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Generating functions of (partially-)massless higher-spin cubic interactions

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Abstract: Generating functions encoding cubic interactions of (partially-)massless higher-spin fields are provided within the ambient-space formalism. They satisfy a system of higher-order partial differential equations that can be explicitly solved due to their factorized form. We find that the number of consistent couplings increases whenever the squares of the field masses take some integer values (in units of the cosmological constant) and satisfy certain conditions. Moreover, it is shown that only the supplemental solutions can give rise to non-Abelian deformations of the gauge symmetries. The presence of these conditions on the masses is a distinctive feature of (A)dS interactions that has in general no direct counterpart in flat space.

Contents

1 Introduction

Since the gravitational-interaction problem of higher-spin (HS) fields[1] was overcome by turning on a cosmological constant [8, 9], many studies have been devoted to understand HS field theories in (A)dS background. The most important result of these efforts are Vasiliev's equations [10, 11], that, together with String Theory, represent the only known frameworks in which interactions of HS particles can be consistently described. However, a deeper understanding of them requires further investigations, especially in relation to the possibility that they be only particular members of a wider class of consistent HS theories. A preliminary step towards this goal would be to construct the most general consistent cubic couplings.[2] Among them, only a subset is expected to be compatible with higher-order interactions [19, 20] and lead eventually to fully non-linear theories.

Finding HS interactions in (A)dS has been the aim of our previous investigations [21–24], where, making use of the ambient-space formalism [25, 26], we provided the transverse and traceless (TT) parts of all possible cubic interactions involving massive and massless totally-symmetric HS fields. The key point of our construction was to recast the consistency conditions for the cubic vertices into a system of linear partial differential equations (PDEs),[3] whose solutions are in one-to-one correspondence to the consistent couplings. However, while for the case of massive and massless fields one gets relatively simple second-order PDEs, when partially-massless (PM) fields[4] enter the interactions, due to their higher-derivative gauge transformations, one has to solve higher-order PDEs. In [22] we only provided some examples of this kind of interactions relying on a numerical algorithm.

This paper is aimed at completing the program initiated in [22] and at deriving the generating functions of cubic interactions involving PM fields. A key point of our result is that the number of consistent couplings depends, in a non-trivial way, on the masses of the interacting fields. More precisely, whenever the squares of the masses take some integer values and satisfy certain conditions, a new class of solutions appears. Since the solutions—which are not of the latter type—do not lead to deformations of the gauge symmetries, the aforementioned conditions on the masses represent necessary conditions for the presence of non-Abelian interactions. After some preliminaries on the ambient-space formulation of HS interactions, we summarize our results in Section 1.2.

[1] See e.g. [1, 2] for recent reviews on HS field theories. See [3, 4] for some reviews on Vasiliev's equations, and [5–7] for AdS/CFT-related issues.

[2] Many efforts have been devoted in this direction: see the references in [1] for an exhaustive list of works, and in particular the latest works [12–16] and [17, 18] for flat-space and (A)dS interactions between totally-symmetric fields of arbitrary spins, respectively.

[3] See the partial result [27] for the generalization of this method to mixed-symmetry HS interactions.

[4] The PM spectrum has been first discovered for lower-spin (spin 2 and 3/2) fields in [28–34], while its HS generalization has been considered in [35–38]. The implications of PM fields to (A)dS/CFT have been discussed in [39, 40]. In [41–44], various formulations for the description of PM fields have been proposed. See [45–48] for the interactions of PM fields, and [48–52] for their connection to conformal theories and massive gravity. Finally, see [53] for the representations of $SO(1,4)$.

1.1 Preliminaries

Ambient-space formalism A way of describing totally-symmetric (A)dS tensor fields, $\varphi_{\mu_1\cdots\mu_s}$, is through ambient-space fields, $\Phi_{M_1\cdots M_s}$, that are subject to the *homogeneity* and *tangentiality* (HT) conditions:

$$\begin{aligned} &\text{Homogeneity}: \qquad (X\cdot\partial_X - U\cdot\partial_U + 2 + \mu)\,\Phi(X,U) = 0\,,\\ &\text{Tangentiality}: \qquad X\cdot\partial_U\,\Phi(X,U) = 0\,, \end{aligned} \tag{1.1}$$

where $\Phi(X,U)$ is the generating function of $\Phi_{M_1\cdots M_s}$:

$$\Phi(X,U) = \sum_{s=0}^{\infty} \frac{1}{s!}\,\Phi_{M_1\ldots M_s}(X)\,U^{M_1}\cdots U^{M_s}\,. \tag{1.2}$$

When $\mu = 0, 1, \ldots, s-1$, the constraints (1.1) allow higher-derivative gauge symmetries:

$$\delta^{(0)}\,\Phi(X,U) = (U\cdot\partial_X)^{\mu+1}\,\Omega(X,U)\,, \tag{1.3}$$

with gauge parameters Ω satisfying

$$(X\cdot\partial_X - U\cdot\partial_U - \mu)\,\Omega(X,U) = 0\,, \qquad X\cdot\partial_U\,\Omega(X,U) = 0\,. \tag{1.4}$$

Massless fields, $\mu = 0$, are the first members of a class of short representations where the other members, $\mu = 1, 2, \ldots, s-1$, are PM fields. For other values of μ, no gauge symmetry is allowed, implying that the corresponding fields are massive. Focussing on unitary representations, the corresponding values of μ are constrained to the regions shown in Figure 1. Let us mention that, contrary to massless fields, PM representations are unitary only in dS.

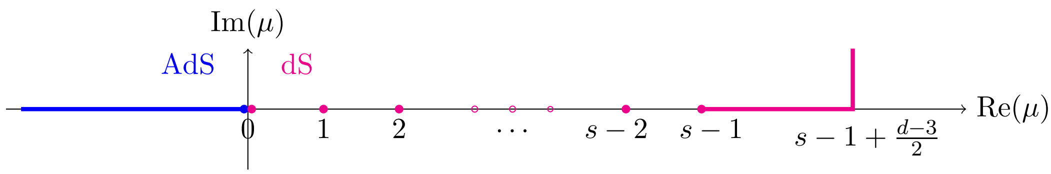

Figure 1. Unitary values of μ (for $s > 0$) .

Cubic-interaction problem The most general expression for the TT parts of the cubic vertices reads

$$S^{(3)} = \frac{1}{3!}\int_{(\mathrm{A})\mathrm{dS}} C(Y,Z)\;\Phi(X_1,U_1)\;\Phi(X_2,U_2)\;\Phi(X_3,U_3)\;\Big|_{\substack{X_i=X\\ U_i=0}}\,, \tag{1.5}$$

where $\int_{(\mathrm{A})\mathrm{dS}}$ is an integral over the codimension-one hypersurface $X^2 = \epsilon\,L^2$ ($\epsilon = 1$ for dS and $\epsilon = -1$ for AdS), while C is an arbitrary function of the following parity-preserving Lorentz invariants:[5]

$$Y_i = \partial_{U_i}\cdot\partial_{X_{i+1}}\,, \qquad Z_i = \partial_{U_{i+1}}\cdot\partial_{U_{i-1}} \qquad [i \simeq i+3]\,. \tag{1.6}$$

[5] Henceforth, for brevity, we shall denote the dependence on all six variables by Y and Z.

Assuming the i-th field to be (partially-)massless ((P)M) (*i.e.* $\mu_i \in \mathbb{N}$), the corresponding compatibility condition of the cubic vertices (1.5) with the gauge symmetries (1.3) is equivalent to imposing

$$\big[\, C(Y,Z)\,,\, (U_i\cdot \partial_{X_i})^{\mu_i+1}\,\big]\Big|_{U_i=0} \approx 0\,. \tag{1.7}$$

Using Leibniz's rule, one can recast the condition (1.7) into a higher-order PDE:

$$\mathcal{L}_i(\bar{\mu}_i-\mu_i)\,\mathcal{L}_i(\bar{\mu}_i-\mu_i+2)\cdots\mathcal{L}_i(\bar{\mu}_i+\mu_i)\;C(Y,Z)=0 \qquad \big[\,\bar{\mu}_i:=\mu_{i-1}-\mu_{i+1}\,\big]\,, \tag{1.8}$$

consisting in the product of the following commuting differential operators:

$$\mathcal{L}_i(x):=Y_{i+1}\,\partial_{Z_{i-1}}-Y_{i-1}\,\partial_{Z_{i+1}}+\tfrac{\hat{\delta}}{L}\left(Y_{i+1}\,\partial_{Y_{i+1}}-Y_{i-1}\,\partial_{Y_{i-1}}+\tfrac{1}{2}\,x\right)\partial_{Y_i}\,. \tag{1.9}$$

Depending on the number of (P)M fields involved in the interactions, one can have up to three PDEs, whose solutions encode all possible consistent couplings.

1.2 Summary of the results

As already mentioned, a peculiar feature underlying (A)dS interactions is the appearance of non-trivial conditions on the mass values for which the number of consistent couplings may get enhanced. More precisely, if the i-th field is at one of its (P)M points $\mu_i \in \{0,1,\ldots,s_i-1\}$, the solution space of the corresponding system of PDEs may become bigger whenever the conditions

$$\mu_i+\mu_{i+1}-\mu_{i-1}\in 2\,\mathbb{Z}\,, \tag{1.10}$$

hold. An explicit analysis shows that, while for arbitrary values of the μ_i's the solutions can be written in terms of arbitrary functions of the following operators:

$$\tilde{H}_i:=\partial_{X_{i+1}}\cdot\partial_{X_{i-1}}\,\partial_{U_{i+1}}\cdot\partial_{U_{i-1}}-\partial_{U_{i-1}}\cdot\partial_{X_{i+1}}\,\partial_{U_{i+1}}\cdot\partial_{X_{i-1}}\,, \tag{1.11}$$

when the condition (1.10) is satisfied, additional solutions involving the operator:

$$G(Y,Z):=Y_1\,Z_1+Y_2\,Z_2+Y_3\,Z_3\,, \tag{1.12}$$

also appear. However, some of the G-couplings can be also expressed in terms of the $\tilde{H}$-couplings, so that the two kinds of solutions may have some overlap. Let us stress that, since the $\tilde{H}$-couplings are trivially gauge invariant, non-Abelian interactions are only among those G-couplings which cannot be written as $\tilde{H}$-couplings. Therefore, a necessary condition for non-Abelian interactions to be present is (1.10). The existence of the latter mass-pattern may have also some interesting consequences for the interactions of one massless and two massive fields: on the mass-pattern, indeed, such interactions can induce deformations of the gauge symmetries related to non-trivial Noether currents involving fields with *different masses*, a novelty of (A)dS interactions.[6]

Our results are summarized in the following framed paragraph.

[6] This point has been omitted in our previous work [22], as, in the analysis of the flat limit (see Appendix D), we overlooked the singular points of the PDEs.

(P)M – Massive – Massive

For arbitrary $\mu_2 - \mu_3$:

$$\tilde{C} = \sum_{\sigma_1=0}^{\mu_1} Y_1^{\sigma_1}\, \tilde{K}^{\sigma_1}(Y_2, Y_3, Z_1, \tilde{H}_2, \tilde{H}_3)\,. \tag{1.13}$$

For $\mu_1 + \mu_2 - \mu_3 \in 2\,\mathbb{Z}$, one also has

$$C = \sum_{(\tau_1,\tau_2,\tau_3)\in\mathscr{L}_1} Z_1^{\tau_1}\, Z_2^{\tau_2}\, Z_3^{\tau_3}\, Y_2^{R\left(\tau_2+\frac{\mu_2-\mu_3-\mu_1}{2}\right)}\, Y_3^{R\left(\tau_3+\frac{\mu_3-\mu_1-\mu_2}{2}\right)} \times$$

$$\times\, e^{-\frac{\hat{\delta}}{L}\mathcal{D}}\, K^{\tau_1\tau_2\tau_3}(Y, G)\Big|_{G=G(Y,Z)}, \tag{1.14}$$

where $R(x) = (x + |x|)/2$ is the ramp function, $\mathcal{D}$ is a differential operator:

$$\mathcal{D} := Z_1\, \partial_{Y_2}\, \partial_{Y_3} + Z_1\, Z_2\, \partial_{Y_3}\, \partial_G + \text{cyclic} + Z_1\, Z_2\, Z_3\, \partial_G^2\,, \tag{1.15}$$

and $\mathscr{L}_i$ is the lattice:

$$\mathscr{L}_i := \left\{\, (\tau_1, \tau_2, \tau_3) \in \mathbb{N}^3 \;\middle|\; \tau_{i+1} + \tau_{i-1} \leq \mu_i \,\right\}. \tag{1.16}$$

(P)M – (P)M – Massive

For arbitrary μ_3 :

$$\tilde{C} = \sum_{\sigma_1=0}^{\mu_1} \sum_{\sigma_2=0}^{\mu_2} Y_1^{\sigma_1}\, Y_2^{\sigma_2}\, \tilde{K}^{\sigma_1\sigma_2}(Y_3, \tilde{H}_1, \tilde{H}_2, \tilde{H}_3)\,. \tag{1.17}$$

For $\mu_i + \mu_{i+1} - \mu_{i-1} \in 2\,\mathbb{Z}$, one also has

$$C = \sum_{(\tau_1,\tau_2,\tau_3)\in\mathscr{L}_1\cap\mathscr{L}_2\cap\mathscr{L}_3} Z_1^{\tau_1}\, Z_2^{\tau_2}\, Z_3^{\tau_3}\, Y_1^{R\left(\tau_1+\frac{\mu_1-\mu_2-\mu_3}{2}\right)}\, Y_2^{R\left(\tau_2+\frac{\mu_2-\mu_3-\mu_1}{2}\right)}\, Y_3^{R\left(\tau_3+\frac{\mu_3-\mu_1-\mu_2}{2}\right)} \times$$

$$\times\, e^{-\frac{\hat{\delta}}{L}\mathcal{D}}\, K^{\tau_1\tau_2\tau_3}(Y, G)\Big|_{G=G(Y,Z)}. \tag{1.18}$$

(P)M – (P)M – (P)M

For arbitrary μ_i's :

$$\tilde{C} = \sum_{\sigma_1=0}^{\mu_1} \sum_{\sigma_2=0}^{\mu_2} \sum_{\sigma_3=0}^{\mu_3} Y_1^{\sigma_1}\, Y_2^{\sigma_2}\, Y_3^{\sigma_3}\, \tilde{K}^{\sigma_1\sigma_2\sigma_3}(\tilde{H}_1, \tilde{H}_2, \tilde{H}_3)\,. \tag{1.19}$$

For $\mu_i + \mu_{i+1} - \mu_{i-1} \in 2\,\mathbb{Z}$, one also has

$$C = \sum_{(\tau_1,\tau_2,\tau_3)\in\mathscr{L}_1\cap\mathscr{L}_2\cap\mathscr{L}_3} Z_1^{\tau_1}\, Z_2^{\tau_2}\, Z_3^{\tau_3}\, Y_1^{R\left(\tau_1+\frac{\mu_1-\mu_2-\mu_3}{2}\right)}\, Y_2^{R\left(\tau_2+\frac{\mu_2-\mu_3-\mu_1}{2}\right)}\, Y_3^{R\left(\tau_3+\frac{\mu_3-\mu_1-\mu_2}{2}\right)} \times$$

$$\times\, e^{-\frac{\hat{\delta}}{L}\mathcal{D}}\, K^{\tau_1\tau_2\tau_3}(Y, G)\Big|_{G=G(Y,Z)}. \tag{1.20}$$

Organization of the paper

Section 2 contains some general discussions about polynomial solutions to PDEs. In Section 3, we provide the most general solutions to one PDE encoding the interactions of one (P)M field. The solutions to the system of PDEs corresponding to general couplings that involve more than one (P)M field are derived in Section 4. Our results are discussed in Section 5. Finally, Appendices A, B, C and D contain some mathematical details on the derivations of the results presented in this paper.

2 General structure of the solutions

Before solving the consistency equation (1.8), let us first discuss in some detail the general idea underlying our way of organizing the polynomial solutions. The latter are in fact the only relevant solutions for the analysis of HS cubic interactions.

2.1 Cubic interactions as a vector space

Restricting the attention to polynomials in Y and Z, the function $C(Y,Z)$ can be expanded as

$$C(Y,Z) = \sum_{\sigma_i,\tau_i\geq 0} C^{\tau_1\tau_2\tau_3}_{\sigma_1\sigma_2\sigma_3}\, Z_1^{\tau_1}\, Z_2^{\tau_2}\, Z_3^{\tau_3}\, Y_1^{\sigma_1}\, Y_2^{\sigma_2}\, Y_3^{\sigma_3}\,, \tag{2.1}$$

where $C^{\tau_1\tau_2\tau_3}_{\sigma_1\sigma_2\sigma_3}$ are arbitrary coefficients. Since the operators $\mathcal{L}_i$ (1.9) preserve the spin degrees:

$$s_i = \sigma_i + \tau_{i+1} + \tau_{i-1}\,, \tag{2.2}$$

any solution to eq. (1.8) can be decomposed into solutions with fixed spins s_i. Moreover, being a linear PDE, for given s_i its solution space is a finite dimensional vector space, whose elements can be labeled as:

$$C^{s_1s_2s_3}(Y,Z) = \sum_{n=1}^{N} K_n\; P_n^{s_1s_2s_3}(Y,Z) \qquad [K_n\in\mathbb{R}]\,. \tag{2.3}$$

Here, N is the dimension of the vector space, namely the number of $s_1-s_2-s_3$ independent couplings, whereas

$$\{\, P_1^{s_1s_2s_3}(Y,Z)\,,\, P_2^{s_1s_2s_3}(Y,Z)\,,\,\ldots\,,\, P_N^{s_1s_2s_3}(Y,Z)\,\}\,, \tag{2.4}$$

is a corresponding basis. Therefore, solving eq. (1.8) is tantamount to finding a set of basis vectors $P_n^{s_1s_2s_3}(Y,Z)$. Let us notice that, since the spins do not enter explicitly the PDE, its solutions are spin independent. Hence, one can first determine the basis $\{\,P_n(Y,Z)\,\}$ without specifying its spin dependence, and then restrict the attention to fixed spins.

2.2 Choice of basis

The basis $\{\,P_n(Y,Z)\,\}$ is of course not unique, and some choices can be more convenient than others. In order to clarify this point, let us first expand the basis solution $P_n(Y,Z)$ in powers of $\hat{\delta}/L$:

$$P_n\big(\tfrac{\hat{\delta}}{L};Y,Z\big) = \sum_{k\geq 0}\,\big(\tfrac{\hat{\delta}}{L}\big)^k\, P_n^{(k)}(Y,Z)\,. \tag{2.5}$$

In physical term, this is an expansion in the number of (ambient-space) derivatives, so that the leading term $P_n^{(0)}$ can be identified with the highest-derivative piece of the coupling $P_n(Y,Z)$. A convenient choice is a basis whose elements $P_n(Y,Z)$ have all different leading terms:

$$P_n^{(0)}(Y,Z) \neq P_m^{(0)}(Y,Z) \qquad \forall\; n \neq m\,. \tag{2.6}$$

Let us notice that this requirement does not fix the basis uniquely. Indeed, denoting by Δ_n the maximum number of derivatives in $P_n(Y,Z)$, one can always add a lower-derivative piece $P_m(Y,Z)$ as

$$Q_n(Y,Z) = P_n(Y,Z) + \left(\tfrac{\hat{\delta}}{L}\right)^{\frac{1}{2}(\Delta_n-\Delta_m)}\, P_m(Y,Z) \qquad [\Delta_n > \Delta_m]\,, \tag{2.7}$$

without spoiling the condition (2.6). Our strategy in order to construct a basis of this type consists in two steps:

- identify all possible leading terms $P_n^{(0)}(Y,Z)$;
- for each $P_n^{(0)}(Y,Z)$, find (if it exists) a corresponding full solution $P_n(Y,Z)$.

As we have just explained, the solution $P_n(Y,Z)$ is not unique but can be always replaced by some other solutions $Q_n(Y,Z)$ differing by lower-derivative solutions. In the next Section we exploit this freedom in the choice of basis in order to simplify our analysis. In particular, we first construct a basis:

$$B_P = \{\, P_n(Y,Z)\,\}\,, \tag{2.8}$$

proving that it spans the entire solution space of eq. (1.8), and then we introduce another set of solutions:

$$B_Q = \{\, Q_n(Y,Z)\,\}\,, \tag{2.9}$$

whose elements satisfy $Q_n^{(0)}(Y,Z) = P_n^{(0)}(Y,Z)$ for all n[7] (so, the index n can be understood as a label for different leading terms). The latter conditions ensure that B_Q be also a basis of the solution space, *i.e.*, $\text{Span}(B_Q) = \text{Span}(B_P)$. The reason for this change of basis is that, while B_P is convenient to prove the completeness of the solution space, B_Q turns out to be more suitable in constructing solutions to more than one equation.

3 Solutions to one equation

3.1 Massless equation

The aim of this Section is to find the general solution to eq. (1.8). Since the latter consists in a product of commuting operators $\mathcal{L}_1$, it is convenient to first analyze the kernel of a single operator. In particular, we start analyzing the massless case ($\mu_1 = 0$), where eq. (1.8) reduces to

$$\mathcal{L}_1(\bar{\mu}_1)\, C(Y,Z) = 0\,, \tag{3.1}$$

with the operator $\mathcal{L}_1$ given by

$$\mathcal{L}_1(\bar{\mu}_1) = Y_2\,\partial_{Z_3} - Y_3\,\partial_{Z_2} + \tfrac{\hat{\delta}}{L}\left(Y_2\,\partial_{Y_2} - Y_3\,\partial_{Y_3} + \tfrac{\bar{\mu}_1}{2}\right)\partial_{Y_1}\,. \tag{3.2}$$

[7] More precisely, we shall provide linear combinations satisfying this equality.

3.1.1 General solution

Eq. (3.1) can be solved as a power series in Y_1 :

$$C_{\sigma_1}(Y,Z) = \sum_{k=0}^{\sigma_1} C^{(k)}_{\sigma_1}(Y_2,Y_3,Z)\,\big(-\tfrac{\hat{\delta}}{L}\,\partial_{Y_1}\big)^k\,Y_1^{\,\sigma_1}\,, \tag{3.3}$$

where $C_{\sigma_1}(Y,Z)$ denote the solutions whose highest power of Y_1 is σ_1. Plugging the expansion (3.3) into eq. (3.1), one ends up with a differential recurrence relation for $C^{(k)}_{\sigma_1}$:

$$(Y_2\,\partial_{Z_3} - Y_3\,\partial_{Z_2})\,C^{(k)}_{\sigma_1}(Y_2,Y_3,Z) = \big(Y_2\,\partial_{Y_2} - Y_3\,\partial_{Y_3} + \tfrac{1}{2}\,\bar{\mu}_1\big)\,C^{(k-1)}_{\sigma_1}(Y_2,Y_3,Z)\,, \tag{3.4}$$

where $C^{(-1)}_{\sigma_1} = 0$. The latter can be solved iteratively starting from the leading term:

$$\begin{aligned} C^{(0)}_{\sigma_1}(Y_2,Y_3,Z) &= K\big(Y_2,Y_3,Z_1,G_1(Y,Z)\big) \\ &= \sum_{\sigma_2,\sigma_3,\tau_1,\upsilon} K^{\tau_1}_{\sigma_1\sigma_2\sigma_3\upsilon}\,Z_1^{\,\tau_1}\,Y_2^{\,\sigma_2}\,Y_3^{\,\sigma_3}\,[G_1(Y,Z)]^{\upsilon}\,, \end{aligned} \tag{3.5}$$

where $G_1(Y,Z) := Y_2\,Z_2 + Y_3\,Z_3$. After summing C_{σ_1} over σ_1, the full solution reads (see Appendix A for details):

$$C(Y,Z) = \sum_{\sigma_1,\sigma_2,\sigma_3,\tau_1,\upsilon} K^{\tau_1}_{\sigma_1\sigma_2\sigma_3\upsilon}\,P^{\tau_1}_{\sigma_1\sigma_2\sigma_3\upsilon}(\bar{\mu}_1;Y,Z)\,, \tag{3.6}$$

where the polynomial functions $P^{\tau_1}_{\sigma_1\sigma_2\sigma_3\upsilon}$ are defined as

$$\begin{aligned} &P^{\tau_1}_{\sigma_1\sigma_2\sigma_3\upsilon}(\bar{\mu}_1;Y,Z) = \\ &= \left[\sum_{p,q\geq 0}^{p+q\leq\sigma_1} \frac{\big[\sigma_2+\frac{\bar{\mu}_1}{2}\big]_p\,\big[\sigma_3-\frac{\bar{\mu}_1}{2}\big]_q}{[\sigma_2+\sigma_3]_{p+q}}\,\frac{\big(-\frac{\hat{\delta}}{L}\,Z_3\,\partial_{Y_1}\,\partial_{Y_2}\big)^p}{p!}\,\frac{\big(-\frac{\hat{\delta}}{L}\,Z_2\,\partial_{Y_3}\,\partial_{Y_1}\big)^q}{q!}\right]\times \\ &\quad\times Z_1^{\,\tau_1}\,Y_1^{\,\sigma_1}\,Y_2^{\,\sigma_2}\,Y_3^{\,\sigma_3}\,[G_1(Y,Z)]^{\upsilon}\,, \end{aligned} \tag{3.7}$$

and $[a]_n := a(a-1)\cdots(a-n+1)$ is the descending Pochhammer symbol. Let us emphasize that, due to the presence of the terms $[\sigma_2+\sigma_3]_{p+q}$ in the denominator, these functions may be ill-defined. More precisely, potentially diverging terms appear for those values of p and q such that

$$\sigma_2+\sigma_3+1 \leq p+q \leq \sigma_1\,. \tag{3.8}$$

Hence, there are only two cases in which the solutions are well-defined:

- When $\sigma_2+\sigma_3 \geq \sigma_1$. In this case, the problematic terms are simply absent and one gets the following set of solutions:

$$B_{P1}(\bar{\mu}_1) = \big\{\,P^{\tau_1}_{\sigma_1\sigma_2\sigma_3\upsilon}(\bar{\mu}_1)\;\big|\;\sigma_2+\sigma_3\geq\sigma_1\,\big\}\,. \tag{3.9}$$

- When the condition (3.8) holds and the residue:

$$\begin{aligned} \big[\sigma_2+\tfrac{\bar{\mu}_1}{2}\big]_p\,\big[\sigma_3-\tfrac{\bar{\mu}_1}{2}\big]_q &= \big(\tfrac{\bar{\mu}_1}{2}+\sigma_2\big)\cdots\big(\tfrac{\bar{\mu}_1}{2}-\sigma_3\big)\times \\ &\times(-1)^q\,\big(\tfrac{\bar{\mu}_1}{2}+q-\sigma_3+1\big)\cdots\big(\tfrac{\bar{\mu}_1}{2}-p+\sigma_2+1\big)\,, \end{aligned} \tag{3.10}$$

of the corresponding diverging piece vanishes. This happens when

$$(\tfrac{\bar{\mu}_1}{2}+\sigma_2)\cdots(\tfrac{\bar{\mu}_1}{2}-\sigma_3)=0\,, \tag{3.11}$$

so that the following set of solutions:

$$B_{P2}(\bar{\mu}_1)=\left\{\,P^{\tau_1}_{\sigma_1\sigma_2\sigma_3\upsilon}(\bar{\mu}_1)\;\middle|\;\sigma_3\geq\tfrac{\bar{\mu}_1}{2}\,,\;\tfrac{\bar{\mu}_1}{2}\in\mathbb{Z}_{\geq 0}\,;\quad\sigma_2\geq|\tfrac{\bar{\mu}_1}{2}|\,,\;\tfrac{\bar{\mu}_1}{2}\in\mathbb{Z}_{<0}\,\right\}, \tag{3.12}$$

are well-defined.

Notice that B_{P2} is non-empty only when $\bar{\mu}_1$ is an even integer. Moreover, it has a non-vanishing intersection with B_{P1}, so much so that, when $\bar{\mu}_1=0$, B_{P1} becomes a subset of B_{P2}. Finally, the union of the two sets:

$$B_P(\bar{\mu}_1)=B_{P1}(\bar{\mu}_1)\cup B_{P2}(\bar{\mu}_1)\,, \tag{3.13}$$

forms a (redundant) basis[8] of the solution space of eq. (3.1). Comparing eq. (3.13) to eq. (2.8), one can see that the index n in $P_n(Y,Z)$ corresponds to the collective index $\{\sigma_1,\sigma_2,\sigma_3,\tau_1,\upsilon\}$ labeling different leading terms.

3.1.2 Change of basis

At this stage, we have explicitly constructed the basis (3.13) of the solution space of eq. (3.1). However, although the form (3.7) in which the basis solutions are written makes the completeness of the solution space manifest, it is not suitable for the analysis of more than one equation. Therefore, in the following we construct other two sets of solutions $B_{\tilde{Q}}$ and B_Q, and, analyzing their leading terms, we prove that their union spans the same space as B_P.

$\tilde{Q}$ solutions Let us notice that the operators:[9]

$$\tilde{H}_i:=\partial_{X_{i+1}}\cdot\partial_{X_{i-1}}\,\partial_{U_{i+1}}\cdot\partial_{U_{i-1}}-\partial_{U_{i-1}}\cdot\partial_{X_{i+1}}\,\partial_{U_{i+1}}\cdot\partial_{X_{i-1}}\,, \tag{3.14}$$

commute with the gradient operators:

$$[\,\tilde{H}_i\,,\,U_j\cdot\partial_{X_j}\,]=0\,, \tag{3.15}$$

without relying on the on-shell conditions. As a consequence, one can easily construct couplings of the form:

$$\tilde{K}(Y_2,Y_3,Z_1,\tilde{H}_2,\tilde{H}_3)\,, \tag{3.16}$$

that are invariant under the gauge transformations associated to the first field. These couplings are not written as functions of Y and Z, but they can be always brought to that form by performing the integrations by parts of all the total-derivative terms present in $\tilde{H}_2$ and $\tilde{H}_3$. Hence, one can consider the following set:

$$B_{\tilde{Q}}(\bar{\mu}_1)=\left\{\,\tilde{Q}^{\tau_1}_{\sigma_2\sigma_3h_2h_3}(\bar{\mu}_1)\,\right\}, \tag{3.17}$$

[8] With a slightly abuse of notation we use the term 'basis' to denote a set of solutions which spans the entire solution space, regardless of the fact that they be linearly independent or not.

[9] These operators can be obtained as deformations of their flat-space counterparts (see Appendix D).

whose elements are given by

$$\tilde{Q}^{\tau_1}_{\sigma_2\sigma_3 h_2 h_3}(\bar{\mu}_1; Y, Z) \simeq Z_1^{\tau_1}\, Y_2^{\sigma_2}\, Y_3^{\sigma_3}\, \tilde{H}_2^{h_2}\, \tilde{H}_3^{h_3}\,, \tag{3.18}$$

where $\simeq$ means equivalence modulo integrations by parts. Although finding the exact form of $\tilde{Q}^{\tau_1}_{\sigma_2\sigma_3 h_2 h_3}$ requires an involved analysis (see Appendix C for details), for our purpose it is sufficient to identify the corresponding leading terms:

$$\tilde{Q}^{\tau_1}_{\sigma_2\sigma_3 h_2 h_3}(\bar{\mu}_1; Y, Z) = Z_1^{\tau_1}\, Y_1^{h_2+h_3}\, Y_2^{\sigma_2+h_3}\, Y_3^{\sigma_3+h_2} + \mathcal{O}\big(\tfrac{\hat{\delta}}{L}\big)\,. \tag{3.19}$$

At this point, it is straightforward to check that these leading terms exactly reproduce the ones of $P^{\tau_1}_{\sigma_1\sigma_2\sigma_3 \upsilon}$ in B_{P1} (3.9) for $\upsilon = 0$. The dependence on G_1 can be recovered by taking proper combinations of them. For instance, the $3-2-3$ coupling starting with the leading term $Y_1\, Y_3\, G_1^2$ is given by

$$\begin{aligned} &\Big[\tfrac{\bar{\mu}_1-2}{2}\,\big(Y_2\,\tilde{H}_2\big)^2 - \bar{\mu}_1\,\big(Y_2\,\tilde{H}_2\big)\big(Y_3\,\tilde{H}_3\big) + \tfrac{\bar{\mu}_1+2}{2}\,\big(Y_3\,\tilde{H}_3\big)^2\Big]\,\tilde{H}_2 \\ &\simeq \big(\tfrac{\hat{\delta}}{L}\big)^2\,\tfrac{\bar{\mu}_1-2}{2}\,\tfrac{\bar{\mu}_1}{2}\,\tfrac{\bar{\mu}_1+2}{2}\,\Big[Y_1\,Y_3\,G_1^2 + \tfrac{\hat{\delta}}{L}\,Z_2\,G_1\,\Big(\tfrac{\bar{\mu}_1-2}{2}\,Y_2\,Z_2 + \tfrac{\bar{\mu}_1-6}{2}\,Y_3\,Z_3\Big)\Big]\,. \end{aligned} \tag{3.20}$$

As one can see from this example, depending on the values of $\bar{\mu}_1$, the relation between the elements of the two basis $B_{\tilde{Q}}(\bar{\mu}_1)$ and $B_{P1}(\bar{\mu}_1)$ can be singular. However, the vanishing coefficient (*i.e.*, $(\bar{\mu}_1+2)\,\bar{\mu}_1\,(\bar{\mu}_1-2)$ in (3.20)) is overall so that one can always normalize the leading term. See Appendix C for the general case. Since all the leading terms of $B_{P1}(\bar{\mu}_1)$ can be reproduced by $\tilde{Q}^{\tau_1}_{\sigma_2\sigma_3 h_2 h_3}(\bar{\mu}_1)$, and $B_{P1}(\bar{\mu}_1)$ covers the whole solution space when (1.10) is not satisfied, one can conclude that[10]

$$\mathrm{Span}\big(B_{\tilde{Q}}(\bar{\mu}_1)\big) = \mathrm{Span}\big(B_{P1}(\bar{\mu}_1)\big)\,. \tag{3.21}$$

Q solutions Let us notice that, when the constant $\bar{\mu}_1/2$ in eq. (3.1) is an integer number, it can be removed factoring terms of the form $Y_2^{-\bar{\mu}_1/2}$ or $Y_3^{\bar{\mu}_1/2}$. Hence, using the ramp function

$$R(x) := (|x| + x)/2\,, \tag{3.22}$$

one can construct the following solutions:

$$C(Y, Z) = Y_2^{R(-\frac{\bar{\mu}_1}{2})}\, Y_3^{R(\frac{\bar{\mu}_1}{2})}\, e^{-\frac{\hat{\delta}}{L}\,\mathcal{D}}\, K(Y_1, Y_2, Y_3, Z_1, G)\,\Big|_{G=G(Y,Z)} \qquad [\text{if } \tfrac{\bar{\mu}_1}{2} \in \mathbb{Z}\,]\,, \tag{3.23}$$

with

$$G(Y, Z) := Y_1\, Z_1 + Y_2\, Z_2 + Y_3\, Z_3\,. \tag{3.24}$$

Here $e^{-\frac{\hat{\delta}}{L}\,\mathcal{D}}\, K$ are the general solutions to the massless equation $\mathcal{L}_1(0)\, C = 0$, where the operator $\mathcal{D}$ is defined as

$$\mathcal{D} := Z_1\, \partial_{Y_2}\, \partial_{Y_3} + Z_1\, Z_2\, \partial_{Y_3}\, \partial_G + \text{cyclic} + Z_1\, Z_2\, Z_3\, \partial_G^2\,. \tag{3.25}$$

[10] By continuity of the solution spaces $B_{P1}(\bar{\mu}_1)$ and $B_{\tilde{Q}}(\bar{\mu}_1)$ in $\bar{\mu}_1$, this statement holds also when the condition (1.10) is satisfied.

The solutions (3.23) can be decomposed in terms of the following functions:

$$\begin{aligned} Q^{\tau_1}_{\sigma_1\sigma_2\sigma_3\upsilon}(\bar\mu_1;Y,Z) &:= Z_1^{\tau_1}\,Y_2^{R(-\frac{\bar\mu_1}{2})}\,Y_3^{R(\frac{\bar\mu_1}{2})}\,e^{-\frac{\hat\delta}{L}\,\mathcal{D}}\,Y_1^{\sigma_1}\,Y_2^{\sigma_2}\,Y_3^{\sigma_3}\,G^{\upsilon}\,\Big|_{G=G(Y,Z)} \\ &= Z_1^{\tau_1}\,Y_1^{\sigma_1}\,Y_2^{\sigma_2+R(-\frac{\bar\mu_1}{2})}\,Y_3^{\sigma_3+R(\frac{\bar\mu_1}{2})}\,[G_1(Y,Z)+Y_1\,Z_1]^{\upsilon}+\mathcal{O}\big(\tfrac{\hat\delta}{L}\big)\,, \end{aligned} \tag{3.26}$$

whose leading terms coincide with those of $P^{\tau_1}_{\sigma_1\sigma_2\sigma_3\upsilon}$ in B_{P2} (3.12). As a consequence, the set:

$$B_Q(\bar\mu_1)=\big\{\,Q^{\tau_1}_{\sigma_1\sigma_2\sigma_3\upsilon}(\bar\mu_1)\,\big\}\,, \tag{3.27}$$

together with $B_{\tilde Q}(\bar\mu_1)$ span the entire solution space:

$$\mathrm{Span}\big(B_{\tilde Q}(\bar\mu_1)\cup B_Q(\bar\mu_1)\big)=\mathrm{Span}\big(B_{P1}(\bar\mu_1)\cup B_{P2}(\bar\mu_1)\big)\,. \tag{3.28}$$

3.2 Partially-massless equation

Let us now move to the case in which one PM field ($\mu_1\in\mathbb{N}$) is involved in the interactions. Then, the corresponding cubic vertices have to satisfy the PDE:

$$\mathcal{L}_1(\bar\mu_1-\mu_1)\,\mathcal{L}_1(\bar\mu_1-\mu_1+2)\cdots\mathcal{L}_1(\bar\mu_1+\mu_1)\,C(Y,Z)=0\,. \tag{3.29}$$

3.2.1 General solutions

The general solutions to the above equation can be decomposed in terms of the functions C_k's satisfying

$$\mathcal{L}_1(\bar\mu_1-\mu_1+2\,k)\,C_k=0\,, \tag{3.30}$$

with $k=0,1,\dots,\mu_1$. The C_k's are given by (3.6) where $\bar\mu_1$ is replaced by $\bar\mu_1-\mu_1+2\,k$, hence the following set of functions:

$$\big\{\,P^{\tau_1}_{\sigma_1\sigma_2\sigma_3\upsilon}(\bar\mu_1-\mu_1)\,\big\}\,\cup\,\big\{\,P^{\tau_1}_{\sigma_1\sigma_2\sigma_3\upsilon}(\bar\mu_1-\mu_1+2)\,\big\}\,\cup\,\cdots\,\cup\,\big\{\,P^{\tau_1}_{\sigma_1\sigma_2\sigma_3\upsilon}(\bar\mu_1+\mu_1)\,\big\}\,, \tag{3.31}$$

forms a complete basis for the solution space of eq. (3.29). However, this decomposition is possible only when σ_2 and σ_3 are regarded as real (non-integer) numbers. Indeed, as in the massless case, the set (3.31) contains solutions that are ill-defined. Therefore, out of them, one needs to select only the ones that are well-behaved as σ_2 and σ_3 approach some integer numbers. In the massless case, for any given leading term there is only one function in the set, so that the selection simply amounts in examining whether each function $P^{\tau_1}_{\sigma_1\sigma_2\sigma_3\upsilon}(\bar\mu_1)$ is well-defined or not. On the other hand, in the PM case one has μ_1+1 solutions $P^{\tau_1}_{\sigma_1\sigma_2\sigma_3\upsilon}(\bar\mu_1-\mu_1+2\,k)$ with the same leading term, so one has to analyze all possible linear combinations of them.

$P1$ solutions Let us consider the first divergent term in the series expansion of $P^{\tau_1}_{\sigma_1\sigma_2\sigma_3\upsilon}(x)$ (3.7). It arises for $p+q=\sigma_2+\sigma_3+1$ and its residue is $(\frac{x}{2}+\sigma_2)\cdots(\frac{x}{2}-\sigma_3)\,(-1)^q$ (see eq. (3.10)). Therefore, it can be cancelled by taking the following linear combination:

$$\big(\tfrac{x}{2}+1+\sigma_2\big)\,P^{\tau_1}_{\sigma_1\sigma_2\sigma_3\upsilon}(x)-\big(\tfrac{x}{2}-\sigma_3\big)\,P^{\tau_1}_{\sigma_1\sigma_2\sigma_3\upsilon}(x+2)\,. \tag{3.32}$$

Although one can get rid of the first divergent term, there is no way to eliminate the remaining divergent pieces that are still present for $p+q \geq \sigma_2+\sigma_3+2$. Hence, the combination (3.32) is well-defined only when the order of the polynomial is low enough in order to prevent those divergent terms from showing up, that is, when $\sigma_2+\sigma_3+1 \geq \sigma_1$. Let us notice that this condition on the leading term is very similar to the one obtained in the massless case (3.9), but for the shift by one on the left-hand side of the inequality. In general, taking the following linear combination:

$$P^{[n]\,\tau_1}_{\sigma_1\sigma_2\sigma_3 \upsilon}(x) := \\ = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left[\tfrac{x}{2}+n+\sigma_2\right]_{n-k} \left(\tfrac{x}{2}-\sigma_3\right)_k P^{\tau_1}_{\sigma_1\sigma_2\sigma_3\upsilon}(x+2k)\,, \tag{3.33}$$

one can cancel the first n divergent terms. Once again, as these functions contain divergent terms for $p+q \geq \sigma_2+\sigma_3+n+1$, only the ones belonging to the set:

$$B^{[n]}_{P1}(x) = \left\{ P^{[n]\,\tau_1}_{\sigma_1\sigma_2\sigma_3\upsilon}(x) \mid \sigma_2+\sigma_3+n \geq \sigma_1 \geq n \right\}, \tag{3.34}$$

are well-defined. Here, the lower bound on σ_1 has been introduced since the solutions $P^{[n]\,\tau_1}_{\sigma_1\sigma_2\sigma_3\upsilon}$ with $\sigma_1 < n$—not having a sufficient number of terms—are simply linear combinations of $P^{[0]\,\tau_1}_{\sigma_1\sigma_2\sigma_3\upsilon}$, $P^{[1]\,\tau_1}_{\sigma_1\sigma_2\sigma_3\upsilon}$, ... , $P^{[n-1]\,\tau_1}_{\sigma_1\sigma_2\sigma_3\upsilon}$. To sum up, starting from the formal basis (3.31), one can extract the following set of well-defined solutions:

$$\boldsymbol{B}_{P1}(\bar\mu_1,\mu_1) := B^{[0]}_{P1}(\bar\mu_1-\mu_1) \cup B^{[1]}_{P1}(\bar\mu_1-\mu_1) \cup \cdots \cup B^{[\mu_1]}_{P1}(\bar\mu_1-\mu_1)\,. \tag{3.35}$$

$P2$ **solutions** On the other hand, precisely as in the massless case (3.12), the divergent terms are harmless when the corresponding residues vanish. Since everything is well-defined in this case, the corresponding solution space is spanned by the union of the massless sets (3.12):

$$B_{P2}(\bar\mu_1-\mu_1) \cup B_{P2}(\bar\mu_1-\mu_1+2) \cup \cdots \cup B_{P2}(\bar\mu_1+\mu_1)\,. \tag{3.36}$$

However, for the succeeding analysis it proves convenient to change basis. For this purpose, let us first introduce the new basis functions:

$$P^{\{n\}\tau_1}_{\sigma_1\sigma_2\sigma_3\upsilon}(x) := \sum_{k=0}^{|n|} (-1)^k \binom{|n|}{k} P^{\tau_1}_{\sigma_1\sigma_2\sigma_3\upsilon}(x+\mathrm{sgn}(n)\,2k)\,, \tag{3.37}$$

which are linear combinations (with binomial coefficients) of $|n|+1$ functions with consecutive arguments starting from x. The functions (3.37) are well-defined for $n \geq 0$ if $\sigma_3 \geq x/2+n$ and for $n<0$ if $\sigma_2 \geq x/2-n$. As we will see in the next Section, these linear combinations allow the compensation of the first $|n|$ terms in the Y_1-expansion, making the link to the PM counterpart of the Q basis function more straightforward. In terms of these functions, one can define the following sets:

$$B^{[n]}_{P2}(\bar\mu_1,\mu_1) = \begin{cases} \left\{ P^{\{n-R(\frac{\bar\mu_1-\mu_1}{2})\}\tau_1}_{\sigma_1\sigma_2\sigma_3\upsilon}\big(R(\bar\mu_1-\mu_1)\big) \mid \sigma_3 \geq n \right\} & [n \geq 0] \\ \left\{ P^{\{n+R(-\frac{\bar\mu_1+\mu_1}{2})\}\tau_1}_{\sigma_1\sigma_2\sigma_3\upsilon}\big(-R(-\bar\mu_1-\mu_1)\big) \mid \sigma_2 \geq |n| \right\} & [n<0] \end{cases}, \tag{3.38}$$

such that their union:

$$\boldsymbol{B}_{P2}(\bar{\mu}_1,\mu_1) := B_{P2}^{[\frac{\bar{\mu}_1-\mu_1}{2}]}(\bar{\mu}_1,\mu_1)\,\cup\, B_{P2}^{[\frac{\bar{\mu}_1-\mu_1+2}{2}]}(\bar{\mu}_1,\mu_1)\,\cup\,\cdots\,\cup\, B_{P2}^{[\frac{\bar{\mu}_1+\mu_1}{2}]}(\bar{\mu}_1,\mu_1)\,, \tag{3.39}$$

spans the same space as (3.36). To be concrete, let us consider the $\mu_1 = 2$ case, where one can distinguish between three different subcases: $\bar{\mu}_1 \geq 2$, $\bar{\mu}_1 \leq -2$ and $\bar{\mu}_1 = 0$. If $\bar{\mu}_1 \geq 2$, then one gets

$$\begin{aligned}
B_{P2}^{[\frac{\bar{\mu}_1-2}{2}]} &= \left\{P^{\{0\}\tau_1}_{\sigma_1\sigma_2\sigma_3 \upsilon}(\bar{\mu}_1-2) = P^{\tau_1}_{\sigma_1\sigma_2\sigma_3 \upsilon}(\bar{\mu}_1-2)\,\Big|\,\sigma_3 \geq \tfrac{\bar{\mu}_1-2}{2}\right\}\,,\\
B_{P2}^{[\frac{\bar{\mu}_1}{2}]} &= \left\{P^{\{1\}\tau_1}_{\sigma_1\sigma_2\sigma_3 \upsilon}(\bar{\mu}_1-2) = P^{\tau_1}_{\sigma_1\sigma_2\sigma_3 \upsilon}(\bar{\mu}_1-2) - P^{\tau_1}_{\sigma_1\sigma_2\sigma_3 \upsilon}(\bar{\mu}_1)\,\Big|\,\sigma_3 \geq \tfrac{\bar{\mu}_1}{2}\right\}\,,\\
B_{P2}^{[\frac{\bar{\mu}_1+2}{2}]} &= \Big\{P^{\{2\}\tau_1}_{\sigma_1\sigma_2\sigma_3 \upsilon}(\bar{\mu}_1-2) = \\
&\quad = P^{\tau_1}_{\sigma_1\sigma_2\sigma_3 \upsilon}(\bar{\mu}_1-2) - 2\,P^{\tau_1}_{\sigma_1\sigma_2\sigma_3 \upsilon}(\bar{\mu}_1) + P^{\tau_1}_{\sigma_1\sigma_2\sigma_3 \upsilon}(\bar{\mu}_1+2)\,\Big|\,\sigma_3 \geq \tfrac{\bar{\mu}_1+2}{2}\Big\}\,,
\end{aligned} \tag{3.40}$$

while, if $\bar{\mu}_1 \leq -2$ the corresponding sets are given by

$$\begin{aligned}
B_{P2}^{[\frac{\bar{\mu}_1-2}{2}]} &= \Big\{P^{\{-2\}\tau_1}_{\sigma_1\sigma_2\sigma_3 \upsilon}(\bar{\mu}_1+2) = \\
&\quad = P^{\tau_1}_{\sigma_1\sigma_2\sigma_3 \upsilon}(\bar{\mu}_1+2) - 2\,P^{\tau_1}_{\sigma_1\sigma_2\sigma_3 \upsilon}(\bar{\mu}_1) + P^{\tau_1}_{\sigma_1\sigma_2\sigma_3 \upsilon}(\bar{\mu}_1-2)\,\Big|\,\sigma_2 \geq |\tfrac{\bar{\mu}_1-2}{2}|\Big\}\,,\\
B_{P2}^{[\frac{\bar{\mu}_1}{2}]} &= \left\{P^{\{-1\}\tau_1}_{\sigma_1\sigma_2\sigma_3 \upsilon}(\bar{\mu}_1+2) = P^{\tau_1}_{\sigma_1\sigma_2\sigma_3 \upsilon}(\bar{\mu}_1+2) - P^{\tau_1}_{\sigma_1\sigma_2\sigma_3 \upsilon}(\bar{\mu}_1)\,\Big|\,\sigma_2 \geq |\tfrac{\bar{\mu}_1}{2}|\right\}\,,\\
B_{P2}^{[\frac{\bar{\mu}_1+2}{2}]} &= \left\{P^{\{0\}\tau_1}_{\sigma_1\sigma_2\sigma_3 \upsilon}(\bar{\mu}_1+2) = P^{\tau_1}_{\sigma_1\sigma_2\sigma_3 \upsilon}(\bar{\mu}_1+2)\,\Big|\,\sigma_2 \geq |\tfrac{\bar{\mu}_1+2}{2}|\right\}\,.
\end{aligned} \tag{3.41}$$

Finally, in the last case $\bar{\mu}_1 = 0$, one ends up with

$$B_{P2}^{[-1]} = \left\{P^{\{-1\}\tau_1}_{\sigma_1\sigma_2\sigma_3 \upsilon}(0)\,\Big|\,\sigma_2 \geq 1\right\}\,,\quad B_{P2}^{[0]} = \left\{P^{\{0\}\tau_1}_{\sigma_1\sigma_2\sigma_3 \upsilon}(0)\right\}\,,\quad B_{P2}^{[1]} = \left\{P^{\{1\}\tau_1}_{\sigma_1\sigma_2\sigma_3 \upsilon}(0)\,\Big|\,\sigma_3 \geq 1\right\}\,. \tag{3.42}$$

As one can see from the above example, the set (3.38) is chosen such that its elements are linear combinations involving always the $P^{\tau_1}_{\sigma_1\sigma_2\sigma_3 \upsilon}(x)$ with $x = \min\{|\bar{\mu}_1-\mu_1|,\ldots,|\bar{\mu}_1+\mu_1|\}$.

All in all, the solution space of the partially-massless equation (3.29) is spanned by the union of the two sets (3.35) and (3.39).

3.2.2 Change of basis

After having generalized the sets B_{P1} and B_{P2} to the sets (3.35) and (3.39), we now aim at finding the analogue of the convenient basis $B_{\tilde{Q}}$ (3.17) and B_Q (3.27) for the partially-massless case.

$\tilde{Q}$ solutions The first observation is that the identity:

$$\big[\,{Y_1}^n\,\tilde{K}(Y_2,Y_3,Z_1,\tilde{H}_2,\tilde{H}_3)\,,\,(U_1\cdot\partial_{X_1})^{\mu_1+1}\,\big] = 0 \qquad [\,n = 0,1,\ldots,\mu_1\,]\,, \tag{3.43}$$

holds because the operator $\tilde{K}(Y_2,Y_3,Z_1,\tilde{H}_2,\tilde{H}_3)$ commutes with $U_1\cdot\partial_{X_1}$ (without on-shell conditions). Thus, let us consider the functions:

$$\tilde{Q}^{\tau_1}_{[n]\sigma_2\sigma_3 h_2 h_3}(Y,Z) \simeq Z_1^{\tau_1}\,{Y_1}^n\,{Y_2}^{\sigma_2}\,{Y_3}^{\sigma_3}\,\tilde{H}_2^{\,h_2}\,\tilde{H}_3^{\,h_3}\,, \tag{3.44}$$

that are of the same form as (3.18) but for an additional factor of Y_1^n. In the previous sections, we have shown that, for $n = 0$, the sets $B_{\tilde{Q}}$ (3.17) reproduce all the leading terms with $\sigma_2 + \sigma_3 \geq \sigma_1$ in B_{P1}. For $n \geq 1$, it is still manifest that the following set:

$$B_{\tilde{Q}}^{[n]}(\bar{\mu}_1, \mu_1) = \left\{ \tilde{Q}^{\tau_1}_{[n]\sigma_2\sigma_3 h_2 h_3} \right\}, \tag{3.45}$$

gives leading terms with $\sigma_2 + \sigma_3 + n \geq \sigma_1 \geq n$ and $\upsilon = 0$, reproducing the same bound (3.34) on σ_1. However, for a given leading term, depending on the values of the σ_i's, there can be more than one solutions—more than one values of n can satisfy the inequality—in both ($P1$ and $\tilde{Q}$ solutions) sides. In order to handle this subtlety as well as the $\upsilon \geq 1$ case, one can consider once again a change of basis in such a way to remove the degeneracy between the leading terms. Concerning the $P1$ solutions, this can be done by taking particular linear combinations of the $P^{[n]\,\tau_1}_{\sigma_1\sigma_2\sigma_3\upsilon}$'s, with $n = \mu_1, \mu_1 - 1, \ldots, \mu_1 - m$, that give rise to the cancelation of the first m leading terms. A similar analysis can be also done for the $\tilde{Q}$ solutions after integrating by parts the total-derivative terms present in $\tilde{H}_2$ and $\tilde{H}_3$ (see Appendix C for more details). In the end, one can see that all the leading terms of the $P1$ solutions in the set (3.34) can be reproduced by the $\tilde{Q}$ solutions in the sets (3.45). Hence, exploiting the completeness of the $P1$ solutions when the condition (1.10) is not satisfied, one can conclude that

$$\mathrm{Span}\big[\boldsymbol{B}_{\tilde{Q}}(\bar{\mu}_1, \mu_1)\big] = \mathrm{Span}\big[\boldsymbol{B}_{P1}(\bar{\mu}_1, \mu_1)\big]\,. \tag{3.46}$$

where

$$\boldsymbol{B}_{\tilde{Q}}(\bar{\mu}_1, \mu_1) := \bigcup_{n=0}^{\mu_1} B_{\tilde{Q}}^{[n]}(\bar{\mu}_1, \mu_1)\,. \tag{3.47}$$

Q solutions Here, we search for a generalization of the solutions (3.23) to the partially-massless case.[11] When $\mu_1 = 1$, due to the identity:

$$\begin{aligned}\mathcal{L}_1(\bar{\mu}_1 - 1)\,\mathcal{L}_1(\bar{\mu}_1 + 1)\,Z_2 &= [Z_2\,\mathcal{L}_1(\bar{\mu}_1 + 1) - 2\,Y_3]\,\mathcal{L}_1(\bar{\mu}_1 - 1)\,,\\ \mathcal{L}_1(\bar{\mu}_1 - 1)\,\mathcal{L}_1(\bar{\mu}_1 + 1)\,Z_3 &= [Z_3\,\mathcal{L}_1(\bar{\mu}_1 - 1) + 2\,Y_2]\,\mathcal{L}_1(\bar{\mu}_1 + 1)\,,\end{aligned} \tag{3.49}$$

one can easily check that the functions:

$$\begin{aligned}C(Y,Z) = {} & Z_2\,Y_2^{R\left(-\frac{\bar{\mu}_1-1}{2}\right)}\,Y_3^{R\left(\frac{\bar{\mu}_1-1}{2}\right)}\,e^{-\frac{\hat{\delta}}{L}\mathcal{D}}\,K_1(Y_1,Y_2,Y_3,Z_1,G)\\ & + Z_3\,Y_2^{R\left(-\frac{\bar{\mu}_1+1}{2}\right)}\,Y_3^{R\left(\frac{\bar{\mu}_1+1}{2}\right)}\,e^{-\frac{\hat{\delta}}{L}\mathcal{D}}\,K_2(Y_1,Y_2,Y_3,Z_1,G)\Big|_{G=G(Y,Z)}\,,\end{aligned} \tag{3.50}$$

solve the PDE $\mathcal{L}_1(\bar{\mu}_1 - 1)\,\mathcal{L}_1(\bar{\mu}_1 + 1)\,C(Y,Z) = 0$. In general, one can consider the set:

$$B_Q^{[\tau_2,\tau_3]}(\bar{\mu}_1, \mu_1) = \left\{ Q^{[\tau_2,\tau_3]\tau_1}_{\sigma_1\sigma_2\sigma_3\upsilon} \right\}, \tag{3.51}$$

[11] Because of (3.28) and (3.36), one is naturally led to consider the basis:

$$B_Q(\bar{\mu}_1 - \mu_1) \cup B_Q(\bar{\mu}_1 - \mu_1 + 2) \cup \cdots \cup B_Q(\bar{\mu}_1 + \mu_1)\,. \tag{3.48}$$

However, the latter is not suitable to study the solution space of more than one PDE.

whose elements are given by

$$Q^{[\tau_2,\tau_3]\tau_1}_{\sigma_1\sigma_2\sigma_3\upsilon}(Y,Z) \,=\, Z_1^{\tau_1}\, Z_2^{\tau_2}\, Z_3^{\tau_3}\, Y_2^{R\left(\tau_2-\frac{\bar\mu_1+\mu_1}{2}\right)}\, Y_3^{R\left(\tau_3+\frac{\bar\mu_1-\mu_1}{2}\right)} \times \\ \times\, e^{-\frac{\hat\delta}{L}\mathcal{D}}\, Y_1^{\sigma_1}\, Y_2^{\sigma_2}\, Y_3^{\sigma_3}\, G^{\upsilon}\Big|_{G=G(Y,Z)}\,, \tag{3.52}$$

with $\tau_2+\tau_3\leq\mu_1$ (see Appendix B for details). The proof that the union of these sets together the $\tilde Q$ solutions span the entire solution space relies once again on the leading term analysis. Using the identity (A.9), one can show that

$$P^{\{n\}\tau_1}_{\sigma_1\sigma_2\sigma_3 n}(x) = \left(\tfrac{\hat\delta}{L}\right)^{|n|}\Big[(Z_2\,\partial_{Y_3}-Z_3\,\partial_{Y_2})^{|n|}\; Z_1^{\tau_1}\,Y_1^{\sigma_1}\,Y_2^{\sigma_2}\,Y_3^{\sigma_3}\,[G_1(Y,Z)]^{\upsilon}+\mathcal{O}\big(\tfrac{\hat\delta}{L}\big)\Big] \\ = \left(\tfrac{\hat\delta}{L}\right)^{|n|}\sum_{\tau_2+\tau_3=|n|}(-1)^{\tau_3}\binom{|n|}{\tau_3}\,[\sigma_2]_{\tau_3}\,[\sigma_3]_{\tau_2}\,\times \\ \times\, Z_1^{\tau_1}\,Z_2^{\tau_2}\,Z_3^{\tau_3}\,Y_1^{\sigma_1}\,Y_2^{\sigma_2-\tau_3}\,Y_3^{\sigma_3-\tau_2}\,[G_1(Y,Z)]^{\upsilon}+\mathcal{O}\Big(\big(\tfrac{\hat\delta}{L}\big)^{|n|+1}\Big)\,. \tag{3.53}$$

In this form, it is straightforward to check that each leading term above can be reproduced by the functions (3.52). As an example, let us assume $\bar\mu_1\geq\mu_1$. In this case, the function (3.53) in $B^{[n+\frac{\bar\mu_1-\mu_1}{2}]}_{P2}(\bar\mu_1,\mu_1)$ has to satisfy the condition $\sigma_3\geq(\bar\mu_1-\mu_1)/2+n$. As a consequence, the minimum power of Y_3 is $(\bar\mu_1-\mu_1)/2+\tau_3$, coinciding with the one in (3.52). All in all, we have

$$\mathrm{Span}\Big[\boldsymbol{B}_{P1}(\bar\mu_1,\mu_1)\,\cup\,\boldsymbol{B}_{P2}(\bar\mu_1,\mu_1)\Big] = \mathrm{Span}\Big[\boldsymbol{B}_{\tilde Q}(\bar\mu_1,\mu_1)\,\cup\,\boldsymbol{B}_{Q}(\bar\mu_1,\mu_1)\Big]\,, \tag{3.54}$$

where

$$\boldsymbol{B}_Q(\bar\mu_1,\mu_1) := \bigcup_{\tau_2+\tau_3\leq\mu_1} B_Q^{[\tau_2,\tau_3]}(\bar\mu_1,\mu_1)\,. \tag{3.55}$$

Let us mention that, though highly redundant, the basis (3.55) proves very convenient in the study of the solution space associated to more than one PDE.

Before moving to the next Section, let us summarize the general solutions to eq. (1.8) in the $\tilde Q$ and Q basis.

Solutions to one equation

For arbitrary $\mu_2-\mu_3$:

$$\tilde C = \sum_{\sigma_1=0}^{\mu_1} Y_1^{\sigma_1}\,\tilde K^{\sigma_1}(Y_2,Y_3,Z_1,\tilde H_2,\tilde H_3)\,. \tag{3.56}$$

For $\mu_1+\mu_2-\mu_3\in 2\,\mathbb{Z}$, one also has

$$C \,=\, \sum_{(\tau_1,\tau_2,\tau_3)\in\mathscr{L}_1} Z_1^{\tau_1}\,Z_2^{\tau_2}\,Z_3^{\tau_3}\,Y_2^{R\left(\tau_2+\frac{\mu_2-\mu_3-\mu_1}{2}\right)}\,Y_3^{R\left(\tau_3+\frac{\mu_3-\mu_1-\mu_2}{2}\right)}\times \\ \times\, e^{-\frac{\hat\delta}{L}\mathcal{D}}\,K^{\tau_1\tau_2\tau_3}(Y,G)\Big|_{G=G(Y,Z)}, \tag{3.57}$$

where

$$\mathscr{L}_i := \big\{\,(\tau_1,\tau_2,\tau_3)\in\mathbb{N}^3\;\big|\;\tau_{i+1}+\tau_{i-1}\leq\mu_i\,\big\}\,. \tag{3.58}$$

4 Intersection of the solution spaces

Let us now consider general interactions involving more than one (P)M field. In these cases, depending on their number, one has to consider the intersections of the solution spaces of the corresponding PDEs. In the following, we carry out this analysis for the $\tilde{Q}$ and Q solutions separately.

4.1 $\tilde{Q}$ solutions

When two (P)M fields (say $i = 1, 2$) are present, one has to take the intersection between the coupling (3.56) and its cyclic permutation:

$$\sum_{\sigma_1=0}^{\mu_1} Y_1^{\sigma_1} \tilde{K}_1^{\sigma_1}(Y_2, Y_3, Z_1, \tilde{H}_2, \tilde{H}_3)\,, \qquad \sum_{\sigma_2=0}^{\mu_2} Y_2^{\sigma_2} \tilde{K}_2^{\sigma_2}(Y_3, Y_1, Z_2, \tilde{H}_3, \tilde{H}_1)\,, \tag{4.1}$$

that is

$$\tilde{C} = \sum_{\sigma_1=0}^{\mu_1} \sum_{\sigma_2=0}^{\mu_2} Y_1^{\sigma_1} Y_2^{\sigma_2} \tilde{K}^{\sigma_1\sigma_2}(Y_3, \tilde{H}_1, \tilde{H}_2, \tilde{H}_3)\,. \tag{4.2}$$

When all three interacting fields are (P)M, one has to intersect the couplings (4.2) with

$$\sum_{\sigma_3=0}^{\mu_3} Y_3^{\sigma_3} \tilde{K}_3^{\sigma_3}(Y_1, Y_2, Z_3, \tilde{H}_1, \tilde{H}_2)\,, \tag{4.3}$$

leading to

$$\tilde{C} = \sum_{\sigma_1=0}^{\mu_1} \sum_{\sigma_2=0}^{\mu_2} \sum_{\sigma_3=0}^{\mu_3} Y_1^{\sigma_1} Y_2^{\sigma_2} Y_3^{\sigma_3} \tilde{K}^{\sigma_1\sigma_2\sigma_3}(\tilde{H}_1, \tilde{H}_2, \tilde{H}_3)\,. \tag{4.4}$$

4.2 Q solutions

For the analysis of the intersections among the Q solutions (3.57), it is convenient to start with the leading terms $C^{(0)}$ of the solutions to the entire system of PDEs. Plugging the expansion (2.5) into eq. (1.8), one discovers that $C^{(0)}$ satisfies the following relatively simple equations:

$$\big(Y_{i+1}\,\partial_{Z_{i-1}} - Y_{i-1}\,\partial_{Z_{i+1}}\big)^{\mu_i+1}\, C^{(0)}(Y, Z) = 0\,, \tag{4.5}$$

which can be solved in terms of an arbitrary function $K^{\tau_1\tau_2\tau_3}(Y, G)$ as

$$C^{(0)}(Y, Z) = \sum_{(\tau_1,\tau_2,\tau_3)\in\mathscr{L}} Z_1^{\tau_1} Z_2^{\tau_2} Z_3^{\tau_3} K^{\tau_1\tau_2\tau_3}\big(Y, G(Y, Z)\big)\,. \tag{4.6}$$

Here, the lattice $\mathscr{L}$ is given by the intersection of the lattices $\mathscr{L}_i$'s (3.58) associated with the i-th PM fields. In the following, we derive the general solutions to the system of eqs. (1.8) starting from the leading terms $C^{(0)}$ of eq. (4.6). For that, we distinguish between the cases where two or three (P)M fields are involved.

When two (P)M fields (say $i = 1, 2$) are present, one has to solve the system:

$$\mathcal{L}_1(\mu_3 - \mu_2 - \mu_1)\, \mathcal{L}_1(\mu_3 - \mu_2 - \mu_1 + 2) \cdots \mathcal{L}_1(\mu_3 - \mu_2 + \mu_1)\; C(Y, Z) = 0\,, \tag{4.7}$$

$$\mathcal{L}_2(\mu_1 - \mu_3 - \mu_2)\, \mathcal{L}_2(\mu_1 - \mu_3 - \mu_2 + 2) \cdots \mathcal{L}_2(\mu_1 - \mu_3 + \mu_2)\; C(Y, Z) = 0\,, \tag{4.8}$$

where the leading terms of the corresponding couplings are given by (4.6) with

$$\mathscr{L} = \mathscr{L}_1 \cap \mathscr{L}_2 \,. \tag{4.9}$$

At this point, it is convenient to split the sum over (τ_1, τ_2, τ_3) into the two regions:

$$\mathscr{L}_1 \cap \mathscr{L}_2 \cap \mathscr{L}_3 \,, \qquad \mathscr{L}_1 \cap \mathscr{L}_2 \cap {\mathscr{L}_3}^c \,, \tag{4.10}$$

where ${\mathscr{L}_i}^c$ denotes the complement of the set $\mathscr{L}_i$. When $(\tau_1, \tau_2, \tau_3) \in \mathscr{L}_1 \cap \mathscr{L}_2 \cap \mathscr{L}_3$, out of the three combinations $\tau_i + \frac{\mu_i - \mu_{i+1} - \mu_{i-1}}{2}$, at most one can be positive. Hence, without loss of generality, one can assume

$$\tau_2 + \tfrac{\mu_2 - \mu_3 - \mu_1}{2} \leq 0 \,, \qquad \tau_3 + \tfrac{\mu_3 - \mu_1 - \mu_2}{2} \leq 0 \,. \tag{4.11}$$

Then, depending on the sign of $\tau_1 + \frac{\mu_1 - \mu_2 - \mu_3}{2}$, there are two subcases. When the latter is non-positive, a comparison with eq. (3.57) and its cyclic permutation directly shows that the coupling:

$$C(Y,Z) = Z_1^{\tau_1}\, Z_2^{\tau_2}\, Z_3^{\tau_3}\, e^{-\frac{\hat{\delta}}{L}\,\mathcal{D}}\, K^{\tau_1 \tau_2 \tau_3}(Y,G)\Big|_{G=G(Y,Z)} \,, \tag{4.12}$$

is a solution to both PDEs (4.7, 4.8). On the other hand, when

$$\tau_1 + \tfrac{\mu_1 - \mu_2 - \mu_3}{2} > 0 \,, \tag{4.13}$$

one can consider the following function:

$$C(Y,Z) = Z_1^{\tau_1}\, Z_2^{\tau_2}\, Z_3^{\tau_3}\, Y_1^{\tau_1 + \frac{\mu_1 - \mu_2 - \mu_3}{2}}\, e^{-\frac{\hat{\delta}}{L}\,\mathcal{D}}\, K^{\tau_1 \tau_2 \tau_3}(Y,G)\Big|_{G=G(Y,Z)} \,, \tag{4.14}$$

that is a solution to eq. (4.8). Although it is not manifest, the latter solves also eq. (4.7). One way of proving it consists in pushing the exponential function to the left, ending up with:

$$C(Y,Z) = \sum_{p,q,r \geq 0} Z_1^{\tau_1}\, Z_2^{\bar{\tau}_2}\, Z_3^{\bar{\tau}_3}\, e^{-\frac{\hat{\delta}}{L}\,\mathcal{D}}\, \bar{K}^{\tau_1 \bar{\tau}_2 \bar{\tau}_3}_{p,q,r}(Y,G)\Big|_{G=G(Y,Z)} \,, \tag{4.15}$$

where $\bar{\tau}_2 = \tau_2 + p + r$, $\bar{\tau}_3 = \tau_3 + q + r$ and

$$\begin{aligned} \bar{K}^{\tau_1 \bar{\tau}_2 \bar{\tau}_3}_{p,q,r}(Y,G) = & \binom{\tau_1 + \frac{\mu_1 - \mu_2 - \mu_3}{2}}{p,q,r} \left(\tfrac{\hat{\delta}}{L}\right)^{p+q+r} \times \\ & \times Y_1^{\tau_1 + \frac{\mu_1 - \mu_2 - \mu_3}{2} - p - q - r}\, \partial_{Y_2}^{q}\, \partial_{Y_3}^{p}\, \partial_G^{r}\, K^{\tau_1 \tau_2 \tau_3}(Y,G) \,. \end{aligned} \tag{4.16}$$

Comparing with (3.57), one can see that the latter is a solution to the first equation provided the conditions:

$$\begin{gathered} \tau_1 + \bar{\tau}_2 \leq \mu_3 \,, \qquad \bar{\tau}_2 + \bar{\tau}_3 \leq \mu_1 \,, \qquad \bar{\tau}_3 + \tau_1 \leq \mu_2 \,, \\ \bar{\tau}_2 + \tfrac{\mu_2 - \mu_3 - \mu_1}{2} \leq 0 \,, \qquad \bar{\tau}_3 + \tfrac{\mu_3 - \mu_1 - \mu_2}{2} \leq 0 \,, \end{gathered} \tag{4.17}$$

are satisfied. These conditions hold for any $(\tau_1,\tau_2,\tau_3)\in\mathscr{L}_1\cap\mathscr{L}_2\cap\mathscr{L}_3$ satisfying conditions (4.11) and (4.13), and therefore the function (4.14) solves eq. (4.7). Finally, the solutions (4.12) and (4.14) can be written at once in a cyclic form as

$$C=\sum_{(\tau_1,\tau_2,\tau_3)\in\mathscr{L}_1\cap\mathscr{L}_2\cap\mathscr{L}_3} Z_1^{\tau_1}\,Z_2^{\tau_2}\,Z_3^{\tau_3}\,Y_1^{R\left(\tau_1+\frac{\mu_1-\mu_2-\mu_3}{2}\right)}\,Y_2^{R\left(\tau_2+\frac{\mu_2-\mu_3-\mu_1}{2}\right)}\,Y_3^{R\left(\tau_3+\frac{\mu_3-\mu_1-\mu_2}{2}\right)}\times$$
$$\times\, e^{-\frac{\hat{\delta}}{L}\mathcal{D}}\,K^{\tau_1\tau_2\tau_3}(Y,G)\Big|_{G=G(Y,Z)}. \tag{4.18}$$

Let us now consider the case $(\tau_1,\tau_2,\tau_3)\in\mathscr{L}_1\cap\mathscr{L}_2\cap\mathscr{L}_3^{\,c}$, where $\mu_3<\tau_1+\tau_2\le s_3$. If one requires that the third field be a unitary massive field, then this type of interactions are ruled out in dS (see Figure 1).[12] On the other hand, in AdS these would correspond to the interactions between two massless and one massive fields with $\mu_3<0$, but we do not find any solution of this type.

Last of all, when all three fields are (P)M, the leading terms of the corresponding couplings are given by (4.6) with

$$\mathscr{L}=\mathscr{L}_1\cap\mathscr{L}_2\cap\mathscr{L}_3\,. \tag{4.19}$$

This situation is nothing but a subcase of the two (P)M interactions that we have considered before, therefore the solution is given by (4.18).

At this point, we have completed the analysis of the (P)M cubic interactions, and the results are summarized in Section 1.2.

5 Discussions

To conclude, we discuss the implications of the condition (1.10) in more details, pointing out the key differences with respect to the flat-space case. Moreover, in order to make contact with our previous work [22], we provide an example.

Non-Abelian interactions Let us recall that the cubic interactions which exist regardless of the condition (1.10) are given by arbitrary functions of $\tilde{H}$ ($\tilde{Q}$ solutions). Since the latter are trivially gauge invariant—their gauge invariance does not rely on the on-shell conditions—they do not lead to any deformation of the gauge transformations. To iterate, they are all Abelian and of the Born-Infeld type, expressible in terms of linear curvatures. When the condition (1.10) is satisfied, besides these $\tilde{H}$-couplings, supplemental G-couplings (Q solutions) appear. However, only a part of them is independent from the $\tilde{H}$-couplings and may lead to non-Abelian deformations of the gauge symmetries (see Figure 2 for a schematic picture). Let us stress that there are many PM interactions which do not satisfy the condition (1.10), including notably the PM spin-2 self-interactions [46, 48]. Those interactions cannot give rise to any non-Abelian deformations of the gauge symmetries.

[12] The third field may be PM, but then one has to impose gauge invariance also under its gauge symmetry. This case is considered later.

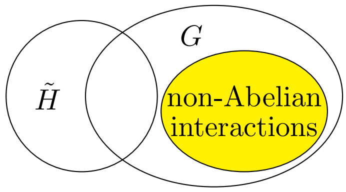

Figure 2. $\tilde{H}$- and G-couplings. Non-abelian couplings are a subset of the G-couplings that can not be written as $\tilde{H}$-couplings.

Interactions of one massless and two massive fields Apart from PM interactions, which do not have any flat-space counterpart, a novel property of (A)dS is present in the cubic interactions of one massless ($\mu_1 = 0$) and two massive ($\mu_3 \geq \mu_2$) fields. According to the condition (1.10), supplemental G solutions appear when $\mu_3 - \mu_2 \in 2\,\mathbb{Z}_{\geq 0}$. Among them, the non-$\tilde{H}$-couplings are the ones satisfying conditions (3.8) and (3.12):

$$\sigma_1 \geq \sigma_2 + \sigma_3 + 1\,, \qquad \sigma_3 \geq \tfrac{\mu_3 - \mu_2}{2}\,, \tag{5.1}$$

where the σ_i's are related to the spins s_i's as

$$s_1 = \sigma_1 + \upsilon\,, \qquad s_2 = \sigma_2 + \upsilon + \tau_1\,, \qquad s_3 = \sigma_3 + \upsilon + \tau_1\,, \tag{5.2}$$

with υ and τ_1 being the powers of G_1 and Z_1 respectively (see (3.7)). This type of interactions is related to non-trivial Noether currents. In particular, the electromagnetic ($1-s-s$) and the gravitational ($2-s-s$) minimal couplings correspond to $\mu_2 = \mu_3$, as in flat space. On the other hand, there are many interactions that satisfy the above conditions for positive even integers $\mu_3 - \mu_2$. For instance, when $\mu_3 - \mu_2 = 2$, one can have $2-s-(s+1)$ interactions involving non-trivial Noether currents made of fields with different masses. This is a novelty with respect to flat space, where the interactions leading to non-trivial conserved currents are only available when the two masses are *equal*: $\mu_2 = \mu_3$. This observation suggests that, in (A)dS, (HS) multiplets may involve not only fields with different spins, but also with different masses. Let us also mention that the minimal-like couplings s_1-s-s require $\mu_3 - \mu_2 \leq s_1 - 1$, so that in this case interactions involving fields with different masses are available only for $s_1 \geq 3$.

Example: $4-4-2$ interactions For concreteness, starting from the general solutions provided in this paper, we show how to recover the example of $4-4-2$ interactions obtained in [22] by means of a numerical algorithm. The latter are the cubic interactions between two spin-4 fields at their first PM points ($\mu_1 = \mu_2 = 1$) and a massless spin-2 field ($\mu_3 = 0$). In this case one can find two $\tilde{Q}$ solutions (1.19):

$$\tilde{H}_1\,\tilde{H}_2\,\tilde{H}_3^3\,, \qquad \tilde{H}_1\,\tilde{H}_2\,\tilde{H}_3^2\,Y_1\,Y_2\,, \tag{5.3}$$

and six Q solutions (1.20):

$$\begin{aligned} &e^{-\frac{\hat{\delta}}{L}\mathcal{D}}\,Y_1^4\,Y_2^4\,Y_3^2\,, && e^{-\frac{\hat{\delta}}{L}\mathcal{D}}\,Y_1^3\,Y_2^3\,Y_3\,G\,, && e^{-\frac{\hat{\delta}}{L}\mathcal{D}}\,Y_1^2\,Y_2^2\,G^2\,, \\ Z_3\,&e^{-\frac{\hat{\delta}}{L}\mathcal{D}}\,Y_1^3\,Y_2^3\,Y_3^2\,, \quad Z_3\,&&e^{-\frac{\hat{\delta}}{L}\mathcal{D}}\,Y_1^2\,Y_2^2\,Y_3\,G\,, \quad Z_3\,&&e^{-\frac{\hat{\delta}}{L}\mathcal{D}}\,Y_1\,Y_2\,G^2\,. \end{aligned} \tag{5.4}$$

However, two of the above Q couplings—the first and the fourth— can be expressed in terms of the $\tilde{Q}$ couplings (5.3) and the remaining Q couplings. Hence, one is left with six independent solutions (two $\tilde{H}$-couplings and four (non-$\tilde{H}$) G-couplings) in complete agreement with the result of [22]. More explicitly, for instance, the lowest-derivative interaction $Z_3\, e^{-\frac{\hat{\delta}}{L}\mathcal{D}}\, Y_1\, Y_2\, G^2$ gives

$$Z_3\, Y_1\, Y_2\, G^2 - \tfrac{\hat{\delta}}{L}\, Z_3^2 \left[G^2 + 2\,(Y_1\, Z_1 + Y_2\, Z_2)\, G + 2\, Y_1\, Y_2\, Z_1\, Z_2\right] + 4\left(\tfrac{\hat{\delta}}{L}\right)^2 Z_1\, Z_2\, Z_3^3\,, \tag{5.5}$$

reproducing the vertex C_6 in eq. (3.40) of [22].

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A Recurrence relation

In this Appendix, we provide some details on the derivation of the solutions to the massless eq. (3.1) and to the PM eq. (3.29).

A.1 Massless equation

As explained in Section 3, plugging the expansion:

$$C_{\sigma_1}(Y,Z) = \sum_{k=0}^{\infty} C^{(k)}_{\sigma_1}(Y_2,Y_3,Z)\,\left(-\tfrac{\hat{\delta}}{L}\,\partial_{Y_1}\right)^k\, Y_1^{\,\sigma_1}\,, \tag{A.1}$$

into the equation:

$$\Big[Y_2\,\partial_{Z_3} - Y_3\,\partial_{Z_2} + \tfrac{\hat{\delta}}{L}\left(Y_2\,\partial_{Y_2} - Y_3\,\partial_{Y_3} + \tfrac{\bar{\mu}_1}{2}\right)\,\partial_{Y_1}\Big]\, C(Y,Z) = 0\,, \tag{A.2}$$

one ends up with a differential recurrence relation for $C^{(k)}_{\sigma_1}$:

$$(Y_2\,\partial_{Z_3} - Y_3\,\partial_{Z_2})\; C^{(k)}_{\sigma_1}(Y_2,Y_3,Z) = \left(Y_2\,\partial_{Y_2} - Y_3\,\partial_{Y_3} + \tfrac{1}{2}\,\bar{\mu}_1\right) C^{(k-1)}_{\sigma_1}(Y_2,Y_3,Z)\,, \tag{A.3}$$

where $C^{(-1)}_{\sigma_1} = 0$. The latter can be solved iteratively starting from $k=0$:

$$\begin{aligned} C^{(0)}_{\sigma_1}(Y_2,Y_3,Z) \;&=\; K\big(Y_2,Y_3,Z_1,G_1(Y,Z)\big) \\ &=\; \sum_{\sigma_2,\sigma_3,\tau_1,\upsilon} K^{\tau_1}_{\sigma_1\sigma_2\sigma_3\upsilon}\, Z_1^{\tau_1}\, Y_2^{\sigma_2}\, Y_3^{\sigma_3}\, [G_1(Y,Z)]^{\upsilon}\,, \end{aligned} \tag{A.4}$$

where $G_1(Y,Z) := Y_2\, Z_2 + Y_3\, Z_3$. For this purpose, one can consider the following ansatz:

$$C_{\sigma_1}(Y,Z) = \sum_{\sigma_2,\sigma_3,\tau_1,\upsilon} K^{\tau_1}_{\sigma_1\sigma_2\sigma_3\upsilon}\, P^{\tau_1}_{\sigma_1\sigma_2\sigma_3\upsilon}(\bar{\mu}_1;Y,Z)\,, \tag{A.5}$$

with

$$P^{(k)\tau_1}_{\sigma_1\sigma_2\sigma_3\upsilon}(\bar{\mu}_1; Y_2, Y_3, Z) = \sum_{\ell=0}^{k} c_{k,\ell}\,(Z_3\,\partial_{Y_2})^{k-\ell}\,(Z_2\,\partial_{Y_3})^{\ell}\,Y_2^{\sigma_2}\,Y_3^{\sigma_3}\,[G_1(Y,Z)]^{\upsilon}\,, \tag{A.6}$$

and turn eq. (A.3) into a recurrence relation for $c_{k,\ell}$:

$$\begin{aligned}&(k-\ell)\,c_{k,\ell} + (\ell+1)\,c_{k,\ell+1} = c_{k-1,\ell}\,,\\ &(k-\ell)\,(\sigma_3-\ell)\,c_{k,\ell} - (\ell+1)\,(\sigma_2-k+\ell+1)\;c_{k,\ell+1} = \tfrac{\bar{\mu}_1}{2}\,c_{k-1,\ell}\,.\end{aligned} \tag{A.7}$$

The latter can be straightforwardly solved as

$$c_{k,\ell} = \frac{1}{(k-\ell)!\,\ell!}\,\frac{\big[\sigma_2+\frac{\bar{\mu}_1}{2}\big]_{k-\ell}\,\big[\sigma_3-\frac{\bar{\mu}_1}{2}\big]_{\ell}}{[\sigma_2+\sigma_3]_k}\,. \tag{A.8}$$

Plugging the coefficients (A.8) into eq. (A.6) and summing over k, one gets eq. (3.7).

To conclude this part of the Appendix, let us provide an identity involving the function $P^{\tau_1}_{\sigma_1\sigma_2\sigma_3 n}$:

$$\begin{aligned}&\sum_{k=0}^{n}(-1)^k\binom{n}{k}\,P^{\tau_1}_{\sigma_1\sigma_2\sigma_3 n}(x-n+2k)\\ &= \big(\tfrac{\hat{\delta}}{L}\big)^n\,(Z_2\,\partial_{Y_3} - Z_3\,\partial_{Y_2})^n\,\times\\ &\times\left[\sum_{p,q\geq 0}\frac{\big[\sigma_2+\frac{x}{2}\big]_p\,\big[\sigma_3-\frac{x}{2}\big]_q}{[\sigma_2+\sigma_3]_{p+q+n}}\,\frac{\big(-\frac{\hat{\delta}}{L}\,Z_3\,\partial_{Y_1}\,\partial_{Y_2}\big)^p}{p!}\,\frac{\big(-\frac{\hat{\delta}}{L}\,Z_2\,\partial_{Y_3}\,\partial_{Y_1}\big)^q}{q!}\right]\times\\ &\times\,Z_1^{\tau_1}\,Y_1^{\sigma_1}\,Y_2^{\sigma_2}\,Y_3^{\sigma_3}\,[G_1(Y,Z)]^{\upsilon}\,.\end{aligned} \tag{A.9}$$

A.2 PM equation

The first step consists in recasting the higher-order PDE:

$$\mathcal{L}_1(\bar{\mu}_1-\mu_1)\,\mathcal{L}_1(\bar{\mu}_1-\mu_1+2)\cdots\mathcal{L}_1(\bar{\mu}_1+\mu_1)\,C(\bar{\mu}_1,\mu_1;Y,Z) = 0\,, \tag{A.10}$$

into the following system of equations:

$$\begin{cases}\mathcal{L}_1(\bar{\mu}_1-\mu_1)\,\bar{C}_{[\mu_1]} = 0\\ \mathcal{L}_1(\bar{\mu}_1-\mu_1+2)\,\bar{C}_{[\mu_1-1]} = \bar{C}_{[\mu_1]}\\ \qquad\vdots\\ \mathcal{L}_1(\bar{\mu}_1+\mu_1)\,C = \bar{C}_{[1]}\end{cases}. \tag{A.11}$$

Afterwards, one expands the C and $\bar{C}_{[n]}$'s as in eq. (A.1), so that the system (A.11) translates into a set of differential recurrence relations for $C^{(k)}_{\sigma_1}$ and $\bar{C}^{(k)}_{[n]\sigma_1}$:

$$\begin{cases}(Y_2\,\partial_{Z_3} - Y_3\,\partial_{Z_2})\;\bar{C}^{(k)}_{[\mu_1]\sigma_1} = \big(Y_2\,\partial_{Y_2} - Y_3\,\partial_{Y_3} + \frac{\bar{\mu}_1-\mu_1}{2}\big)\,\bar{C}^{(k-1)}_{[\mu_1]\sigma_1}\\ (Y_2\,\partial_{Z_3} - Y_3\,\partial_{Z_2})\;\bar{C}^{(k)}_{[\mu_1-1]\sigma_1} = \Big(Y_2\,\partial_{Y_2} - Y_3\,\partial_{Y_3} + \frac{\bar{\mu}_1-\mu_1+2}{2}\Big)\,\bar{C}^{(k-1)}_{[\mu_1-1]\sigma_1} + \bar{C}^{(k)}_{[\mu_1]\sigma_1}\\ \qquad\vdots\\ (Y_2\,\partial_{Z_3} - Y_3\,\partial_{Z_2})\;C^{(k)}_{\sigma_1} = \big(Y_2\,\partial_{Y_2} - Y_3\,\partial_{Y_3} + \frac{\bar{\mu}_1+\mu_1}{2}\big)\,C^{(k-1)}_{\sigma_1} + \bar{C}^{(k)}_{[1]\sigma_1}\end{cases}, \tag{A.12}$$

where $\bar{C}^{(-1)}_{[n]\sigma_1} = 0$. The solutions $C^{(k)}_{\sigma_1}$ to the above system can be written as

$$C^{(k)}_{\sigma_1}(\bar{\mu}_1, \mu_1; Y, Z) = \sum_{n=0}^{\mu_1} \sum_{\ell=0}^{n} \frac{(-1)^{n-\ell}}{(n-\ell)!\,\ell!}\, C^{(k+n)}_{[n]\sigma_1}(\bar{\mu}_1 + \mu_1 - 2\,\ell, 0; Y, Z)\,, \tag{A.13}$$

where the $C^{(k)}_{[n]\sigma_1}$'s are the expansion coefficients of $C_{[n]\sigma_1}$, satisfying

$$\mathcal{L}_1(x)\, C_{[n]\sigma_1}(x; Y, Z) = 0\,. \tag{A.14}$$

Let us stress once again that, since the $C_{[n]\sigma_1}$'s may involve singular terms, the expression (A.13) is only a formal solution. Hence, for any (formal) solution C_{σ_1} to the PM equation, there exist (formal) solutions $C_{[n]\sigma_1+n}$ to the massless equation such that

$$C_{\sigma_1}(\bar{\mu}_1, \mu_1; Y, Z) = \sum_{n=0}^{\mu_1} \sum_{\ell=0}^{n} (-1)^{\ell} \binom{n}{\ell}\, C_{[n]\,\sigma_1+n}(\bar{\mu}_1 + \mu_1 - 2\,\ell; Y, Z)\,. \tag{A.15}$$

B Shift solutions

In this Appendix, we prove that the function (3.52) is a solution to the PM equation (1.8). For that, let us first generalize the identities (3.49) to

$$\begin{aligned}
&\mathcal{L}_1(x-n)\,\cdots\,\mathcal{L}_1(x+n-2)\,\mathcal{L}_1(x+n)\,Z_2\\
&= \big[Z_2\,\mathcal{L}_1(x+n) - (n+1)\,Y_3\big]\,\mathcal{L}_1(x-n)\,\cdots\,\mathcal{L}_1(x+n-2)\,,\\
&\mathcal{L}_1(x-n)\,\mathcal{L}_1(x-n+2)\,\cdots\,\mathcal{L}_1(x+n)\,Z_3\\
&= \big[Z_3\,\mathcal{L}_1(x-n) + (n+1)\,Y_2\big]\,\mathcal{L}_1(x-n+2)\,\cdots\,\mathcal{L}_1(x+n)\,.
\end{aligned} \tag{B.1}$$

One can see that the effect of Z_2 (or Z_3) is to remove the differential operator $\mathcal{L}_1$ with the largest (or the smallest) argument. Hence, considering generic powers $Z_2^{\tau_2}\,Z_3^{\tau_3}$ with $\tau_2 + \tau_3 \leq \mu_1$, any solution to the differential equation:

$$\mathcal{L}_1(\bar{\mu}_1 - \mu_1 + 2\,\tau_3)\,\cdots\,\mathcal{L}_1(\bar{\mu}_1 + \mu_1 - 2\,\tau_2)\,C^{\tau_2\tau_3}(Y, Z) = 0\,, \tag{B.2}$$

is also a solution to

$$\mathcal{L}_1(\bar{\mu}_1 - \mu_1)\,\cdots\,\mathcal{L}_1(\bar{\mu}_1 + \mu_1)\,Z_2^{\tau_2}\,Z_3^{\tau_3}\,C^{\tau_2\tau_3}(Y, Z) = 0\,. \tag{B.3}$$

Moreover, as any solution to a single $\mathcal{L}_1$ equation:

$$\mathcal{L}_1(\bar{\mu}_1 - \mu_1 + 2\,\tau_3 + 2\,n)\,C^{\tau_2\tau_3}(Y, Z) = 0\,, \tag{B.4}$$

also solves the equation (B.2) for any $n = 0, 1, \ldots, \mu_1 - \tau_2 - \tau_3$, one can choose the value of n which minimizes $|\bar{\mu}_1 - \mu_1 + 2\,\tau_3 + 2\,n|$. This corresponds to the solution (3.52).

C H solutions

In this appendix we show how to turn a generic function of $\tilde{H}$ into a function of Y and Z after integrating by parts all the total-derivative terms present in (3.14). Let us start considering the integration by parts of a single $\tilde{H}_i$: $\tilde{H}_i\, K(Y,Z,\tilde{H}) \simeq \hat{H}_i\, K(Y,Z,\tilde{H})$. Here the $\hat{H}_i$'s are operators defined as

$$\hat{H}_i := Y_{i-1}\, Y_{i+1} - \tfrac{\hat{\delta}}{L}\, N_i\, Z_i\,, \tag{C.1}$$

where

$$N_i := Y_{i+1}\,\partial_{Y_{i+1}} + Y_{i-1}\,\partial_{Y_{i-1}} - Y_i\,\partial_{Y_i} + Z_i\,\partial_{Z_i} + \tfrac{\mu_i-\mu_{i+1}-\mu_{i-1}}{2}\,, \tag{C.2}$$

and $\simeq$ means equality under the integral sign and modulo TT. The above identity suffices to integrate by parts any function of $\tilde{H}$ as

$$K(Y,Z,\tilde{H}) = \sum_{h_1,h_2,h_3} \tilde{H}_1^{h_1}\, \tilde{H}_2^{h_2}\, \tilde{H}_3^{h_3}\, K_{h_1 h_2 h_3}(Y,Z) \simeq \sum_{h_1,h_2,h_3} \hat{H}_1^{h_1}\, \hat{H}_2^{h_2}\, \hat{H}_3^{h_3}\, K_{h_1 h_2 h_3}(Y,Z)\,. \tag{C.3}$$

The operators $\hat{H}_i$ and $\hat{H}_j$ commutes when $i \neq j$, while $\hat{H}_i^{h_i}$ gives

$$\hat{H}_i^{h_i} = \sum_{k=0}^{h_i} \binom{h_i}{k}\, [N_i + h_i]_k \left(-\tfrac{\hat{\delta}}{L}\, Z_i\right)^k (Y_{i+1}\, Y_{i-1})^{h_i-k}\,. \tag{C.4}$$

Using the above identity, one can recast eq. (C.3) into a compact form as

$$K(Y,Z,\tilde{H}) \simeq \left[\prod_{i=1}^{3} \left(1 - \tfrac{\hat{\delta}}{L}\, Z_i\, \partial_{H_i}\right)^{N_i + H_i \partial_{H_i}}\right] K(Y,Z,H)\Big|_{H_i = Y_{i-1} Y_{i+1}}\,. \tag{C.5}$$

In the following, we find a set of functions of $\tilde{H}$ which explicitly give all possible leading terms.

One massless − two massive case We aim at finding the functions:

$$K(Y_2, Y_3, \tilde{H}_2, \tilde{H}_3, Z_1)\,, \tag{C.6}$$

that, after integration by parts, have leading terms involving G_1^{n+1}. Starting from the identity (C.5) and requiring that the first $n+1$ leading terms cancel, one gets

$$\begin{aligned}
& Z_1^{\tau_1}\, Y_2^{\sigma_2}\, Y_3^{\sigma_3}\, \tilde{H}_2^{h_2}\, \tilde{H}_3^{h_3} \times \\
& \times (Y_2\, \tilde{H}_2 - Y_3\, \tilde{H}_3) \sum_{k=0}^{n} \binom{n}{k} (x-n)_k\, [x+n]_{n-k}\, (Y_2\, \tilde{H}_2)^k\, (-Y_3\, \tilde{H}_3)^{n-k} \\
& \simeq \left(\tfrac{\hat{\delta}}{L}\right)^{n+1} [x+n]_{2n+1}\, Z_1^{\tau_1}\, \hat{H}_2^{h_2}\, \hat{H}_3^{h_3}\, Y_2^{\sigma_2}\, Y_3^{\sigma_3}\, [G_1(Y,Z)]^{n+1} \\
& = \left(\tfrac{\hat{\delta}}{L}\right)^{n+1} [x+n]_{2n+1}\, Z_1^{\tau_1}\, Y_1^{h_2+h_3}\, Y_2^{\sigma_2+h_3}\, Y_3^{\sigma_3+h_2}\, [G_1(Y,Z)]^{n+1} + \mathcal{O}(\hat{\delta}^{n+2})\,.
\end{aligned} \tag{C.7}$$

Here, $[a]_n := a(a-1)\cdots(a-n+1)$ and $(a)_n := a(a+1)\cdots(a+n-1)$ are the descending and ascending Pochhammer symbols respectively, while $x = (\mu_3-\mu_2)/2 + \sigma_2 - \sigma_3$. Let us notice that the results come with factors which vanish when $x \in \{-n, -n+1, \ldots, n\}$. Hence, one has to normalize the corresponding leading terms in order to have non-vanishing couplings also for these values of x.

Two massless − one massive case In this case, we look for the functions:

$$K(Y_3, \tilde{H}_1, \tilde{H}_2, \tilde{H}_3)\,, \tag{C.8}$$

which give rise to leading terms proportional to G^{n+1}, and we get

$$\begin{aligned}
&Y_3^{\sigma_3}\, \tilde{H}_1^{h_1}\, \tilde{H}_2^{h_2}\, \tilde{H}_3^{h_3} \times \\
&\times (\tilde{H}_1\, \tilde{H}_2 - Y_3^2\, \tilde{H}_3) \sum_{k=0}^{n} \binom{n}{k} (x-2n-1)_k\, [x-1]_{n-k}\, (\tilde{H}_1\, \tilde{H}_2)^k\, (-Y_3^2\, \tilde{H}_3)^{n-k} \\
&\simeq \left(\tfrac{\hat{\delta}}{L}\right)^{n+1} [x-1]_{2n+1}\, Y_1^{h_2+h_3}\, Y_2^{h_3+h_1}\, Y_3^{\sigma_3+h_1+h_2+n+1}\, [G(Y,Z)]^{n+1} + \mathcal{O}(\hat{\delta}^{n+2})\,,
\end{aligned} \tag{C.9}$$

with $x = \mu_3/2 - \sigma_3$.

One PM − two massive case In the PM case, a generic $\tilde{Q}$ coupling is of the form:

$$\sum_{\sigma_1=0}^{\mu_1} Y_1^{\sigma_1}\, \tilde{K}^{\sigma_1}(Y_2, Y_3, \tilde{H}_2, \tilde{H}_3, Z_1)\,. \tag{C.10}$$

On the other hand, the leading terms of the corresponding $P1$ solutions, containing a given number n of Z_2 or Z_3 (which can be obtained by using the identity (A.9)), are

$$(Z_2\partial_{Y_3} - Z_3\partial_{Y_2})^k\, G_1^{n-k}\, Y_1^{\sigma_1}\, Y_2^{\sigma_2+k}\, Y_3^{\sigma_3+k}\, Z_1^{\tau_1}\,, \qquad k = 0, \ldots, \min\{n, \mu_1\}\,, \tag{C.11}$$

where the additional inequality

$$\sigma_2 + \sigma_3 + n \geq \sigma_1\,, \tag{C.12}$$

is enforced. In general, it is rather involved to find a combination of $\tilde{H}$ giving rise to each of the leading terms (C.11) since in this case the solutions will be of the form

$$Z_1^{\tau_1}\, Y_1^{\sigma_1-h}\, Y_2^{\sigma_2-h_3}\, Y_3^{\sigma_3-h+h_3}\, \tilde{H}_2^{h-h_3}\, \tilde{H}_3^{h_3}\, \mathcal{P}_n^{(k)}(Y_2\, \tilde{H}_2, Y_3\, \tilde{H}_3, Y_1\, Y_2\, Y_3) \tag{C.13}$$

$$\simeq \left(\tfrac{\hat{\delta}}{L}\right)^n Z_1^{\tau_1} (Z_2\partial_{Y_3} - Z_3\partial_{Y_2})^k\, G_1^{n-k}\, Y_1^{\sigma_1}\, Y_2^{\sigma_2+k}\, Y_3^{\sigma_3+k} + \mathcal{O}(\hat{\delta}^{n+1})\,. \tag{C.14}$$

Here, we have introduced the combination:

$$\mathcal{P}_n^{(k)}(Y_2\, \tilde{H}_2, Y_3\, \tilde{H}_3, Y_1\, Y_2\, Y_3) := \sum_{a+b+c=n} p_{abc}^{(k)}\, (Y_2\, \tilde{H}_2)^a (Y_3\, \tilde{H}_3)^b (Y_1\, Y_2\, Y_3)^c\,, \tag{C.15}$$

where the coefficients $p_{abc}^{(k)}$ have to satisfy a system of linear equations. Let us notice that the inequality (C.12) is automatically satisfied due to the form of $\tilde{H}$. For a given order n in Z_2 and Z_3, one can count $n+1$ different polynomials in $Y_2\, Z_2$ and $Y_3\, Z_3$. Hence, the total number of polynomials with at most n powers of $Y_2\, Z_2$ and $Y_3\, Z_3$ is $\frac{(n+1)(n+2)}{2}$, matching the number of different coefficients $p_{abc}^{(k)}$ entering the combination (C.15). This is equivalent to saying that, allowing enough powers of Y_1, any leading term in (C.11) can be reproduced after integration by parts by a suitable choice of the coefficients in eq. (C.15). In the PM case, one has to impose also the bounds $c \leq \mu_1$ in eq. (C.15) and $k \leq \mu_1$ for the leading terms of eq. (C.11). Hence, the counting of available leading terms with at most n powers of $Y_2\,Z_2$ and $Y_3\,Z_3$ is

$$\begin{aligned}(\text{number of leading terms}) &= \tfrac{(\mu_1+1)(\mu_1+2)}{2} + (\mu_1+1)(n-\mu_1) \\ &= \tfrac{1}{2}\,(\mu_1+1)(2\,n+2-\mu_1)\,,\end{aligned} \tag{C.16}$$

while the number of free coefficients in (C.15) satisfying $c \le \mu_1$ is

$$\big(\text{number of } p^{(k)}_{abc}\big) = \sum_{\ell=0}^{\mu_1} (m-\ell+1) = \tfrac{1}{2}\,(\mu_1+1)(2\,n+2-\mu_1)\,. \tag{C.17}$$

Therefore, as the number of solutions matches, using completeness of the $P1$ solutions when (1.10) is not satisfied, one can conclude that the $\tilde{Q}$ solutions span the same space.

In order to clarify the latter discussion, we provide the explicit example of $n=2$. The combination proportional to G_1^2 is given by

$$\begin{aligned}&Z_1^{\tau_1}\,Y_1^{\sigma_1-h}\,Y_2^{\sigma_2-h_3}\,Y_3^{\sigma_3-h+h_3}\,\tilde{H}_2^{h-h_3}\,\tilde{H}_3^{h_3}\,\Big\{\,[x+y+1]_3\,(x-y+1)\,(Y_2\,\tilde{H}_2)^2 \\ &\quad -2\,(x+y)_2\,[x-y]_2\,(Y_2\,\tilde{H}_2)\,(Y_3\,\tilde{H}_3) + (x+y-1)\,(x-y-1)_3\,(Y_3\,\tilde{H}_3)^2 \\ &\quad -4\,y\,\Big[(x+y)_2(x-y)\,(Y_2\,\tilde{H}_2) - (x+y)\,[x-y]_2\,(Y_3\,\tilde{H}_3)\Big]\,(Y_1\,Y_2\,Y_3) \\ &\quad +2\,y\,(2y-1)\,(x+y+1)\,(x-y-1)\,(Y_1\,Y_2\,Y_3)^2\Big\}\,,\end{aligned} \tag{C.18}$$

while the one proportional to $G_1\,(Y_2\,Z_2 - Y_3\,Z_3)$ is

$$\begin{aligned}&Z_1^{\tau_1}\,Y_1^{\sigma_1-h}\,Y_2^{\sigma_2-h_3}\,Y_3^{\sigma_3-h+h_3}\,\tilde{H}_2^{h-h_3}\,\tilde{H}_3^{h_3}\,\times \\ &\times\Big\{\,[x+y+1]_3\,(x-y+1)\,(Y_2\,\tilde{H}_2)^2 - (x+y-1)\,(x-y-1)_3\,(Y_3\,\tilde{H}_3)^2 \\ &\quad -2\,y\,\Big[(x+y-1)_3\,(x-y)\,(Y_2\,\tilde{H}_2) - (x+y)\,(x-y-1)_3\,(Y_3\,\tilde{H}_3)\Big]\,(Y_1\,Y_2\,Y_3) \\ &\quad +2\,(2y-1)\,x\,(x+y+1)\,(x-y-1)\,(Y_1\,Y_2\,Y_3)^2\Big\}\,.\end{aligned} \tag{C.19}$$

Finally, the combination proportional to $(Y_2\,Z_2 - Y_3\,Z_3)^2$ reads

$$\begin{aligned}&Z_1^{\tau_1}\,Y_1^{\sigma_1-h}\,Y_2^{\sigma_2-h_3}\,Y_3^{\sigma_3-h+h_3}\,\tilde{H}_2^{h-h_3}\,\tilde{H}_3^{h_3}\,\Big\{\,[x+y+1]_3\,(x-y+1)\,(Y_2\,\tilde{H}_2)^2 \\ &+2\,(x+y)_2\,[x-y]_2\,(Y_2\,\tilde{H}_2)\,(Y_3\,\tilde{H}_3) + (x+y-1)\,(x-y-1)_3\,(Y_3\,\tilde{H}_3)^2 \\ &-4\,y\,\Big[(x-1)\,(x+y)_2\,(x-y)\,(Y_2\,\tilde{H}_2) + (x+1)\,(x+y)\,[x-y]_2\,(Y_3\,\tilde{H}_3)\Big]\,(Y_1\,Y_2\,Y_3) \\ &+2\,(2\,x^2-y)\,(x-y-1)\,(x+y+1)\,(Y_1\,Y_2\,Y_3)^2\Big\}\,.\end{aligned} \tag{C.20}$$

Here, x and y are defined as

$$x = \tfrac{\mu_2-\mu_3}{2} + \sigma_3 - \sigma_2\,, \qquad y = \tfrac{\mu_1}{2} - \sigma_1\,. \tag{C.21}$$

As one can see from the above example, if no bound on the powers of Y_1 is imposed in (C.10), then all possible leading terms can be reproduced. However, for given values of y, only a subset of the above solutions satisfy $c \le \mu_1$. For instance, when $\mu_1 = 0$ one can see that only the leading terms proportional to G_1^n are leftover.

D Flat-space limit

In this appendix, we present another argument for the equivalence between the $\tilde{Q}$ solutions and $P1$ solutions. It relies on the limiting process where both L and μ go to infinity in such a way that their ratio stays finite:

$$\lim_{L\to\infty} \frac{|\mu|}{L} = M\,. \tag{D.1}$$

Let us notice that, as the supplemental Q and $P2$ solutions have to satisfy the conditions (1.10), they cannot be taken into account in this limit so that effectively we are restricting the attention to the $\tilde{Q}$ and $P1$ solutions only. In the (D.1) limit, the PDE (3.29) reduces to

$$\Big[Y_2\,\partial_{Z_3} - Y_3\,\partial_{Z_2} + \tfrac{\hat{\delta}}{2}\,(M_2 - M_3)\,\partial_{Y_1}\Big]^{\mu_1+1} C(Y,Z) = 0\,, \tag{D.2}$$

which, for $\mu_1 = 0$, corresponds to the usual flat Noether procedure equation for one massless and two massive fields. However, in general, it does not have a direct physical interpretation. The PDE (D.2) can be easily solved as

$$C = \sum_{\sigma_1=0}^{\mu_1} Y_1^{\sigma_1}\,K^{\sigma_1}(Y_2, Y_3, Z_1, H_2, H_3)\,, \tag{D.3}$$

where the H_i's are given by

$$\begin{aligned} H_i &= Y_{i+1}\,Y_{i-1} + \tfrac{\hat{\delta}}{2}\,(M_i - M_{i+1} - M_{i-1})\,Z_i\,, \\ &= Y_{i+1}\,Y_{i-1} - \tfrac{1}{2}\,\partial_X\cdot\big(\partial_{X_i} - \partial_{X_{i+1}} - \partial_{X_{i-1}}\big)\,Z_i\,. \end{aligned} \tag{D.4}$$

The (A)dS counterpart of (D.4) can be obtained by adding proper total-derivative terms to the $Y_{i\pm1}$'s, ending up with $\tilde{H}_i$ (3.14). In this way, keeping the number of independent solutions unchanged, one recovers the following (A)dS vertices:

$$C = \sum_{\sigma_1=0}^{\mu_1} Y_1^{\sigma_1}\,\tilde{K}^{\sigma_1}(Y_2, Y_3, Z_1, \tilde{H}_2, \tilde{H}_3)\,, \tag{D.5}$$

in agreement with eq. (3.56). Notice that, in the limit, the combinations (C.7) and (C.9), giving rise to G_1^n- and G^n-type leading terms, become proportional to

$$(Y_2\,H_2 - Y_3\,H_3)^n\,, \qquad (H_1\,H_2 - Y_3^{\,2}\,H_3)^n\,, \tag{D.6}$$

respectively, that are nothing but the corresponding flat combinations.

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