix was derived independently in two ways. First, Shastry observed that the Hubbard model at zero coupling decomposes into two non-interacting free fermions[1] and then made an Ansatz for the full R-matrix using a linear combination of (twisted) free fermion R-matrices. Secondly, in the context of the AdS/CFT correspondence, Beisert noted in [2] that the S-matrix of the gauge theory spin chain model decomposes into a tensor product

[1]Also known as XX models.

$S_1 \otimes S_2$ of two matrices, each one of which is completely fixed by $\mathrm{su}(2|2) \ltimes \mathbb{R}^2$ symmetry, up to an overall phase which was found by other means [3]. Amazingly enough, each of the S_i's coincides, see [2, 4], with the Hubbard R-matrix by a similarity transformation under a certain unitarity condition. For the past several years, this factor S_i of the full S-matrix, which for simplicity we will call for the rest of this paper somewhat sloppily "the AdS/CFT S-matrix", was studied intensively in relation to the superalgebra $\mathrm{su}(2|2) \ltimes \mathbb{R}^2$ and its Yangian (see [5, 6, 7] and references therein). Interestingly, prior to these discoveries, the Hubbard model had already appeared, see [8], in the study of the AdS/CFT correspondence, though in a seemingly quite different context.

Originally, it was not clear that the R-matrix proposed by Shastry was a solution to the Yang-Baxter equation. This issue was clarified by Shiroishi and Wadati in [9], who not only showed that Shastry's R-matrix satisfies the Yang-Baxter equation, but later on also generalized the construction, based on a formalism due to Korepanov [10, 11] that he developed for Zamolodchikov's Tetrahedron algebra [12]. We find that their generalized R-matrix exactly coincides with the AdS/CFT S-matrix of [13], up to some similarity transformations and a condition on the parameters, without assuming the unitarity condition, as [4] implicitly do. Furthermore, we argue that this generalized R-matrix is for generic values of its parameters *not* equivalent to the AdS/CFT S-matrix. Instead, it contains it as a special case[2].

In the present paper, we investigate the quantum symmetry of the Hubbard model and relate it to $\mathrm{su}(2|2) \ltimes \mathbb{R}^2$ on the AdS/CFT side. In the process, the four-layer structure of the AdS/CFT S-matrix is made apparent. While the broad lines of the relationship between the Hubbard model R-matrix and the AdS/CFT S-matrix are clear, several points either remain obscure and/or indicate possible new directions of investigation:

- Shastry's R-matrix has by construction an obvious two layer structure that is not at all apparent in the AdS/CFT formulation. In the present article, we strive to remedy this and to rewrite the S_i matrices in a fashion to expose their two layer properties as completely as possible.

[2]When we compare the S-matrix in [2] with the R-matrix in [14], we have to correct a few minor typos in page 33 in [2]: U in the second and the third equations in eq. (4.7) has to be removed; A_{12} in the last equation in eq. (4.10) should be D_{12}.

- As mentioned, Shastry's construction was later extended by Shiroishi and Wadati based on Korepanov's formalism of the tetrahedron Zamolodchikov algebra. In the process, they obtained a generalized R-matrix, referred to here as R. We suspect that for generic values of its parameters R is not equivalent to the AdS/CFT S-matrix. Rather, it seems that R is generically less symmetric and thus strictly contains it as a special case. One of our goals is to carefully investigate the symmetries of R and to find the precise conditions for its reduction to the S-matrix.

- Each of the two layers of the Hubbard model R-matrix enjoys an quantum affine sl(2) symmetry. As the layers are glued together in Shastry's construction, half of this symmetry is lost, but the remaining one can then be identified with a part of the su(2|2) $\ltimes \mathbb{R}^2$ symmetry of the S-matrix. We carefully spell out this identification and in the process investigate and clarify the origins and importance of the R-matrix symmetries.

- An important, so far overlooked direction of research opened by the introduction of the tetrahedron Zamolodchikov algebra is the investigation of the three-dimensional integrable structures in the context of the AdS/CFT correspondence. Specifically, one finds a natural object in Shastry's construction, referred to as $\mathbb{S}$ in this text, which obeys the tetrahedron Zamolodchikov equation, a three-dimensional generalization of the Yang-Baxter equation. While outside the scope of the present work, we lay here the ground work for further research and hope that our results give a cue on how to reveal a hidden three-dimensional integrable structure in the context of the AdS/CFT correspondence.

Having thus spelled out our scope and goals, we are ready to begin with the main body of our investigation.

2 The free fermion model

2.1 Preliminaries

We start our journey by writing down the free fermion model using oscillators and by describing the tetrahedron Zamolodchikov algebra. For this purpose, we define the fermionic creation operators $\mathbf{c}_j^\dagger$ as well as the annihilation operators $\mathbf{c}_j$, where $j \in \mathbb{Z}$ labels the lattice site. They obey the canonical anti-commutation relations

$$\left\{\mathbf{c}_j \, , \, \mathbf{c}_k^\dagger\right\} = \delta_{jk}, \qquad j, k \in \mathbb{Z}. \tag{1}$$

It is useful to define the (bosonic) compound operators $\mathbf{n}_j := \mathbf{c}_j^\dagger \mathbf{c}_j$ and $\mathbf{m}_j := \mathbf{c}_j \mathbf{c}_j^\dagger$. The R-matrix of the model is a special case of the XXZ one and has the shape

$$\mathsf{R}^{\mathsf{f}}_{jk}(A) = -a\mathbf{n}_j\mathbf{n}_k - ib\mathbf{n}_j\mathbf{m}_k - ic\mathbf{m}_j\mathbf{n}_k + d\mathbf{m}_j\mathbf{m}_k + \mathbf{c}_j^\dagger\mathbf{c}_k + \mathbf{c}_k^\dagger\mathbf{c}_j, \tag{2}$$

where the parameters satisfy the *free fermion condition*, i.e. $ad - bc = 1$, meaning that

$$A := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C}). \tag{3}$$

Thus, in a sense the R-matrix of the free fermion model has a $\mathrm{SL}(2, \mathbb{C})$ spectral parameter, see [15]. The Hamiltonian density of the spin chain is then obtained by choosing a curve in $\mathrm{SL}(2, \mathbb{C})$, i.e. making A depend on a parameter $u \in \mathbb{C}$ such that for $u = u_0$ the coefficients become $a = d = 1$ and $b = c = 0$, implying the relation $\mathsf{R}^{\mathsf{f}}_{jk}(A(u_0)) = \mathsf{P}_{jk}$, where the graded permutation is defined by

$$\mathsf{P}_{jk} = -\mathbf{n}_j\mathbf{n}_k + \mathbf{m}_j\mathbf{m}_k + \mathbf{c}_j^\dagger\mathbf{c}_k + \mathbf{c}_k^\dagger\mathbf{c}_j. \tag{4}$$

One then constructs the transfer matrix as a supertrace of the monodromy matrix over an auxiliary space, i.e. $\tau(u) = \mathrm{str}_a \left(\mathsf{R}^{\mathsf{f}}_{aN}(u) \cdots \mathsf{R}^{\mathsf{f}}_{a1}(u)\right)$ and derives the Hamiltonian $\mathsf{H}^{\mathsf{f}} = \tau(u_0)^{-1} \frac{d}{du}\tau(u)_{|u=u_0}$. One computes that the nearest neighbor Hamiltonian is

$$\mathsf{H}^{\mathsf{f}}_{i,i+1} = \frac{d}{du}\check{\mathsf{R}}^{\mathsf{f}}_{i,i+1}(A(u))_{|u=u_0}. \tag{5}$$

Here, we introduce the notation $\check{\mathsf{R}}^{\mathsf{f}}_{jk} = \mathsf{P}_{jk}\mathsf{R}^{\mathsf{f}}_{jk}$. The simplest example for such a curve is obtained by setting the parameters to $a = d = \cos u$ and $b = c = i \sin u$ leading to the purely hopping XX model Hamiltonian density $\mathsf{H}^{\rm XX}_{i,i+1} = \mathbf{c}_i^\dagger \mathbf{c}_{i+1} + \mathbf{c}_{i+1}^\dagger \mathbf{c}_i$. Integrability is ensured by the Yang-Baxter equation $\mathsf{R}^{\rm f}_{12}(A)\mathsf{R}^{\rm f}_{13}(B)\mathsf{R}^{\rm f}_{23}(C) = \mathsf{R}^{\rm f}_{23}(C)\mathsf{R}^{\rm f}_{13}(B)\mathsf{R}^{\rm f}_{12}(A)$ for the operator (2), which is obeyed if the SL$(2,\mathbb{C})$ matrices fulfill the product relation $B = CA$, see [15]. In order to have this conditions be automatic, we define a new operator R^0 as

$$\mathsf{R}^0_{jk}(A_j, A_k) := \mathsf{R}^{\rm f}_{jk}(A_k A_j^{-1}), \tag{6}$$

so that now $\mathsf{R}^0_{12}\mathsf{R}^0_{13}\mathsf{R}^0_{23} = \mathsf{R}^0_{23}\mathsf{R}^0_{13}\mathsf{R}^0_{12}$ for any three SL$(2,\mathbb{C})$ matrices A_i. The above relation (6) represents a SL$(2,\mathbb{C})$ analogue of the difference property on the spectral parameters. Furthermore, one can easily prove the property $\mathsf{P}_{23}\mathsf{R}^0_{12}(A_1, A_2)\mathsf{P}_{23} = \mathsf{R}^0_{13}(A_1, A_2)$. Note that the right hand side is not $\mathsf{R}^0_{13}(A_1, A_3)$. As a final remark, we have the inversion relation

$$\mathsf{R}^{\rm f}_{jk}(A)\mathsf{R}^{\rm f}_{kj}(A^{-1}) = ad\ . \tag{7}$$

2.2 The quantum affine symmetry

In this section, we want to investigate the symmetries of the R-matrix of (2) and see the extension to which it is constrained by symmetry. A main object of concern is the quantum group $\mathrm{U}_q(\mathrm{sl}(2))$ at a root of unity i of the deformation parameter q (see for example, chapter 7 of [16]). For brevity of notation, we let $\mathfrak{A}$ denote the quantum group $\mathrm{U}_q(\mathrm{sl}(2))$, while $\hat{\mathfrak{A}}$ stands for its affine counterpart. We set the deformation parameter q of the quantum group to i. The quantum group $\hat{\mathfrak{A}}$ is defined as generated by the operators k_r, e_r and f_r for $r = 0, 1$ that for $q = i$ obey the relations

$$\mathsf{k}_r\mathsf{k}_s = \mathsf{k}_s\mathsf{k}_r, \quad \mathsf{k}_r\mathsf{e}_s = -\mathsf{e}_s\mathsf{k}_r, \quad \mathsf{k}_r\mathsf{f}_s = -\mathsf{f}_s\mathsf{k}_r, \quad [\mathsf{e}_r\ ,\ \mathsf{f}_s] = \delta_{rs}\frac{\mathsf{k}_r - \mathsf{k}_r^{-1}}{2i}, \tag{8}$$

together with the Serre relations that we have omitted. Furthermore, we introduce the operators h_r by the relation $\mathsf{k}_r = q^{\mathsf{h}_r}$, which implies

$$[\mathsf{h}_r\ ,\ \mathsf{e}_s] = \mathbf{A}_{rs}\mathsf{e}_s, \quad [\mathsf{h}_r\ ,\ \mathsf{f}_s] = -\mathbf{A}_{rs}\mathsf{f}_s, \tag{9}$$

where $\mathbf{A} = (\mathbf{A}_{rs})_{0\le r,s\le 1}$ is the Cartan matrix of affine sl(2), namely

$$\mathbf{A} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}. \tag{10}$$

The Hopf algebra $\hat{\mathfrak{A}}$ for $q = i$ has a family of two-dimensional representations (akin to nilpotent representations), denoted by $\widehat{V}_{\mu;x,y}$, which we write using the fermionic oscillators of (1) as

$$\begin{aligned} &\mathsf{k}_0 = \lambda^{-1}(\mathbf{n}-\mathbf{m}), \quad && \mathsf{e}_0 = -\varphi x^{-1}\,\mathbf{c}^\dagger, \quad && \mathsf{f}_0 = \varphi x\,\mathbf{c}, \quad && \mathsf{h}_0 = \mu - \mathbf{m} + \mathbf{n}, \\ &\mathsf{k}_1 = \lambda(\mathbf{n}-\mathbf{m}), && \mathsf{e}_1 = \varphi y^{-1}\,\mathbf{c}, && \mathsf{f}_1 = -\varphi y\,\mathbf{c}^\dagger, && \mathsf{h}_1 = -\mu + \mathbf{m} - \mathbf{n}, \end{aligned} \tag{11}$$

where λ, μ, x and y are complex parameters, we have introduced the element φ through the equation $\varphi^2 = \frac{\lambda-\lambda^{-1}}{2i}$ and the lattice site index is omitted. Since $\mathsf{k}_r = q^{\mathsf{h}_r}$, the parameter λ is fixed by the equation $\lambda = i^{-1-\mu}$.

At this point, a couple of remarks are in order. First, the structure of the representations only depends on the product of x and y, since $\widehat{V}_{\mu;x,y}$ is isomorphic to $\widehat{V}_{\mu;xy,1}$ thanks to the similarity transformation $\mathsf{J} \mapsto \mathsf{g}\mathsf{J}\mathsf{g}^{-1}$, $\mathsf{g} = \mathbf{n} + y\mathbf{m}$, where J are the generators in (11). We choose to keep the label y nonetheless, because it introduces a twisting by an inner automorphism that will be useful later on. Note further that the representation $\widehat{V}_{\mu;xy,1}$ is based on a homogeneous gradation of $\hat{\mathfrak{g}}$ and that the squares of some generators are central elements, specifically we have non-trivial central elements $\mathsf{k}_0^2 = \lambda^{-2}, \mathsf{k}_1^2 = \lambda^2$ and trivial ones $\mathsf{e}_r^2 = \mathsf{f}_r^2 = 0$.

Second, the $\hat{\mathfrak{A}}$ modules $\widehat{V}_{\mu;x,y}$ are also representations of the Hopf subalgebra $\mathfrak{A} \subset \hat{\mathfrak{A}}$ that is generated by the operators k_0, e_0, f_0 and h_0. We shall refer to them as V_μ since they do not depend on the parameters x or y. Lastly, there is an ambiguity in the sign of φ due to the square root in its definition. This does not lead to different representations however, because one can change the sign of φ via the inner automorphism $\mathsf{J} \mapsto \mathsf{k}_0\mathsf{J}\mathsf{k}_0^{-1}$.

In defining the coproduct, we introduce two additional operators, the first being the grading operator $\mathsf{F} := (-1)^{\mathbf{n}} = \mathbf{m} - \mathbf{n}$, while the second one is an additional central element Z. At first we define the coproduct for the diagonal elements

$$\Delta(\mathsf{k}_r) = \mathsf{k}_r \otimes \mathsf{k}_r, \quad \Delta(\mathsf{Z}) = \mathsf{Z} \otimes \mathsf{Z}, \quad \Delta(\mathsf{F}) = \mathsf{F} \otimes \mathsf{F}, \quad \Delta(\mathsf{h}_r) = \mathsf{h}_r \otimes \mathbb{1} + \mathbb{1} \otimes \mathsf{h}_r, \tag{12}$$

while for the non-diagonal one we set

$$\begin{aligned} &\Delta(\mathsf{e}_0) = \mathsf{e}_0 \otimes \mathsf{Z} + \mathsf{k}_0\mathsf{F} \otimes \mathsf{e}_0, \qquad && \Delta(\mathsf{f}_0) = \mathsf{f}_0 \otimes \mathsf{k}_0^{-1}\mathsf{Z}^{-1} + \mathsf{F} \otimes \mathsf{f}_0, \\ &\Delta(\mathsf{e}_1) = \mathsf{e}_1 \otimes \mathbb{1} + \mathsf{Z}\mathsf{k}_1\mathsf{F} \otimes \mathsf{e}_1, && \Delta(\mathsf{f}_1) = \mathsf{f}_1 \otimes \mathsf{k}_1^{-1} + \mathsf{Z}^{-1}\mathsf{F} \otimes \mathsf{f}_1. \end{aligned} \tag{13}$$

The presence of the grading operator F can be explained by the fermionic nature of the operators[3] e_r and f_r. Were we to represent the $\hat{\mathfrak{A}}$ generators with matrices instead of oscillators, we would not need F. Note that the tensor product is graded, so that for instance $(\mathbb{1} \otimes \mathsf{f}_0)(\mathsf{e}_0 \otimes \mathbb{1}) = -\mathsf{e}_0 \otimes \mathsf{f}_0$. The central element Z is used to twist the coproduct[4] and we impose on it the requirement that its eigenvalue z be the same in every space. This twisting will become necessary in section 4 in order to connect the quantum symmetry to the S-matrix of (2).

We now look for an intertwiner r^0_{12} acting on the space $\widehat{V}_{\mu_1;x_1,y_1} \otimes \widehat{V}_{\mu_2;x_2,y_2}$ subject to the condition

$$\Delta'(\mathsf{J})\mathsf{r}^0_{12} = \mathsf{r}^0_{12}\Delta(\mathsf{J}) \qquad \text{for} \quad \forall \mathsf{J} \in \hat{\mathfrak{A}} \,, \tag{14}$$

where[5] $\Delta' := \sigma \circ \Delta$ is the opposite coproduct to the one of (12) and (13). For example, one has $\Delta'(\mathsf{e}_0) = \mathsf{Z} \otimes \mathsf{e}_0 + \mathsf{e}_0 \otimes \mathsf{k}_0\mathsf{F}$. The solution to the symmetry constraints (14) is unique up to normalization and can be written explicitly as

$$\begin{aligned} \mathsf{r}^0_{12} &= (x_1y_1\lambda_1\lambda_2 - x_2y_2)\mathbf{n}_1\mathbf{n}_2 + z^{-1}(x_2y_2\lambda_1 - x_1y_1\lambda_2)\mathbf{n}_1\mathbf{m}_2 + z(x_2y_2\lambda_2 - x_1y_1\lambda_1)\mathbf{m}_1\mathbf{n}_2 \\ &+ (x_1y_1 - x_2y_2\lambda_1\lambda_2)\mathbf{m}_1\mathbf{m}_2 - \sqrt{(\lambda_1 - \lambda_1^{-1})(\lambda_2 - \lambda_2^{-1})}\left(x_1y_2\lambda_2\mathbf{c}_2^\dagger\mathbf{c}_1 + x_2y_1\lambda_1\mathbf{c}_1^\dagger\mathbf{c}_2\right) . \end{aligned} \tag{15}$$

The operator r^0_{12} obeys the Yang-Baxter equation automatically since the tensor product of three $\hat{\mathfrak{A}}$ representations $\widehat{V}_{\mu_r;x_r,y_r}$ is generically irreducible. This means that there can be only one $\hat{\mathfrak{A}}$-invariant intertwiner and since both $\mathsf{r}^0_{12}\mathsf{r}^0_{13}\mathsf{r}^0_{23}$ and $\mathsf{r}^0_{23}\mathsf{r}^0_{13}\mathsf{r}^0_{12}$ are by construction invariant intertwiners, they must be equal up to a multiplicative constant that is easily seen to be one. One remarks that the representation labels μ_r only enter

[3] It can also be explained by first defining the coproduct in the usual way and then performing a Jordan-Wigner transformation.

[4] Let us consider an automorphism of $\hat{\mathfrak{A}}$ defined by $[\mathsf{B}_i, \mathsf{e}_j] = \delta_{ij}\mathsf{e}_j$, $[\mathsf{B}_i, \mathsf{f}_j] = -\delta_{ij}\mathsf{f}_j$, $[\mathsf{B}_i, \mathsf{h}_j] = 0$, $i, j = 0, 1$. Then we can remove the central element Z from the co-product (13) by the following map

$$\Delta(\mathsf{X}) \mapsto (1 \otimes \mathsf{Z})^{-\mathsf{B}_0 \otimes 1}(\mathsf{Z} \otimes 1)^{-1 \otimes \mathsf{B}_1}\Delta(\mathsf{X})(\mathsf{Z} \otimes 1)^{1 \otimes \mathsf{B}_1}(1 \otimes \mathsf{Z})^{\mathsf{B}_0 \otimes 1} \quad \text{for} \quad \mathsf{X} \in \hat{\mathfrak{g}}.$$

Moreover this map can be realized as a composition of a change of the basis and a Reshetikhin-twist [17] for the R-matrix (see the discussion in section 6 of [18]). This type of map may be useful to connect the AdS/CFT S-matrix with Shiroishi-Wadati's generalized Hubbard R-matrix. In fact, as we will see later, the central element is not free but effectively $\pm i$ for the AdS/CFT S-matrix.

[5] Here $\sigma(X \otimes Y) := Y \otimes X$ for any elements X, Y of the algebra.

(15) through the λ_r, but they will play a more explicit role later on. Furthermore, the y_i dependence of (15) can be removed by a similarity transformation

$$\mathsf{r}^0_{12} \mapsto \mathsf{g}_1\mathsf{g}_2\mathsf{r}^0_{12}\mathsf{g}_2^{-1}\mathsf{g}_1^{-1}, \qquad \mathsf{g}_i = \mathbf{n}_i + y_i\mathbf{m}_i, \quad i = 1,2. \tag{16}$$

and the rescaling of the spectral parameter $x_i \mapsto x_i y_i^{-1}$.

If we now define the operators $\mathsf{G}_s := \mathbf{m}_s + \sqrt{\frac{y_s}{x_s}\lambda_s}\mathbf{n}_s$, then it turns out that r^0_{12} can be identified with the free fermion operator R^0 of (6) in the following way

$$\mathsf{R}^0_{12} = -\frac{1}{\sqrt{(\lambda_1-\lambda_1^{-1})(\lambda_2-\lambda_2^{-1})x_1y_1x_2y_2\lambda_1\lambda_2}}\mathsf{G}_1^{-1}\mathsf{G}_2^{-1}\mathsf{r}^0_{12}\mathsf{G}_1\mathsf{G}_2\ , \tag{17}$$

provided that we relate the parameters as

$$a_r = \sqrt{\frac{(x_ry_r)^{-1}\lambda_r}{\lambda_r-\lambda_r^{-1}}}, \quad b_r = \frac{1}{iz}\sqrt{\frac{x_ry_r\lambda_r^{-1}}{\lambda_r-\lambda_r^{-1}}}, \quad c_r = iz\sqrt{\frac{(x_ry_r)^{-1}\lambda_r^{-1}}{\lambda_r-\lambda_r^{-1}}}, \quad d_r = \sqrt{\frac{x_ry_r\lambda_r}{\lambda_r-\lambda_r^{-1}}}. \tag{18}$$

It is here that the need for the introduction of the central operator Z becomes apparent. We remind that z is the eigenvalue of the Z and that it is a global number, i.e. it does not depend on the lattice index. We observe that with the identification (18) we get the relation

$$a_rb_r = -z^{-2}c_rd_r \quad \text{for all} \quad r. \tag{19}$$

This means that z is necessary in order to cover the full space of parameters $\mathrm{SL}(2,\mathbb{C})$ of the operator R^0_{12}. Lastly, as we have noted before, the role of the labels y_r is to twist the J_1 generators relative to the J_0 ones. When such a twist is not necessary, we will simply set $y_r = x_r$, which corresponds to the principal gradation of $\hat{\mathfrak{g}}$. For convenience, in the next section we summarize all the relations between the free fermion variables and the quantum group variables in table 1.

2.3 The tetrahedron Zamolodchikov algebra

In the previous sections, we described how the XX R-matrix is uniquely fixed by the requirement (14) of invariance under $\hat{\mathfrak{A}}$. It will however turn out to be useful to relax that condition. Let us consider the quantum group $\mathfrak{A} \subset \hat{\mathfrak{A}}$ that is generated by the elements k_0, e_0, f_0 and h. We denote a representation given by restricting the generators in (11) to the ones for $\mathfrak{A}$ as[6] V_μ. Although (11) depends on the spectral parameter x, this representation is essentially independent of it since it can be removed by a similarity transformation.

If we now consider (14) only for the quantum group $\mathfrak{A} \subset \hat{\mathfrak{A}}$, then the space of solutions to the equation

$$\Delta'(\mathsf{J})\mathsf{r}_{12} = \mathsf{r}_{12}\Delta(\mathsf{J}) \quad \forall \mathsf{J} \in \mathfrak{A} \tag{20}$$

is two-dimensional, since the tensor product of two $\mathfrak{A}$ modules V_{μ_r} decomposes into two irreducible pieces[7] as

$$V_{\mu_1} \otimes V_{\mu_2} \cong V_{\mu_1+\mu_2+1} \oplus V_{\mu_1+\mu_2-1}. \tag{21}$$

A basis for this space is given by the set $\{\mathsf{r}^0_{12}, \mathsf{r}^1_{12}\}$. The first of these operators is the operator of the previous section r^0_{12}, which is the solution (15) to the affine invariance equations (14) for the representations $\widehat{V}_{\mu_1;x_1,x_1} \otimes \widehat{V}_{\mu_2;x_2,x_2}$. The second one, we call r^1_{12} and it is also a solution of (14), but for the representations $\widehat{V}_{\mu_1;x_1,x_1} \otimes \widehat{V}_{\mu_2;x_2,-x_2}$, i.e. the action on the second factor has been twisted by a minus sign[8].

A rescaling of both of these operators by the same factor in (17), followed by a similarity transformation as in (17) with $\mathsf{G}_r = \mathbf{m}_r + \sqrt{\lambda_r}\mathbf{n}_r$ and a change of variables as in (18) but with $y_r = x_r$ leads to the basis of operators $\{R^0, R^1\}$ with

$$\mathsf{R}^0_{12}(A_1, A_2) := \mathsf{R}^{\mathsf{f}}_{12}(A_2 A_1^{-1}), \quad \mathsf{R}^1_{12}(A_1, A_2) := \mathsf{R}^{\mathsf{f}}_{12}(A_2\sigma_3 A_1^{-1}\sigma_3)(\mathbf{n}_2 - \mathbf{m}_2), \tag{22}$$

where $\sigma_3 = \mathrm{diag}(1, -1)$ and the A_i are elements of $\mathrm{SL}(2, \mathbb{C})$. We want to use these two operators to describe the space of $\mathfrak{A}$-invariant intertwiners on the the tensor product of three modules V_{μ_r}, which decomposes generically as

$$V_{\mu_1} \otimes V_{\mu_2} \otimes V_{\mu_3} \cong V_{\mu_1+\mu_2+\mu_3+2} \oplus 2V_{\mu_1+\mu_2+\mu_3} \oplus V_{\mu_1+\mu_2+\mu_3-2}. \tag{23}$$

[6] This is a two-dimensional irreducible $\mathfrak{A}$ module generated by the highest weight vector v defined by $\mathsf{e}_0 v = 0$, $\mathsf{h}v = (\mu+1)v$, $\mathsf{k}_0 v = \lambda^{-1}v$.

[7] Unless $\mu_1 + \mu_2 = 0 \bmod 2$.

[8] Let us consider an automorphism $\mathsf{X} \mapsto e^{i\pi\mathsf{B}_1}\mathsf{X}e^{-i\pi\mathsf{B}_1}$ for any $\mathsf{X} \in \mathfrak{A}$. This induces a map $\widehat{V}_{\mu;x,y} \mapsto \widehat{V}_{\mu;x,-y}$ on the representation. Thus the above twisting can be interpreted as a consequence of this map.

Thanks to the invariance properties of the R^0 and R^1 it is clear that the 16 operators $\mathsf{R}_{12}^{\alpha}\mathsf{R}_{13}^{\beta}\mathsf{R}_{23}^{\gamma}$ and $\mathsf{R}_{23}^{\alpha}\mathsf{R}_{13}^{\beta}\mathsf{R}_{12}^{\gamma}$ for $\alpha, \beta, \gamma \in \{0, 1\}$ are all $\mathfrak{A}$-invariant. However, (23) tells us that there at most six of them can be linearly independent, since the dimension of the space of such invariant intertwiners is by Schur's lemma equal to $1^2 + 2^2 + 1^2 = 6$. The relationships between the various intertwiners is described by two equations: the tetrahedron Zamolodchikov algebra and the linear dependence equations. In order to write them down, it turns out to be useful to perform a change of basis to "light-cone" operators $\mathsf{R}^{\pm} := \frac{1}{2}(\mathsf{R}^0 \pm \mathsf{R}^1)$, explicitly written using oscillators as

$$\begin{aligned}
\mathsf{R}_{jk}^{+}(A_j, A_k) &= (a_k \mathbf{n}_j + i c_k \mathbf{m}_j)(-d_j \mathbf{n}_k + i b_j \mathbf{m}_k) + \mathbf{c}_j^{\dagger}\mathbf{c}_k \\
\mathsf{R}_{jk}^{-}(A_j, A_k) &= (b_k \mathbf{n}_j + i d_k \mathbf{m}_j)(c_j \mathbf{n}_k - i a_j \mathbf{m}_k) + \mathbf{c}_k^{\dagger}\mathbf{c}_j \ .
\end{aligned} \tag{24}$$

The tetrahedron Zamolodchikov algebra in this basis is then defined as the set of relations

$$\mathsf{R}_{23}^{\alpha}\mathsf{R}_{13}^{\beta}\mathsf{R}_{12}^{\gamma} = \sum_{\bar\alpha,\bar\beta,\bar\gamma=\pm} \mathbb{S}_{\bar\alpha\bar\beta\bar\gamma}^{\alpha\beta\gamma}\, \mathsf{R}_{12}^{\bar\gamma}\mathsf{R}_{13}^{\bar\beta}\mathsf{R}_{23}^{\bar\alpha} \ , \tag{25}$$

where the coefficients $\mathbb{S}_{\bar\alpha\bar\beta\bar\gamma}^{\alpha\beta\gamma}$ are given by (cf. [14])

$$\begin{aligned}
&\mathbb{S}_{\bar\alpha\,\bar\beta\,\bar\gamma}^{\alpha\,\beta\,\gamma} = \delta_{\bar\alpha\,\bar\beta\,\bar\gamma}^{\alpha\,\beta\,\gamma} \qquad \text{for } (\alpha,\beta,\gamma) \notin \{(+,-,+),(-,+,-)\} \\
&\mathbb{S}_{\bar\alpha\,\bar\beta\,\bar\gamma}^{+-+} = \delta_{\bar\alpha\,\bar\beta\,\bar\gamma}^{-+-} + \mathbb{F}_1^{23}\left(\frac{b_3d_3}{b_2d_2}\delta_{\bar\alpha\,\bar\beta\,\bar\gamma}^{++-} - \frac{a_3c_3}{a_2c_2}\delta_{\bar\alpha\,\bar\beta\,\bar\gamma}^{--+}\right) - \mathbb{F}_3^{21}\left(\frac{b_1d_1}{b_2d_2}\delta_{\bar\alpha\,\bar\beta\,\bar\gamma}^{+--} - \frac{a_1c_1}{a_2c_2}\delta_{\bar\alpha\,\bar\beta\,\bar\gamma}^{-++}\right), \\
&\mathbb{S}_{\bar\alpha\,\bar\beta\,\bar\gamma}^{-+-} = \delta_{\bar\alpha\,\bar\beta\,\bar\gamma}^{+-+} - \mathbb{F}_1^{23}\left(\frac{b_3d_3}{b_2d_2}\delta_{\bar\alpha\,\bar\beta\,\bar\gamma}^{++-} - \frac{a_3c_3}{a_2c_2}\delta_{\bar\alpha\,\bar\beta\,\bar\gamma}^{--+}\right) + \mathbb{F}_3^{21}\left(\frac{b_1d_1}{b_2d_2}\delta_{\bar\alpha\,\bar\beta\,\bar\gamma}^{+--} - \frac{a_1c_1}{a_2c_2}\delta_{\bar\alpha\,\bar\beta\,\bar\gamma}^{-++}\right).
\end{aligned} \tag{26}$$

Here, we use the abbreviations $\delta_{\bar\alpha\,\bar\beta\,\bar\gamma}^{\alpha\,\beta\,\gamma} := \delta_{\bar\alpha}^{\alpha}\delta_{\bar\beta}^{\beta}\delta_{\bar\gamma}^{\gamma}$ and $\mathbb{F}_i^{jk} := \frac{a_id_i - a_jd_j}{a_id_i - a_kd_k}$. Formally, the matrix elements $\mathbb{S}_{\bar\alpha\,\bar\beta\,\bar\gamma}^{\alpha\,\beta\,\gamma}$ define an endomorphism $\mathbb{S}$ of the space of $\mathfrak{A}$-invariant operators. As follows from (23), the eight operators $\mathsf{R}_{12}^{\alpha}\mathsf{R}_{13}^{\beta}\mathsf{R}_{23}^{\gamma}$ are not linearly independent, but span a six-dimensional space. One finds that they are subject to the following two linear dependence relations

$$\sum_{\alpha,\beta,\gamma=\pm} \mathbb{T}_{\alpha\beta\gamma}^{(i)}\mathsf{R}_{12}^{\alpha}\mathsf{R}_{13}^{\beta}\mathsf{R}_{23}^{\gamma} = 0 \qquad \text{for } i = 1, 2. \tag{27}$$

Here, the coefficients are explicitly

$$\mathbb{T}_{+++}^{(1)} = \frac{a_1b_3}{b_1a_3}, \quad \mathbb{T}_{--+}^{(1)} = \frac{b_1c_2}{a_1d_2}, \quad \mathbb{T}_{+--}^{(1)} = \frac{d_2a_3}{c_2b_3}, \quad \mathbb{T}_{+-+}^{(1)} = 1. \tag{28}$$

The $\mathbb{T}^{(2)}$ coefficients are obtained by making the formal parameter exchanges $a_r \leftrightarrow c_r$ and $b_r \leftrightarrow d_r$, while the remaining ones follow from the relation $\mathbb{T}^{(i)}_{-\alpha,-\beta,-\gamma} = (\mathbb{T}^{(i)}_{\alpha\beta\gamma})^{-1}$. Using both equations (27), we can write the eight vectors $\mathsf{R}^{\alpha}_{12}\mathsf{R}^{\beta}_{13}\mathsf{R}^{\gamma}_{23}$ in terms of six linear independent vectors, which we do not specify. We denote the 6×8 matrix for this change of basis as $\mathbb{P}$. Note that, due to (27), the coefficients appearing in (26) are not unique. In fact, one has a 16 parameter freedom since any $\mathbb{S}'$ with

$$(\mathbb{S}')^{\alpha\,\beta\,\gamma}_{\bar{\alpha}\,\bar{\beta}\,\bar{\gamma}} = \mathbb{S}^{\alpha\,\beta\,\gamma}_{\bar{\alpha}\,\bar{\beta}\,\bar{\gamma}} + \sum_{i=1}^{2} c^{\alpha\,\beta\,\gamma}_{(i)} \mathbb{T}^{(i)}_{\bar{\gamma}\,\bar{\beta}\,\bar{\alpha}} \tag{29}$$

will obey the tetrahedron Zamolodchikov algebra (25) for any value of the complex parameters $c^{\alpha\,\beta\,\gamma}_{(i)}$.

We end this section by considering the application of the tetrahedron Zamolodchikov algebra (25) to the product of six operators $\mathsf{R}^{\pm}$. Specifically, one finds that there are two a priori inequivalent ways of transforming the product $\mathsf{R}^{a}_{34}\mathsf{R}^{b}_{24}\mathsf{R}^{c}_{14}\mathsf{R}^{d}_{23}\mathsf{R}^{e}_{13}\mathsf{R}^{f}_{12}$ into a linear combination of products of six $\mathsf{R}^{\pm}_{ij}$ with the reverse lattice order. We make apparent the fact that the coefficients in (26) depend explicitly on the free fermion parameters $\{a_k, b_k, c_k, d_k\}_{k=1}^{3}$ by writting them as $(\mathbb{S}_{123})^{\alpha\beta\gamma}_{\bar{\alpha}\bar{\beta}\bar{\gamma}}$. We define analogously the coefficients of $\mathbb{S}_{124}$, $\mathbb{S}_{134}$ and $\mathbb{S}_{234}$, and find:

$$\begin{aligned}\Big(& \big(\mathbb{S}_{123}\big)^{def}_{d'e'f'} \big(\mathbb{S}_{124}\big)^{bcf'}_{b'c'f''} \big(\mathbb{S}_{134}\big)^{ac'e'}_{a'c''e''} \big(\mathbb{S}_{234}\big)^{a'b'd'}_{a''b''d''} \\ & - \big(\mathbb{S}_{234}\big)^{abd}_{a'b'd'} \big(\mathbb{S}_{134}\big)^{a'ce}_{a''c'e'} \big(\mathbb{S}_{124}\big)^{b'c'f}_{b''c''f'} \big(\mathbb{S}_{123}\big)^{d'e'f'}_{d''e''f''} \Big) \mathsf{R}^{f''}_{12}\mathsf{R}^{e''}_{13}\mathsf{R}^{d''}_{23}\mathsf{R}^{c''}_{14}\mathsf{R}^{b''}_{24}\mathsf{R}^{a''}_{34} = 0\end{aligned} \tag{30}$$

where the Einstein summation convention applies. However, because of the linear dependence equations of (27), we cannot simply set the coefficients in the sum of (30) to zero. One needs to use a transformation of the kind (29) in order to obtain a tensor $\mathbb{S}'_{ijk}$ that obeys the *tetrahedron Zamolodchikov equations*[9] :

$$\mathbb{S}'_{123}\mathbb{S}'_{124}\mathbb{S}'_{134}\mathbb{S}'_{234} = \mathbb{S}'_{234}\mathbb{S}'_{134}\mathbb{S}'_{124}\mathbb{S}'_{123}. \tag{31}$$

[9]This equation should be interpreted as an equation in $\mathrm{End}((\mathbb{C}^2)^{\otimes 6})$. Let us introduce 2×2 matrix units E_{ab} whose (i,j) elements are given by $(E_{ab})_{ij} = \delta_{ai}\delta_{bj}$, and define $E^{(12)}_{ab} = E_{ab} \otimes 1^{\otimes 5}$, $E^{(13)}_{ab} = 1 \otimes E_{ab} \otimes 1^{\otimes 4}$, $E^{(23)}_{ab} = 1^{\otimes 2} \otimes E_{ab} \otimes 1^{\otimes 3}$, $E^{(14)}_{ab} = 1^{\otimes 3} \otimes E_{ab} \otimes 1^{\otimes 2}$, $E^{(24)}_{ab} = 1^{\otimes 4} \otimes E_{ab} \otimes 1$, $E^{(34)}_{ab} = 1^{\otimes 5} \otimes E_{ab}$. Then the tensors in (31) are defined by $\mathbb{S}'_{ijk} = \sum_{a,b,c,d,e,f=\pm} (\mathbb{S}'_{ijk})^{abc}_{def} E^{(ij)}_{cf} E^{(ik)}_{be} E^{(jk)}_{ad}$.

Unfortunately, such a solution seems to be known only in the symmetric case for which $a_k = d_k = \cos u_k$ and $b_k = c_k = -i \sin u_k$, see [11, 19]. Whether a solution exists for the general case seems to be an open problem.

3 The Shastry-Shiroishi-Wadati R-matrix

Here, we wish to review the construction of the R-matrix of the one-dimensional Hubbard model and generalizations thereof. We start first by taking a look at the Hamilton operator for the one-dimensional Hubbard model, which reads

$$\mathsf{H}^{\text{Hub}} = -\sum_{j=1}^{N} \sum_{\sigma=\uparrow,\downarrow} \left(\mathbf{c}^{\dagger}_{j+1,\sigma} \mathbf{c}_{j,\sigma} + \mathbf{c}^{\dagger}_{j,\sigma} \mathbf{c}_{j+1,\sigma} \right) + \frac{U}{4} \sum_{j=1}^{N} \left(\mathbf{m}_{j,\uparrow} - \mathbf{n}_{j,\uparrow} \right) \left(\mathbf{m}_{j,\downarrow} - \mathbf{n}_{j,\downarrow} \right), \tag{32}$$

where we introduced two copies of the fermionic oscillators $\mathbf{c}_{j,\sigma}$ and $\mathbf{c}^{\dagger}_{j,\sigma}$ for $\sigma =\uparrow, \downarrow$ that satisfy

$$\left\{ \mathbf{c}_{j,\sigma} \, , \, \mathbf{c}^{\dagger}_{k,\tau} \right\} = \delta_{jk} \delta_{\sigma\tau}, \quad \sigma, \tau =\uparrow \text{ or } \downarrow, \tag{33}$$

and defined the bosonic compound operators $\mathbf{n}_{j,\sigma} := \mathbf{c}^{\dagger}_{j,\sigma} \mathbf{c}_{j,\sigma}$ and $\mathbf{m}_{j,\sigma} := \mathbf{c}_{j,\sigma} \mathbf{c}^{\dagger}_{j,\sigma}$. The non-negative integer N is the number of one-dimensional lattice sites. We impose the periodic boundary conditions $\mathbf{c}^{\dagger}_{N+1,\sigma} = \mathbf{c}^{\dagger}_{1,\sigma}$, $\mathbf{c}_{N+1,\sigma} = \mathbf{c}_{1,\sigma}$, $\sigma =\uparrow, \downarrow$. Here, the number U is the coupling constant. One sees that at $U = 0$, the spin chain decomposes into two non-interacting XX models, described by the oscillators $\mathbf{c}_{j,\uparrow}$ and $\mathbf{c}_{j,\downarrow}$ respectively. The R-matrix of this product model is of course simply the product $\mathsf{R}^0_{jk,\uparrow} \mathsf{R}^0_{jk,\downarrow}$ of the R matrices of the individual models[10]. This observation led Shastry in [1] into making an Ansatz for all U, later generalized by Shiroishi and Wadati [14], of the following form

$$\mathsf{R}_{jk} := \mathsf{R}^0_{jk,\uparrow} \mathsf{R}^0_{jk,\downarrow} + \alpha_{jk} \left(\mathsf{R}^0_{jk,\uparrow} \mathsf{R}^1_{jk,\downarrow} + \mathsf{R}^1_{jk,\uparrow} \mathsf{R}^0_{jk,\downarrow} \right) + \beta_{jk} \mathsf{R}^1_{jk,\uparrow} \mathsf{R}^1_{jk,\downarrow}, \tag{34}$$

where $j, k \in \{1, 2, \ldots, N\}$, and the unknown coefficients $\alpha_{jk}, \beta_{jk} \in \mathbb{C}$ are to be determined by the requirements that (34) obeys the Yang-Baxter equation:

$$\mathsf{R}_{12} \mathsf{R}_{13} \mathsf{R}_{23} = \mathsf{R}_{23} \mathsf{R}_{13} \mathsf{R}_{12} \tag{35}$$

[10] We denote the operators $\mathcal{O}$ obtained by replacing the fermionic oscillators $\mathbf{c}_j, \mathbf{c}^{\dagger}_j$ therein with $\mathbf{c}_{j,\alpha}, \mathbf{c}^{\dagger}_{j,\alpha}$ for $\alpha =\uparrow$ or $\downarrow$ by $\mathcal{O}_{,\alpha}$.

and that they both vanish in the free fermion limit $U \to 0$. One can formulate these constraints better by rewriting the operator R_{jk} in the basis $\mathsf{R}^{\pm}$ of (24) as $\mathsf{R}_{jk} = \sum_{\alpha,\beta=\pm} \gamma_{jk;\alpha\beta} \mathsf{R}^{\alpha}_{jk,\uparrow} \mathsf{R}^{\beta}_{jk,\downarrow}$. Then, making use of the tetrahedron Zamolodchikov algebra (25) and of the linear dependence (27) leads to the set of equations[11]

$$\gamma_{12}\gamma_{13}\gamma_{23} \left(\mathbb{1} \otimes \mathbb{1} - \mathbb{S} \otimes \mathbb{S}\right) \mathbb{P} \otimes \mathbb{P} = 0 \quad \text{and} \quad \gamma_{jk;\alpha\beta}(U=0) = 1 \quad \text{for} \quad \forall j,k,\alpha,\beta. \tag{36}$$

These equations can be simplified if one assumes certain symmetries of the coefficients, for instance $\alpha_{jk} = 0$ in the original application to the Hubbard model. The most general known solution to (36) has been found[12] in [14] and leads to an R_{jk} matrix that depends, not on just one constant U, but rather on two complex parameters that we name Θ and Ξ. Explicitly, the most general non-trivial solution has the form

$$\mathsf{R}_{jk} = \frac{v_k + \frac{b_j c_j}{a_j d_j} v_j}{\frac{b_k c_k}{a_k d_k} v_k + v_j} \left(\frac{a_j b_k}{b_j a_k} \mathsf{R}^{+}_{jk,\uparrow} \mathsf{R}^{+}_{jk,\downarrow} + \frac{d_j c_k}{c_j d_k} \mathsf{R}^{-}_{jk,\uparrow} \mathsf{R}^{-}_{jk,\downarrow} \right) + \mathsf{R}^{+}_{jk,\uparrow} \mathsf{R}^{-}_{jk,\downarrow} + \mathsf{R}^{-}_{jk,\uparrow} \mathsf{R}^{+}_{jk,\downarrow} \,, \tag{37}$$

where we have suppressed the dependence of the operators[13] $\mathsf{R}^{\pm}_{jk,\uparrow\downarrow}$ on the matrices A_j and A_k. We note though that the $\uparrow$ and $\downarrow$ layers both use the same matrices. The new variables v_j, referred to as *gluing parameters*, are not free but depend on the constants Θ and Ξ through the *gluing conditions*:

$$i \frac{\Theta^2}{a_k d_k} v_k - i \frac{\Xi^2}{b_k c_k} v_k^{-1} = 1 \quad \text{for} \quad k = 1, 2. \tag{38}$$

The obvious two-layer structure of the operator R leads to a number of interesting relations, which we elaborate on in the appendix A. The R-matrix of the Hubbard model is a special case of (37) for which the global parameters take the values $\Theta^2 = -\Xi^2 = -iU^{-1}$, the $\mathrm{SL}(2,\mathbb{C})$ parameters are specialized to $a_j = d_j = \cos u_j$, $b_j = c_j = -i \sin u_j$ and we replace the gluing parameters with h_j via $v_j = e^{2h_j} \cot u_j$. This then reduces the

[11] Let us introduce unit row vectors v_a which satisfy $v_c E_{ab} = \delta_{ca} v_b$. Then the tensors and vectors in (36) are defined by $\mathbb{S} = \sum_{a,b,c,d,e,f=\pm} \mathbb{S}^{abc}_{def} E_{cf} \otimes E_{be} \otimes E_{ad}$, $\gamma_{12} = \sum_{a,b=\pm} \gamma_{12;ab} v_a \otimes 1 \otimes 1 \otimes v_b \otimes 1 \otimes 1$, $\gamma_{13} = \sum_{a,b=\pm} \gamma_{13;ab} 1 \otimes v_a \otimes 1 \otimes 1 \otimes v_b \otimes 1$, $\gamma_{23} = \sum_{a,b=\pm} \gamma_{23;ab} 1 \otimes 1 \otimes v_a \otimes 1 \otimes 1 \otimes v_b$. Then (36) correspond to the equations (4.3)-(4.7) in [14].

[12] To be precise, the solution in [14] is written in terms of some matrices rather than fermionic oscillators.

[13] From now on, we often use a shorthand notation on indices for any operators $\mathcal{O}$: $\mathcal{O}_{\uparrow\downarrow}$ denotes $\mathcal{O}_{\uparrow}$ or $\mathcal{O}_{\downarrow}$. When we need to write the matrix dependence of R_{jk}, we use the notation $\mathsf{R}_{jk}(A_j, A_k)$.

gluing equation of (38) to the well-known formula $\sinh 2h_j = \frac{U}{4}\sin 2u_j$, which relates the extra parameters appearing in the R-matrix to the coupling constant U and the spectral parameter. One should note that in general, the solutions to (38) involve elliptic functions (see [20] for more details on the use of elliptic functions in the context of the AdS/CFT S-matrix[14]). This does not however mean that the model is elliptic in the usual sense, since (37) lacks a difference property.

We can make several observations regarding the symmetries of the R-matrix (37). First, one notes that the R-matrix is invariant under the exchange of the two layers $\uparrow$ and $\downarrow$. Second, one sees that part of the quantum symmetry of the free fermion building blocks survives. Specifically, we still have a $\mathfrak{A}\oplus\mathfrak{A}$ quantum group symmetry[15] generated by the elements $\mathsf{k}_{0,\uparrow\downarrow}, \mathsf{e}_{0,\uparrow\downarrow}, \mathsf{f}_{0,\uparrow\downarrow}$ and $\mathsf{h}_{0,\uparrow\downarrow}$, namely

$$\left(\mathsf{G}^{-1}\Delta'(\mathsf{J})\mathsf{G}\right)\mathsf{R}_{12} = \mathsf{R}_{12}\left(\mathsf{G}^{-1}\Delta(\mathsf{J})\mathsf{G}\right) \text{ with } \mathsf{J} \in \{\mathsf{e}_{0,\uparrow\downarrow}, \mathsf{f}_{0,\uparrow\downarrow}, \mathsf{k}_{0,\uparrow\downarrow}, \mathsf{h}_{0,\uparrow\downarrow}\}. \tag{39}$$

Here we have defined the operator $\mathsf{G} := \mathsf{G}_{1,\uparrow}\mathsf{G}_{1,\downarrow}\mathsf{G}_{2,\uparrow}\mathsf{G}_{2,\downarrow}$, see (17). Thus, the two-layer form of the R-matrix makes apparent its invariance under the four fermionic generators $\mathsf{e}_{0,\uparrow\downarrow}, \mathsf{f}_{0,\uparrow\downarrow}$. In establishing the connection (39), it is useful to keep in mind to relationships between the free fermion variables and the quantum group variables, summarized in table 1. In the next section, we shall see how the quantum symmetry is related to the larger algebra $\mathrm{su}(2|2) \ltimes \mathbb{R}^2$, which contains eight fermionic generators.

4 Symmetries

In this section, we want to investigate the symmetry properties of the operator (37). First, we shall connect the results of [9] to those of [2, 5, 13] by defining the superalgebra $\mathrm{su}(2|2) \ltimes \mathbb{R}^2$ and showing that it is sufficient to determine the R-matrix in a certain limit. In a further part, we connect the quantum symmetry of the XX model to the superalgebra.

---

[14] We thank V.Kazakov and especially A.Zabrodin for interesting discussions on the elliptic parametrization of the the AdS/CFT S-matrix.

[15] This corresponds to (14) at each layer.

Quantum group variables	Free fermion variables
$x_r,\ \lambda_r,\ z,\ \varphi_r,\ \mu_r$	$a_r,\ b_r,\ c_r,\ d_r$
$\lambda_r = i^{-1-\mu_r}$	$a_r d_r - b_r c_r = 1$
$\varphi_r^2 = \frac{\lambda_r - \lambda_r^{-1}}{2i}$	$a_r = x_r^{-1}\sqrt{\frac{\lambda_r}{\lambda_r - \lambda_r^{-1}}}$
$\lambda_r^2 = \frac{a_r d_r}{b_r c_r}$	$b_r = \frac{x_r}{iz}\sqrt{\frac{\lambda_r^{-1}}{\lambda_r - \lambda_r^{-1}}}$
$x_r^2 = \frac{d_r}{a_r}$	$c_r = izx_r^{-1}\sqrt{\frac{\lambda_r^{-1}}{\lambda_r - \lambda_r^{-1}}}$
$z^2 = -\frac{c_r d_r}{a_r b_r}$	$d_r = x_r\sqrt{\frac{\lambda_r}{\lambda_r - \lambda_r^{-1}}}$
Gluing parameters and variables: Θ, Ξ, v_r	
$\Theta^2\lambda_r^{-1}v_r - \Xi^2\lambda_r v_r^{-1} = -\frac{1}{2}\varphi_r^{-2}$	$i\frac{\Theta^2}{a_r d_r}v_r - i\frac{\Xi^2}{b_r c_r}v_r^{-1} = 1$

Table 1: The connections between the free fermion variables and the quantum group variables as well as the gluing conditions in both languages.

4.1 Realization of the centrally extended superalgebra

As written for example in [2], the superalgebra $\mathfrak{g}_0 := \mathrm{psu}(2|2)$ is spanned by the six[16] even generators $\mathcal{L}^\alpha{}_\beta$ and $\mathcal{R}^a{}_b$ and the eight odd generators $\mathcal{Q}^\alpha{}_a$ and $\mathcal{S}^a{}_\alpha$, where the indices α, β, a, b run over $\{1, 2\}$. One can extend this superalgebra by adding three even central charges $\mathcal{C}$, $\mathcal{P}$ and $\mathcal{K}$ to get $\mathfrak{g} := \mathrm{su}(2|2) \ltimes \mathbb{R}^2$. The commutation relations for $\mathfrak{g}$ can be summarized as follows:

$$
\begin{aligned}
&\left[\mathcal{L}^\alpha{}_\beta\,,\,\mathcal{L}^\gamma{}_\xi\right] = \delta^\gamma_\beta\mathcal{L}^\alpha{}_\xi - \delta^\alpha_\xi\mathcal{L}^\gamma{}_\beta, && [\mathcal{R}^a{}_b\,,\,\mathcal{R}^c{}_d] = \delta^c_b\mathcal{R}^a{}_d - \delta^a_d\mathcal{R}^c{}_b,\\
&\left[\mathcal{L}^\alpha{}_\beta\,,\,\mathcal{Q}^\gamma{}_b\right] = \delta^\gamma_\beta\mathcal{Q}^\alpha{}_b - \frac{1}{2}\delta^\alpha_\beta\mathcal{Q}^\gamma{}_b, && \left[\mathcal{L}^\alpha{}_\beta\,,\,\mathcal{S}^a{}_\gamma\right] = -\delta^\alpha_\gamma\mathcal{S}^a{}_\beta + \frac{1}{2}\delta^\alpha_\beta\mathcal{S}^a{}_\gamma,\\
&\left[\mathcal{R}^a{}_b\,,\,\mathcal{S}^c{}_\beta\right] = \delta^c_b\mathcal{S}^a{}_\beta - \frac{1}{2}\delta^a_b\mathcal{S}^c{}_\beta, && [\mathcal{R}^a{}_b\,,\,\mathcal{Q}^\alpha{}_c] = -\delta^a_c\mathcal{Q}^\alpha{}_b + \frac{1}{2}\delta^a_b\mathcal{Q}^\alpha{}_c,\\
&\left\{\mathcal{Q}^\alpha{}_a\,,\,\mathcal{Q}^\beta{}_b\right\} = \epsilon^{\alpha\beta}\epsilon_{ab}\mathcal{P}, && \left\{\mathcal{S}^a{}_\alpha\,,\,\mathcal{S}^b{}_\beta\right\} = \epsilon^{ab}\epsilon_{\alpha\beta}\mathcal{K},\\
&\left\{\mathcal{Q}^\alpha{}_a\,,\,\mathcal{S}^b{}_\beta\right\} = \delta^b_a\mathcal{L}^\alpha{}_\beta + \delta^\alpha_\beta\mathcal{R}^b{}_a + \delta^b_a\delta^\alpha_\beta\mathcal{C}.
\end{aligned}
\tag{40}
$$

In the above relations, ϵ is the standard antisymmetric tensor with $\epsilon^{12} = \epsilon_{12} = 1$. Using two fermionic oscillators $\mathbf{c}_\uparrow$ and $\mathbf{c}_\downarrow$ we can easily obtain an important class of

[16] We have $\mathcal{L}^1{}_1 = -\mathcal{L}^2{}_2$ as well as $\mathcal{R}^1{}_1 = -\mathcal{R}^2{}_2$.

four-dimensional representations of $\mathfrak{g}$. The even part of the algebra is represented as

$$\begin{aligned}
&\mathcal{R}^1{}_1 = -\mathcal{R}^2{}_2 = \frac{1}{2}\left(1 - \mathbf{n}_\uparrow - \mathbf{n}_\downarrow\right), && \mathcal{R}^1{}_2 = \mathbf{c}_\downarrow \mathbf{c}_\uparrow, && \mathcal{R}^2{}_1 = \mathbf{c}^\dagger_\uparrow \mathbf{c}^\dagger_\downarrow, \\
&\mathcal{L}^1{}_1 = -\mathcal{L}^2{}_2 = \frac{1}{2}\left(\mathbf{n}_\uparrow - \mathbf{n}_\downarrow\right), && \mathcal{L}^1{}_2 = \mathbf{c}^\dagger_\uparrow \mathbf{c}_\downarrow, && \mathcal{L}^2{}_1 = \mathbf{c}^\dagger_\downarrow \mathbf{c}_\uparrow,
\end{aligned} \tag{41}$$

while the odd one takes the form

$$\begin{aligned}
&\mathcal{Q}^1{}_1 = (\mathfrak{a}\mathbf{m}_\downarrow + \mathfrak{b}\mathbf{n}_\downarrow)\mathbf{c}^\dagger_\uparrow, && \mathcal{Q}^2{}_1 = (\mathfrak{a}\mathbf{m}_\uparrow + \mathfrak{b}\mathbf{n}_\uparrow)\mathbf{c}^\dagger_\downarrow, \\
&\mathcal{Q}^1{}_2 = -(\mathfrak{b}\mathbf{m}_\uparrow + \mathfrak{a}\mathbf{n}_\uparrow)\mathbf{c}_\downarrow, && \mathcal{Q}^2{}_2 = (\mathfrak{b}\mathbf{m}_\downarrow + \mathfrak{a}\mathbf{n}_\downarrow)\mathbf{c}_\uparrow, \\
&\mathcal{S}^1{}_1 = (\mathfrak{d}\mathbf{m}_\downarrow + \mathfrak{c}\mathbf{n}_\downarrow)\mathbf{c}_\uparrow, && \mathcal{S}^2{}_1 = -(\mathfrak{c}\mathbf{m}_\uparrow + \mathfrak{d}\mathbf{n}_\uparrow)\mathbf{c}^\dagger_\downarrow, \\
&\mathcal{S}^1{}_2 = (\mathfrak{d}\mathbf{m}_\uparrow + \mathfrak{c}\mathbf{n}_\uparrow)\mathbf{c}_\downarrow, && \mathcal{S}^2{}_2 = (\mathfrak{c}\mathbf{m}_\downarrow + \mathfrak{d}\mathbf{n}_\downarrow)\mathbf{c}^\dagger_\uparrow,
\end{aligned} \tag{42}$$

where $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ and $\mathfrak{d}$ are a priori free complex parameters. Closure of the algebra requires that $\mathfrak{a}\mathfrak{d} - \mathfrak{b}\mathfrak{c} = 1$ and the central charges take the values

$$\mathcal{C} = \frac{\mathfrak{a}\mathfrak{d} + \mathfrak{b}\mathfrak{c}}{2}, \quad \mathcal{P} = \mathfrak{a}\mathfrak{b}, \quad \mathcal{K} = \mathfrak{c}\mathfrak{d}. \tag{43}$$

We group the central charges in the vector $\vec{\mathcal{C}} = (\mathcal{C}, \mathcal{P}, \mathcal{K})$ and denote by $V(\vec{\mathcal{C}})$ the four-dimensional representation of $\mathfrak{g}$ generated by the operators (42) on a vacuum $|\,0\rangle$ annihilated by $\mathbf{c}_{\uparrow\downarrow}$. The condition $\mathfrak{a}\mathfrak{d} - \mathfrak{b}\mathfrak{c} = 1$ translates to $\mathcal{C}^2 - \mathcal{P}\mathcal{K} = \frac{1}{4}$ for the central charges. The outer automorphism group of the centrally extended superalgebra $\mathfrak{g}$ is isomorphic to $\mathrm{SL}(2,\mathbb{C})$ and it acts on the representations $V(\vec{\mathcal{C}})$ by sending

$$\mathfrak{D} \mapsto \mathfrak{D}\mathfrak{E}, \text{ for } \mathfrak{D} := \begin{pmatrix} \mathfrak{a} & \mathfrak{b} \\ \mathfrak{c} & \mathfrak{d} \end{pmatrix} \text{ and } \mathfrak{E} \in \mathrm{SL}(2,\mathbb{C}). \tag{44}$$

It turns out that we can represent one of the generators[17] of this outer automorphism group, $\mathcal{B}$, using the fermionic oscillators as follows

$$\mathcal{B} = \mathbf{n}_\uparrow \mathbf{n}_\downarrow + \mathbf{m}_\uparrow \mathbf{m}_\downarrow. \tag{45}$$

The operator $\mathcal{B}$ commutes with the even generators and generates the transformation (44) with $\mathfrak{E} = \mathrm{diag}(e^{-i\phi}, e^{i\phi})$, where $\phi \in \mathbb{C}$, namely $e^{i\phi\mathcal{B}}\mathcal{J}(\mathfrak{D})e^{-i\phi\mathcal{B}} = \mathcal{J}(\mathfrak{D}\mathfrak{E})$, where $\mathcal{J}(\mathfrak{D})$ is the generator in (41)-(42) as a function of the matrix $\mathfrak{D}$ in (44).

[17]This is the one extra generator that turns sl(2|2) into gl(2|2).

One of the original motivations for the investigations of the relationships between the AdS/CFT S-matrix and the Hubbard model R-matrix was to determine in what way the $\mathrm{SL}(2,\mathbb{C})$ group appearing here as the outer automorphism group of the algebra $\mathfrak{g}$ is related to the $\mathrm{SL}(2,\mathbb{C})$ group that plays an important role for the free fermion R-matrix, see (3). This will be the subject of the subsections 4.3 and 4.4.

4.2 Invariance of the R matrix under bosonic transformations

One can check easily that the operator (37) obeys the following invariance equations[18].

$$\Big[\mathsf{R}_{jk}\,,\,\big(\mathcal{L}^{\alpha}{}_{\beta}\big)_j+\big(\mathcal{L}^{\alpha}{}_{\beta}\big)_k\Big]=0,\quad \Big[\mathsf{R}_{jk}\,,\,\big(\mathcal{R}^{1}{}_{1}\big)_j+\big(\mathcal{R}^{1}{}_{1}\big)_k\Big]=0\;, \tag{46}$$

independently of the gluing conditions (38). The invariance under the raising or lowering generators of the other sl(2) is tricky and leads to anticommutation relations. One finds that

$$\Big\{\mathsf{R}_{jk}\,,\,\frac{b_j}{c_j}\big(\mathcal{R}^{1}{}_{2}\big)_j-\frac{b_k}{c_k}\big(\mathcal{R}^{1}{}_{2}\big)_k\Big\}=0,\quad \Big\{\mathsf{R}_{jk}\,,\,\frac{c_j}{b_j}\big(\mathcal{R}^{2}{}_{1}\big)_j-\frac{c_k}{b_k}\big(\mathcal{R}^{2}{}_{1}\big)_k\Big\}=0, \tag{47}$$

if and only if the following equation that we refer to as *symmetry conditions* is fulfilled:

$$a_k b_k = c_k d_k \qquad \forall k. \tag{48}$$

We remark that this corresponds to the condition that the eigenvalue z of the central element Z coincides with i or $-i$ (see (19)). Note that the Hubbard model satisfies this condition (48), since it obeys the even stronger condition $a_r = d_r$ and $b_r = c_r$. We wish to rewrite the invariance conditions in a way that would make them more apparent and that would make the symmetry of the Hamilton operator transparent while strengthening the connection to the works [2, 13]. We start by defining the gauge transformation matrices U_r and V_r

$$\mathsf{U}_r := \mathbf{m}_{r,\uparrow}\mathbf{m}_{r,\downarrow}+t_r\mathbf{m}_{r,\uparrow}\mathbf{n}_{r,\downarrow}+t_r\mathbf{n}_{r,\uparrow}\mathbf{m}_{r,\downarrow}+\frac{c_r}{b_r}\mathbf{n}_{r,\uparrow}\mathbf{n}_{r,\downarrow},\quad \mathsf{V}_r := (\mathbf{m}_{r,\uparrow}-i\mathbf{n}_{r,\uparrow})(\mathbf{m}_{r,\downarrow}-i\mathbf{n}_{r,\downarrow})\;, \tag{49}$$

[18] We denote the operators $\mathcal{O}$ acting on the lattice site j, namely $\mathcal{O}$ obtained by replacing the fermionic oscillators $\mathbf{c}_\alpha, \mathbf{c}_\alpha^\dagger$ therein with $\mathbf{c}_{j,\alpha}, \mathbf{c}_{j,\alpha}^\dagger$ for $\alpha =\uparrow$ or $\downarrow$ by $(\mathcal{O})_j$.

where we have introduces a new parameter $t_r \in \mathbb{C}$, that will remain unconstrained for now. We use this operators to perform a similarity transformation in order to get the new operator $\check{\mathsf{R}}'_{jk} := \mathsf{P}_{jk}\left(\mathsf{U}_j^{-1}\mathsf{U}_k^{-1}\mathsf{R}_{jk}\mathsf{U}_j\mathsf{U}_k\right)$ where the two-layer permutation operator is defined as $\mathsf{P}_{12} := \mathsf{P}_{12,\uparrow}\mathsf{P}_{12,\downarrow}$. This new operator obeys the same invariance conditions (46) with $\mathcal{L}^\alpha{}_\beta$ and $\mathcal{R}^1{}_1$ as R but the remaining ones, specifically (47), turn to

$$\left[\check{\mathsf{R}}'_{jk}\,,\,\left(\mathcal{R}^1{}_2\right)_j-\left(\mathcal{R}^1{}_2\right)_k\right]=0,\quad \left[\check{\mathsf{R}}'_{jk}\,,\,\left(\mathcal{R}^2{}_1\right)_j-\left(\mathcal{R}^2{}_1\right)_k\right]=0. \tag{50}$$

We emphasize again that the above equations are true only if the symmetry conditions (48) are obeyed. Furthermore, the operator $\check{\mathsf{R}}'$ inherits from the R the property

$$\check{\mathsf{R}}'_{12}(A_2,A_3)\check{\mathsf{R}}'_{23}(A_1,A_3)\check{\mathsf{R}}'_{12}(A_1,A_2)=\check{\mathsf{R}}'_{23}(A_1,A_2)\check{\mathsf{R}}'_{12}(A_1,A_3)\check{\mathsf{R}}'_{23}(A_2,A_3). \tag{51}$$

We want to use this residual symmetry in order to get rid of the minus signs in (50). For this purpose, we perform a similarity transformation of the $\check{\mathsf{R}}'$, leading to the definition of the operator $\check{\mathsf{R}}$ as

$$\check{\mathsf{R}}_{j,j+1}=\begin{cases}\mathsf{V}_j^{-1}\check{\mathsf{R}}'_{j,j+1}\mathsf{V}_j & \text{if } j \text{ is even}\\ \mathsf{V}_{j+1}^{-1}\check{\mathsf{R}}'_{j,j+1}\mathsf{V}_{j+1} & \text{if } j \text{ is odd}\end{cases} \tag{52}$$

Finally, we get the invariance equations that we want, namely

$$\left[\check{\mathsf{R}}_{j,j+1}\,,\,\left(\mathcal{L}^\alpha{}_\beta\right)_j+\left(\mathcal{L}^\alpha{}_\beta\right)_{j+1}\right]=0,\quad \left[\check{\mathsf{R}}_{j,j+1}\,,\,\left(\mathcal{R}^a{}_b\right)_j+\left(\mathcal{R}^a{}_b\right)_{j+1}\right]=0, \tag{53}$$

for all $\alpha,\beta,a,b\in\{1,2\}$ and for all $j\in\mathbb{Z}$. Thus, we get the proper invariance under the bosonic symmetry provided that we look at the operator $\check{\mathsf{R}}$ between nearest neighbor sites. Since the nearest neighbor Hamiltonian of the integrable system associated to R is simply obtained, following the algebraic Bethe Ansatz, as in (5) for the XX model by taking the logarithmic derivative of the operator $\check{\mathsf{R}}$, the $\mathrm{sl}(2)\oplus\mathrm{sl}(2)$ symmetry is guaranteed. The only possible problem comes from boundary conditions. If we put the model on a periodic chain with a odd number of lattice sites, then (53) will not be true for the last site and the global $\mathcal{R}$ symmetry will be broken. In the Hubbard model language, the $\mathcal{L}$ symmetry corresponds to the spin SU(2) symmetry, while the $\mathcal{R}$ one is the η-pairing SU(2) symmetry. It was noted in [21] that the R-matrix of the Hubbard model in the fermionic formulation anticommutes with the η-pairing ladder generators, which reflected the well established fact that this second SU(2) symmetry was only present for chains of even length.

4.3 Invariance under fermionic transformations

The invariance of the operator $\check{\mathsf{R}}_{12}$ under fermionic transformations introduced in (42) is quite subtle. The action of $\check{\mathsf{R}}$ transforms the central charges in a non-linear way and one is furthermore faced with the challenge of finding the relations between the $\mathfrak{g}$ labels $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ and $\mathfrak{d}$ on the one hand and the free fermion variables a, b, c and d that enter formula (52) on the other. The picture is further muddied by the presence of the gluing parameters v_i and of the extra coefficients t_i appearing in the similarity transformations.

The identifications are done as follows. Given two $\mathrm{SL}(2,\mathbb{C})$ matrices A_j written as in (3) with parameters that obey the symmetry condition (48), we define the following sets of matrices

$$\mathfrak{B}_j := \sqrt{\Theta\Xi}e^{-i\frac{\pi}{4}} \begin{pmatrix} \frac{\Theta}{\Xi}\frac{c_j}{a_j d_j}\frac{v_j}{t_j} & \frac{\Xi}{\Theta}\frac{t_j}{c_j v_j} \\ -\frac{1}{b_j t_j} & -\frac{t_j}{c_j} \end{pmatrix} , \quad \mathfrak{C}_j := \begin{pmatrix} \frac{c_j}{a_j} & 0 \\ 0 & \frac{a_j}{c_j} \end{pmatrix} . \tag{54}$$

The determinants of these matrices is one because of the gluing conditions (38). Thus these are also elements of $\mathrm{SL}(2,\mathbb{C})$. Setting then $\mathfrak{D}_1 = \mathfrak{C}_2\mathfrak{B}_1$, $\mathfrak{D}_2 = \mathfrak{B}_2$, $\mathfrak{D}'_2 = \mathfrak{C}_1\mathfrak{B}_2$ and $\mathfrak{D}'_1 = \mathfrak{B}_1$, we find the following invariance condition, valid for all non-zero values of the parameters t_r:

$$\check{\mathsf{R}}_{12}(A_1, A_2)\,[\mathcal{J}_1(\mathfrak{D}_1) + \mathcal{J}_2(\mathfrak{D}_2)] = [\mathcal{J}_1(\mathfrak{D}'_2) + \mathcal{J}_2(\mathfrak{D}'_1)]\,\check{\mathsf{R}}_{12}(A_1, A_2)\ , \tag{55}$$

where $\mathcal{J}_i(\mathfrak{D})$ is any of the fermionic operator in (42) acting on the lattice site $i \in \{1, 2\}$. Here we regard this as a function of the matrix $\mathfrak{D}$ in (44). The action of the fermionic generators can be understood in the following way. On the left hand side of (55), the $\mathcal{J}_1+\mathcal{J}_2$ act on the tensor product of two $\mathfrak{g}$ representations $V(\vec{\mathcal{C}_1})\otimes V(\vec{\mathcal{C}_2})$. In our notation, it is understood that $\mathcal{J}_i$ always acts on the i^{th} factor in a tensor product. However, on the right hand side of (55), the $\mathcal{J}_1+\mathcal{J}_2$ act on the tensor product $V(\vec{\mathcal{C}'_2})\otimes V(\vec{\mathcal{C}'_1})$ because the operator $\check{\mathsf{R}}_{12}$ has exchanged the labels of the central charges, meaning that $\check{\mathsf{R}}_{12}$ is to be seen as an operator

$$\check{\mathsf{R}}_{12} : V(\vec{\mathcal{C}_1}) \otimes V(\vec{\mathcal{C}_2}) \to V(\vec{\mathcal{C}'_2}) \otimes V(\vec{\mathcal{C}'_1}). \tag{56}$$

We remind that the central charges $\vec{\mathcal{C}_r}$, respectively $\vec{\mathcal{C}_r}'$ are related to the matrices $\mathfrak{D}_r$, respectively $\mathfrak{D}'_r$ via (43). From the explicit expression of (54), we find the relations:

$$\mathcal{C}_1 = i\frac{\Theta^2 v_1}{a_1 d_1} - \frac{1}{2}, \qquad \mathcal{P}_1 = \frac{\Theta\Xi}{ia_1 d_1}\left(\frac{c_2}{a_2}\right)^2, \qquad \mathcal{K}_1 = \frac{\Theta\Xi}{ib_1 c_1}\left(\frac{a_2}{c_2}\right)^2,$$
$$\mathcal{C}_2 = i\frac{\Theta^2 v_2}{a_2 d_2} - \frac{1}{2}, \qquad \mathcal{P}_2 = \frac{\Theta\Xi}{ia_2 d_2}, \qquad \mathcal{K}_2 = \frac{\Theta\Xi}{ib_2 c_2}. \tag{57}$$

From the above, we find the the transformation law relating the central charges of the representations before and after the action of $\check{\mathsf{R}}$:

$$\mathcal{C}'_i = \mathcal{C}_i, \quad \mathcal{P}'_i = \mathcal{K}_i\frac{\mathcal{P}_1+\mathcal{P}_2}{\mathcal{K}_1+\mathcal{K}_2}, \quad \mathcal{K}'_i = \mathcal{P}_i\frac{\mathcal{K}_1+\mathcal{K}_2}{\mathcal{P}_1+\mathcal{P}_2}, \quad i=1,2, \tag{58}$$

in complete agreement with (3.6) of [2].

At this point, we are able to relate the variables of the free fermion formulation of the $\check{\mathsf{R}}_{12}$ operator with the variables commonly found in the AdS/CFT literature. One can explicitly check that the operator $\mathsf{P}_{12}\check{\mathsf{R}}_{12}$ is to be identified with the AdS/CFT infinite volume S-matrix $S_{12}(p_1,p_2)$ in the string basis provided in [13]. Comparisons of (55) with the fermionic invariance equations (4.11) of the same article leads to the following expressions for the matrices $\mathfrak{D}_j$:

$$\mathfrak{D}_1 = \sqrt{g}\begin{pmatrix} e^{i\frac{p_2}{2}}\eta_1 & e^{i\frac{p_2}{2}}\frac{i}{\eta_1}\left(\frac{x_1^+}{x_1^-}-1\right) \\ -e^{-i\frac{p_2}{2}}\frac{\eta_1}{x_1^+} & e^{-i\frac{p_2}{2}}\frac{x_1^+}{i\eta_1}\left(1-\frac{x_1^-}{x_1^+}\right) \end{pmatrix}, \quad \mathfrak{D}_2 = \sqrt{g}\begin{pmatrix} \eta_2 & \frac{i}{\eta_2}\left(\frac{x_2^+}{x_2^-}-1\right) \\ -\frac{\eta_2}{x_2^+} & \frac{x_2^+}{i\eta_2}\left(1-\frac{x_2^-}{x_2^+}\right) \end{pmatrix}. \tag{59}$$

The matrices $\mathfrak{D}'_j$ are then obtained by performing the formal exchange of indices $1 \leftrightarrow 2$ on the right hand sides of the above equations. Hence, comparing (59) with (54) allows us to find the relations between the free fermion/gluing variables entering $\check{\mathsf{R}}$ and the AdS/CFT variables. Specifically, we get

$$x_k^+ = \frac{\Theta}{\Xi}\frac{b_k c_k}{a_k d_k}v_k, \quad x_k^- = \frac{\Theta}{\Xi}v_k, \quad e^{i\frac{p_k}{2}} = \frac{c_k}{a_k}, \quad \eta_k = e^{-i\frac{\pi}{4}}\frac{\Theta}{\Xi}\frac{c_k v_k}{t_k a_k d_k}, \quad g = \Theta\Xi\ . \tag{60}$$

In the above, we have not restricted ourselves to the unitary case, hence the parameters η_i are unconstrained. We remark that the parameters η_i are to be identified with the $\eta(p_i)$ appearing in [13], even though here they are independent parameters and not functions of the momenta. With the identifications (60), the gluing condition (38) becomes the mass shell condition:

$$x_k^+ + \frac{1}{x_k^+} - x_k^- - \frac{1}{x_k^-} = \frac{i}{g}. \tag{61}$$

4.4 Connections to the quantum symmetry

In the previous sections, we established that the $\check{\mathsf{R}}$ matrix of (52) that we obtained from the Shastry-Shiroishi-Wadati construction of (37) is invariant under the centrally extended superalgebra $\mathfrak{g}$ =su(2|2) $\ltimes \mathbb{R}^2$. In fact, from [2, 13], we know that it is uniquely determined by the symmetry requirement. On the other hand, in section 2.2 we found that the building blocks of the R matrix are invariant under the affine quantum group $\hat{\mathfrak{A}}$, broken to just $\mathfrak{A}$ by the construction in (52). Now, we would like to carefully connect the two. It turns out that we can relate all non-diagonal generators of $\hat{\mathfrak{A}} \oplus \hat{\mathfrak{A}}$ to the fermionic generators of su(2|2) $\ltimes \mathbb{R}^2$. Using (11), (42) as well as table 1, we find

$$
\begin{aligned}
&\mathcal{S}^1{}_1(\mathfrak{B}) = \sqrt{2\Theta\Xi}\,\mathsf{t}^{-1}\mathsf{f}_{0,\uparrow}\mathsf{t}, && \mathcal{S}^2{}_1(\mathfrak{B}) = -\sqrt{2\Theta\Xi}\,\mathsf{t}^{-1}\mathsf{e}_{0,\downarrow}\mathsf{k}_{0,\downarrow}^{-1}\mathsf{t},\\
&\mathcal{S}^1{}_2(\mathfrak{B}) = \sqrt{2\Theta\Xi}\,\mathsf{t}^{-1}\mathsf{f}_{0,\downarrow}\mathsf{t}, && \mathcal{S}^2{}_2(\mathfrak{B}) = \sqrt{2\Theta\Xi}\,\mathsf{t}^{-1}\mathsf{e}_{0,\uparrow}\mathsf{k}_{0,\uparrow}^{-1}\mathsf{t},\\
&\mathcal{Q}^1{}_1(\mathfrak{B}) = \sqrt{\frac{2}{\Theta\Xi}}\,\mathsf{t}^{-1}\mathsf{x}^{-1}\mathsf{f}_{1,\uparrow}\mathsf{x}\mathsf{t}, && \mathcal{Q}^1{}_2(\mathfrak{B}) = \sqrt{\frac{2}{\Theta\Xi}}\,\mathsf{t}^{-1}\mathsf{y}^{-1}\mathsf{e}_{1,\downarrow}\mathsf{k}_{1,\downarrow}^{-1}\mathsf{y}\mathsf{t},\\
&\mathcal{Q}^2{}_1(\mathfrak{B}) = \sqrt{\frac{2}{\Theta\Xi}}\,\mathsf{t}^{-1}\mathsf{x}^{-1}\mathsf{f}_{1,\downarrow}\mathsf{x}\mathsf{t}, && \mathcal{Q}^2{}_2(\mathfrak{B}) = -\sqrt{\frac{2}{\Theta\Xi}}\,\mathsf{t}^{-1}\mathsf{y}^{-1}\mathsf{e}_{1,\uparrow}\mathsf{k}_{1,\uparrow}^{-1}\mathsf{y}\mathsf{t},
\end{aligned}
\tag{62}
$$

where the matrices $\mathfrak{B}$ were defined in (54). Here, we have suppressed the lattice indices in order to improve readability. Furthermore, in (62), we introduced the operators $\mathsf{t}_r := \mathsf{G}_{r,\uparrow}\mathsf{G}_{r,\downarrow}\mathsf{U}_r$ as well as x_r and y_r:

$$
\begin{aligned}
\mathsf{x}_r &:= \Theta^2\Xi^2 x_r^{-2}\mathbf{m}_{r,\uparrow}\mathbf{m}_{r,\downarrow} + \Xi^2\lambda_r v_r^{-1}\mathbf{m}_{r,\uparrow}\mathbf{n}_{r,\downarrow} + \Xi^2\lambda_r v_r^{-1}\mathbf{n}_{r,\uparrow}\mathbf{m}_{r,\downarrow} + x_r^2\mathbf{n}_{r,\uparrow}\mathbf{n}_{r,\downarrow},\\
\mathsf{y}_r &:= x_r^{-2}\mathbf{m}_{r,\uparrow}\mathbf{m}_{r,\downarrow} + \Xi^2\lambda_r v_r^{-1}\mathbf{m}_{r,\uparrow}\mathbf{n}_{r,\downarrow} + \Xi^2\lambda_r v_r^{-1}\mathbf{n}_{r,\uparrow}\mathbf{m}_{r,\downarrow} + \Theta^2\Xi^2 x_r^2\mathbf{n}_{r,\uparrow}\mathbf{n}_{r,\downarrow}.
\end{aligned}
\tag{63}
$$

Having thus established a direct connection between the algebras, we would like to relate the invariance conditions (39) and (55). We described in section 4.2 how to write the operator R_{12} as a function of $\check{\mathsf{R}}_{12}$ and inserting the result into (39) leads to

$$
\check{\mathsf{R}}_{12}\left(\mathsf{T}^{-1}\Delta(\mathsf{J})\mathsf{T}\right) = \left(\mathsf{V}_2^{-1}\mathsf{P}_{12}\mathsf{V}_2\right)\left(\mathsf{T}^{-1}\Delta'(\mathsf{J})\mathsf{T}\right)\left(\mathsf{V}_2^{-1}\mathsf{P}_{12}\mathsf{V}_2\right)\check{\mathsf{R}}_{12}, \quad \forall \mathsf{J} \in \mathfrak{A}\oplus\mathfrak{A}, \tag{64}
$$

with $\mathsf{T} := \mathsf{t}_1\mathsf{t}_2\mathsf{V}_2$. The appearance of the similarity transformation V acting only on the second lattice site can be tracked back to the central operator Z that we had to introduce in the quantum group coproduct back in (13). We can now identify the coproducts of the quantum group elements to the left hand side of the invariance equation (55). A straightforward computation using (11), (13), and (42) leads to

$$\begin{aligned}
\left(\mathcal{S}^1{}_1\right)_1(\mathfrak{D}_1) + \left(\mathcal{S}^1{}_1\right)_2(\mathfrak{D}_2) &= -i\sqrt{2\Theta\Xi}\left(\mathsf{T}^{-1}\Delta(\mathsf{f}_{0,\uparrow})\Delta(\mathsf{F}_\uparrow)\mathsf{T}\right),\\
\left(\mathcal{S}^1{}_2\right)_1(\mathfrak{D}_1) + \left(\mathcal{S}^1{}_2\right)_2(\mathfrak{D}_2) &= -i\sqrt{2\Theta\Xi}\left(\mathsf{T}^{-1}\Delta(\mathsf{f}_{0,\downarrow})\Delta(\mathsf{F}_\downarrow)\mathsf{T}\right),\\
\left(\mathcal{S}^2{}_1\right)_1(\mathfrak{D}_1) + \left(\mathcal{S}^2{}_1\right)_2(\mathfrak{D}_2) &= i\sqrt{2\Theta\Xi}\left(\mathsf{T}^{-1}\Delta(\mathsf{e}_{0,\downarrow}\mathsf{k}_{0,\downarrow}^{-1})\Delta(\mathsf{F}_\downarrow)\mathsf{T}\right),\\
\left(\mathcal{S}^2{}_2\right)_1(\mathfrak{D}_1) + \left(\mathcal{S}^2{}_2\right)_2(\mathfrak{D}_2) &= -i\sqrt{2\Theta\Xi}\left(\mathsf{T}^{-1}\Delta(\mathsf{e}_{0,\uparrow}\mathsf{k}_{0,\uparrow}^{-1})\Delta(\mathsf{F}_\uparrow)\mathsf{T}\right),
\end{aligned}\tag{65}$$

where we have made the choice[19] $z = i$ for the central element of the quantum group. Similar equations can also be written for the Δ' factors. These formulas thus establish a direct link between the invariance of the Shiroishi and Wadati operator R under the quantum group symmetry $\mathfrak{A}\oplus\mathfrak{A}$, (39), and the invariance of the AdS/CFT S-matrix under half of the $\mathfrak{g}$ fermionic generators, (55).

We are then led to the question of whether there exists a similar connection between the remaining fermionic generators of $\mathfrak{g}$ and the broken symmetries of the affine quantum group $\hat{\mathfrak{A}}\oplus\hat{\mathfrak{A}}$. It turns out that, if we define $\mathsf{X} := \mathsf{x}_1\mathsf{x}_2$ and $\mathsf{Y} := \mathsf{y}_1\mathsf{y}_2$, we find for the remaining fermionic generators the relations:

$$\begin{aligned}
\left(\mathcal{Q}^1{}_1\right)_1(\mathfrak{D}_1) + \left(\mathcal{Q}^1{}_1\right)_2(\mathfrak{D}_2) &= \sqrt{\frac{2}{\Theta\Xi}}\left(\mathsf{T}^{-1}\mathsf{X}^{-1}\Delta(\mathsf{f}_{1,\uparrow})\Delta(\mathsf{F}_\uparrow)\mathsf{X}\mathsf{T}\right),\\
\left(\mathcal{Q}^2{}_1\right)_1(\mathfrak{D}_1) + \left(\mathcal{Q}^2{}_1\right)_2(\mathfrak{D}_2) &= \sqrt{\frac{2}{\Theta\Xi}}\left(\mathsf{T}^{-1}\mathsf{X}^{-1}\Delta(\mathsf{f}_{1,\downarrow})\Delta(\mathsf{F}_\downarrow)\mathsf{X}\mathsf{T}\right),\\
\left(\mathcal{Q}^1{}_2\right)_1(\mathfrak{D}_1) + \left(\mathcal{Q}^1{}_2\right)_2(\mathfrak{D}_2) &= -\sqrt{\frac{2}{\Theta\Xi}}\left(\mathsf{T}^{-1}\mathsf{Y}^{-1}\Delta(\mathsf{e}_{1,\downarrow}\mathsf{k}_{1,\downarrow}^{-1})\Delta(\mathsf{F}_\downarrow)\mathsf{Y}\mathsf{T}\right),\\
\left(\mathcal{Q}^2{}_2\right)_1(\mathfrak{D}_1) + \left(\mathcal{Q}^2{}_2\right)_2(\mathfrak{D}_2) &= \sqrt{\frac{2}{\Theta\Xi}}\left(\mathsf{T}^{-1}\mathsf{Y}^{-1}\Delta(\mathsf{e}_{1,\uparrow}\mathsf{k}_{1,\uparrow}^{-1})\Delta(\mathsf{F}_\uparrow)\mathsf{Y}\mathsf{T}\right).
\end{aligned}\tag{66}$$

Hence, up to these similarity transformations, we can relate every generator of $\hat{\mathfrak{A}}\oplus\hat{\mathfrak{A}}$ to an odd element of $\mathfrak{g}$. To summarize, we find that the $\mathcal{S}$ elements of $\mathfrak{g}$ are directly linked to the unbroken $\mathfrak{A}\oplus\mathfrak{A}$ generators, while the $\mathcal{Q}$ ones become symmetries of the Shastry-Shiroishi-Wadati R-matrix only after an appropriate similarity transformation has been applied.

[19]We remind that the symmetry conditions (48) imply $z^2 = -1$ for the quantum group variables.

5 The two layer structure in the AdS variables

Our goal in this section is to provide a direct connection between the two-layer formulation and the one commonly used in the AdS/CFT literature. For that, we wish to rewrite the operator $\check{\mathsf{R}}$ of (52) using the variable identifications of (60). After a rescaling, we identify $\mathsf{P}\check{\mathsf{R}}$ with the matrix S of [13] and notice that $\check{\mathsf{R}}$ depends on more parameters than S. Thus, we can set some of our parameters to special values without damaging the essence of the identification. A very symmetric choice is the following:

$$\Theta = \Xi = \sqrt{g} \quad \text{and} \quad t_k = \sqrt{\frac{x_k^+}{\eta_k}}. \tag{67}$$

This allows us to invert (60) and get

$$v_k = x_k^-, \quad a_k = \frac{\sqrt{i\eta_k x_k^-}}{x_k^- - x_k^+}, \quad b_k = \sqrt{\frac{x_k^+}{i\eta_k}}, \quad c_k = \frac{\sqrt{i\eta_k x_k^+}}{x_k^- - x_k^+}, \quad d_k = \sqrt{\frac{x_k^-}{i\eta_k}}. \tag{68}$$

Plugging the above into the definition of the $\mathsf{R}^{\pm}$ operators of (24), we get for each layer

$$\begin{aligned}
\mathsf{R}^+_{12} &= -\sqrt{\frac{\eta_2}{\eta_1}}\frac{1}{x_2^- - x_2^+}\left(\sqrt{x_2^-}\mathbf{n}_1 + i\sqrt{x_2^+}\mathbf{m}_1\right)\left(\sqrt{x_1^-}\mathbf{n}_2 - i\sqrt{x_1^+}\mathbf{m}_2\right) + \mathbf{c}_1^\dagger\mathbf{c}_2, \\
\mathsf{R}^-_{12} &= \sqrt{\frac{\eta_1}{\eta_2}}\frac{1}{x_1^- - x_1^+}\left(\sqrt{x_2^+}\mathbf{n}_1 + i\sqrt{x_2^-}\mathbf{m}_1\right)\left(\sqrt{x_1^+}\mathbf{n}_2 - i\sqrt{x_1^-}\mathbf{m}_2\right) + \mathbf{c}_2^\dagger\mathbf{c}_1 \ .
\end{aligned} \tag{69}$$

Furthermore, the similarity transformation matrices of (49) become

$$\begin{aligned}
\mathsf{U}_k &= \mathbf{m}_{k,\uparrow}\mathbf{m}_{k,\downarrow} + \sqrt{\frac{x_k^+}{\eta_k}}\left(\mathbf{m}_{k,\uparrow}\mathbf{n}_{k,\downarrow} + \mathbf{n}_{k,\uparrow}\mathbf{m}_{k,\downarrow}\right) + \frac{i\eta_k}{x_k^+ - x_k^-}\mathbf{n}_{k,\uparrow}\mathbf{n}_{k,\downarrow} \\
\mathsf{V}_k &= (\mathbf{m}_{k,\uparrow} - i\mathbf{n}_{k,\uparrow})(\mathbf{m}_{k,\downarrow} - i\mathbf{n}_{k,\downarrow}) \ ,
\end{aligned} \tag{70}$$

After expressing the operator $\check{\mathsf{R}}_{12}$ in the new variables, we rescale it by the factor $\frac{x_1^+x_2^+ - x_1^-x_2^-}{(x_1^- - x_1^+)(x_2^- - x_2^+)}\frac{x_1^- - x_2^+}{x_1^- + x_2^+}$. The final expression then reads

$$\begin{aligned}
\check{\mathsf{R}}_{12} &= \frac{(x_1^- - x_1^+)(x_2^- - x_2^+)}{x_1^+x_2^+ - x_1^-x_2^-}\mathsf{V}_2^{-1}\mathsf{P}_{12}\mathsf{U}_1^{-1}\mathsf{U}_2^{-1}\left\{\frac{x_2^- + x_1^+}{x_1^- - x_2^+}\left(\frac{x_2^+\eta_1}{x_1^+\eta_2}\frac{e^{i\frac{p_2}{2}} - e^{-i\frac{p_2}{2}}}{e^{i\frac{p_1}{2}} - e^{-i\frac{p_1}{2}}}\mathsf{R}^+_{12,\uparrow}\mathsf{R}^+_{12,\downarrow}\right.\right. \\
&\left.\left. + \frac{x_1^-\eta_2}{x_2^-\eta_1}\frac{e^{i\frac{p_1}{2}} - e^{-i\frac{p_1}{2}}}{e^{i\frac{p_2}{2}} - e^{-i\frac{p_2}{2}}}\mathsf{R}^-_{12,\uparrow}\mathsf{R}^-_{12,\downarrow}\right) + \frac{x_2^+ + x_1^-}{x_1^- - x_2^+}\left(\mathsf{R}^+_{12,\uparrow}\mathsf{R}^-_{12,\downarrow} + \mathsf{R}^-_{12,\uparrow}\mathsf{R}^+_{12,\downarrow}\right)\right\}\mathsf{U}_1\mathsf{U}_2\mathsf{V}_2 \ ,
\end{aligned} \tag{71}$$

where as usual the worldsheet momentum is given by the rapidity variables as $e^{ip} = \frac{x_k^+}{x_k^-}$. Thus, the above equation (71) finally makes completely explicit the two-layer structure of the AdS/CFT S-matrix.

6 Outlook

In this paper we have (re)constructed the S-matrix for $AdS_5 \times S^5$ without relying on the central extension of $su(2|2)$ [2]. It turned out that the AdS/CFT S-matrix [5, 13] is essentially a special case of Shiroishi and Wadati's generalized Hubbard R-matrix [14] which appeared about 10 years earlier. We had to impose the symmetry condition (48) on Shiroishi and Wadati's R-matrix to obtain the AdS/CFT S-matrix. The only known S-matrix[20] which contains the AdS/CFT S-matrix other than Shiroishi and Wadati's R-matrix is the q-deformed S-matrix proposed in [22]. In this context, whether relaxing the symmetry condition corresponds to the q-deformation or not is an interesting open question.

An ambitious goal will be to construct an infinite-dimensional R-matrix for the AdS/CFT correspondence in a multilayer approach, possibly in a four layer model, by generalizing our formalism. This R-matrix would of course need to realize all the intrinsic structures suggested by the asymptotic Bethe Ansatz equations [23], the Y-system [24], thermodynamic Bethe Ansatz equations [25], nonlinear integral equations [26] and a group theoretical argument on characters [27]. Some help in that direction might come by a better understanding of the role that the tetrahedron Zamolodchikov equations, as well as the three-dimensional integrability structures they allude to, play in the AdS/CFT correspondence. Bazhanov and Sergeev obtained in [28] solutions of the tetrahedron equations systematically and uncovered a relation to the quantum affine algebra $\mathrm{U}_q(\widehat{\mathrm{sl}}(n))$. It will be very interesting to see how their method fits into our problem.

[20] that intertwines between two four-dimensional vector spaces.

Acknowledgments

The authors thank Vladimir Bazhanov, Vladimir Kazakov and Anton Zabrodin for discussions. We also thank Takuya Matsumoto for a correspondence on [18]. The work of ZT is supported by SFB 647 "Space-Time-Matter". The research leading to these results has received funding from the European Union Marie Curie International Research Staff Exchange Scheme UNIFY under grant agreement nº 269217.

A The double free fermion condition

In this appendix, we would like to present another aspect of the connection between the Shastry-Shiroishi-Wadati R-matrix of (37) and the AdS/CFT S-matrix. As pointed out in [15], the R-matrix of the free fermion model satisfies the condition (3). We want to introduce a generalization of this condition, the double free fermion condition, and to write it for the AdS/CFT S-matrix. Unlike in the rest of the main text, here we prefer to work with ordinary matrices instead of oscillators. To that end, by using the Jordan-Wigner transformation[21], the operator $R_{12}(A)$ in (2) is mapped to the R-matrix of the free fermion 6-vertex model:

$$R(A) = \begin{pmatrix} -a & 0 & 0 & 0 \\ 0 & -ib & 1 & 0 \\ 0 & 1 & -ic & 0 \\ 0 & 0 & 0 & -d \end{pmatrix}, \qquad ad - bc = 1. \tag{A1}$$

Then we can introduce the matrices $R^0(A_1, A_2) := R(A_2 A_1^{-1})$ as well as $R^1(A_1, A_2) := R(A_2\sigma_3 A_1^{-1}\sigma_3)(1 \otimes \sigma_3)$ for $A_1, A_2 \in \mathrm{SL}(2,\mathbb{C})$, which correspond to their oscillator counterparts of (22). On the linear space of 4×4 complex matrices, we can introduce the following bilinear form:

$$(R, R') := R_{11}R'_{44} + R_{44}R'_{11} + R_{22}R'_{33} + R_{33}R'_{22} - R_{23}R'_{32} - R_{32}R'_{23} \; . \tag{A2}$$

Then we find that the matrices R^i satisfy the relations

$$\left(R^0, R^0\right) = \left(R^1, R^1\right) = \left(R^0, R^1\right) = 0 \; . \tag{A3}$$

[21] One has to apply the Jordan-Wigner transformation to $\check{R}_{12}$ first and then multiply it by the non-graded permutation matrix to obtain (A1). See the discussion in the appendix of [21].

The first two follow from $\det A_1 = \det A_2 = 1$. We refer to the third relation as the *compatibility equation*. It does not depend on the condition that the determinant of the matrices A_1 and A_2 is one, but follows instead from the specific form of the R^i matrices. Let us now introduce a 16×16-dimensional matrix $\mathsf{R} = \sum_{i,j,k,l=1}^{4} \mathsf{R}^{ik}{}_{jl} E_{ij}\otimes E_{kl}$, where E_{ab} is a 4×4 matrix unit $(E_{ab})_{ij} = \delta_{ai}\delta_{bj}$. We require that R has only 36 non-vanishing entries by imposing:

$$\mathsf{R}^{ik}{}_{jl} = 0 \text{ if } (i,j) \text{ or } (k,l) \notin \{(1,1),(2,2),(2,3),(3,2),(3,3),(4,4)\}\,. \tag{A4}$$

We also require that R satisfies the free fermion condition both in the first space and the second space in the tensor product. In components, this can be written as

$$\mathsf{R}^{i1}{}_{j1}\mathsf{R}^{i4}{}_{j4} + \mathsf{R}^{i2}{}_{j2}\mathsf{R}^{i3}{}_{j3} - \mathsf{R}^{i2}{}_{j3}\mathsf{R}^{i3}{}_{j2} = \mathsf{R}^{1k}{}_{1l}\mathsf{R}^{4k}{}_{4l} + \mathsf{R}^{2k}{}_{2l}\mathsf{R}^{3k}{}_{3l} - \mathsf{R}^{2k}{}_{3l}\mathsf{R}^{3k}{}_{2l} = 0\,. \tag{A5}$$

We call this condition the *double free fermion condition*. Taken together, (A3) imply that any matrix of the form

$$\mathsf{R} \;=\; \sum_{r,s=0}^{1} c_{rs} R^r(A_1,A_2)\otimes R^s(A_3,A_4), \tag{A6}$$

with $A_r \in \mathrm{SL}(2,\mathbb{C})$, satisfies the double free fermion condition for any value of the complex parameters c_{rs}. Since the Shastry-Shiroish-Wadati R-matrix of (37), written here using matrices instead of oscillators, is precisely of the form (A6), we know that the double free fermion condition is obeyed. We would now like to find the implications of this for the AdS/CFT correspondence. First, we transcribe the S-matrix of [2] by setting $|\,\phi^1\rangle = e_1$, $|\,\phi^2\rangle = e_4$, $|\,\psi^1\rangle = e_2$, $|\,\psi^2\rangle = e_3$, where e_i are canonical basis vectors. Then, using the variable identification (60), one can relate the Shastry-Shiroish-Wadati R to S. We define

$$W := \sigma_3\otimes \mathrm{diag}(1,-i,-i,1)\otimes \mathbb{1}_2, \quad \bar{P}_{12} := \mathbb{1}_2 \otimes \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix} \otimes \mathbb{1}_2 \tag{A7}$$

and obtain

$$S_{12} = W P_{12}\big(U_1\otimes U_2\big)\bar{P}_{12}\mathsf{R}\bar{P}_{12}\big(U_1^{-1}\otimes U_2^{-1}\big)W^{-1}, \tag{A8}$$

where the precise expression of the diagonal matrices U_i does not matter for what follows. The matrices W were introduced following [29] to account for the difference of grading, since S_{12} graded, while the Shastry-Shiroishi-Wadati R-matrix of (A6) isn't. Inserting R into (A5) and making use of (A8) to express everything in the coefficients of S, leads to the following[22] three quadratic equations:

$$
\begin{aligned}
A_{12}D_{12} &= H_{12}K_{12} - G_{12}L_{12} \\
B_{12}E_{12} - C_{12}F_{12} &= H_{12}K_{12} - G_{12}L_{12} \\
A_{12}E_{12} + B_{12}D_{12} &= 2\left(H_{12}K_{12} + G_{12}L_{12}\right) \ .
\end{aligned}
\tag{A9}
$$

We dispensed in (A8) with writing down the precise form of U_i, since its only effect is to multiply the above equation by global factor. It turns out that the first two equations are to be found in the article [22], where they were pointed out in (3.6) as a curious occurrence in the context of unitarity of the R matrix. The third one on the other hand follows for the others by making use of the mass-shell condition (61). Here, we derive these three equations as a direct consequence of the two-layer structure of the model.

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