Inverse Scattering and the Geroch Group

Despoina Katsimpouri[1], Axel Kleinschmidt[1,2]

and Amitabh Virmani[1]

[1]*Max-Planck-Institut für Gravitationsphysik (Albert-Einstein-Institut)*
Am Mühlenberg 1, DE-14476 Potsdam, Germany

[2]*International Solvay Institutes*
ULB-Campus Plaine CP231, BE-1050 Brussels, Belgium

We study the integrability of gravity-matter systems in $D = 2$ spatial dimensions with matter related to a symmetric space G/K using the well-known linear systems of Belinski–Zakharov (BZ) and Breitenlohner–Maison (BM). The linear system of BM makes the group structure of the Geroch group manifest and we analyse the relation of this group structure to the inverse scattering method of the BZ approach in general. Concrete solution generating methods are exhibited in the BM approach in the so-called soliton transformation sector where the analysis becomes purely algebraic. As a novel example we construct the Kerr–NUT solution by solving the appropriate purely algebraic Riemann–Hilbert problem in the BM approach.

Contents

1 Introduction

In late 1970s and early 1980s a large variety of solution generating techniques for the four-dimensional vacuum Einstein equations and Einstein-Maxwell equations were explored, for an overview see [1]. It was later realized in the case of two commuting and hypersurface orthogonal Killing vectors that all these approaches are nothing but different manifestations of the integrability of the corresponding effectively two-dimensional system that can be exhibited by means of a linear system or Lax pair [2, 3, 4]. Several authors made efforts to find interrelations between these different methods. Cosgrove [5] took a computational approach whereas Breitenlohner and Maison [3] concentrated on unraveling the group theoretical structure behind these techniques, taking up ideas of Geroch [6] and Julia [7]. Other relevant work includes [8, 9, 10, 11, 12, 13, 14]. In the present work, we revisit these studies to further elucidate the interrelation between the various techniques.

The best developed technique for generating solutions is that of Belinski and Zakharov [2, 4], henceforth BZ for short and often called the inverse scattering method. It has been very successful in constructing novel solutions in both four- and five-dimensional vacuum gravity. The method involves some rather special adjustments to certain quantities before new physical solutions can be obtained. However, there are problems when applying this method to different gravity-matter systems like those of interest in supergravity where the implementation of the same adjustments fails and it is not guaranteed that an inverse scattering transformation preserves all features required of a solution to the gravity-matter equations [15]. In the group theoretical framework of Breitenlohner and Maison (BM) [3] this problem does not arise since the solution generating transformations form the so-called Geroch group (an affine group) and by the group property any transformation will generate a new solution. The drawback of the BM method is that it is not as easy to implement and does not always operate directly on the physical quantities. Despite these shortcomings, the promise of the BM approach is that it can be applied to various general settings of interest.

In order to illustrate this point, we consider for concreteness $D = 4$ gravity with a space-like and a time-like Killing vector such that the system is effectively two-dimensional and we are in the realm of so-called stationary axisymmetric solutions. The infinite-dimensional affine symmetry group can be viewed as the closure of two finite-dimensional symmetry groups that act on the space of solutions [6, 3]. The first one is typically called the Matzner–Misner group $\mathrm{SL}(2, \mathbb{R})_{\mathrm{MM}}$ and consists of area preserving diffeomorphisms of the orbit space of the two Killing vectors (the 'torus' that one reduced on, the Killing vector orbits are not necessarily compact). The other group is called the Ehlers group $\mathrm{SL}(2, \mathbb{R})_{\mathrm{E}}$ and is a hidden symmetry already of the three-dimensional model. The fields that it acts on are partly formed from dualising a Kaluza–Klein vector field (in $D = 3$) to a scalar field; therefore it does not act directly on the metric components. Combining $\mathrm{SL}(2, \mathbb{R})_{\mathrm{MM}}$ and $\mathrm{SL}(2, \mathbb{R})_{\mathrm{E}}$ yields the infinite-dimensional affine Geroch group [6, 3].

The linear systems of BZ and BM display the integrability of the $D = 2$ theory in terms of generating functions that depend on spectral parameters besides the dependence on the two-dimensional space(-time) coordinates x^m. There are in principle four choices for the linear system and generating functions at this point that can be summarized in the following diagram:

	Ehlers	Matzner–Misner
BZ	$\Psi_{\mathrm{E}}(\lambda, x) \quad \longleftrightarrow$	$\Psi_{\mathrm{MM}}(\lambda, x)$
	$\updownarrow$	$\updownarrow$
BM	$\mathcal{V}_{\mathrm{E}}(t, x) \quad \longleftrightarrow$	$\mathcal{V}_{\mathrm{MM}}(t, x)$

The horizontal arrows in this diagram correspond to duality transformations between some of the two-dimensional scalar fields. At the level of the Lie group this corresponds to the action of an (outer) automorphism, now known as the Kramer–Neugebauer mapping [16, 3, 17]. The vertical lines relate the two linear systems and are the main subject of the present work. Note that the spectral parameters of the generating functions are different in the two rows. The

traditional inverse scattering method of BZ operates on the physical variables and so is located in the Matzner–Misner column of the diagram. In fact it uses a slight extension of the Matzner–Misner variables in that it also treats the volume of the two-dimensional orbit space and so corresponds to $\mathrm{GL}(2,\mathbb{R})_{\mathrm{MM}}$ rather than $\mathrm{SL}(2,\mathbb{R})_{\mathrm{MM}}$. In our work, we will focus mainly on the left column where the group theoretical structures are a bit nicer, e.g., empty Minkowski space corresponds to the identity function. We will always display the subscripts E and MM to avoid any confusion.

It follows from the work of BM that any gravity-coupled matter system in $D = 3$ with symmetric space target $\mathrm{G_E}/\mathrm{K_E}$ becomes integrable when an additional Killing symmetry is present. Here, $\mathrm{G_E}$ is the generalization of the Ehlers symmetry $\mathrm{SL}(2,\mathbb{R})_{\mathrm{E}}$ to other systems. The group of symmetry transformations form the (non-twisted) affine extension of $\mathrm{G_E}$ and we will restrict to simple $\mathrm{G_E}$ for simplicity. Beyond this general fact, it is quite hard to construct actual solutions explicitly using this method since one has to solve a matrix valued Riemann–Hilbert problem [1, 3, 18], sometimes also referred to as Birkhoff factorization. In the case analogous to the soliton transformations of BZ, however, the problem reduces to a linear algebra system that can be solved as shown in [19] which we will review below since the work is not published.

Our chief motivation to study the different formulations of integrability is to find their interrelation and, by this, to make new methods available for generating solutions of (super-)gravity beyond the cases that have been covered so far. Of particular interest are cases when $\mathrm{G_E}$ is an exceptional symmetry group and other cases that arise in string theory. There are no known established systematic techniques for constructing non-supersymmetric solutions that exploit the integrability structure of supergravity theories. With such techniques at hand one can construct a variety of new solutions, in particular, new black hole solutions generalizing [20, 21] and new fuzzball solutions generalizing [22, 23].

The structure of this article is as follows. In sections 2 and 3 we introduce the linear systems of Breitenlohner–Maison and of Belinski–Zakharov in Ehlers form and elucidate their interrelation. In section 4 we study meromorphic generating functions and solve the related linear systems algebraically. Section 5 gives the explicit example of the Kerr–NUT solution in both cases. We conclude in section 6. Certain technical computations have been relegated to the appendices.

2 Breitenlohner–Maison linear system

In this section we review the linear system of Breitenlohner and Maison (BM) [3, 24, 25, 26]. This linear system arises from considering a $D = 3$ gravity-matter system with target being a symmetric space $\mathrm{G_E}/\mathrm{K_E}$ where $\mathrm{G_E}$ is the global symmetry group that we refer to as Ehlers symmetry. Our conventions are such that the three-dimensional space has signature $+++$ and we consider the space of solutions admitting an additional (axial) symmetry so that the theory becomes effectively two-dimensional. This two-dimensional system can be shown to be integrable.

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2.1 $D = 3$ model with G_E/K_E matter

Consider a three-dimensional Euclidean gravity-matter system with action

$$S_E = \int d^3x \sqrt{-g} \left(R - \frac{1}{2} \langle P_{E,\mu} | P_{E,\nu} \rangle g^{\mu\nu} \right), \tag{2.1}$$

where $\langle \cdot | \cdot \rangle$ is the symmetric invariant bilinear form on the Lie algebra $\mathfrak{g}_E$ of the real Lie group G_E. The group K_E is a subgroup of G_E with dimension equal to that of the maximal compact subgroup and the coset space G_E/K_E is a (pseudo-)Riemannian symmetric space. The subgroup K_E is fixed by some involutive automorphism τ on G_E [27]. This induces an involution θ on the Lie algebra (equal to the Cartan involution when K_E is maximally compact) which splits $\mathfrak{g}_E = \mathfrak{k}_E \oplus \mathfrak{p}_E$. Let $V_E \in G_E/K_E$ be a coset representative (e.g. in Borel gauge according to the Iwasawa theorem in a patch where it applies); then we decompose

$$\partial_\mu V_E V_E^{-1} = P_{E,\mu} + Q_{E,\mu} \tag{2.2}$$

with

$$Q_{E,\mu} = \frac{1}{2} \left(\partial_\mu V_E V_E^{-1} - (\partial_\mu V_E V_E^{-1})^\# \right), \tag{2.3a}$$

$$P_{E,\mu} = \frac{1}{2} \left(\partial_\mu V_E V_E^{-1} + (\partial_\mu V_E V_E^{-1})^\# \right), \tag{2.3b}$$

where we have defined the 'generalized transpose' $x^\# = -\theta(x)$ on the Lie algebra. (For $\mathfrak{sl}(n, \mathbb{R})$ it is the standard transpose when $\mathfrak{k} = \mathfrak{so}(n)$.) The two components in the splitting (2.3) satisfy

$$Q_E^\# = -Q_E, \qquad P_E^\# = P_E. \tag{2.4}$$

The map $\#$ is an anti-involution and its group version will be denoted by the same symbol and is also an anti-involution: For $g, h \in G_E$ one has $g^\# = \tau(g^{-1}) = \tau(g)^{-1}$ and $(gh)^\# = h^\# g^\#$.

The symmetry transformations acting on V_E are

$$V_E(x) \to k_E(x) V_E(x) g_E \tag{2.5}$$

with constant $g_E \in G_E$ (global transformations) and varying $k_E \in K_E$ (gauge transformations). Under these transformations, the Lie algebra valued quantities transform as

$$Q_{E,\mu} \to k_E Q_{E,\mu} k_E^{-1} + \partial_\mu k_E k_E^{-1}, \tag{2.6a}$$

$$P_{E,\mu} \to k_E Q_{E,\mu} k_E^{-1}, \tag{2.6b}$$

i.e., Q_E as a K_E-connection and P_E transforms K_E-covariantly.

A useful quantity associated with $V_E \in G_E/K_E$ is the 'monodromy matrix'

$$M_E = V_E^\# V_E \tag{2.7}$$

that transforms as $M_E \to g_E^\# M_E g_E$ under (2.5) since $k_E^\# k_E = \mathbb{1}$. It is hence insensitive to the K_E-gauge chosen for the coset representative V_E and only transforms under the global G_E transformation.

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The $D = 3$ equations of motion derived from (2.1) are

$$R_{\mu\nu} - \frac{1}{2}\langle P_{\mathrm{E},\mu}|P_{\mathrm{E},\nu}\rangle = 0, \tag{2.8a}$$

$$\partial_\mu\left(\sqrt{-g}g^{\mu\nu}P_{\mathrm{E},\nu}\right) - \sqrt{-g}g^{\mu\nu}[Q_{\mathrm{E},\mu}, P_{\mathrm{E},\nu}] = 0. \tag{2.8b}$$

For convenience, we introduce the K_{E}-covariant derivative

$$D_{\mathrm{E},\mu} = \partial_\mu - [Q_{\mathrm{E},\mu}, \cdot] \tag{2.9}$$

in terms of which (2.8b) becomes

$$D_{\mathrm{E},\mu}(\sqrt{-g}P_{\mathrm{E}}^\mu) = 0. \tag{2.10}$$

2.2   Reduction to $D = 2$

If the system admits an axial isometry ∂_ϕ we reduce the metric according to

$$ds_3^2 = f_{\mathrm{E}}^2 ds_2^2 + \rho^2 d\phi^2. \tag{2.11}$$

The function f_{E} will be referred to as the conformal factor of the effective two-dimensional metric. We label the two-dimensional coordinates as x^m. The two-dimensional metric ds_2^2 is assumed to be flat by appropriate choice of coordinates. Note that there is no Kaluza–Klein vector A_m of the type $(d\phi + A_m dx^m)^2$ in (2.11) since it carries no degrees of freedom and can be set to zero without loss of generality.

With this ansatz, the equations of motion (2.8) then imply

$$\Box\rho = 0, \tag{2.12}$$

which we solve by choosing Weyl canonical coordinates $x^m = (\rho, z)$ on the flat two-dimensional space so that $ds_2^2 = d\rho^2 + dz^2$. We let z be the conjugate variable to ρ such that $\partial_\rho = \epsilon^{\rho z}\partial_z = \star_2\partial_z$. It is often useful to combine the two real variables into a single complex variable (and its complex conjugate) which we denote by

$$x^\pm = \frac{1}{2}(z \mp i\rho) \tag{2.13}$$

in analogy with light-cone coordinates that would arise if the two-dimensional base was Minkowskian as for colliding plane wave solutions. Note that (2.13) implies $\partial_\pm\rho = \pm i$ and $\star_2\partial_\pm = \pm i\partial_\pm$.

With this choice the remaining equations are equivalent to

$$\pm i f_{\mathrm{E}}^{-1}\partial_\pm f_{\mathrm{E}} = \frac{\rho}{2}\langle P_{\mathrm{E},\pm}|P_{\mathrm{E},\pm}\rangle, \tag{2.14a}$$

$$D_m(\rho\, P_{\mathrm{E}}^m) = 0. \tag{2.14b}$$

The first equation is a constraint and yields the conformal factor by a single integration and is therefore of secondary interest. The main equation of interest in this paper is the last equation (2.14b). A key result of [3] is that this equation is integrable and has an underlying symmetry structure associated with the affine extension of $\mathrm{G_E}$.

2.3 BM linear system in general

Consider the generalization of $V_E \in G_E/K_E$ to also depend on some spectral parameter t (we suppress the space-time dependence) $V_E \longrightarrow \mathcal{V}_E(t)$, where

$$\mathcal{V}_E(t) = V_E^{(0)} + t V_E^{(1)} + \frac{1}{2} t^2 V_E^{(2)} + \dots . \tag{2.15}$$

importantly is regular in t around $t = 0$ and the limit $t \to 0$ gives back the original V_E:

$$\lim_{t \to 0} \mathcal{V}_E(t) = \mathcal{V}_E(0) = V_E^{(0)} := V_E. \tag{2.16}$$

Consider the linear system [3, 26]

$$\partial_\pm \mathcal{V}_E \mathcal{V}_E^{-1} = \frac{1 \mp it}{1 \pm it} P_{E,\pm} + Q_{E,\pm}, \tag{2.17}$$

where Q_E and P_E are independent of the spectral parameter t and are defined in terms of the t-independent V_E as in (2.3).[1] The integrability condition for (2.17) is equivalent to the equation of motion (2.14b) if and only if the spectral parameter t satisfies the differential equation

$$t^{-1} \partial_\pm t = \frac{1 \mp it}{1 \pm it} \rho^{-1} \partial_\pm \rho. \tag{2.18}$$

This differential equation can be integrated [3] to an equation for t which we write in the more conventional form in terms of the Weyl coordinates (ρ, z) (cf. (2.13))

$$t^2 - \frac{2t}{\rho}(z - w) - 1 = 0, \tag{2.19}$$

where w is an integration constant. This quadratic equation has two solution branches

$$t_\pm = \frac{1}{\rho} \left[(z - w) \pm \sqrt{(z-w)^2 + \rho^2} \right]. \tag{2.20}$$

Equation (2.19) defines a two-sheeted Riemann surface over the two-dimensional flat base. We take the solution with the $+$ sign to be the physical sheet and when we write t we always mean t_+ unless indicated otherwise. We will refer to t as the *space-time dependent spectral parameter* and to w as the *constant spectral parameter*.[2]

We will refer to (2.17) as the *BM linear system (in Ehlers form)* and we have just reviewed how its integrability condition gives rise to the equations of motion of the $D = 3$ gravity-matter

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[1]In a general coordinate system, the linear system takes the form

$$\partial_m \mathcal{V}_E \mathcal{V}_E^{-1} = Q_{E,m} + \frac{1 - t^2}{1 + t^2} P_{E,m} - \frac{2t}{1 + t^2} \epsilon_{mn} P_E^n.$$

[2]When written in terms of the 'light-cone' coordinates (2.13), equation (2.20) becomes

$$t_\pm = -i \left[\frac{\sqrt{w - 2x^+} \mp \sqrt{w - 2x^-}}{\sqrt{w - 2x^+} \pm \sqrt{w - 2x^-}} \right].$$

system (2.1) in the presence of a Killing isometry. This establishes the integrability of the equation (2.14b); the conformal factor f_E can then be obtained by integrating (2.14a) [3]. The linear system (2.17) is vastly underdetermined since it represents two differential equations for a function of three variables. There is an infinity of integration constants associated with this system.

Besides giving the integrability of (2.14b) the BM linear system also serves to unveil the group theory underlying the system. The original t-independent $V_E \in G_E/K_E$ transformed under global Ehlers transformations $g_E \in G_E$ as $V_E \to k_E V_E g_E$ (cf. (2.5)), where $k_E \in K_E$ is the usual local compensator required to restore a chosen gauge for the coset representative. The presence of the spectral parameter now suggests to enlarge the set of global symmetry transformations by allowing g_E to depend on the constant spectral parameter w:

$$\mathcal{V}_E(t) \to k_E(t)\mathcal{V}_E(t)g_E(w). \tag{2.21}$$

As indicated, the compensator is now also t-dependent as it has to be chosen such that the transformed $\mathcal{V}_E(t)$ is regular around $t = 0$ as in (2.15). This enlarged set of global transformations consists therefore of functions $g_E(w)$, i.e., maps of the type $\mathbb{C} \to G_E$, where we impose that $g_E(w)$ admits an expansion around $w = \infty$ in order to remain expandable as in (2.15). These maps include transformations from $S^1 \subset \mathbb{C}$ into G_E and (under additional regularity assumptions) will lead to the loop group $\hat{G}_E$ associated with the Ehlers group G_E. Therefore the group underlying the integrability in $D = 2$ includes the infinite-dimensional loop group; in fact the extension to the full affine group is active [7, 3] where the central extension acts on the conformal factor f_E (see below). We note that besides the affine group one can also define the action of the (centerless) Virasoro algebra which arises from arbitrary reparametrisations of the constant spectral parameter [28, 29, 24]. Together with the infinitesimal affine transformations one obtains a semi-direct product in the standard way. We will not use the Virasoro symmetry in this paper.

The involution # extends to functions $\mathcal{V}_E(t)$ by

$$(\mathcal{V}_E(t))^{\#} = \mathcal{V}_E^{\#}\left(-\frac{1}{t}\right). \tag{2.22}$$

One can use # to split the loop algebra into an invariant and an anti-invariant part (generalizing $\mathfrak{p}_E$ and $\mathfrak{k}_E$ above). Now, it is important that the right hand side of (2.17) is anti-invariant under (the Lie algebra version of) # and therefore belongs to the 'compact' subalgebra of the affine algebra based on $\mathfrak{g}_E$. This anti-invariance implies that $((\mathcal{V}_E(t))^{\#})^{-1}$ is a solution to the linear system if $\mathcal{V}_E(t)$ is a solution. In general, the two solutions related by this involutive mapping will be different. The mapping implies that the monodromy matrix

$$\mathcal{M}_E = \mathcal{M}_E(w) = (\mathcal{V}_E(t))^{\#} \, \mathcal{V}_E(t) = \mathcal{V}_E^{\#}\left(-\frac{1}{t}\right)\mathcal{V}_E(t) \tag{2.23}$$

is independent of the space-time coordinates x^m and therefore a function of w alone. The matrix $\mathcal{M}_E(w)$ is invariant under the application of # and simultaneously exchanging $t \to -1/t$. We note that this is evident from (2.19) which implies that $w(t, x)$ is invariant under $t \to -1/t$.

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Constructing new solutions of the linear system by means of the Geroch symmetry proceeds along the following chain of steps[3]

$$V_{\rm E} \to \mathcal{V}_{\rm E}(t) \to \mathcal{M}_{\rm E}(w) \to \mathcal{M}_{\rm E}^g(w) \to \mathcal{V}_{\rm E}^g(t) \to V_{\rm E}^g, \tag{2.24}$$

where we introduced the notation

$$\mathcal{M}_{\rm E}(w) \to \mathcal{M}_{\rm E}^g(w) := g_{\rm E}^\#(w)\mathcal{M}_{\rm E}(w)g_{\rm E}(w) \tag{2.25}$$

for the transformed solution. The individual steps in (2.24) starting from a given seed solution $V_{\rm E}$ are: (i) find a corresponding generating function $\mathcal{V}_{\rm E}(t)$ that solves the linear system (2.17), (ii) compute the associated monodromy, (iii) transform the monodromy under a global transformation $g(w)$ as in (2.25), (iv) factorize the new monodromy into a new generating function $\mathcal{V}_{\rm E}^g(t)$ and (v) take the limit $t \to 0$ to find the new solution.

For practical purposes, the main difficulty resides in step (iv) in factorizing the transformed $\mathcal{M}_{\rm E}^g(w)$ as

$$\mathcal{M}_{\rm E}^g(w) = (\mathcal{V}_{\rm E}^g(t))^\# \mathcal{V}_{\rm E}^g(t) \tag{2.26}$$

with the new $\mathcal{V}_{\rm E}^g(t)$ having an expansion as in (2.15). This is a Riemann–Hilbert problem [3] whose solution is in general hard to obtain. In the particular case of meromorphic $\mathcal{M}_{\rm E}^g(w)$ with single poles in w of certain simple type one can reduce the problem to a set of linear algebraic equations. This is the case of soliton charging transformations that will be discussed further in section 4. Once the new $\mathcal{V}_{\rm E}^g(t)$ has been obtained, one can recover the solution to the gravity-matter system (2.14) by taking the limit $t \to 0$ and obtain $V_{\rm E}^g \in {\rm G}_{\rm E}/{\rm K}_{\rm E}$ that characterises the physical fields.

Besides the knowledge of the coset 'scalars' $V_{\rm E}^g \in {\rm G}_{\rm E}/{\rm K}_{\rm E}$ one also requires the new conformal factor $f_{\rm E}^g$ in (2.14). This can be obtained from a simple integration of (2.14a) but it also follows from group theoretic properties using the central extension. This is discussed in detail in [3] to which we refer for the general expression. In section 4 we will present the formula in the case of soliton transformations.

We note that a trivial solution of the equations (2.17) and (2.14a) is given by

$$\mathcal{V}_{\rm E}(t) = \mathbb{1} \quad \text{and} \quad f_{\rm E} = 1. \tag{2.27}$$

This solution will be referred to as flat space as it corresponds to the Minkowski vacuum in the four-dimensional case.

3 Belinski–Zakharov linear system

In this section we present the linear system used by Belinski and Zakharov (BZ) [2, 4]. We will not present it in the standard form which uses what was called the Matzner–Misner formulation

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[3]We present here the solution generating method based on the monodromy matrix $\mathcal{M}_{\rm E}(w)$. Alternatively, one could work at the level of the generating function $\mathcal{V}_{\rm E}(t)$ and the transformation (2.21); however, the step of finding the compensator $k_{\rm E}(t)$ in (2.21) is typically very hard.

in the introduction. Rather, we will use the Ehlers description to make contact with the discussion in the preceding section. (The Matzner–Misner version and its relation to BM is discussed in appendix B.)

3.1 BZ Ehlers linear system

Equation (2.14b) for the G_E/K_E coset fields admits an alternative Lax pair that can be written as

$$D_1 \Psi_E = \frac{\rho V - \lambda U}{\lambda^2 + \rho^2} \Psi_E, \qquad D_2 \Psi_E = \frac{\rho U + \lambda V}{\lambda^2 + \rho^2} \Psi_E, \qquad (3.1)$$

where λ is the (space-time independent) spectral parameter of BZ and $\Psi_E(\lambda, \rho, z)$ is the generating function such that the matrix $M_E = V_E^\# V_E$ of (2.7) is recovered for $\lambda = 0$:

$$M_E(\rho, z) = \Psi_E(0, \rho, z). \qquad (3.2)$$

The matrices U, V are defined as $U = \rho \partial_\rho M_E M_E^{-1}$, $V = \rho \partial_z M_E M_E^{-1}$, and the differential operators D_1, D_2 are

$$D_1 = \partial_z - \frac{2\lambda^2}{\lambda^2 + \rho^2} \partial_\lambda, \qquad D_2 = \partial_\rho + \frac{2\lambda\rho}{\lambda^2 + \rho^2} \partial_\lambda. \qquad (3.3)$$

The operators D_1 and D_2 commute and the associated integrability condition of the linear system (3.1) is equivalent to the desired non-linear equation (2.14b).

Solutions of the BZ linear system (3.1) are constructed using the inverse scattering method [2]. One starts from a 'seed' $\Psi_{E,0}$, that is 'dressed' to obtain a new solution Ψ_E through

$$\Psi_E(\lambda) = \chi(\lambda) \Psi_{E,0}(\lambda) \qquad (3.4)$$

where χ is called the dressing matrix and it depends on the spectral parameter λ. The seed $\Psi_{E,0}$ corresponds to a solution of (3.1) for a seed solution $M_{E,0}$. We can take it to be the identity matrix $\Psi_{E,0} = \mathbb{1}$, which corresponds to taking the seed solution to be flat space[4]. In order for the 'dressed' Ψ_E to also solve the linear system (3.1) the dressing matrix has to satisfy its own linear system

$$D_1 \chi = \frac{\rho V - \lambda U}{\lambda^2 + \rho^2} \chi - \chi \frac{\rho V_0 - \lambda U_0}{\lambda^2 + \rho^2}, \qquad D_2 \chi = \frac{\rho U + \lambda V}{\lambda^2 + \rho^2} \chi - \chi \frac{\rho U_0 + \lambda V_0}{\lambda^2 + \rho^2}. \qquad (3.5)$$

In addition, the matrix χ must satisfy further constraints in order to ensure that the new solution $M_E(\rho, z) = \Psi(0, \rho, z)$ is real, satisfies $M_E^\# = M_E$ and is a representative of the coset G_E/K_E [2, 15, 30].

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[4]Note that this differs from the more common (and complicated) expression for flat space in the Matzner–Misner form [2, 4].

3.2 Relation between the two linear systems

Compared to the discussion of the BM linear system, the differential operators D_1 and D_2 of (3.3) can be demystified by thinking of λ as space-time dependent, so that [3]

$$D_1 = \partial_z = \partial_z|_{\lambda \text{ fixed}} + \partial_z \lambda \partial_\lambda, \quad D_2 = \partial_\rho = \partial_\rho|_{\lambda \text{ fixed}} + \partial_\rho \lambda \partial_\lambda. \tag{3.6}$$

If the spacetime dependence of the spectral parameter λ is given by

$$\partial_z \lambda = -\frac{2\lambda^2}{\lambda^2 + \rho^2}, \quad \partial_\rho \lambda = \frac{2\lambda\rho}{\lambda^2 + \rho^2} \tag{3.7}$$

one recovers (3.3). The solution to (3.7) is given by

$$\lambda(\rho, z) = (w - z) \mp \sqrt{(z - w)^2 + \rho^2}, \tag{3.8}$$

where w is an integration constant. Comparing to (2.20), it follows that from this viewpoint the relation of the BZ spectral parameter λ to t in the BM approach [3, 15] is[5]

$$\lambda(\rho, z) = -\rho t(\rho, z). \tag{3.9}$$

The relation between the two generating functions $\mathcal{V}_{\mathrm{E}}$ of (2.17) and Ψ_{E} of (3.1) is given by

$$\Psi_{\mathrm{E}}(\lambda, x) = V_{\mathrm{E}}^{\#}(x)\mathcal{V}_{\mathrm{E}}(t, x) \tag{3.10}$$

where one also has to use (3.9). Note that on the right-hand side we have once the spectral parameter independent $V_{\mathrm{E}}(x) = \mathcal{V}_{\mathrm{E}}(0, x)$ and once the full $\mathcal{V}_{\mathrm{E}}(t, x)$. This obscures the action (2.21) of the full affine Geroch group since the transformation of $V_{\mathrm{E}}(x)$ under affine elements is complicated.

In the following we restrict to $\mathrm{G}_{\mathrm{E}} = \mathrm{SL}(n, \mathbb{R})$ for concreteness. In that case M_{E} has to be a symmetric matrix. For other groups, there will be different conditions on some of the quantities introduced below.

If the matrices M_{E} and $M_{\mathrm{E},0}$ obtained by the $\lambda \to 0$ limits of Ψ_{E} and $\Psi_{\mathrm{E},0}$ are symmetric, then

$$\chi'(\lambda) = M_{\mathrm{E}}\chi^{T-1}\left(-\frac{\rho^2}{\lambda}\right)M_{\mathrm{E},0}^{-1}, \tag{3.11}$$

solves exactly the same linear system (3.5) as χ [2]. Given this observation, one has that $\chi'(\lambda)$ is related to $\chi(\lambda)$ through some arbitrary matrix $C(w)$ via

$$\chi'(\lambda)\Psi_{\mathrm{E},0} = \chi(\lambda)\Psi_{\mathrm{E},0}C(w). \tag{3.12}$$

This reflects the fact that the linear system (3.5) is underdetermined and $C(w)$ corresponds to a gauge freedom of (3.1). However, Belinski and Zakharov demand

$$\chi'(\lambda) = \chi(\lambda), \tag{3.13}$$

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[5]The sign in this relation in [3] appears incorrect.

which corresponds to fixing the gauge freedom of (3.12). In addition, they do *not* require $\chi(\lambda)$ to satisfy the coset constraint $\det \chi(\lambda) = 1$. Since $\det \chi \neq 1$ one has that the new matrix M_{E} does not have unit determinant and so does not represent a physical configuration. To obtain the 'physical' matrix $M_{\mathrm{E}}^{(\mathrm{phys})}$ that fullfills the determinant condition, one rescales the matrix M_{E} appropriately.

Condition (3.13) automatically ensures that the final M_{E} is symmetric, but it is a rather strong assumption. Relation (3.13) is a central equation in the BZ inverse scattering framework and it fixes an infinite ambiguity in the dressing matrix that corresponds roughly to the Borel part of the Geroch group. In other words, demanding (3.13) in the BM framework amounts to choosing finely tuned integration constants for all the dual potentials $V_{\mathrm{E}}^{(n)}$ with $n \geq 1$ introduced through (2.15). The transformation (3.11) is similar to the one discussed below (2.22) in the BM framework. There one normally does not fix this freedom and so there is no direct analogue of (3.13) in the BM approach.

Due to the complications of $\det \chi \neq 1$ and the issues mentioned around (3.10), it is impractical to find a satisfactory embedding of the full BZ solution generating technique in the Geroch group[6]. The best one can do is to find a representative relation between the BZ-generating function $\Psi_{\mathrm{E}}(\lambda)$ and the group-theoretic BM generating function $\mathcal{V}_{\mathrm{E}}(t)$. This relation is precisely equation (3.10) for the Ehlers coset and is obtained in Appendix B for the Matzner–Misner coset, see (B.13).

3.3 Solitonic solutions

So-called *solitonic solutions* for the matrix M_{E} correspond to a dressing matrix $\chi(\lambda)$ with simple poles in the complex λ-plane. The general N-soliton solution is obtained by dressing the seed solution with a matrix χ of the form

$$\chi = \mathbb{1} + \sum_{k=1}^{N} \frac{R_k}{\lambda - \mu_k}. \tag{3.14}$$

The matrices R_k and the pole trajectories μ_k are functions of ρ, z only. For each soliton, there exist two possible solutions for the pole trajectory μ_k

$$\mu_k = -(z - w_k) \pm \sqrt{(z - w_k)^2 + \rho^2}, \tag{3.15}$$

where the parameters w_k may generally be complex but for the examples considered here we will take them to be real. The pole trajectories with a "+" sign are referred to as solitons and the ones with a "−" sign as antisolitons.

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[6]Naively one might conclude from (3.12)–(3.13) that it simply corresponds to taking $C(w) = \mathbb{1}$ from the Geroch group point of view. However, this interpretation is not correct. This is because the dressed BZ matrix $\Psi_{\mathrm{E}}(\lambda) = \chi(\lambda)\Psi_{\mathrm{E},0}$ does not directly give the physical matrix $M_{\mathrm{E}}^{(\mathrm{phys})}$. In order to have an interpretation of $C(w)$ in the Geroch group, one first needs to construct $\chi^{(\mathrm{phys})}(\lambda)$. A procedure to do this was suggested in [5]. Requiring something like $\chi^{(\mathrm{phys})\prime}(\lambda) = \chi^{(\mathrm{phys})}(\lambda)$ will indeed be more amenable to the group theoretic interpretation, but it is not the BZ technique.

In order to construct the N-soliton dressing matrix, one needs to parametrise the residue matrices R_k. Here, one has the freedom of introducing certain arbitrary *constant* parameters $m_{0b}^{(k)}$ (with $b = 1, \ldots, n$ when Ψ is represented as an $n \times n$-matrix) for each soliton μ_k as follows[7]. Defining

$$m_a^{(k)} = m_{0b}^{(k)} \left[\Psi_{\mathrm{E},0}^{-1}(\mu_k, \rho, z) \right]_{ba}, \tag{3.16}$$

and the symmetric matrix Γ_{BZ} as

$$(\Gamma_{\mathrm{BZ}})_{kl} = \frac{m_a^{(k)} (M_{\mathrm{E},0})_{ab} m_b^{(l)}}{\rho^2 + \mu_k \mu_l}, \tag{3.17}$$

the elements of the residue matrices R_k are given by

$$(R_k)_{ab} = m_a^{(k)} \sum_{l=1}^{N} \frac{(\Gamma_{\mathrm{BZ}}^{-1})_{lk} \, m_c^{(l)} \, (M_{\mathrm{E},0})_{cb}}{\mu_l}. \tag{3.18}$$

The new matrix $M_{\mathrm{E}}(\rho, z) = \chi(0, \rho, z)\Psi_{\mathrm{E},0}(0, \rho, z)$ now reads

$$(M_{\mathrm{E}})_{ab} = (M_{\mathrm{E},0})_{ab} - \sum_{k,l=1}^{N} \frac{(M_{\mathrm{E},0})_{ac} \, m_c^{(k)} \, (\Gamma_{\mathrm{BZ}}^{-1})_{kl} \, m_d^{(l)} \, (M_{\mathrm{E},0})_{db}}{\mu_k \mu_l}. \tag{3.19}$$

The symmetry of this expression is ensured by (3.13). A problem that arises at this stage is that possibly the new matrix M_{E} does not satisfy the coset constraint $\det M_{\mathrm{E}} = 1$, i.e. is not an element of the group $\mathrm{SL}(n, \mathbb{R})_{\mathrm{E}}$. In fact the determinant of the new matrix is given by

$$\det M_{\mathrm{E}} = (-1)^N \rho^{2N} \left(\prod_{k=1}^{N} \mu_k^{-2} \right) \det M_{\mathrm{E},0}. \tag{3.20}$$

In order to obtain an N-soliton solution that remains in the group $\mathrm{SL}(n, \mathbb{R})_{\mathrm{E}}$, the new matrix M_{E} must be multiplied by an overall factor[8]

$$M_{\mathrm{E}}^{(\mathrm{phys})} = \pm \left(\frac{1}{\pm \det M_{\mathrm{E}}} \right)^{\frac{1}{n}} M_{\mathrm{E}}. \tag{3.21}$$

The overall sign in this expression should be chosen in order to ensure the right metric signature. Thus obtained $M_{\mathrm{E}}^{(\mathrm{phys})}$ fulfils the constraint $\det M_{\mathrm{E}}^{(\mathrm{phys})} = 1$. Finally, following the discussion in [2], the conformal factor for the dressed solution can also be obtained. We find for $\mathrm{SL}(2, \mathbb{R})_{\mathrm{E}}$

$$(f_{\mathrm{E}}^{(\mathrm{phys})})^2 = k_{\mathrm{BZ}} \cdot \rho^{N - \frac{N^2}{2}} \cdot \left(\prod_{k=1}^{N} \mu_k \right)^N \cdot \left[\prod_{k,l=1,\, k>l}^{N} (\mu_k - \mu_l)^2 \right]^{-1} \cdot \det \Gamma_{\mathrm{BZ}} \cdot f_{\mathrm{E},0}^2, \tag{3.22}$$

where k_{BZ} is an arbitrary numerical constant. For $\mathrm{SL}(n, \mathbb{R})_{\mathrm{E}}$ a similar but more complicated expression can also be written [4]. However, note that for $n > 2$ the rescaling (3.21) introduces fractional powers of ρ from (3.20) that typically lead to singular solutions. For this reason it is more useful to employ the so-called Pomeransky trick [31] for $n > 2$. In this approach one can write a general expression for the conformal factor valid for $n \geq 2$ [31, 15].

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[7]The normalization of each of the vectors $m_{0b}^{(k)}$ is arbitrary. Rescaling them by arbitrary constants does not change any of the final expressions.

[8]This formula differs slightly from the standard expression in [2] since we are working in the Ehlers description.

4 BM Soliton transformations

In this section we present an algebraic method of generating new solutions of the BM linear system (2.17) from a given seed solution. Our discussion closely follows that of [19], see also [32].

The method makes use of the constant group element $g_E(w)$ of the Geroch group. We take the seed solution to be flat space (2.27) since it is believed that the Geroch group action is transitive on the space of solutions and all solutions are related to flat space [18, 3, 13]. As mentioned around (2.26), the action of the Geroch group generally leads to a matrix valued Riemann–Hilbert problem. In the case when the matrix functions to be factorized are meromorphic in the spectral parameter w, the problem can be solved algebraically. This is the case that we focus on and we term it the solitonic case. There are a number of (formal) similarities and at the same time a number of differences (in details) with the procedure of Belinski and Zakharov [2] that we briefly reviewed in section 3.3. In this section we also restrict ourselves to the Ehlers $SL(n, \mathbb{R})_E$ of $D = 2 + n$ vacuum gravity. In this case the generalized transpose $\#$ (at the group level) simply becomes the usual matrix transpose.

4.1 Riemann–Hilbert problem

In section 2 we presented the construction of $\mathcal{M}_E(w)$ starting with $V_E(x)$. We start with $V_E(x)$, solve the linear system (2.17) to find $\mathcal{V}_E(t, x)$, and then construct $\mathcal{M}_E(w)$. We now ask, following the steps of (2.24), if we can reverse the process and reconstruct $V_E(x)$ *from* $\mathcal{M}_E(w)$, i.e., we solve the Riemann–Hilbert problem to factorize $\mathcal{M}_E(w)$ as in (2.23). It is not guaranteed that for a general $\mathcal{M}_E(w)$ such a factorization exists. In fact, one can construct explicit examples where it does not exist. This is however a technical problem that will not concern us here. Certain aspects of this have been studied in the literature, see e.g. [18]. We only work with those matrices $\mathcal{M}_E(w)$ for which the Riemann–Hilbert problem admits a solution.

Let us start with a real symmetric unit determinant matrix $\mathcal{M}_E(w)$ assuming suitable analyticity properties. In particular we assume $\mathcal{M}_E(\infty) = \mathbb{1}$. We wish to factorize it as

$$\mathcal{M}_E(w) = A_-^T(t, x) M_E(x) A_+(t, x), \tag{4.1}$$

with $A_-(t, x) = A_+\left(-\frac{1}{t}, x\right)$ and $M_E(x)$ symmetric and real. Moreover, we require

$$\det A_\pm(t, x) = 1. \tag{4.2}$$

Next, we factorize $M_E(x)$ as $M_E(x) = V_E^T(x) V_E(x)$ with a triangular matrix $V_E(x)$ to obtain

$$\mathcal{V}_E(t, x) = V_E(x) A_+(t, x). \tag{4.3}$$

The factorization problem (4.1) can be viewed in two different ways. (i) We start with an appropriate $\mathcal{M}_E(w)$ and solve for $\mathcal{V}_E(t, x)$, (ii) we start with a seed $\mathcal{V}_E(t, x)$ and act with an element $g_E(w)$ and attempt to determine the transformed $\mathcal{V}_E^g(t, x)$. We take the first viewpoint in what follows.[9] It is, however, convenient to have the second viewpoint in the back of one's

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[9] The second viewpoint was taken in [19], however, some of their assumptions about pole structures seem too restrictive to make that method directly applicable to interesting solutions.

mind and relate it to the first one by taking the seed to be flat space. At the level of equations this means

$$\mathcal{V}_E^g(t,x) = Z_+^g(t,x)\mathcal{V}_E(t,x) = Z_+^g(t,x) \cdot \mathbb{1} = Z_+^g(t,x), \tag{4.4}$$

where $Z_+^g(t,x)$ is a triangular 'dressing' matrix in the sense of (2.15) that is determined by the Geroch group element $g_E(w)$. In terms of the monodromy matrix one similarly has

$$\mathcal{M}_E^g(w) = \mathcal{V}_E^T\left(-\frac{1}{t},x\right)\left[Z_+^g\left(-\frac{1}{t},x\right)\right]^T Z_+^g(t,x)\mathcal{V}_E(t,x). \tag{4.5}$$

Introducing

$$\mathcal{Z}^g(t,x) = \left[Z_+^g\left(-\frac{1}{t},x\right)\right]^T Z_+^g(t,x), \tag{4.6}$$

one also has

$$\mathcal{Z}^g(t,x) = \left[\mathcal{V}_E^T\left(-\frac{1}{t},x\right)\right]^{-1} \mathcal{M}_E^g(w)\left[\mathcal{V}_E(t,x)\right]^{-1} \tag{4.7}$$

$$= \mathbb{1} \cdot \mathcal{M}_E^g(w) \cdot \mathbb{1} = \mathcal{M}^g(w). \tag{4.8}$$

When $\mathcal{V}_E(t,x) \neq \mathbb{1}$ one should take $\mathcal{Z}^g(t,x)$ in (4.7) to be the left hand side of equation (4.1) and solve the corresponding factorization problem. In this paper we always work with flat space (2.27) as seed solution. Consequently, for notational simplicity we drop the superscript g from now on and just think of being given a monodromy $\mathcal{M}_E(w)$ that needs to be factorized as in (4.1).

4.2 Multisoliton solutions

The factorization problem can be solved algebraically when the matrix functions to be factorized are meromorphic. We now present this factorization explicitly by adapting [19]. We assume that $\mathcal{M}_E(w)$ has simple poles with residues of rank one. Since $\det\mathcal{M}_E(w) = 1$ and $\mathcal{M}_E(\infty) = \mathbb{1}$ the inverse matrix $\mathcal{M}_E^{-1}(w)$ also has poles at the same points with residues of rank one. If we have N poles at points w_k with $k = 1, 2, \ldots, N$ we can express $\mathcal{M}_E(w)$ and $\mathcal{M}_E^{-1}(w)$ in the form

$$\mathcal{M}_E(w) = \mathbb{1} + \sum_{k=1}^N \frac{A_k}{w - w_k}, \tag{4.9a}$$

$$\mathcal{M}_E^{-1}(w) = \mathbb{1} - \sum_{k=1}^N \frac{B_k}{w - w_k}, \tag{4.9b}$$

with symmetric (since $\mathcal{M}_E$ is symmetric) and *constant* residue matrices A_k and B_k of rank one. This means that we can factorize these matrices as the outer product of vectors

$$A_k = a_k\alpha_k a_k^T, \qquad B_k = b_k\beta_k b_k^T. \tag{4.10}$$

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One could absorb α_k and β_k in the definition of the constant vectors a_k and b_k respectively but we leave them explicit on purpose. They play a very important role: for a given set of a_k and b_k we can tune the α_k and β_k appropriately to ensure that the matrices $\mathcal{M}_{\mathrm{E}}(w)$ and $\mathcal{M}_{\mathrm{E}}^{-1}(w)$ have unit determinant. Despite this, there is an ambiguity in the factorization (4.10) related to the normalization of the vectors a_k and b_k. Nothing must depend on this choice of normalization. This will indeed be the case as will be apparent shortly. At this stage we just remark that the constant vectors a_k are the analog of the constant vectors $m_0^{(k)}$ of (3.16) in the BZ method. The ambiguity related to the factorization of rank one matrices in vectors is directly related to the ambiguity in the normalization of the vectors $m_0^{(k)}$ in the BZ method (cf. footnote 7). As is well known in the BZ method, nothing depends on the overall normalization of those vectors.

In order to factorize $\mathcal{M}_{\mathrm{E}}(w)$ as in (4.1) we have to change from the constant spectral parameter w to the space-time dependent parameter t through (2.20), which implies

$$\frac{1}{w - w_k} = \nu_k \left(\frac{t_k}{t - t_k} + \frac{1}{1 + t t_k} \right), \tag{4.11}$$

where the moving poles t_k are determined by (2.20) evaluated at w_k with the plus sign and

$$\nu_k = -\frac{2 t_k}{\rho \left(1 + t_k^2\right)}. \tag{4.12}$$

As a function of t, $\mathcal{M}_{\mathrm{E}}(t, x)$ has a total of $2N$ poles: N poles at $t = t_k$ and N poles at $t = -1/t_k$. These have to be distributed among the factors A_+ and A_- in (4.1). The analytic properties of the Riemann–Hilbert problem are such that the poles at $t = -1/t_k$ come from $A_+(t)$ and those at $t = t_k$ from A_-. One therefore makes the ansätze [19]

$$A_+(t) = \mathbb{1} - \sum_{k=1}^{N} \frac{c_k t a_k^T}{1 + t t_k}, \tag{4.13a}$$

$$A_+^{-1}(t) = \mathbb{1} + \sum_{k=1}^{N} \frac{b_k t d_k^T}{1 + t t_k}, \tag{4.13b}$$

where the second equation arises in the factorization of $\mathcal{M}_{\mathrm{E}}^{-1}(w)$. These two equations introduce two new sets of vectors that we call c_k and d_k.

The vectors a_k, b_k, c_k and d_k are not all independent and determining their relation amounts to solving the Riemann–Hilbert problem. We first note that the constant matrices A_k and B_k are not independent since the two matrices $\mathcal{M}_{\mathrm{E}}(w)$ and $\mathcal{M}_{\mathrm{E}}^{-1}(w)$ are inverses of each other. This determines the vectors b_k from the a_k up to scaling, a freedom that is reflected in the β_k in (4.9b).

We can use the pole structure of $\mathcal{M}_{\mathrm{E}}(t, x) \mathcal{M}_{\mathrm{E}}^{-1}(t, x)$ to deduce some properties of the a_k and b_k. To start with, the absence of double poles at $t = -1/t_k$ in the product implies that the vectors a_k and b_k are orthogonal:

$$a_k^T b_k = 0 \qquad \text{for each } k. \tag{4.14}$$

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From the absence of single poles at $t = -1/t_k$ in the product $\mathcal{M}_{\mathrm{E}}(t,x)\mathcal{M}_{\mathrm{E}}^{-1}(t,x)$ one deduces the relations

$$a_k \alpha_k a_k^T \mathcal{A}^k = \mathcal{A}_k b_k \beta_k b_k^T, \tag{4.15}$$

with the definitions

$$\mathcal{A}^k = \left[\mathcal{M}_{\mathrm{E}}^{-1}(t,x) + \frac{b_k \nu_k \beta_k b_k^T}{1 + tt_k} \right]_{t=-\frac{1}{t_k}}, \qquad \mathcal{A}_k = \left[\mathcal{M}_{\mathrm{E}}(t,x) - \frac{a_k \nu_k \alpha_k a_k^T}{1 + tt_k} \right]_{t=-\frac{1}{t_k}}. \tag{4.16}$$

Equation (4.15) is satisfied if there exist γ_k such that

$$a_k^T \mathcal{A}^k = \gamma_k \nu_k \beta_k b_k^T \qquad \text{and} \qquad \mathcal{A}_k b_k = \gamma_k \alpha_k \nu_k a_k. \tag{4.17}$$

Then both sides of (4.15) are equal to $\alpha_k \nu_k \beta_k \gamma_k (a_k b_k^T)$. We note that the γ_k defined uniquely by (4.17) depend on space-time.

The next non-trivial step is to determine the vectors c_k. With the ansatz (4.13a) for $A_+(t)$, the requirement that $A_+(t)\mathcal{M}_{\mathrm{E}}^{-1}(t,x)$ have no poles at $t = -1/t_k$ gives the vector equation

$$c_k \Gamma_{kl} = b_l, \tag{4.18}$$

where Γ_{kl} is an $N \times N$ matrix with elements

$$\Gamma_{kl} = \begin{cases} \frac{\gamma_k}{t_k} & \text{for} & k = l \\ \frac{1}{t_k - t_l} a_k^T b_l & \text{for} & k \neq l. \end{cases} \tag{4.19}$$

Solving equation (4.18) for the c_k we obtain the matrix $A_+(t)$. A similar argument is used to construct $A_+^{-1}(t)$ in (4.13b). One finds the equation for the vectors d_k to be $\Gamma_{kl} d_l = a_k$. Solving this equation for the d_k we can readily construct the monodromy matrix M_{E} by taking the limit $t \to \infty$ in (4.1) and using that $\mathcal{M}_{\mathrm{E}}(\infty) = A_+(\infty) = \mathbb{1}$. The result is

$$M_{\mathrm{E}} = A_+^{-1}(\infty) = \mathbb{1} + \sum_{k,l=1}^N b_k t_k^{-1} (\Gamma^{-1})_{kl} a_l^T. \tag{4.20}$$

Finally we factorize $M_{\mathrm{E}} = V_{\mathrm{E}}^T V_{\mathrm{E}}$ and obtain the space-time fields. If needed, we can also construct explicitly the generating function $\mathcal{V}_{\mathrm{E}}(t,x)$ from equation (4.3).

At this stage it is instructive to investigate how the final answers (4.13a) (in conjunction with (4.18)) and (4.20) are insensitive to the ambiguity related to the rescaling of the vectors. Note that if we rescale the vectors a_k to $r_k a_k$ and b_k to $s_k b_k$, then we must rescale α_k and β_k as $r_k^{-2} \alpha_k$ and $s_k^{-2} \beta_k$ respectively in order to preserve the products (4.10). This then means that the γ_k scale as $r_k s_k \gamma_k$. It then immediately follows that the matrix Γ_{kl} scales as $r_k s_l \Gamma_{kl}$. The inverse matrix $(\Gamma^{-1})_{kl}$ naturally scales with the inverse factor $r_k^{-1} s_l^{-1}$ and hence we see that the final answers (4.13a) (in conjunction with (4.18)) and (4.20) are insensitive to such rescalings.

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4.3 Conformal factor

For solitonic solutions of the previous subsection the conformal factor f_E can also be obtained in a closed form. The final result is

$$f_E^2 = k_{BM} \cdot \prod_{k=1}^{N} (t_k \nu_k) \cdot \det \Gamma, \tag{4.21}$$

where k_{BM} is an arbitrary numerical constant. This constant needs to be chosen appropriately in order to ensure certain physical properties (say, asymptotic flatness) of the final space-time. A derivation of expression (4.21) is given in appendix A.

5 Example: Kerr–NUT solution

In this section we present a concrete implementation of the method of section 4 by constructing the Kerr–NUT metric. The construction illustrates all the steps of the BM solitonic method. More complicated examples are certainly doable; we leave such a line of investigation for the future.

5.1 Construction of general 2-soliton solution

The main difficulty in constructing the general multi-soliton solutions using the BM group theoretic approach lies in finding meromorphic matrices $\mathcal{M}_E(w)$ that satisfy the coset constraints. The analog of this problem does not arise in the approach of BZ because they relax this constraint and consequently have to renormalize the resulting matrices. This works well for $SL(2, \mathbb{R})$ but already for $SL(3, \mathbb{R})$ it gives spacetimes that do not represent black holes. A clever solution of this problem in the BZ approach was found by Pomeransky [31]. In this respect the BZ method supplemented with the Pomeransky trick remains the most effective and powerful method for constructing solutions of vacuum four- and five-dimensional gravity. For a concise review and further references see [33]. The Pomeransky trick works well for $SL(n, \mathbb{R})$ but has no known analog for other coset models. As mentioned in the introduction, despite the initial complications, the promise of the BM method lies in its generality; it can be taken over to other coset models.

It turns out that for $SL(n, \mathbb{R})_E$ monodromy matrices with a maximum of two poles, it is rather straightforward to take into account the coset constraints explicitly. When more poles are present one can perhaps set up a recursive algorithm for finding the appropriate meromorphic matrices. We have not attempted this and leave this line of investigation for the future. Here we present a discussion of a two soliton $SL(2, \mathbb{R})_E$ matrix. The most general such configuration describes the Kerr-NUT solution as we show now.

We start with the general form of $\mathcal{M}_E(w)$ (compare (4.9a)),

$$\mathcal{M}_E(w) = \mathbb{1} + \frac{a_1 \alpha_1 a_1^T}{w - c} + \frac{a_2 \alpha_2 a_2^T}{w + c}, \tag{5.1}$$

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where a_1 and a_2 are two-dimensional column vectors. The poles can be chosen in this way by a shift of axis, see (2.19). Given the constant 2×2 matrix $a = (a_1, a_2)$ and the 2×2 matrix $\xi = a^T a$ we must choose

$$\alpha = \frac{2c}{\det \xi} \begin{pmatrix} \xi_{22} & 0 \\ 0 & -\xi_{11} \end{pmatrix}, \qquad \alpha = \mathrm{diag}\{\alpha_1, \alpha_2\}, \tag{5.2}$$

in order to satisfy the constraint $\det \mathcal{M}_{\mathrm{E}}(w) = 1$. The matrix $\mathcal{M}_{\mathrm{E}}(w)$ is symmetric and is of determinant one, hence it is in the coset $\mathrm{SL}(2, \mathbb{R})_{\mathrm{E}}/\mathrm{SO}(2)_{\mathrm{E}}$. For the parametrization of the inverse $\mathcal{M}_{\mathrm{E}}(w)^{-1}$ we can choose $b = a\xi^{-1}\epsilon$ and $\beta = -\alpha \det \xi$ with

$$\epsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{5.3}$$

Due to the scaling freedom for the vectors, we can choose without any loss of generality

$$a_1 = \begin{pmatrix} 1 \\ \zeta_1 \end{pmatrix}, \qquad a_2 = \begin{pmatrix} \zeta_2 \\ 1 \end{pmatrix}. \tag{5.4}$$

Thus

$$\xi = \begin{pmatrix} 1 + \zeta_1^2 & \zeta_1 + \zeta_2 \\ \zeta_1 + \zeta_2 & 1 + \zeta_2^2 \end{pmatrix}, \tag{5.5}$$

and

$$\alpha_1 = \frac{2c(1 + \zeta_2^2)}{(1 - \zeta_1\zeta_2)^2} \qquad \alpha_2 = -\frac{2c(1 + \zeta_1^2)}{(1 - \zeta_1\zeta_2)^2}. \tag{5.6}$$

For the b_k vectors we have

$$b_1 = \frac{1}{1 - \zeta_1\zeta_2} \begin{pmatrix} -\zeta_1 \\ 1 \end{pmatrix}, \qquad b_2 = \frac{1}{1 - \zeta_1\zeta_2} \begin{pmatrix} -1 \\ \zeta_2 \end{pmatrix}, \tag{5.7}$$

and for β_k

$$\beta_1 = -2c(1 + \zeta_2^2), \qquad \beta_2 = 2c(1 + \zeta_1^2). \tag{5.8}$$

From the above expressions we see that $a_1^T b_1 = 0 = a_2^T b_2$, as expected. Furthermore we have $a_2^T b_1 = -a_1^T b_2 = 1$. With the above choices we find

$$\Gamma = \frac{1}{t_2 - t_1} \begin{pmatrix} \frac{\xi_{12}}{\xi_{22}} \frac{t_2(t_1 + t_1^{-1})}{1 + t_1 t_2} & 1 \\ 1 & \frac{\xi_{12}}{\xi_{11}} \frac{t_1(t_2 + t_2^{-1})}{1 + t_1 t_2} \end{pmatrix}. \tag{5.9}$$

From Γ we obtain the c_k vectors by (4.18) and from there by looking at the limiting value of the $A_+^{-1}(t)$ at $t = \infty$, cf. equation (4.20), we obtain $\mathcal{M}_{\mathrm{E}}(x)$. We find

$$\mathcal{M}_{\mathrm{E}}(x) = \mathbb{1} + a(\Gamma T \epsilon^{-1}\xi)^{-1}a^T, \qquad \text{where} \qquad T = \mathrm{diag}\{t_1, t_2\}. \tag{5.10}$$

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We also observe that

$$\Gamma T \epsilon^{-1} \xi = (\alpha \nu)^{-1} - \frac{t_1 t_2}{1 + t_1 t_2} \xi, \qquad \text{where} \qquad (\alpha \nu) = \text{diag}\{\alpha_1 \nu_1, \alpha_2 \nu_2\}. \qquad (5.11)$$

From $M_{\rm E}(x)$ one can read off the physical fields. The conformal factor, given by (4.21), is

$$f_{\rm E}^2 = k_{\rm BM} \frac{t_1 \nu_1 t_2 \nu_2}{(t_2 - t_1)^2} \left[\frac{\xi_{12}^2}{\xi_{11} \xi_{22}} \frac{(1 + t_1^2)(1 + t_2^2)}{(1 + t_1 t_2)^2} - 1 \right]. \qquad (5.12)$$

5.2 Interpretation as Kerr–NUT metric

For four-dimensional vacuum gravity with $\mathrm{SL}(2, \mathbb{R})_{\rm E}$ symmetry we can parametrise the monodromy $M_{\rm E}$ as [3]

$$M_{\rm E} = V_{\rm E}^T V_{\rm E} = \begin{pmatrix} \Delta + \Delta^{-1} \tilde{\psi}^2 & \Delta^{-1} \tilde{\psi} \\ \Delta^{-1} \tilde{\psi} & \Delta^{-1} \end{pmatrix}. \qquad (5.13)$$

Here, $\tilde{\psi}$ is dual to the metric function ψ by the duality relation[10]

$$\star_2 d\tilde{\psi} = -\frac{\Delta^2}{\rho} d\psi. \qquad (5.14)$$

The $D = 4$ metric is given by

$$ds_4^2 = -\Delta \left(dt + \psi d\phi\right)^2 + \Delta^{-1}(f_{\rm E}^2 \left(d\rho^2 + dz^2\right) + \rho^2 d\phi^2). \qquad (5.15)$$

(The form of the conformal factor is due to the change from Ehlers to Matzner–Misner variables to describe the physical space-time.)

To write explicit expressions for the scalars it is convenient to introduce prolate spheroidal coordinates (u, v)

$$z = uv, \qquad \rho = \sqrt{(u^2 - c^2)(1 - v^2)}, \qquad c \le u < \infty, \qquad -1 \le v \le 1. \qquad (5.16)$$

These coordinates allow us to write the pole trajectories t_1 and t_2 as

$$t_1 = \frac{(u - c)(1 + v)}{\sqrt{(u^2 - c^2)(1 - v^2)}}, \qquad t_2 = \frac{(u + c)(1 + v)}{\sqrt{(u^2 - c^2)(1 - v^2)}}. \qquad (5.17)$$

The inverse relations are

$$\frac{t_1}{t_2} = \frac{u - c}{u + c}, \qquad t_1 t_2 = \frac{1 + v}{1 - v}. \qquad (5.18)$$

In these new coordinates we have

$$\Delta = \frac{1}{D} \left[v^2 c^2 (\zeta_1 + \zeta_2)^2 + u^2 (1 - \zeta_1 \zeta_2)^2 - c^2 (1 + \zeta_1^2)(1 + \zeta_2^2)\right] \qquad (5.19)$$

---

[10]From the point of view of the Geroch group, this duality with its non-linear prefactor is at the heart of the infinite-dimensional symmetry and integrability of the system: $\tilde{\psi}$ is the first in an infinite set of so-called dual potentials on which the infinite symmetry acts.

and

$$\tilde{\psi} = \frac{1}{D}\left[2cu(\zeta_2 - \zeta_1)(1 - \zeta_1\zeta_2) - 2c^2v(\zeta_1 + \zeta_2)(1 + \zeta_1\zeta_2)\right], \tag{5.20}$$

where the common denominator of these expressions is

$$D = v^2c^2(\zeta_1 + \zeta_2)^2 + 2vc^2(\zeta_2^2 - \zeta_1^2) + u^2(1 - \zeta_1\zeta_2)^2 + c^2(1 + \zeta_1^2)(1 + \zeta_2^2) + 2cu(1 - \zeta_1^2\zeta_2^2). \tag{5.21}$$

By applying the duality relation (5.14) one can write an expression for ψ.[11] It is slightly more complicated

$$\psi = \frac{N_\psi}{(1 - \zeta_1\zeta_2)D_\psi}, \tag{5.22a}$$

$$D_\psi = u^2(1 - \zeta_1\zeta_2)^2 - c^2(1 + \zeta_1^2)(1 + \zeta_2^2) - c^2v^2(\zeta_1 + \zeta_2)^2, \tag{5.22b}$$

$$\begin{aligned}
N_\psi &= -4c^3\zeta_1(1 + \zeta_1^2)(1 + \zeta_2^2) - 2c^2(\zeta_1 + \zeta_2)(1 - \zeta_1^2\zeta_2^2)u + 2c(\zeta_1 - \zeta_2)(1 - \zeta_1\zeta_2)^2u^2 \\
&\quad - 2c^3(\zeta_1 - \zeta_2)(1 - \zeta_1\zeta_2)^2v + 2c(\zeta_1 - \zeta_2)(1 - \zeta_1\zeta_2)^2u^2v \\
&\quad + 2c^3(\zeta_1 + \zeta_2)(1 + 2\zeta_1^2 + \zeta_1^2\zeta_2^2)v^2 + 2c^2(\zeta_1 + \zeta_2)(1 - \zeta_1^2\zeta_2^2)uv^2. \tag{5.22c}
\end{aligned}$$

These expressions look somewhat cumbersome. To compare them with the corresponding expressions for the Kerr–NUT metric in the standard Boyer-Lindquist coordinates, let us recall that the latter takes the form (see e.g. [4])

$$\begin{aligned}
ds^2 &= -\frac{1}{\Sigma}\left(\Xi - a^2\sin^2\theta\right)dt^2 + \frac{\Sigma}{\Xi}dr^2 + \Sigma d\theta^2 + \frac{1}{\Sigma}\left((\Sigma + a\chi)^2\sin^2\theta - \chi^2\Xi\right)d\phi^2 \\
&\quad + \frac{2}{\Sigma}\left(\chi\Xi - a(\Sigma + a\chi)\sin^2\theta\right)dtd\phi, \tag{5.23}
\end{aligned}$$

with

$$\Sigma = r^2 + (n + a\cos\theta)^2 \tag{5.24a}$$

$$\Xi = r^2 - 2mr - n^2 + a^2 \tag{5.24b}$$

$$\chi = a\sin^2\theta - 2n(1 + \cos\theta). \tag{5.24c}$$

The Boyer-Lindquist coordinates (r, θ) are related to the prolate spheroidal coordinates (see e.g. appendix G of [34]) simply as,

$$u = r - m, \qquad v = \cos\theta. \tag{5.25}$$

We find that with the identifications

$$\zeta_1 = \frac{c - m}{a + n}, \qquad \zeta_2 = -\frac{a + n}{c + m}, \qquad c = \sqrt{m^2 + n^2 - a^2}, \tag{5.26}$$

---

[11]Alternatively, one could construct the generating function $\mathcal{V}_E(t)$ and apply the algebraic Kramer–Neugebauer transformation (relating the Ehlers and Matzner–Misner description) to obtain $\mathcal{V}_{MM}(t)$ that directly contains ψ [3].

the tt, $t\phi$, and $\phi\phi$ part of the metric matches with corresponding expressions obtained through (5.19), (5.20) and (5.22). Relations (5.26) can be inverted to read

$$a = -m\frac{\zeta_1 + \zeta_2}{1 + \zeta_1\zeta_2}, \qquad n = m\frac{\zeta_1 - \zeta_2}{1 + \zeta_1\zeta_2}, \qquad m = c\frac{1 + \zeta_1\zeta_2}{1 - \zeta_1\zeta_2}. \qquad (5.27)$$

When $\zeta_1 = \zeta_2$ we obtain the Kerr solution and the corresponding expressions match those of [25] and [19] (when certain minor typos and misprints are fixed in those references). When $\zeta_1 = \zeta_2 = 0$ we obtain the Schwarzschild solution as in [25, 32].

The conformal factor can also be easily computed using the formula (4.21). We find

$$f_E^2 = -k_{BM}\frac{u^2 - m^2 - n^2 + a^2v^2}{4(m^2 + n^2)(u^2 - c^2v^2)}. \qquad (5.28)$$

Choosing the constant k_{BM} to be $-4(m^2 + n^2)$ so that $f_E \to 1$ as $r \to \infty$, we have

$$f_E^2 = \frac{u^2 - m^2 - n^2 + a^2v^2}{(u^2 - c^2v^2)}, \qquad (5.29)$$

which in the MM coset allows us to match directly with the metric (5.23)

$$\Delta^{-1}f_E^2 = \frac{(m + u)^2 + (n + av)^2}{u^2 - c^2v^2}. \qquad (5.30)$$

For completeness, we record the form of the (Ehlers) monodromy matrix $\mathcal{M}_E(w)$ expressed in terms of the physical quantities

$$\mathcal{M}_E(w) = \frac{1}{w^2 - c^2}\begin{pmatrix} (m + w)^2 + (n + a)^2 & 2(am - nw) \\ 2(am - nw) & (w - m)^2 + (a - n)^2 \end{pmatrix} \qquad (5.31)$$

with c as in (5.26).

5.3 BZ Ehlers Construction

The Kerr–NUT metric has been reconstructed by several authors in the context of the standard BZ method, see e.g. [4, 33]. In this section we revisit this computation and perform it using the Ehlers generating function $\Psi_E(\lambda)$. To the best of our knowledge this has not been presented before.

The Killing part of the metric of flat space translates into M_E being the identity matrix. This implies that the seed $\Psi_E(\lambda, x)$ is also identity. Now we add solitons at $\lambda = \mu_1$ with BZ vectors $m_0^{(1)} = (A_1, A_2)$ and at $\lambda = \mu_2$ with BZ vectors $m_0^{(2)} = (B_1, B_2)$. The Γ_{BZ} matrix is found to be

$$\Gamma_{BZ} = \begin{pmatrix} \frac{A_1^2 + A_2^2}{\mu_1^2 + \rho^2} & \frac{A_1B_1 + A_2B_2}{\rho^2 + \mu_1\mu_2} \\ \frac{A_1B_1 + A_2B_2}{\rho^2 + \mu_1\mu_2} & \frac{B_1^2 + B_2^2}{\mu_2^2 + \rho^2} \end{pmatrix} \qquad (5.32)$$

and we need to rescale the resulting M_E by $-\frac{\mu_1\mu_2}{\rho^2}$ [2, 4] to ensure the determinant condition. With the choice of parameters $w_1 = -c$ and $w_2 = +c$ for the pole locations μ_1 and μ_2 in (3.15),

and $(A_1, A_2) = (-\zeta_1, 1)$ and $(B_1, B_2) = (1, -\zeta_2)$ for the BZ vectors, the final expression for the rescaled M_E matches precisely with the M_E obtained from the BM method in section 5. The conformal factor obtained via equation (3.22) also matches with the one obtained in section 5, with the constant k_{BZ} in (3.22) being $k_{BZ} = 4(m+c)^2$. It is intriguing to note that up to overall normalizations the BZ vectors are precisely the vectors b_1 and b_2 of the BM construction above.

6   Conclusions

In this paper we studied the integrability of two-dimensional gravity-matter systems with matter related to a symmetric space G_E/K_E. The integrability of these set-ups can be exhibited either using the Belinski–Zakharov (BZ) linear system or through the Breitenlohner–Maison (BM) linear system. As emphasized, the approach of Breitenlohner–Maison makes the group structure of the Geroch group manifest. We analysed the relation between the BZ and the BM linear systems and presented explicit relations between the generating functions appearing in these two linear systems.

An embedding of the Belinski–Zakharov solution generating technique in the Geroch group was also studied in general. We pointed out that it is impractical to find a satisfactory general embedding of the full BZ solution generating technique in the Geroch group. Relation (3.10) provides in principle the link between the generating functions. However, one must keep in mind that the left hand side of (3.10) must be a 'physical' generating function $\Psi_E(\lambda)$. Here, by a 'physical' generating function we mean a generating function that gives a representative of the coset G_E/K_E upon taking the limit $\lambda \to 0$. This does not happen automatically in the BZ technique where the dressing matrix $\chi(\lambda)$ is considered in a more general context.

On the other hand, following the unpublished work of Breitenlohner and Maison [19], we exhibited a novel solution generating method where the group theoretical interpretation is clear from the beginning to the end. In our approach we solve the requisite Riemann–Hilbert problem algebraically. Since only algebraic manipulations are involved, our technique is akin to the BZ technique. As a novel example, we constructed the Kerr-NUT solution in this approach.

Our main interest in performing the analysis of integrability in these gravity-matter systems is to make the means for constructing new solutions available in situations where the standard inverse scattering method of BZ is not applicable. This is typically the case for extended supergravity theories that have a string theory origin. For minimal $D = 5$ supergravity with exceptional Ehlers symmetry $G_E = G_{2(2)}$ this problem was pointed out in [15] and also arises for the STU model [35] with $G_E = SO(4, 4)$ or maximal supergravity with $G_E = E_{8(8)}$ [36]. The method explained in section 4 is still applicable in those cases as long as one finds a way to parametrise the G_E valued monodromy $\mathcal{M}_E(w)$ in a way similar to (5.1) and (5.2). There are several ways in which this could be achieved: (i) One could use global elements $k \in K_E \subset G_E \subset$ (Geroch group) to rotate the vectors in $\mathcal{M}(w)$ in (5.1) into canonical positions. In the example of section 5 this would correspond to setting $\zeta_1 = \zeta_2$. The solution for canonical vectors can then be generalised by applying conventional K_E charging transformations. These charging transformations use only a very small subset of the full power of the Geroch group. (ii) One could embed G_E in $GL(n, \mathbb{R})$ for n large enough and then solve the constraints on the vectors

for the embedding explicitly. We expect a combination of these two techniques to be the most promising line of attack. Alternatively to (i) and (ii), one could perhaps hope to develop some general algorithms to find the appropriate monodromy matrices by combining ideas from uniqueness proofs for black holes, see e.g. [37], and the fact that the monodromy matrices are closely related to the behavior of solutions on the z-axis, see e.g. section 4 of [3]. In future work we plan to explore these sets of ideas. See also [30] for a slightly different but related viewpoint on this problem. It will be very interesting to see if the algebraic Riemann–Hilbert factorization approach can be used to construct new black hole solutions generalizing [20, 21] and new fuzzball solutions generalizing [22, 23].

Another interesting aspect of the integrable structure in two-dimensional models is their possible relation to the recently studied infinite-dimensional symmetries of string and M-theory [38, 39]. These symmetry groups are extensions of the Geroch group and are conjectured to be symmetries of the unreduced theory. First steps in investigating this relation were undertaken in [40] in the restricted case of polarized Gowdy space-times. The relation (2.18) between a spectral parameter and a space-time coordinate suggests that a mapping between Lie algebraic and geometric data might be possible. If taken seriously, this approach would allow the treatment of space-time as a concept that fully emerges from symmetry considerations.

Acknowledgements

A   Computation of the BM conformal factor

In this appendix we present a derivation of equation (4.21) for meromorphic monodromy matrices following [19]. Using the light cone coordinates (2.13) with the property $\star_2 \partial_\pm = \pm i \partial_\pm$ we can write the differential equations for the conformal factor f_E as

$$\partial_\pm \ln \rho \, \partial_\pm \ln f_\mathrm{E} = \frac{1}{2} \mathrm{Tr}(P_{\mathrm{E},\pm} P_{\mathrm{E},\pm}). \tag{A.1}$$

Next we wish to write $\mathrm{Tr}(P_{\mathrm{E},\pm} P_{\mathrm{E},\pm})$ in terms of the matrix $A_+(t)$ introduced in (4.3) in section 4. To this end we evaluate the residue of the poles at $t = \pm i$ in the Lax equation (2.17). For evaluating the residue on the l.h.s. of (2.17) we use the relation

$$\partial_\pm \mathcal{V}_\mathrm{E}(t,x) = \partial_\pm \mathcal{V}_\mathrm{E}(t,x)|_t + (\partial_\pm t) \dot{\mathcal{V}}_\mathrm{E}(t,x), \tag{A.2}$$

where $\dot{\mathcal{V}}_\mathrm{E}(t,x) = \frac{\partial \mathcal{V}_\mathrm{E}(t,x)}{\partial t}$. These relations together with (2.18) give

$$\pm i \partial_\pm \ln \rho \, \dot{\mathcal{V}}_\mathrm{E}(\pm i) = P_{\mathrm{E},\pm} \mathcal{V}_\mathrm{E}(\pm i). \tag{A.3}$$

Now replacing (4.3) in (A.3) we obtain an expression for $P_{\text{E},\pm}$ in terms of $A_+(\pm i)$ and $\dot{A}_+(\pm i)$. Substituting that expression in (A.1) we obtain

$$\partial_\pm \ln f_{\text{E}} = -\frac{1}{2}(\partial_\pm \ln \rho)\text{Tr}\left(A_+^{-1}(\pm i)\dot{A}_+(\pm i)\right)^2. \tag{A.4}$$

Using the explicit form of $A_+(t)$ from equation (4.13a) together with (4.18) and the identity $a_k^T b_k = \Gamma_{kl}(t_k - t_l)$ we get,

$$A_+^{-1}(t)\dot{A}_+(t) = -b\frac{\mathbb{1}}{\mathbb{1} + tT}\Gamma^{-1}\frac{\mathbb{1}}{\mathbb{1} + tT}a^T. \tag{A.5}$$

In writing this equation we have used a convenient matrix notation, where T is a diagonal matrix with entries t_k. Differentiating for $k \neq l$ the identity (cf. (4.19))

$$a_k^T b_l = \Gamma_{kl}(t_k - t_l), \tag{A.6}$$

with respect to the light cone coordinates we obtain the components with $k \neq l$ of the equation

$$\partial_\pm \Gamma = -(\partial_\pm \ln \rho)\frac{\mathbb{1}}{\mathbb{1} \pm iT}\left[\Gamma \mp iT\Gamma \mp i\Gamma T + T\Gamma T\right]\frac{\mathbb{1}}{\mathbb{1} \pm iT}. \tag{A.7}$$

Looking at the definition of the diagonal components Γ_{kk} in (4.19), we note that we need $\partial_\pm \gamma_k$ in order to obtain the corresponding expression for the diagonal components of Γ. We hence differentiate the relation $\mathcal{A}_k b_k = \gamma_k \alpha_k \nu_k a_k$ from (4.17). An important intermediate result for this is that $\mathcal{A}_k b_k$ is constant. From this it is easy to deduce that (A.7) holds as well for the diagonal components.

Using these formulas, we first substitute (A.5) into (A.4) and then manipulate the new r.h.s. to bring out terms that are total derivatives using (A.7). As a result, we can rewrite equation (A.4) in the form

$$\partial_\pm \ln f_{\text{E}} = \frac{1}{2}\text{Tr}\left(\Gamma^{-1}\partial_\pm \Gamma\right) + \frac{1}{2}\text{Tr}\left((T\nu)^{-1}\partial_\pm(T\nu)\right), \tag{A.8}$$

where $(T\nu)$ is the diagonal matrix with entries $t_k \nu_k$. Equation (A.8) can now be readily integrated to give the final result (4.21)

$$f_{\text{E}}^2 = k_{\text{BM}} \cdot \prod_{k=1}^{N}(t_k \nu_k) \cdot \det \Gamma, \tag{A.9}$$

with k_{BM} an integration constant. (More generally, the conformal factor is related to a cocycle calculation in the affine group [3].)

B   BZ Matzner–Misner

The Belinski–Zakharov (BZ) approach is a well established solution generating technique for vacuum gravity. The method is applicable in any dimension, though only in four and five dimensions can the generated solutions be asymptotically flat [33]. In this appendix we focus

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on $D = 4$. We assume that the space-time admits two commuting Killing vectors, one spacelike (angular) and one timelike. In this case the four-dimensional metric admits the following form in the Weyl canonical coordinates (cf. (5.15))

$$ds^2 = e^{2\nu}\left(d\rho^2 + dz^2\right) + g_{ab}dx^a dx^b, \tag{B.1}$$

where the indices a, b run over the Killing coordinates ϕ and t. The vacuum Einstein equations can be used to choose without any loss of generality the coordinates such that [34]

$$\det g = -\rho^2. \tag{B.2}$$

For this class of metrics, the Einstein equations divide in two groups: one for the Killing part g of the metric

$$\partial_\rho U + \partial_z V = 0, \qquad \text{with} \qquad U = \rho(\partial_\rho g)g^{-1} \quad \text{and} \quad V = \rho(\partial_z g)g^{-1} \tag{B.3}$$

and the second group for the conformal factor ν. From the discussion in the main text of the paper it is clear that equation (B.3) are the equations for a GL(2,$\mathbb{R}$) integrable sigma model. The equations for the conformal factor ν read

$$\partial_\rho \nu = -\frac{1}{2\rho} + \frac{1}{8\rho}\text{Tr}(U^2 - V^2), \qquad \partial_z \nu = \frac{1}{4\rho}\text{Tr}(UV). \tag{B.4}$$

Note that the equations for g do not contain the function ν. Once one obtains a solution of (B.3), ν can be obtained by a line integral.

The Belinski–Zakharov spectral equations for the above GL(2,$\mathbb{R}$) model (B.3) are [2][12]

$$D_1\Psi = \frac{\rho V - \lambda U}{\lambda^2 + \rho^2}\Psi, \qquad D_2\Psi = \frac{\rho U + \lambda V}{\lambda^2 + \rho^2}\Psi, \tag{B.5}$$

where λ is the spectral parameter and D_1 and D_2 are the two commuting differential operators of (3.3).[13] The generating matrix $\Psi(\lambda, \rho, z)$ is such that in the limit $\lambda \to 0$ it gives the volumeful metric g. Using the above linear system (B.5) one can construct an infinite class of new solutions by dressing seed solutions. This procedure has been reviewed at several places, see e.g. [4, 33].

The matrix g can also be written in terms of the SL$(2, \mathbb{R})_{\text{MM}}$ unimodular matrix M_{MM}

$$g = \rho M_{\text{MM}}, \tag{B.6}$$

where the subscripts MM stand for Matzner–Misner. In terms of the coset variables it takes the form $M_{\text{MM}} = V_{\text{MM}}^T \eta V_{\text{MM}}$, where V_{MM} is the Matzner–Misner coset representative of the quotient

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[12]Note that there is neither an Ehlers nor a Matzner–Misner subscript on this generating function Ψ since it agrees with neither. It is, however, closely related to the Matzner–Misner version as will become clear in the sequel.

[13]In 'light-cone' coordinates the spectral equations read

$$D_\pm\Psi = \frac{\pm i\rho\,(\partial_\pm g)\,g^{-1}}{\lambda \pm i\rho}\Psi, \qquad \text{with} \qquad D_\pm = \partial_\pm - \frac{2\lambda}{\lambda \pm i\rho}\partial_\lambda.$$

$\mathrm{SL}(2,\mathbb{R})_{\mathrm{MM}}/\mathrm{SO}(1,1)_{\mathrm{MM}}$ and η is the $\mathrm{SO}(1,1)_{\mathrm{MM}}$ invariant metric $\eta = \mathrm{diag}\{1,-1\}$. Following [3] we take the parameterization for M_{MM} to be

$$M_{\mathrm{MM}} = V_{\mathrm{MM}}^T \eta V_{\mathrm{MM}} = \begin{pmatrix} \frac{\rho}{\Delta} - \frac{\Delta}{\rho}\psi^2 & -\frac{\Delta}{\rho}\psi \\ -\frac{\Delta}{\rho}\psi & -\frac{\Delta}{\rho} \end{pmatrix} \qquad \text{with} \qquad \eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad (\text{B.7})$$

where we emphasize that in our conventions the second index denotes the time component. We use this (non-standard) convention to facilitate comparison with [3].

Equations (B.3) and (B.4) can now be brought to the form

$$\partial_m \left(\rho M_{\mathrm{MM}}^{-1} \partial^m M_{\mathrm{MM}}\right) = 0, \qquad (\text{B.8})$$

and

$$\xi^{-1}\partial_\rho \xi = \frac{\rho}{8}\left(\mathrm{Tr}\left(M_{\mathrm{MM}}^{-1}\partial_\rho M_{\mathrm{MM}}\right)^2 - \mathrm{Tr}\left(M_{\mathrm{MM}}^{-1}\partial_z M_{\mathrm{MM}}\right)^2\right), \qquad (\text{B.9})$$

$$\xi^{-1}\partial_z \xi = \frac{\rho}{4}\mathrm{Tr}\left(M_{\mathrm{MM}}^{-1}\partial_\rho M_{\mathrm{MM}} M_{\mathrm{MM}}^{-1}\partial_z M_{\mathrm{MM}}\right), \qquad (\text{B.10})$$

where we have used $\Box\rho = 0$ and $\xi(\rho, z)$ is defined as $\xi = e^\nu \rho^{\frac{1}{4}}$. Defining as before

$$P_{\mathrm{MM},m} = \frac{1}{2}\left(\partial_m V_{\mathrm{MM}} V_{\mathrm{MM}}^{-1} + \left(\partial_m V_{\mathrm{MM}} V_{\mathrm{MM}}^{-1}\right)^T\right), \qquad (\text{B.11})$$

we can rewrite the above equations in the coordinates $x^\pm$ of (2.13) as

$$\pm i\xi^{-1}\partial_\pm \xi = \frac{\rho}{2}\mathrm{Tr}\left(P_{\mathrm{MM},\pm} P_{\mathrm{MM},\pm}\right), \qquad (\text{B.12a})$$

$$D_m(\rho P_{\mathrm{MM}}^m) = 0. \qquad (\text{B.12b})$$

Equations (B.12a) and (B.12b) are formally identical to equations (2.14a) and (2.14b). The BZ Lax pair can be written for these equations [15] as well.

If one wants to relate the BZ-generating function $\Psi(\lambda)$ to the group-theoretic BM-generating function $\mathcal{V}_{\mathrm{MM}}(t)$ additional care has to be taken because of the factor of ρ in (B.6). A convenient choice is

$$\Psi(\lambda, x) = \sqrt{2\rho t w}V_{\mathrm{MM}}^T(x)\eta\mathcal{V}_{\mathrm{MM}}(t,x). \qquad (\text{B.13})$$

(Note that this differs from what was given in [3]; their choice does not map the linear systems into each other away from $t = \lambda = 0$.)

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