

# Asymptotics of relative heat traces and determinants on open surfaces of finite area

Clara L. Aldana

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**Abstract** The goal of this article is to prove that on surfaces with asymptotically cusp ends the relative determinant of pairs of Laplace operators is well defined. We consider a surface with cusps  $(M, g)$  and a metric  $h$  on the surface that is a conformal transformation of the initial metric  $g$ . We prove the existence of the relative determinant of the pair  $(\Delta_h, \Delta_g)$  under suitable conditions on the conformal factor. The core of the paper is the proof of the existence of an asymptotic expansion of the relative heat trace for small times. We find the decay of the conformal factor at infinity for which this asymptotic expansion exists and the relative determinant is defined. Following the paper by B. Osgood, R. Phillips, and P. Sarnak about extremal of determinants on compact surfaces, we prove Polyakov's formula for the relative determinant and discuss the extremal problem inside a conformal class. We discuss necessary conditions for the existence of a maximizer.

**Keywords** Surfaces with asymptotically cusp ends · Heat kernels · Asymptotic expansion of heat traces · Relative determinants

## 1 Introduction

In this paper, we study the relative determinant of Laplace operators on surfaces with asymptotically cusp ends and the asymptotic expansion of the corresponding relative heat traces for small values of time. A surface with asymptotically cusp ends is defined in Sect. 2.4.

Regularized determinants of elliptic operators play an important role in many fields of mathematics and mathematical physics. They were initially introduced by Ray and Singer [18] in relation to  $R$ -torsion. The regularized determinant of the Laplace operator on a compact Riemannian manifold is defined via a zeta function regularization process. It is an important spectral invariant. For instance, in the two-dimensional case, B. Osgood, R. Phillips, and

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C. L. Aldana (✉)  
Max Planck Institute for Gravitational Physics, (Albert Einstein Institute), MPG, Potsdam, Germany  
e-mail: clara.aldana@aei.mpg.de

P. Sarnak (OPS)<sup>1</sup> showed in [16] that the determinant, considered as a functional on the space of metrics, has very interesting extremal properties. They proved the following result: Let  $M$  be a closed surface, then in a given conformal class, among all metrics of unit area, there exists a unique metric of constant curvature at which the regularized determinant attains a maximum. They also proved a corresponding statement for compact surfaces with boundary and suitable conditions at the boundary.

Relative determinants were introduced in a general setting by Müller in [14] as a way to generalize regularized determinants in the compact case. Previously, a relative determinant for admissible surfaces was introduced by Lundelius in [11], and for Dirac operators in  $\mathbb{R}^n$  by Bruneau in [4]. A good example of a non-compact space is a surface with cusps. A surface with cusps is a two-dimensional complete Riemannian manifold  $(M, g)$  of finite area such that outside a compact set the metric is hyperbolic. The hyperbolic ends are called cusps. The Laplace operator  $\Delta_g$  associated to the metric  $g$  on  $M$  has continuous spectrum. Therefore, its zeta regularized determinant cannot be defined in the same way as in the compact case. Here is when relative determinants enter into the play. The relative determinant is defined for a pair of non-negative self-adjoint operators  $(A, B)$  in a Hilbert space provided they satisfy certain conditions. It is defined through a zeta function using the trace of the relative heat semigroup  $\text{Tr}(e^{-tA} - e^{-tB}), t > 0$ .

For surfaces with cusps in [14], Müller proved that the relative determinant of the Laplacian is well defined when the Laplacian is compared with a model operator defined on the cusps. In this paper, we extend this result to surfaces with asymptotically cusp ends. We also prove Polyakov's formula for metrics for which the relative determinant of the corresponding Laplacians is defined. The analysis of the extremal of the determinant in this case is performed in the same way as in OPS [16]. Unfortunately, the maximizer (the metric of constant curvature) is not always among the class of metrics for which we can define the relative determinant.

The paper is organized as follows:

We start by fixing a surface with cusps and a class of metrics on  $M$  that are conformal to  $g$  and that satisfy suitable conditions. Let  $h = e^{2\varphi}g$  be a metric in the conformal class of  $g$ ; if the cusps are “kept” but the metric  $h$  is not hyperbolic on them, then we say that  $(M, h)$  is a surface with asymptotically cusp ends. Associated to the metric  $h$ , there is a Laplacian which we denote by  $\Delta_h$ . We will consider the relative determinant of pairs of the form  $(\Delta_h, \Delta_g)$  and  $(\Delta_h, \tilde{\Delta}_{1,0})$ , where  $\tilde{\Delta}_{1,0}$  is a model operator over  $M$  that is associated to the cusps.

In Sect. 2, we introduce all the notation and background theory that we need throughout the paper. In Sect. 3, we prove the trace class property of the relative heat operator for all positive values of  $t$ , when the conformal factor  $\varphi$  as well as its derivatives up to second-order decay as  $O(y^{-\alpha})$ ,  $\alpha > 0$  as  $y$  goes to infinity; here, we are using coordinates  $(y, x)$  in the cusps  $Z = [1, \infty) \times S^1$ .

In Sect. 4, we prove the existence of an asymptotic expansion of the relative heat trace for small values of  $t$ . Theorem 4.6 gives precise conditions for the existence of such an expansion up to order  $\nu \geq 1$ . The expansion exists if the function  $\varphi|_Z(y, x)$  and its derivatives up to second order are  $O(y^{-k})$  as  $y$  goes to infinity, with  $k \geq 5\nu + 8$ ; although if  $\nu \geq 3$ , more derivatives of  $\varphi$  should decay at infinity as well. The precise decay of the higher derivatives is given in the statement of the theorem.

The proof of this result is very technical but uses classical methods such as parametrices, Duhamel's principle, upper bounds of heat kernels, universal coverings, very particular inequalities, and the explicit form of the local heat invariants. The idea of the proof is to

<sup>1</sup> From now on, we abbreviate B. Osgood, R. Phillips, and P. Sarnak as OPS.

write the relative heat trace as an integral over the manifold, and to split this integral into three areas of integration: the compact part, a cutoff of the cusps, and the end of the cusps. The cutoff is done at a height  $a > 1$  that is fixed at the beginning. The conditions on the conformal factor come from assumptions in different parts of the proof. Along the paper, we will explain each of these assumptions in detail. The main point is that later in the proof we let  $a$  be a function of  $t$  and take the limit as  $t \rightarrow 0$ . Then, the integral over the cutoff will have a complete asymptotic expansion as  $t \rightarrow 0$  (as  $a \rightarrow \infty$ ). The integral on the end of the cusps is estimated by a term  $t^\nu$ ,  $\nu > 0$ . The estimation is obtained using the trace norm of some auxiliary operators. The order  $k \geq 5\nu + 8$  in the decay condition of the conformal factor comes from this bound.

In Sect. 5.1, we use the previous results to define the relative determinant of the pairs  $(\Delta_h, \Delta_g)$  and  $(\Delta_h, \Delta_{1,0})$  using relative zeta functions. In spite of not having an optimal result in Sect. 4, the result is good enough to have a well-defined relative determinant for a pair of metrics  $(h, g)$  satisfying the conditions above.

In Sect. 5.2, we study  $\det(\Delta_h, \Delta_{1,0})$  as a functional on metrics of a given area in a conformal class and look for its extremal values.

We give a proof of a Polyakov’s-type formula for  $\det(\Delta_h, \Delta_{1,0})$ . The proof of this formula follows the same lines as the proof of OPS in the compact case in [16] and the formula is the same as the one obtained by Lundelius in [11] for heights of pairs of admissible surfaces. However, let us point out that our methods are different from the ones in [11]. In the same way as in [11] and [16], we see that if there exists a maximum it is attained at the metric of constant curvature. The equation relating the curvature of the metrics  $g$  and  $h = e^{2\varphi}g$  is  $R_h = e^{-2\varphi}(\Delta_g\varphi + R_g)$ . The study of the associated differential equation for  $\varphi$ , together with the constant curvature condition in the cusps for  $g$  and constant curvature everywhere for  $h$ , leads to a precise decay for the function  $\varphi$  at infinity. Unfortunately, this decay is not included in the conditions required to define the relative determinant. Therefore, the metric of constant curvature will not be in the conformal class under consideration unless we start with a metric of constant curvature.

In relation with this problem, there is a recent paper by Albin et al. [1]. We worked with renormalized integrals to define renormalized determinants of Laplacians on surfaces that have asymptotically hyperbolic ends, cusps as well as funnels (funnels involve infinite area).

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## 2 Notation and definitions

### 2.1 Relative determinants

Let us recall the definition of relative determinants introduced by Müller in [14]: The relative determinant is defined for two self-adjoint, non-negative linear operators,  $H_1$  and  $H_0$ , in a separable Hilbert space  $\mathcal{H}$  satisfying the following assumptions:

- (1) For each  $t > 0$ ,  $e^{-tH_1} - e^{-tH_0}$  is a trace class operator.
- (2) As  $t \rightarrow 0$ , there is an asymptotic expansion for the relative trace of the form:

$$\text{Tr}(e^{-tH_1} - e^{-tH_0}) \sim \sum_{j=0}^{\infty} \sum_{k=0}^{k(j)} a_{jk} t^{\alpha_j} \log^k t,$$

where  $-\infty < \alpha_0 < \alpha_1 < \dots$  and  $\alpha_k \rightarrow \infty$ . Moreover, if  $\alpha_j = 0$ , we assume that  $a_{jk} = 0$  for  $k > 0$ .

- (3)  $\text{Tr}(e^{-tH_1} - e^{-tH_0}) = h + O(e^{-ct})$ , as  $t \rightarrow \infty$  for some constant  $c > 0$  where  $h = \dim \ker H_1 - \dim \ker H_0$ .

These properties allow us to define the relative zeta function as

$$\zeta(s; H_1, H_0) = \frac{1}{\Gamma(s)} \int_0^\infty (\text{Tr}(e^{-tH_1} - e^{-tH_0}) - h)t^{s-1} dt. \tag{2.1}$$

Using the meromorphic continuation of  $\zeta(s; H_1, H_0)$  to the complex plane, the relative determinant is defined as

$$\det(H_1, H_0) := e^{-\zeta'(0; H_1, H_0)}.$$

In a more general setting, condition (3) is replaced by an asymptotic expansion as  $t \rightarrow \infty$ . In that case, in order to define the relative zeta function, the integral in (2.1) has to be split in two parts, see [14].

### 2.2 Surfaces with cusps

A surface with cusps (swc)<sup>2</sup> is a two-dimensional Riemannian manifold that is complete, non-compact, has finite volume, and is hyperbolic in the complement of a compact set. Therefore, it admits a decomposition of the form

$$M = M_0 \cup Z_1 \cup \dots \cup Z_m,$$

where  $M_0$  is a compact surface with smooth boundary and for each  $i = 1, \dots, m$  we assume that

$$Z_i \cong [a_i, \infty) \times S^1, \quad g|_{Z_i} = y_i^{-2}(dy_i^2 + dx_i^2), \quad a_i > 0$$

The subsets  $Z_i$  are called cusps. Sometimes we denote  $Z_i$  by  $Z_{a_i}$  to indicate the “starting point”  $a_i$ . For simplicity, by  $S^1$  we mean the circle with radius  $1/2\pi$  with length 1. Instances of surfaces with cusps are quotients of the form  $\Gamma(N) \backslash \mathbb{H}$ , where  $\mathbb{H}$  is the upper half plane and  $\Gamma(N) \subseteq \text{SL}_2(\mathbb{Z})$  is a congruence subgroup, i.e.,  $\Gamma(N) = \{\gamma \in \text{SL}_2(\mathbb{Z}) | \gamma \equiv \text{Id} \pmod{N}\}$ . These quotients play an important role in the theory of automorphic forms.

To any surface with cusps  $(M, g)$ , we can associate a compact surface  $\bar{M}$  such that  $(M, g)$  is diffeomorphic to the complement of  $m$  points in  $\bar{M}$ . Let  $p$  denote the genus of the compact surface  $\bar{M}$ ; then the pair  $(p, m)$  is called the conformal type of  $M$ .

We use later the following estimate of the Riemannian distance in the cusp  $Z$

$$d_{g_0}(z, z') \geq |\log(y/y')|,$$

for  $z = (y, x), z' = (y', x')$ , see, for example, [12].

For any oriented Riemannian manifold  $(M, g)$ , the Laplace-Beltrami operator on functions is defined as  $\Delta f = -\text{div grad } f$ . It is equal to  $\Delta = d^*d$ . We consider positive Laplacians. If  $(M, g)$  is complete,  $\Delta$  has a unique closed extension that we denote by  $\Delta_g$ .

On a cusp  $Z$ , the Laplacian is given by

$$\Delta_Z = -y^2 \left( \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2} \right).$$

<sup>2</sup> From now on, we abbreviate “surface with cusps” as swc.

Let us consider the following operators:

**Definition 2.1** Let  $a > 0$ , let  $\Delta_{a,0}$  denote the self-adjoint extension of the operator

$$-y^2 \frac{\partial^2}{\partial y^2} : C_c^\infty((a, \infty)) \rightarrow L^2([a, \infty), y^{-2} dy)$$

with respect to Dirichlet boundary conditions at  $y = a$ . The domain of  $\Delta_{a,0}$  is then given by  $\text{Dom}(\Delta_{a,0}) = H_0^1([a, \infty)) \cap H^2([a, \infty))$ , where  $H_0^1([a, \infty)) = \{f \in H^1([a, \infty)) : f(a) = 0\}$ .

Let  $\bar{\Delta}_{a,0} = \oplus_{j=1}^m \Delta_{a_j,0}$  be defined as the direct sum of the self-adjoint operators  $\Delta_{a_j,0}$  defined above. The operator  $\bar{\Delta}_{a,0}$  acts on a subspace of  $\oplus_{j=1}^m L^2([a_j, \infty), y_j^{-2} dy_j)$ .

Now, let  $a > 0$ , let  $Z_a$  be endowed with the hyperbolic metric  $g$  and let  $\Delta_{Z_a,D}$  be the self-adjoint extension of

$$-y^2 \left( \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2} \right) : C_c^\infty((a, \infty) \times S^1) \rightarrow L^2(Z_a, dA_g)$$

with respect to Dirichlet boundary conditions at  $\{a\} \times S^1$ . It is known that the operator  $\Delta_{Z_a,D}$  can be decomposed as follows: Put

$$L_0^2(Z_a) = \{f \in L^2(Z_a, dA_g) \mid \int_{S^1} f(y, x) dx = 0 \text{ for a. e. } y \geq a\}. \tag{2.2}$$

The orthogonal complement of  $L_0^2(Z_a)$  in  $L^2(Z_a, dA_g)$  consists of functions that are independent of  $x \in S^1$ .

Then we can decompose  $L^2(Z_a, dA_g)$  as the orthogonal direct sum

$$L^2(Z_a, dA_g) = L^2([a, \infty), y^{-2} dy) \oplus L_0^2(Z_a).$$

This decomposition is invariant under  $\Delta_{Z_a,D}$  so in terms of this decomposition, we can write  $\Delta_{Z_a,D} = \Delta_{a,0} \oplus \Delta_{Z_a,1}$ , where  $\Delta_{Z_a,1}$  acts on  $L_0^2(Z_a)$ .

*Remark 2.2* The operator  $\Delta_{Z_a,1}$  has compact resolvent; in particular, it has only point spectrum, see Lemma 7.3 in [15]. In addition, the counting function for  $\Delta_{Z_a,1}$ ,  $N_{\Delta_{Z_a,1}}(\lambda) = \#\{\tilde{\lambda}_j \leq \lambda\}$ , where  $\{\tilde{\lambda}_j\}$  are the eigenvalues of  $\Delta_{Z_a,1}$ , satisfies  $N_{\Delta_{Z_a,1}}(\lambda) \sim \frac{\lambda}{4\pi} A_g$ . See [9, Theorem 6]. This implies that the heat operator  $e^{-t\Delta_{Z_a,1}}$  is trace class.

### 2.3 Spectral theory of surfaces with cusps

For the spectral theory of manifolds with cusps, we refer the reader to Müller [12], Colin de Verdière [9], and the references therein. The results in [12] hold for any dimension. For surfaces in particular, we refer to [13]. Here, we recall only the main facts and definitions that we use in this article.

For a surface with cusps  $(M, g)$ , the spectrum of the Laplacian  $\sigma(\Delta_g)$  is the union of the point spectrum  $\sigma_p$  and the continuous spectrum  $\sigma_c$ . The point spectrum consists of a sequence of eigenvalues

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$$

Each eigenvalue has finite multiplicity, and the counting function  $N(\Lambda) = \#\{\lambda_j | \lambda_j \leq \Lambda^2\}$  for  $\Lambda > 0$  satisfies  $\limsup N(\Lambda)\Lambda^{-2} \leq A_g(4\pi)^{-1}$ , where  $A_g$  denotes the area of  $(M, g)$ . Depending on the metric, the set of eigenvalues may be infinite or not.

The continuous spectrum  $\sigma_c$  of  $\Delta_g$  is the interval  $[\frac{1}{4}, \infty)$  with multiplicity equal to the number of cusps of  $M$ . For a proof of this fact, see for example [12, p. 206]. The spectral decomposition of the absolutely continuous part of  $\Delta_g$  is described by the generalized eigenfunctions  $E_j(z, s)$ , for  $j = 1, \dots, m$  with  $z \in M, s \in \mathbb{C}$ . To each cusp, we can associate such generalized eigenfunctions; they are also called Eisenstein functions by analogy with the Eisenstein series on hyperbolic surfaces. They are closely related to the wave operators  $W_{\pm}(\Delta_g, \bar{\Delta}_{a,0})$  and to the scattering matrix  $S(\lambda)$ . For details, see [12, Sect. 7]. The main properties of the Eisenstein functions and the scattering matrix can be found in [12, Theorem 7.24].

### 2.4 Conformal transformations

In this section, we give few properties of metrics that are conformal to each other.

A conformal transformation of a metric  $g$  on  $M$  is a metric  $h$  defined as  $h = \rho g$  where  $\rho \in C^\infty(M)$  and  $\rho > 0$ . In this paper, we write the function  $\rho$  as  $\rho = e^{2\varphi}$  with  $\varphi \in C^\infty(M)$ . We call the function  $\varphi$  the conformal factor. Depending on the case, the conformal factor may have compact support or not. If the support is not compact we require  $\varphi$  as well as some of its derivatives to decay at infinity. In what follows the metric  $h$  will always denote a conformal transformation of  $g$ .

Two metrics  $g_1, g_2$  are said to be quasi-isometric if there exist constants  $C_1, C_2 > 0$  such that

$$C_1 g_1(z) \leq g_2(z) \leq C_2 g_1(z), \quad \text{for all } z \in M,$$

in the sense of positive definite forms.

Quasi-isometric metrics have equivalent geodesic distances. The associated  $L^2$ -spaces coincide as sets, though the inner product is not the same.

*Remark 2.3* Let  $h = e^{2\varphi} g$ . If the function  $\varphi$  is bounded on  $M$ , the metrics  $g$  and  $h$  are quasi-isometric and the geodesic distances,  $d_g$  and  $d_h$ , are equivalent. If in addition the metric  $g$  is complete, so is the metric  $h$ .

Let us first give a handwaving definition of what we mean by a surface with asymptotically cusps ends. The reason to do that is that we need flexibility in the conditions on the conformal factors:

A surface with asymptotically cusp ends (swac)<sup>3</sup> is a surface  $(M, h)$  where the metric  $h$  is a conformal transformation of the metric on a swc  $(M, g)$  such that the conformal factor as well as some of its derivatives have a suitable decay in the cusps.

Now, let  $(M, g)$  be a swc and  $h$  be as above. A point  $z = (y, x)$  in a cusp has injectivity radius  $\text{inj}_g(z) \sim \frac{1}{y}$ . If we assume that  $\Delta_g \varphi = O(1)$  as  $y \rightarrow \infty$ , the surface  $(M, h)$  has bounded Gaussian curvature. Then by [15, Prop. 2.1], the injectivity radius of both metrics are comparable. Thus, the injectivity radius of a swac also vanishes.

Let  $A_g$  denote the area of  $(M, g)$ ,  $dA_g$  the volume element, and  $R_g(z)$  its Gaussian curvature. Let  $A_h, dA_h$  and  $R_h$  be the quantities corresponding to  $(M, h)$ , for any conformal transformation  $h$  of  $g$ . Let  $\Delta_h$  be the Laplacian associated to  $h$ . Then the following relations hold

$$dA_h = e^{2\varphi} dA_g, \quad \Delta_h = e^{-2\varphi} \Delta_g, \quad R_h = e^{-2\varphi} (\Delta_g \varphi + R_g)$$

<sup>3</sup> From now on, we abbreviate “surface with asymptotically cusp ends” as swac.

The domains of the Laplacians  $\Delta_g$  and  $\Delta_h$  lie in different Hilbert spaces. Thus, sometimes it is necessary to consider a unitary map between the spaces  $L^2(M, dA_g)$  and  $L^2(M, dA_h)$ . From the definition of the metrics and the transformation of the area element, the unitary map is given by

$$T : L^2(M, dA_g) \rightarrow L^2(M, dA_h), \quad f \mapsto e^{-\varphi} f. \tag{2.3}$$

The Laplacians transform in the following way:

$$\begin{aligned} T^{-1}\Delta_h T f &= e^{-2\varphi} \left( \Delta_g f + 2\langle \nabla_g f, \nabla_g \varphi \rangle_g - (\Delta_g \varphi + |\nabla_g \varphi|_g^2) f \right) \\ T \Delta_g T^{-1} f &= e^{2\varphi} \left( \Delta_h f - 2\langle \nabla_h \varphi, \nabla_h f \rangle_h + (\Delta_h \varphi - |\nabla_h \varphi|_h) f \right) \end{aligned} \tag{2.4}$$

Note that the operators  $T^{-1}\Delta_h T$  and  $T \Delta_g T^{-1}$  are self-adjoint in the corresponding transformed domain.

Let us finish this section recalling Gauss–Bonnet theorem on a swc. The Euler characteristic of a surface  $M$  with  $m$  cusps is given by  $\chi(M) = (2 - 2p - m)$ , where  $p$  is the genus of the compact surface  $\bar{M}$  defined in Sect. 2.2. A Gauss–Bonnet formula is valid in this setting:

$$\int_M R_g dA_g = 2\pi \chi(M),$$

where  $R_g$  denotes the Gaussian curvature of the metric  $g$ . The same formula is valid for the metric  $h = e^{2\varphi} g$  when  $\varphi$  and  $\Delta_g \varphi$  suitably decay at infinity, since

$$\int_M R_h dA_h = \int_M e^{-2\varphi} (\Delta_g \varphi + R_g) e^{2\varphi} dA_g = \int_M R_g dA_g.$$

## 2.5 Heat kernels and their estimates

### 2.5.1 Heat kernels

The heat semigroup associated to a closed self-adjoint operator can be constructed using the spectral theorem. For the existence and uniqueness of the heat kernel on a complete open manifold with Ricci curvature bounded from below see Dodziuk [10]. For the main properties of heat kernels, see [6] and [10].

Let  $(M, g)$  and  $h = e^{2\varphi} g$  be as above, and let  $e^{-t\Delta_h}, e^{-t\Delta_g}, e^{-t\Delta_{a,0}}$  denote the heat semigroups associated to the Laplacians  $\Delta_h, \Delta_g$  and  $\Delta_{a,0}$ , respectively. Since the Laplacians are positive, the heat equation is  $\Delta + \partial_t = 0$ . Let  $K_h(z, z', t)$  and  $K_g(z, z', t)$  denote the heat kernels corresponding to  $\Delta_h$  and  $\Delta_g$ , respectively. The heat kernel on a surface with cusps was constructed by Müller in [12].

Like the Laplacians, the heat semigroups act on different spaces. The operator  $e^{-t\Delta_h}$  may act on  $L^2(M, dA_g)$ , but it is not self-adjoint with respect to this inner product. To make  $e^{-t\Delta_h}$  and  $e^{-t\Delta_g}$  act on the same space and preserve self-adjointness, we use the unitary map  $T$  defined by (2.3). The transformed operators  $T^{-1}e^{-t\Delta_h}T$  and  $T e^{-t\Delta_g} T^{-1}$  are self-adjoint on the corresponding space. The integral kernel of the transformed operator  $T^{-1}e^{-t\Delta_h}T : L^2(M, dA_g) \rightarrow L^2(M, dA_g)$  is given by  $K_{T^{-1}e^{-t\Delta_h}T}(z, z', t) = e^{\varphi(z)} K_h(z, z', t) e^{\varphi(z')}$ .

### 2.5.2 Estimates of the heat kernels

In this section, we recall the bounds of the heat kernels, since we use them repeatedly. If the manifold is closed, there exists a constant  $c > 0$  such that for any fixed  $0 < \tau < \infty$ , the heat kernel satisfies the following bounds:

$$K(x, y, t) \ll t^{-n/2} e^{-\frac{cd(x,y)^2}{t}}, \quad \text{for } t \leq \tau. \tag{2.5}$$

If the manifold has a boundary, consider the closed self-adjoint extension of the Laplacian with respect to Dirichlet boundary conditions. In this case, let  $K \subset M$  be compact and a  $\tau > 0$ , then there exist positive constants  $c, c'$  such that

$$K_D(x, y, t) \leq c' t^{-n/2} \left( e^{-\frac{cd(x,y)^2}{t}} + e^{-\frac{d(y,\partial M)^2}{8t}} \right),$$

for  $(x, y, t) \in K \times M \times (0, \tau]$ , see [6, Chap. VII].

Now, let  $\tilde{Z} = \mathbb{R}^+ \times S^1$  be the complete cusp. Let us consider the hyperbolic metric on it,  $g_0 = y^{-2}(dy^2 + dx^2)$ . Then  $(\tilde{Z}, g_0)$  is a complete Riemannian manifold and it is called a *horn*. Let  $\Delta_1$  be the unique self-adjoint extension of the Laplacian defined on  $C_c^\infty(\mathbb{R}^+ \times S^1)$ . The notation  $\Delta_1$  is arbitrary. The construction of the heat kernel for  $\Delta_1$  on  $\mathbb{R}^+ \times S^1$  can be found in [12]. We denote this heat kernel by  $K_1$ .

Let  $\tau > 0$  be arbitrary, then there exist constants  $C, c > 0$  such that for  $0 < t < \tau, y, y' \geq 1$ , and  $k, l, m \in \mathbb{N}$  one has

$$\left| \frac{\partial^k}{\partial t^k} d_z^l d_{z'}^m K_1(z, z', t) \right| \leq C(y y')^{\frac{1}{2}} t^{-1-k-l-m} \exp\left(-\frac{cd_{g_0}^2(z, z')}{t}\right), \tag{2.6}$$

where  $d_{g_0}$  the hyperbolic distance in the horn, and the constants depend on  $\tau$ , see [12, Prop.2.32].

Let  $(M, g)$  be a surface with one cusp that we denote by  $Z, Z = [a, \infty) \times S^1$  for some  $a \geq 1$ . Let  $i(z)$  be the function given by

$$i(z) = \begin{cases} 1, & \text{if } z \in M \setminus Z; \\ y, & \text{if } z \in Z \text{ and } z = (y, x). \end{cases} \tag{2.7}$$

Given  $\tau > 0$ , there exist  $C, c > 0$  such that

$$|K_g(z, z', t)| \leq C(i(z)i(z'))^{\frac{1}{2}} t^{-1} \exp\left(-\frac{cd_g^2(z, z')}{t}\right) \tag{2.8}$$

for  $0 < t < \tau$ , where  $d_g$  is the Riemannian distance in  $(M, g)$ , see [12, Eq. (4.12)].

Let us now go back to the metric  $h = e^{2\varphi}g$ . Its restriction to  $Z$  can be extended to a metric on the horn  $\tilde{Z}$  in the following way: On  $\tilde{Z}$ , we have the hyperbolic metric  $g_0$ , and  $g|_Z = g_0$ . We start by extending the function  $\varphi|_Z$  to a smooth function  $\tilde{\varphi}$  on  $\tilde{Z}$  that vanishes in a small neighborhood of zero. Then on  $(0, \infty) \times S^1$ , we define  $h$  as  $h := e^{2\tilde{\varphi}}g_0$ . It is a complete metric and  $h = g_0$  close to the boundary  $\{0\} \times S^1$ . In this way, we can define the Laplacian on  $(\tilde{Z}, h)$ . Denote its unique self-adjoint extension by  $\Delta_{1,h}$ . Clearly  $\Delta_{1,h} = e^{-2\tilde{\varphi}}\Delta_1$ . The heat kernel associated to  $\Delta_{1,h}$  is denoted by  $K_{1,h}(z, z', t)$ , for  $z, z' \in \tilde{Z}$  and  $t > 0$ .

The estimates of the heat kernel of the operator  $\Delta_{1,h}$  can be derived from Theorems 4, 6, and 7 in Cheng et al.'s paper [8]. However, in the estimates appears the injectivity radius to a power  $\alpha$  that depends only on the dimension of the manifold; from the proof in [8], it is not clear how to determine the value  $\alpha = 1$  that we need. In order to pin down the value of  $\alpha$  in this particular case, we prove in Appendix A the following lemma.



**Lemma 2.4** *Let  $h$  and  $g$  be as above and such that  $\varphi$  and  $\Delta_g\varphi$  decay in the cusp. Then the heat kernel  $K_h$  satisfies:*

$$K_h(z, z', t) \ll (i(z)i(z'))^{\frac{1}{2}}t^{-1} \exp\left(-\frac{\tilde{c}d_h^2(z, z')}{t}\right) \tag{2.9}$$

for  $0 < t < \tau$ , where  $\tilde{c} > 0$  is a constant.

Let  $*$  denote the metric  $g$  or  $h$ , then derivatives of the heat kernel  $K_*$  satisfy:

$$|\nabla K_*(z, z', t)| \leq c (i(z)i(z'))^{1/2}t^{-3/2} \exp\left(-\frac{\tilde{c}d_*^2(z, z')}{t}\right), \text{ and} \tag{2.10}$$

$$|\Delta_* K_*(z, z', t)| \leq C (i(z)i(z'))^{1/2}t^{-2} \exp\left(-\frac{\tilde{c}d_*^2(z, z')}{t}\right), \tag{2.11}$$

where the constants  $c, C$  depend on  $\tau$ , the curvature, and the covariant derivatives of the curvature. Even more, we can exchange the distances  $d_g$  and  $d_h$  in the exponentials on the right-hand side by adjusting the constant in the exponential. In the same way, the heat kernel  $K_{1,h}$  and its derivatives satisfy the same estimates as  $K_h$  above.

For a surfaces with hyperbolic cusps, the estimates in the lemma above were established in [12].

### 2.5.3 Heat kernels for other operators

In this part, we introduce the other heat operators that we will use throughout this article.

For  $a > 1$ , let  $\Delta_{a,0}$  be the operator defined in Definition 2.1. The heat kernel  $p_a(y, y', t)$  associated to  $\Delta_{a,0}$  can be computed explicitly, see [5, Sect. 14.2] or [12, p.258]. It is given by

$$p_a(y, y', t) = \frac{e^{-t/4}}{\sqrt{4\pi t}} (yy')^{1/2} \left\{ e^{-(\log(y/y'))^2/4t} - e^{-(\log(yy')-\log(a^2))^2/4t} \right\}, \tag{2.12}$$

for  $y, y' > a$ . This is easy to verify by direct computation. Also, note that for  $1 \leq y \leq a, p_a(y, y', t) = 0$ .

The operator  $e^{-t\Delta_{a,0}}$  acts on  $L^2([a, \infty), y^{-2}dy)$ . However, we can regard it as an operator acting on  $L^2([1, \infty), y^{-2}dy)$  by considering the corresponding inclusion and restriction. Similarly, the operator  $e^{-t\Delta_{1,0}}$  can be regarded as acting on  $L^2([a, \infty), y^{-2}dy)$ .

Now, let us assume that  $M$  can be decomposed as  $M = M_0 \cup Z$  with  $Z = [1, \infty) \times S^1$ . Then we can make the operator  $e^{-t\Delta_{a,0}}$  act on  $L^2(M, dA_g)$  in the following way:

$$e^{-t\Delta_{a,0}} f(z) = \int_a^\infty \int_{S^1} p_a(y, y', t) f|_{Z_a}(y', x') dx' \frac{dy'}{y'^2} \text{ for } z = (y, x) \in Z_a,$$

and zero otherwise. From the symmetry of  $p_a(y, y', t)$ , is clear that the operator  $e^{-t\Delta_{a,0}}$  acting on  $L^2(M, dA_g)$  is symmetric.

Recall the operator  $\Delta_{Z,D}$  defined in Sect. 2.2. The kernel of the operator  $e^{-t\Delta_{Z,D}}$  is constructed by a classical method (see [6, Chap. VII]) and it is given by

$$K_{Z,D}((y, x), (y', x'), t) = K_1((y, x), (y', x'), t) + p_{1,D}((y, x), (y', x'), t), \tag{2.13}$$

where  $y, y' \geq 1, x, x' \in S^1, t > 0$ , and  $p_{1,D}((y, x), (y', x'), t)$  is a function that satisfies: for every  $\tau > 0$ , there exist constants  $C, c > 0$  such that

$$|p_{1,D}(z, z', t)| \leq Ct^{-1}(i(z)i(z'))^{1/2}e^{-\frac{c(d_g(z,\partial Z)+d_g(z',\partial Z))^2}{t}} \tag{2.14}$$

for all  $z, z' \in Z$  and  $0 < t < \tau$ .

Now let  $\Delta_{Z,h}$  be the self-adjoint extension of the operator

$$-e^{-2\varphi}y^2 \left( \partial_y^2 + \partial_x^2 \right) : C_c^\infty(Z) \rightarrow L^2(Z, dA_h)$$

obtained after imposing Dirichlet boundary conditions at  $\{1\} \times S^1$ . Let  $K_{Z,h}$  denote the kernel of the operator  $e^{-t\Delta_{Z,h}}$ . As in the case of the heat kernel associated to the operator  $\Delta_{Z,D}$ , given in (2.13), the kernel  $K_{Z,h}$  is given by

$$K_{Z,h}(z, z', t) = K_{1,h}(z, z', t) + p_{h,D}(z, z', t), \tag{2.15}$$

for  $z, z' \in Z$  and  $t > 0$  where the term  $p_{h,D}(z, z', t)$  is determined by the boundary condition. In the same way as above,  $p_{h,D}(z, z', t)$  satisfies, up to some constants, the same estimate as the one in Eq. (2.14).

### 2.5.4 Duhamel’s principle

There are several ways to state and use Duhamel’s principle, see, for example, [6, VII. 3].

Duhamel’s principle can be applied in the non-compact setting under certain assumptions on the decay of the functions. This is the case of the heat kernels on surfaces with cusps and asymptotically cusp ends. In terms of the operators, Duhamel’s principle can be stated as

$$T^{-1}e^{-t\Delta_h}T - e^{-t\Delta_g} = \int_0^t T^{-1}e^{-s\Delta_h}T(\Delta_g - T^{-1}\Delta_hT)e^{-(t-s)\Delta_g} ds. \tag{2.16}$$

## 3 Trace class property of relative heat operators

In this section, we prove Theorem 3.1, which says that the difference of the heat operators corresponding to the metrics  $g$  and  $h$  is trace class. As we know, none of the heat operators  $e^{-t\Delta_h}$  nor  $e^{-t\Delta_g}$  is trace class, which is the reason why we consider their difference. This is the first step to define the relative determinant of the pair  $(\Delta_h, \Delta_g)$ .

In the second part, we consider other relative heat traces that are naturally associated to a surface with cusps.

### 3.1 Trace class property

Let  $(M, g), M_0, Z$  as well as  $\Delta_g, \Delta_{Z,D}$ , and  $\Delta_1$  be as in Sect. 2. For simplicity, we assume that  $M$  has only one cusp so it can be decomposed as  $M = M_0 \cup Z$  with  $M_0$  compact and  $Z = [1, \infty) \times S^1$ .

**Theorem 3.1** *Let  $h = e^{2\varphi}g$ , and assume that on the cusp  $Z$  the functions  $\varphi(y, x), |\nabla_g\varphi(y, x)|$  and  $\Delta_g\varphi(y, x)$  are  $O(y^{-\alpha})$  with  $\alpha > 0$ , as  $y \rightarrow \infty$ . Let  $T$  be the unitary map defined in Eq. (2.3). Then for any  $t > 0$  the operator*

$$T^{-1}e^{-t\Delta_h}T - e^{-t\Delta_g}$$

is trace class.

To prove this statement, we follow a procedure similar to that used by Müller and Salomonsen in [15]. We use Duhamel’s principle which was stated in Sect. 2.5.4.

Let  $\|\cdot\|$  denote the operator norm and  $\|\cdot\|_{1,g}$ , ( $\|\cdot\|_{1,h}$ , resp.), denote the trace norm in  $L^2(M, dA_g)$ , (in  $L^2(M, dA_h)$ , resp.). From Eq. (2.16), we have

$$\begin{aligned} \|T^{-1}e^{-t\Delta_h}T - e^{-t\Delta_g}\|_{1,g} &\leq \int_0^{t/2} \|(\Delta_g - T^{-1}\Delta_hT)e^{-(t-s)\Delta_g}\|_{1,g} ds \\ &\quad + \int_{t/2}^t \|e^{-s\Delta_h}(T\Delta_gT^{-1} - \Delta_h)\|_{1,h} ds \end{aligned} \tag{3.1}$$

When considering the trace of the operator on the right-hand side of (2.16) as an integral using heat kernels and their estimates, one has to take two aspects into account. One is related with the time singularity at  $t = 0$  and the other one is related with the convergence of the space integral. The idea of breaking up the integral in Eq. (3.1) comes from the need to avoid the time singularities coming from the heat kernel  $K_h(z, z', s)$  ( $K_g(z, z', t - s)$ ) close to  $s = 0$  ( $t - s = t$ ) that do not integrate to something finite in a neighborhood of zero (of  $t$ ). Equation (3.1) reduces the proof of Theorem 3.1 to the following Proposition:

**Proposition 3.2** *Let  $0 < a < b < \infty$ , under the same conditions of Theorem 3.1 we have that for each  $t \in [a, b]$ , the operators*

$$(\Delta_g - T^{-1}\Delta_hT)e^{-t\Delta_g} \quad \text{and} \quad e^{-t\Delta_h}(T\Delta_gT^{-1} - \Delta_h)$$

are trace class and each trace norm is uniformly bounded on  $[a, b]$ .

*Proof* The proof follows in several steps. The idea is to decompose each operator as the product of two Hilbert–Schmidt (HS)<sup>4</sup> operators whose norms are uniformly bounded on  $t$  at the corresponding interval. To prove the HS property, we use that if  $R$  is an integral operator on  $M$  with kernel  $r$ , its HS norm is given by the  $L^2(M \times M)$ -norm of  $r$ . Let  $\alpha$  be as in the statement of Theorem 3.1, i.e.,  $\alpha$  denotes the decay of the conformal factor  $\varphi$ . Let  $\beta = \alpha/2$ , if  $\alpha \in (0, 1)$ , and  $\beta = 1/2$  if  $\alpha \geq 1$ ; so that  $0 < \beta \leq 1/2$ . Let us define an auxiliary function  $\phi$  that we will use repeatedly, such that  $\phi \in C^\infty(M)$  satisfies  $\phi > 0$  and

$$\phi(y, x) = y^{-\beta}, \quad (y, x) \in Z. \tag{3.2}$$

Let  $M_\phi$  and  $M_\phi^{-1}$  denote the operators multiplication by  $\phi$  and  $\phi^{-1}$ , respectively. The motivation to introduce the function  $\phi$  is the fact that the heat operator  $e^{-t\Delta_g}$  itself is not HS but when multiplied by  $\phi$  it becomes HS. The proof is given below.

**Step 1.** To proof the trace class property of  $(\Delta_g - T^{-1}\Delta_hT)e^{-t\Delta_g}$ , we write

$$(\Delta_g - T^{-1}\Delta_hT)e^{-t\Delta_g} = \left( (\Delta_g - T^{-1}\Delta_hT)e^{-(t/2)\Delta_g}M_\phi^{-1} \right) \circ \left( M_\phi e^{-(t/2)\Delta_g} \right),$$

and prove that for every  $t > 0$ ,  $(\Delta_g - T^{-1}\Delta_hT)e^{-t\Delta_g}M_\phi^{-1}$  and  $M_\phi e^{-t\Delta_g}$  are HS operators.

**Step 1.1.**  $(\Delta_g - T^{-1}\Delta_hT)e^{-t\Delta_g}M_\phi^{-1}$  is HS. Equation (2.4) implies:

<sup>4</sup> We use HS to abbreviate Hilbert–Schmidt.

$$(\Delta_g - T^{-1} \Delta_h T) e^{-t \Delta_g} M_\phi^{-1} = \left( (1 - e^{-2\varphi(z)}) \Delta_g \right) e^{-t \Delta_g} M_\phi^{-1} + e^{-2\varphi} \left( -2 \langle \nabla_g \varphi, \nabla_g \cdot \rangle_g + (\Delta_g \varphi + |\nabla_g \varphi|_g^2) \right) e^{-t \Delta_g} M_\phi^{-1}.$$

Let us start with the term  $((1 - e^{-2\varphi(z)}) \Delta_g) e^{-t \Delta_g} M_\phi^{-1}$ ; to prove that it is HS, we just need to prove that the following integral is finite:

$$\int_M \int_M |(1 - e^{-2\varphi(z)}) \Delta_{g,z} K_g(z, z', t) \phi(z')^{-1}|^2 dA_g(z) dA_g(z').$$

Let us use the decomposition of  $M$  as  $M = M_0 \cup Z$  to split the integral as

$$\begin{aligned} \int_M \int_M \dots dA_g(z) dA_g(z') &= \int_{M_0} \int_{M_0} \dots dA_g(z) dA_g(z') \\ &+ \int_{M_0} \int_Z \dots dA_g(z) dA_g(z') + \int_Z \int_{M_0} \dots dA_g(z) dA_g(z') + \int_Z \int_Z \dots dA_g(z) dA_g(z'). \end{aligned} \tag{3.3}$$

Now, we use the estimates of the derivatives of heat kernel  $K_g(z, z', t)$  given in (2.11), the fact that  $1 - e^{-2\varphi(z)}$  decays as  $y^{-\alpha}$  at infinity, and the definition of the function  $i(z)$  given in (2.7). To estimate the resulting integrals, we use the equations in Observation B.1. For simplicity, let us just write  $c$  instead of  $2c$  for the constant in the exponential factor of the estimates of the heat kernels.

For the first term in the sum in Eq. (3.3) which involves  $z \in M_0$  and  $z' \in M_0$  we have

$$\begin{aligned} &\int_{M_0} \int_{M_0} |(1 - e^{-2\varphi(z)}) \Delta_{g,z} K_g(z, z', t) \phi(z')^{-1}|^2 dA_g(z) dA_g(z') \\ &\ll \int_{M_0} \int_{M_0} t^{-4} e^{-\frac{c}{t} d_g^2(z, z')} dA_g(z) dA_g(z') \ll t^{-4}. \end{aligned}$$

For the second term in the sum in (3.3) which involves  $z' \in M_0$  and  $z \in Z$  we have:

$$\begin{aligned} &\int_{M_0} \int_Z |(1 - e^{-2\varphi(z)}) \Delta_{g,z} K_g(z, z', t) \phi(z')^{-1}|^2 dA_g(z) dA_g(z') \\ &\ll t^{-4} \int_{M_0} \int_{S^1} \int_1^\infty \frac{1}{y^{1+2\alpha}} e^{-\frac{c}{t} d_g^2((y,x), z')} dy dx dA_g(z') \ll t^{-4}. \end{aligned}$$

The third term in the sum in Eq. (3.3) involves variables  $z \in M_0$  and  $z' \in Z$ . In this case, we use that the Riemannian distance satisfies  $d_g(z, z') \geq d_g(\partial Z, z') \geq |\log(y')|$  from which we infer

$$\begin{aligned} & \int_Z \int_{M_0} |(1 - e^{-2\varphi(z)})\Delta_{g,z}K_g(z, z', t)\phi(z')^{-1}|^2 dA_g(z)dA_g(z') \\ & \ll \int_1^\infty \int_{S^1} \int_{M_0} y'^{1+2\beta} t^{-4} e^{-\frac{c}{t}d_g^2(z, (y', x'))} dA_g(z) dx' \frac{dy'}{y'^2} \\ & \ll t^{-4} \int_1^\infty e^{-\frac{c}{t}(\log(y'))^2} dy' = t^{-4} \int_0^\infty e^{-\frac{c}{t}u^2} e^u du \ll t^{-7/2} e^{t/c}. \end{aligned}$$

Finally, for the last term in the sum in (3.3) in which the variables  $z, z'$  lie in  $Z$  we have:

$$\begin{aligned} & \int_Z \int_Z |(1 - e^{-2\varphi(z)})\Delta_{g,z}K_g(z, z', t)\phi(z')^{-1}|^2 dA_g(z)dA_g(z') \\ & \ll t^{-4} \int_1^\infty \int_1^\infty y^{-1-2\alpha} y'^{-1+2\beta} e^{-\frac{c}{t}(\log(y/y'))^2} dy dy' \ll t^{-7/2} e^{\frac{t}{c}}, \end{aligned}$$

since  $\alpha > \beta$ . Thus, we obtain:

$$\|(1 - e^{-2\varphi})\Delta_g e^{-t\Delta_g} M_\phi^{-1}\|_2^2 \ll t^{-4} (1 + t^{1/2} e^{t/c}).$$

We proceed now with the operators  $e^{-2\varphi} \langle \nabla_g \varphi, \nabla_g \cdot \rangle_g e^{-t\Delta_g} M_\phi^{-1}$  and  $e^{-2\varphi} (\Delta_g \varphi + |\nabla_g \varphi|_g^2) e^{-t\Delta_g} M_\phi^{-1}$ . Their integral kernels are given by

$$\begin{aligned} & e^{-2\varphi(z)} \langle \nabla_{g,z} \varphi(z), \nabla_{g,z} K_g(z, z', t) \rangle_g \phi^{-1}(z'), \quad \text{and} \\ & e^{-2\varphi(z)} (\Delta_g \varphi(z) + |\nabla_{g,z} \varphi(z)|_g^2) K_g(z, z', t) \phi^{-1}(z'), \end{aligned}$$

respectively. For which we have, respectively, the following estimates:

$$\begin{aligned} & |e^{-2\varphi(z)} \langle \nabla_{g,z} \varphi(z), \nabla_{g,z} K_g(z, z', t) \rangle_g \phi^{-1}(z')|^2 \\ & \ll t^{-3} i(z) i(z') |\nabla_g \varphi(z)|^2 e^{-\frac{c}{t}d_g^2(z, z')} \phi^{-1}(z')^2, \quad \text{and} \\ & |e^{-2\varphi(z)} (\Delta_g \varphi(z) + |\nabla_{g,z} \varphi(z)|_g^2) K_g(z, z', t) \phi^{-1}(z')|^2 \\ & \ll t^{-2} (|\Delta_g \varphi(z)| + |\nabla_g \varphi(z)|_g^2)^2 i(z) i(z') e^{-\frac{c}{t}d_g^2(z, z')} \phi^{-1}(z')^2. \end{aligned}$$

We split the integrals on  $M \times M$  in the same way as in Eq. (3.3), and the integrals obtained are very similar to those carried out in the previous part for the operator  $(1 - e^{-2\varphi})\Delta_g e^{-t\Delta_g}$ . The main difference occurs in the power of  $t$ .

For the operator  $e^{-2\varphi} \langle \nabla_g \varphi, \nabla_g \cdot \rangle_g e^{-t\Delta_g} M_\phi^{-1}$ , we use the estimates in (2.10) and the decay of the function  $|\phi|$  at infinity.

Now, for the operator  $e^{-2\varphi} (\Delta_g \varphi + |\nabla_g \varphi|_g^2) e^{-t\Delta_g} M_\phi^{-1}$ , we use the estimate of the heat kernel given in Eq. (2.8) and the decay of the functions involving  $\varphi$ . Let us only show the integral on  $Z \times Z$ . For  $z \in Z$  we have  $(\Delta_g \varphi(z) + |\nabla_{g,z} \varphi(z)|_g^2)^2 \ll (y^{-\alpha} + y^{-2\alpha})^2 \ll y^{-2\alpha}$ . Then

$$\begin{aligned} & \int_Z \int_Z |e^{-2\varphi(z)} (\Delta_g \varphi(z) + |\nabla_{g,z} \varphi(z)|_g^2) K_g(z, z', t) \phi^{-1}(z')|^2 dA_g(z)dA_g(z') \\ & \ll t^{-2} \int_1^\infty \int_1^\infty y^{-1-2\alpha} y'^{-1+2\beta} e^{-\frac{c}{t}(\log(y/y'))^2} dy dy' \ll t^{-3/2} e^{\frac{t}{c}}. \end{aligned}$$

Thus, in the same way as above, we obtain

$$\begin{aligned} \|e^{-2\varphi} \langle \nabla_g \varphi, \nabla_g \cdot \rangle_g e^{-t\Delta_g} M_\phi^{-1} \|_2^2 &\ll t^{-3} (1 + t^{1/2} e^{t/c}), \text{ and} \\ \|e^{-2\varphi} (\Delta_g \varphi + |\nabla_g \varphi|_g^2) e^{-t\Delta_g} M_\phi^{-1} \|_2^2 &\ll t^{-2} (1 + t^{1/2} e^{t/c}). \end{aligned}$$

**Step 1.2.** The operator  $M_\phi e^{-t\Delta_g}$  is HS. To see this, we have to prove that the following integral is finite:

$$\int_M \int_M |\phi(z) K_g(z, z', t)|^2 dA_g(z) dA_g(z').$$

We decompose the integral as in Eq. (3.3), and proceed in the same way as above, using in this case the estimates of  $K_g(z, z', t)$  given in (2.8) and the definition of the functions  $\phi$  and  $i(z)$ . Again, for the sake of simplicity, we just write  $c$  instead of  $2c$  in the exponential factor of the heat estimates. The computations are very similar to those in the previous case.

The integrals over  $M_0 \times M_0$ ,  $M_0 \times Z$ , and  $Z \times M_0$  do not have any problem. As for the last term, whose variables  $z, z'$  lie in  $Z$ , we have

$$\begin{aligned} \int_Z \int_Z |\phi(z) K_g(z, z', t)|^2 dA_g(z') dA_g(z) &\ll \int_1^\infty \int_1^\infty y^{1-2\beta} y' t^{-2} e^{\frac{-c}{t} (\log(y/y'))^2} \frac{dy'}{y'^2} \frac{dy}{y^2} \\ &= t^{-2} \int_1^\infty \int_1^\infty y^{-1-2\beta} y'^{-1} e^{\frac{-c}{t} (\log(y/y'))^2} dy dy' \\ &\leq t^{-3/2} e^{c't}. \end{aligned} \tag{3.4}$$

Therefore,

$$\|M_\phi e^{-t\Delta_g}\|_2^2 \ll t^{-2} + t^{-3/2} e^{t/4c}.$$

In this way, we have that  $(\Delta_g - T^{-1} \Delta_h T) e^{-t\Delta_g}$  is a trace class operator and the trace norm satisfies:

$$\begin{aligned} \|(\Delta_g - T^{-1} \Delta_h T) e^{-t\Delta_g}\|_{1,g} &\leq \|(\Delta_g - T^{-1} \Delta_h T) e^{-(t/2)\Delta_g} M_\phi^{-1}\|_2 \cdot \|M_\phi e^{-(t/2)\Delta_g}\|_2 \\ &\ll (t^{-2} + t^{-3} + t^{-4})^{1/2} (1 + t^{1/2} e^{t/c})^{1/2} (t^{-2} + t^{-3/2} e^{t/c'})^{1/2}; \end{aligned}$$

the last expression is integrable for  $t$  in compact subsets of  $(0, \infty)$ .

**Step 2.** In this step, we prove that the operator  $e^{-t\Delta_h} (T \Delta_g T^{-1} - \Delta_h)$  is trace class. The proof is very similar to the proof for  $(\Delta_g - T^{-1} \Delta_h T) e^{-t\Delta_g}$  since the heat kernels satisfy the same estimates, and the metrics are quasi-isometric. Let us write

$$e^{-t\Delta_h} (T \Delta_g T^{-1} - \Delta_h) = (e^{-(t/2)\Delta_h} M_\phi) \circ (M_\phi^{-1} e^{-(t/2)\Delta_h} (T \Delta_g T^{-1} - \Delta_h)),$$

where  $\phi \in C^\infty(M)$  is as above. Then we have to prove that for every  $t > 0$ , the kernels of the operators  $e^{-t\Delta_h} M_\phi$  and  $M_\phi^{-1} e^{-t\Delta_h} (T \Delta_g T^{-1} - \Delta_h)$  are square integrable.

The operator  $M_\phi^{-1}e^{-t\Delta_h}(T\Delta_g T^{-1} - \Delta_h)$  is HS. First of all, let us consider the kernel of the operator  $e^{-t\Delta_h}(T\Delta_g T^{-1} - \Delta_h)$ . For  $f \in C_c^\infty(M)$ , we have that

$$\begin{aligned} (e^{-t\Delta_h}(T\Delta_g T^{-1} - \Delta_h)f)(z) &= \int_M K_h(z, z', t) \cdot (T\Delta_{g,z'}T^{-1} - \Delta_{h,z'})f(z')dA_h(z') \\ &= \int_M ((T\Delta_{g,z'}T^{-1} - \Delta_{h,z'})K_h(z, z', t)) \cdot f(z')dA_h(z'), \end{aligned}$$

since the operators  $T\Delta_{g,z'}T^{-1}$  and  $\Delta_h$  are symmetric on  $L^2(M, dA_h)$ . Now, let us use the equation

$$T\Delta_g T^{-1} - \Delta_h = (e^{2\varphi} - 1)\Delta_h - 2e^{2\varphi}\langle \nabla_h \varphi, \nabla_h \cdot \rangle_h + (\Delta_g \varphi - |\nabla_g \varphi|_g^2) \tag{3.5}$$

to write

$$\begin{aligned} M_\phi^{-1}(T\Delta_g T^{-1} - \Delta_h)e^{-t\Delta_h} &= M_\phi^{-1}e^{-t\Delta_h} \{ (e^{2\varphi} - 1)\Delta_h - 2e^{2\varphi}\langle \nabla_h \varphi, \nabla_h \cdot \rangle_h \\ &\quad + (\Delta_g \varphi - |\nabla_g \varphi|_g^2) \}. \end{aligned}$$

It follows that  $M_\phi^{-1}e^{-t\Delta_h}(T\Delta_g T^{-1} - \Delta_h)$  is HS if the following functions

- (1)  $\phi(z)^{-1}(e^{2\varphi}(z') - 1)\Delta_{h,z'}K_h(z, z', t)$ ,
- (2)  $\phi(z)^{-1}e^{2\varphi(z')}\langle \nabla_{h,z'}\varphi, \nabla_{h,z'}K_h \rangle_h$  and
- (3)  $\phi(z)^{-1}(\Delta_g \varphi(z') - |\nabla_{g,z'}\varphi|_g^2)K_h(z, z', t)$

are in  $L^2(M \times M, dA_h dA_h)$ .

We split again the integral in the same way as in Eq. (3.3) and use the estimates of the heat kernel  $K_h(z, z', t)$  and its derivatives given in Eqs. (2.9), (2.10), and (2.11). We also use that for any function  $f \in L^1(M, dA_h)$  we have

$$\int_M |f|dA_h \ll \int_M |f|dA_g.$$

For the first function listed above, the integrals are almost the same as the ones corresponding to the operator  $(1 - e^{-2\varphi})\Delta_g e^{-t\Delta_g} M_\phi^{-1}$ . Then,

$$\int_M \int_M |\phi(z)^{-1}(e^{2\varphi(z')} - 1)\Delta_{h,z'}K_h(z, z', t)|^2 dA_h(z)dA_h(z') \ll t^{-4} + t^{-7/2}e^{t/c}$$

for some constant  $c > 0$ .

Similarly, for the other two functions, we get bounds by  $t^{-3}(1 + t^{1/2}e^{t/c})$  and  $t^{-2}(1 + t^{1/2}e^{t/c})$ , respectively. Combining these estimates, we obtain

$$\|M_\phi^{-1}e^{-t\Delta_h}(T\Delta_g T^{-1} - \Delta_h)\|_2^2 \ll (t^{-4} + t^{-3} + t^{-2})(1 + t^{1/2}e^{t/c}).$$

In the same way as in Step 1.2, we can prove that  $e^{-t\Delta_h}M_\phi$  is HS with HS norm satisfying:

$$\|e^{-t\Delta_h}M_\phi\|_2^2 \ll t^{-2} \left( 1 + t^{1/2}e^{\frac{t}{c}} \right).$$

Finally, for the operator  $e^{-t\Delta_h}(T\Delta_g T^{-1} - \Delta_h)$ , we obtain

$$\begin{aligned} \|e^{-t\Delta_h}(T\Delta_g T^{-1} - \Delta_h)\|_{1,h} &\leq \|e^{-(t/2)\Delta_h}M_\phi\|_2 \cdot \|M_\phi^{-1}e^{-(t/2)\Delta_h}(T\Delta_g T^{-1} - \Delta_h)\|_2 \\ &\ll t^{-1} (t^{-4} + t^{-3} + t^{-2})^{1/2} (1 + t^{1/2}e^{t/c}) \end{aligned}$$

This expression is clearly integrable for  $t$  on compact subsets of  $(0, \infty)$ .

This finishes the proofs of Proposition 3.2 and Theorem 3.1. □

**Corollary 3.3** *Let  $\psi$  satisfy the same conditions as  $\varphi$  in the statement of Theorem 3.1. Then, for any  $t > 0$ , the operator  $\psi e^{-t\Delta_h}$  is trace class.*

*Proof* To prove this corollary, we follow the same method as above. Namely, we use the semigroup property of  $e^{-t\Delta_h}$  to decompose the operator  $\psi e^{-t\Delta_h}$  as

$$\psi e^{-t\Delta_h} = \psi e^{-(t/2)\Delta_h} M_{\phi^{-1}} M_{\phi} e^{-(t/2)\Delta_h},$$

where  $\phi$  is the function given by Eq. (3.2) and  $M_{\phi}$  denotes the multiplication operator by  $\phi$ . We already proved that the operators  $\psi e^{-t/2\Delta_h} M_{\phi^{-1}}$  and  $M_{\phi} e^{-t/2\Delta_h}$  are HS. □

### 3.2 Relative trace for other heat operators

In this section, we consider relative heat traces of some operators naturally associated to the surface with cusps.

**Proposition 3.4** *The operator  $e^{-t\Delta_g} - e^{-t\Delta_{Z,D}}$  is trace class for all  $t > 0$ , where  $e^{-t\Delta_{Z,D}}$  is considered as acting on  $L^2(M, dA_g)$ .*

This is a corollary of Proposition 6.4 in [12]. The statement of that proposition can be rewritten in our notation as follows:

Assume that  $M$  can be decomposed as  $M = M_0 \cup Z$  with  $Z = [1, \infty) \times S^1$ . Let  $P_0$  be the orthogonal projection of  $L^2(M, dA_g)$  onto  $L^2([1, \infty), y^{-2}dy)$ . Then for every  $t > 0$ ,  $e^{-t\Delta_g} - e^{-t\Delta_{1,0}} P_0$  is a trace class operator.

To see that Proposition 3.4 follows from this statement, recall what we explained in Sect. 2.2: the operator  $\Delta_{Z,D}$  can be decomposed as  $\Delta_{Z,D} = \Delta_{1,0} \oplus \Delta_{Z,1}$ , where the heat operator  $e^{-t\Delta_{Z,1}}$  is trace class. So, we have

$$\|e^{-t\Delta_g} - e^{-t\Delta_{Z,D}}\|_1 = \|e^{-t\Delta_g} - e^{-t\Delta_{1,0}}\|_1 + \|e^{-t\Delta_{Z,1}}\|_1$$

Now, let us consider the operator  $\Delta_{a,0}$  for  $a > 1$ . To see that  $e^{-t\Delta_g} - e^{-t\Delta_{a,0}}$  is trace class, we will proceed by writing the difference as

$$e^{-t\Delta_g} - e^{-t\Delta_{a,0}} = e^{-t\Delta_g} - e^{-t\Delta_{1,0}} + e^{-t\Delta_{a,0}} - e^{-t\Delta_{1,0}}.$$

By Proposition 3.4, the first difference is trace class, so it suffices to show that  $e^{-t\Delta_{a,0}} - e^{-t\Delta_{1,0}}$  is trace class.

**Proposition 3.5** *For any  $a > 1$  and  $t > 0$ , the operator  $e^{-t\Delta_{a,0}} - e^{-t\Delta_{1,0}}$  acting on  $L^2([1, \infty), y^{-2}dy)$  is trace class and the trace is given by*

$$\text{Tr}(e^{-t\Delta_{a,0}} - e^{-t\Delta_{1,0}}) = -\frac{1}{\sqrt{4\pi t}} e^{-t/4} \log(a).$$

As an operator on  $L^2([a, \infty), y^{-2}dy)$ , the trace is given by

$$\text{Tr}(e^{-t\Delta_{a,0}} - e^{-t\Delta_{1,0}}) = -\frac{e^{-t/4}}{\sqrt{4\pi}} \text{Erf}\left(\frac{\log(a)}{\sqrt{t}}\right),$$

where  $\text{Erf}(s) = \int_0^s e^{-v^2} dv$ .



*Proof* Let us just sketch the proof. For the complete proof, see [2].

We use the explicit expression of each heat kernel given by Eq. (2.12) to prove that, for each  $t > 0$ ,  $e^{-t\Delta_{a,0}} - e^{-t\Delta_{1,0}}$  is a HS operator. We prove this by direct computation, showing that the difference of the heat kernels is in  $L^2([1, \infty) \times [1, \infty), \frac{dy' dy}{y^2})$ . The computations are tiresome and involve functions of the form  $\exp\left(-\frac{\log(yy'/a^2)^2}{4t}\right)$  and  $\exp\left(-\frac{\log(y/y')^2}{2t}\right)$  that should be properly bounded.

The second step is to decompose the difference as the following sum:

$$e^{-t\Delta_{a,0}} - e^{-t\Delta_{1,0}} = e^{-(t/2)\Delta_{a,0}} M_\phi \cdot M_\phi^{-1} (e^{-(t/2)\Delta_{a,0}} - e^{-(t/2)\Delta_{1,0}}) + (e^{-(t/2)\Delta_{a,0}} - e^{-(t/2)\Delta_{1,0}}) M_\phi^{-1} \cdot M_\phi e^{-(t/2)\Delta_{1,0}},$$

where  $M_\phi$  is multiplication by the function  $\phi$  defined in Eq. (3.2) with  $\beta = 1/2$ . We then prove that each term is HS in a similar fashion as we did in Sect. 3.1.

Now, let us compute the trace:

$$\begin{aligned} \text{Tr}(e^{-t\Delta_{a,0}} - e^{-t\Delta_{1,0}}) &= \int_1^\infty (p_a(y, y, t) - p_1(y, y, t)) \frac{dy}{y^2} \\ &= \frac{e^{-t/4}}{\sqrt{4\pi t}} \int_a^\infty \left( e^{-(\log(y^2))^2/4t} - e^{-(\log(y^2) - \log(a^2))^2/4t} \right) \frac{dy}{y} \\ &\quad - \frac{e^{-t/4}}{\sqrt{4\pi t}} \int_1^a \left( 1 - e^{-(\log(y^2))^2/4t} \right) \frac{dy}{y} \\ &= -\frac{e^{-t/4}}{\sqrt{4\pi t}} \log(a). \end{aligned}$$

If we consider  $e^{-t\Delta_{a,0}} - e^{-t\Delta_{1,0}}$  as an operator acting on  $L^2([a, \infty), y^{-2}dy)$  we have that

$$\begin{aligned} \text{Tr}(e^{-t\Delta_{a,0}} - e^{-t\Delta_{1,0}}) &= \int_a^\infty (p_a(y, y, t) - p_1(y, y, t)) \frac{dy}{y^2} \\ &= -\frac{e^{-t/4}}{\sqrt{4\pi t}} \int_1^a e^{-(\log(y))^2/t} \frac{dy}{y}. \end{aligned}$$

□

*Remark 3.6* The trace of  $e^{-t\Delta_{a,0}} - e^{-t\Delta_{1,0}}$  as an operator on  $L^2([a, \infty), y^{-2}dy)$  has an asymptotic expansion for small values of  $t$ . This follows from Proposition 3.5 and the fact that  $\text{Erf}(x)$  has an expansion for  $x \gg 1$ . Taking into account only the first term, we have that  $\text{Erf}(x) = \frac{\sqrt{\pi}}{2} + O(x^{-1})$ , as  $x \rightarrow \infty$  from which we infer that

$$\text{Tr}(e^{-t\Delta_{a,0}} - e^{-t\Delta_{1,0}})_{L^2([a, \infty), y^{-2}dy)} = -\frac{1}{4} + O(\sqrt{t}) \quad \text{as } t \rightarrow 0.$$

*Remark 3.7* Let us study the case when the manifold  $M$  can be decomposed as  $M = M_0 \cup Z_a$  with  $a \geq 1$  and we want to compare the operators  $e^{-t\Delta_g}$  and  $e^{-t\Delta_{1,0}}$ . In this case, we could consider the operator  $e^{-t\Delta_{1,0}}$  acting on  $L^2(M, dA_g)$  in the way explained in Sect. 2.5.3. However, it is more convenient and accurate to consider the extended space:

$$\begin{aligned} L^2(M, dA_g) \oplus L^2([1, a], y^{-2}dy) &= L^2(M_0, dA_g) \oplus L^2_0(Z_a) \oplus L^2([a, \infty), y^{-2}dy) \\ &\quad \oplus L^2([1, a], y^{-2}dy) \\ &= L^2(M_0, dA_g) \oplus L^2_0(Z_a) \oplus L^2([1, \infty), y^{-2}dy), \end{aligned}$$

where  $L^2_0(Z_a)$  is the space defined in equation (2.2). Then the operators  $e^{-t\Delta_g}$  and  $e^{-t\Delta_{1,0}}$  act on the extended space by being null where they are not defined. In this way, we have that

$$\begin{aligned} &\text{Tr}(e^{-t\Delta_g} - e^{-t\Delta_{1,0}})_{L^2(M) \oplus L^2([1, a])} \\ &= \text{Tr}(e^{-t\Delta_g} - e^{-t\Delta_{a,0}})_{L^2(M)} + \text{Tr}(e^{-t\Delta_{a,0}} - e^{-t\Delta_{1,0}})_{L^2([1, \infty))}, \end{aligned} \tag{3.6}$$

where for the sake of simplicity we dropped the densities in the notation of the  $L^2$  spaces.

### 4 Asymptotics of relative heat traces for small time

In this section, we prove the existence of an asymptotic expansion in  $t$  of the relative heat trace  $\text{Tr}(T^{-1}e^{-t\Delta_h}T - e^{-t\Delta_g})$  for small time. More precisely, we prove that for any  $\nu \geq 1$ , there exists an expansion up to order  $\nu$  of the relative heat trace as  $t \rightarrow 0$ . By an expansion up to order  $\nu$ , we mean that the remainder term is an  $O(t^\nu)$ .

We give explicit conditions on the decay of the conformal factor and its derivatives that guarantee the existences of such expansion.

#### 4.1 Asymptotics for non-compactly supported perturbations

Let  $(M, g)$  be a swc. For the sake of simplicity, we assume that  $(M, g)$  has only one cusp  $Z \cong [1, \infty) \times S^1$  with the hyperbolic metric on it. We take  $g$  as the background metric on  $M$ . Let  $h = e^{2\varphi}g$ . To start with, let us assume that for  $(y, x) \in Z$ , the functions  $\varphi(y, x)$  and  $\Delta_g\varphi(y, x)$  are  $O(y^{-1})$  as  $y \rightarrow \infty$ .

Let  $n > 1$ , let us introduce the following notation:

$$M_n := M_0 \cup ([1, n] \times S^1), \quad Z'_n = [1, n] \times S^1, \quad Z_n = [n, \infty) \times S^1. \tag{4.1}$$

We start by constructing the kernel of a parametrix  $Q_h(z, w, t)$  of the heat operator associated to  $\Delta_h$  by patching together suitable heat kernels over  $Z'_3 = M_3 \cap Z = [1, 3] \times S^1$ . Let us consider the following kernels:

- $K_{1,h}(z, w, t)$ : the heat kernel of  $\Delta_{1,h}$  on the horn  $\tilde{Z} = \mathbb{R}^+ \times S^1$ , as was defined in Sect. 2.5.
- $K_{Z,h}(z, w, t)$ : the heat kernel for  $\Delta_{Z,h}$ , as defined in Sect. 2.5.3.  $K_{Z,h}$  is given by Eq. (2.15).
- For the compact part, we consider a closed manifold  $W$  containing  $M_2$  isometrically. Let  $\Delta_{W,h}$  be the Laplacian on  $W$  and  $K_{W,h}(z, w, t)$  be the kernel of the corresponding heat operator  $e^{-t\Delta_W}$ .

For any two constants  $1 < b < c$ , let  $\phi_{(b,c)}$  be a smooth function on  $[1, \infty) \times S^1$  that is constant in the second variable, is non-decreasing in the first variable, and satisfies  $\phi_{(b,c)}(y, x) = 0$  for  $y \leq b$ , and  $\phi_{(b,c)}(y, x) = 1$  for  $y \geq c$ . Let  $\psi_2 = \phi_{(\frac{5}{4}, 2)}$  and  $\psi_1 = 1 - \psi_2$ ; then  $\{\psi_1, \psi_2\}$  is a partition of unity on  $[1, 2] \times S^1$ . Let  $\varphi_2 = \phi_{(1, \frac{9}{8})}$  and  $\varphi_1 = 1 - \phi_{(\frac{5}{2}, 3)}$ , so that  $\varphi_i = 1$  on the support of  $\psi_i$ ,  $i = 1, 2$ . Extend these functions to  $M$  in the obvious way. Note that  $|\nabla_h\varphi_i(z)| \ll 1$  and  $|\Delta_h\varphi_i(z)| \ll 1$ , for  $i = 1, 2$ . For this choice of functions, we have that

- $\text{supp} \nabla_h \varphi_1 \subseteq [\frac{5}{2}, 3] \times S^1$ , and,  $\text{supp} \psi_1 \subseteq M_2$ .
- $\text{supp} \nabla_h \varphi_2 \subseteq [1, \frac{9}{8}] \times S^1$ , and,  $\text{supp} \psi_2 \subseteq [\frac{5}{4}, \infty) \times S^1$ .

Now, we put

$$Q_h(z, w, t) = \varphi_1(z)K_{W,h}(z, w, t)\psi_1(w) + \varphi_2(z)K_{1,h}(z, w, t)\psi_2(w). \tag{4.2}$$

From the properties of the heat kernels,  $K_{W,h}$  and  $K_{1,h}$ , and the construction of the gluing functions it is easy to see that  $Q_h(z, w, t) \rightarrow \delta_{w-z}$ , as  $t \rightarrow 0$ .

**Lemma 4.1** *There exist constants  $C \geq 0$  and  $c > 0$  such that*

$$\left| \left( \frac{\partial}{\partial t} + \Delta_{h,z} \right) Q_h(z, w, t) \right| \leq C e^{-c/t}, \quad \text{for } 0 < t \leq 1.$$

*Proof* We use the estimates of the heat kernels given by Eqs. (2.9), (2.10), and (2.11) as well as Theorem 3.1 and the equivalence of the geodesic distances  $d_g$  and  $d_h$ . From the definition of  $Q_h$  and the properties of the heat kernels, it follows that

$$\begin{aligned} \left| \left( \frac{\partial}{\partial t} + \Delta_{h,z} \right) Q_h(z, w, t) \right| &\ll |(\langle \nabla \varphi_1, \nabla_z K_{W,h} \rangle + (\Delta_h \varphi_1) K_{W,h}) \psi_1(w)| \\ &\quad + |(\langle \nabla \varphi_2, \nabla_z K_{1,h} \rangle + (\Delta_h \varphi_2) K_{1,h}) \psi_2(w)|. \end{aligned}$$

Note that  $\left| \left( \frac{\partial}{\partial t} + \Delta_{h,z} \right) Q_h(z, w, t) \right|$  has compact support in  $z$ . We consider the following terms separately:

$$\begin{aligned} S_1 &:= |(\langle \nabla \varphi_1, \nabla_z K_{W,h} \rangle + (\Delta_h \varphi_1) K_{W,h}) \psi_1(w)|, \\ S_2 &:= |(\langle \nabla \varphi_2, \nabla_z K_{1,h} \rangle + (\Delta_h \varphi_2) K_{1,h}) \psi_2(w)|. \end{aligned}$$

$S_1 = 0$  unless  $z \in \text{supp} \nabla \varphi_1$  and  $w \in \text{supp} \psi_1$ . In this case  $d_g(z, w) \geq \log(5/4)$ , then that taking  $c'_1 = c \log(5/4)$  we obtain:

$$\begin{aligned} S_1 &\leq (|\nabla \varphi_1(z)| |\nabla_z K_{W,h}(z, w, t)| + |\Delta_h \varphi_1(z)| |K_{W,h}(z, w, t)|) \chi_{\text{supp} \psi_1}(w) \\ &\ll t^{-3/2} e^{-cd_g^2(z,w)/t} + t^{-1} e^{-cd_g^2(z,w)/t} \ll e^{-c'_1/2t} \text{ for } t \in (0, 1]. \end{aligned}$$

In the same way as above,  $S_2 = 0$  unless  $z \in \text{supp} \nabla \varphi_2$  and  $w = (v, u) \in \text{supp} \psi_2 = [\frac{5}{4}, \infty) \times S^1$ . In this case  $d_g(z, w) \geq \log(v/(9/8)) \geq \log(10/9)$ . Therefore,

$$S_2 \ll v^{1/2} e^{-c(\log(8v/9))^2/2t} (t^{-3/2} + t^{-1}) e^{-c'_2/2t} \ll e^{-c'_2/4t},$$

where  $c'_2 = c \log(10/9)$ . This finishes the proof of the lemma. □

*Remark 4.2* Note that

$$\left. \left( \frac{\partial}{\partial t} + \Delta_{h,z} \right) Q_h(z, w, t) \right|_{w=z} = 0.$$

In order that the expression above does not vanish, we need that

$$d_g(z, w) \geq \min\{\log(5/4), \log(10/9)\} > 0.$$

We now prove that in the expression of asymptotic expansion of the relative heat trace, we can replace the heat kernel  $K_h$  by the parametrix  $Q_h$  defined above.

**Lemma 4.3** *There exist constants  $C \geq 0$  and  $c_3 > 0$  such that, for any  $0 < t \leq 1$ :*

$$\int_M |\mathcal{Q}_h(z, z, t) - K_h(z, z, t)| dA_h(z) \leq C e^{-\frac{c_3}{t}}.$$

*Proof* Applying Duhamel’s principle to the heat kernel  $K_h$  and the parametrix  $\mathcal{Q}_h$ , we obtain:

$$\mathcal{Q}_h(z, z', t) - K_h(z, z', t) = \int_0^t \int_M K_h(z, w, s) \left( \frac{\partial}{\partial t} + \Delta_{h,w} \right) \mathcal{Q}_h(w, z', t - s) dA_h(w) ds.$$

Remark 4.2 implies that

$$\begin{aligned} & \int_M |\mathcal{Q}_h(z, z, t) - K_h(z, z, t)| dA_h(z) \\ & \leq \int_0^t \int_M \int_M |K_h(z, w, s) \left( \frac{\partial}{\partial t} + \Delta_{h,w} \right) \mathcal{Q}_h(w, z, t - s)| dA_h(w) dA_h(z) ds \\ & = \int_0^t \left( \int_{M_2 [\frac{5}{2}, 3] \times S^1} \cdot dA_h(w) dA_h(z) + \int_{Z_{\frac{5}{4}} [1, \frac{9}{8}] \times S^1} \cdot dA_h(w) dA_h(z) \right) ds. \end{aligned}$$

The first integral on the right-hand side is bounded by

$$\begin{aligned} & \int_0^t \int_{M_2 [\frac{5}{2}, 3] \times S^1} i(z)^{1/2} s^{-1} e^{-\frac{c_2}{s}} e^{-\frac{c'}{t-s}} dA_h(w) dA_h(z) ds \\ & \ll \left( \int_0^t e^{-\frac{c_2}{2s}} e^{-\frac{c'}{t-s}} ds \right) \left( \int_{\frac{5}{2}}^3 \frac{dv}{v^2} \right) \ll t e^{-\frac{c_3}{t}} \ll e^{-\frac{c_3}{t}} \end{aligned}$$

since  $0 < t \leq 1$ .

For the second integral on the right-hand side above, recall that  $\text{supp } \psi_2 \subset [5/4, \infty) \times S^1$ .

Thus

$$\begin{aligned} & \int_0^t \int_{Z_{\frac{5}{4}} [1, \frac{9}{8}] \times S^1} |K_h(z, w, s) \left( \frac{\partial}{\partial t} + \Delta_{h,w} \right) \mathcal{Q}_h(w, z, t - s)| dA_h(w) dA_h(z) ds \\ & \ll \int_0^t \int_{\frac{5}{4}}^{\infty} \int_1^{\frac{9}{8}} y^{1/2} e^{-\frac{c_2}{2s}} e^{-\frac{c_1}{t-s}} \frac{dv}{v^2} \frac{dy}{y^2} ds \leq t e^{-\frac{c_3}{t}} \leq e^{-\frac{c_3}{t}}. \end{aligned}$$

□

Since the function  $e^{-2\varphi}$  is bounded, the derivatives of the gluing functions  $\varphi_1$  and  $\varphi_2$  with respect to the metric  $g$  satisfy the same bounds as the derivatives with respect to the metric  $h$ . Then we can perform the same construction for the kernel  $K_g(z, w, t)$  to replace it by  $\mathcal{Q}_g(z, w, t)$ .

The relative heat trace is given by

$$\text{Tr} (T^{-1}e^{-t\Delta_h}T - e^{-t\Delta_g}) = \int_M (K_h(z, z, t)e^{2\varphi(z)} - K_g(z, z, t)) dA_g(z).$$

Using Lemma 4.3, we obtain:

$$\left| \int_M (K_h(z, z, t)e^{2\varphi(z)} - K_g(z, z, t))dA_g(z) - \int_M (Q_h(z, z, t)e^{2\varphi(z)} - Q_g(z, z, t))dA_g(z) \right| \ll e^{-c_3/t}.$$

Therefore, we have to determine the asymptotic expansion of the integral:

$$\int_M Q_h(z, z, t)e^{2\varphi(z)} - Q_g(z, z, t)dA_g(z).$$

The definitions of  $Q_h$  and  $Q_g$  induce a natural decomposition of the integral into two regions of integration, the compact part and the cusp. However, when we use the local expansion of the heat kernel in the cusp, we need to integrate the remainder term uniformly. For this purpose, we decompose the cusp as in (4.1): Let  $a > 1$ , then

$$Z = Z'_a \cup Z_a.$$

Therefore, the integral decomposes as

$$\int_M Q_h(z, z, t)e^{2\varphi(z)} - Q_g(z, z, t)dA_g(z) = I_0(t) + I_1(t) + I_2(t),$$

where

$$I_0(t) = \int_M \psi_1(z)(K_{W,h}(z, z, t)e^{2\varphi(z)} - K_{W,g}(z, z, t))dA_g(z), \tag{4.3}$$

$$I_1(t) = \int_{Z'_a} \psi_2(z)(K_{1,h}(z, z, t)e^{2\varphi(z)} - K_{1,g}(z, z, t))dA_g(z), \tag{4.4}$$

$$I_2(t) = \int_{Z_a} \psi_2(z)(K_{1,h}(z, z, t)e^{2\varphi(z)} - K_{1,g}(z, z, t))dA_g(z). \tag{4.5}$$

For the moment we consider  $a$  fixed, but later we will assign to it a value depending on  $t$ .

The integral  $I_0$  has a complete asymptotic expansion in  $t$ . To see that, note that in the local expansions of the kernels  $K_{W,g}(z, z, t)$  and  $K_{W,h}(z, z, t)$  the corresponding remainder terms are uniformly bounded on compact sets, therefore, they can be integrated.

The other two integrals can be rewritten as traces of the operators:

$$A(t) = M_{\chi_{Z'_a}} M_{\psi_2} (T^{-1}e^{-t\Delta_{1,h}}T - e^{-t\Delta_{1,g}}) \text{ and}$$

$$B(t) = M_{\chi_{Z_a}} M_{\psi_2} (T^{-1}e^{-t\Delta_{1,h}}T - e^{-t\Delta_{1,g}}),$$

respectively. Propositions 4.4 and 4.5 below take care of these integrals.

**Proposition 4.4** *Under the conditions of Theorem 3.1, there is a complete asymptotic expansion as  $t \rightarrow 0$  of the integral  $I_1(t)$  in Eq. (4.3). The asymptotic expansion has the following form:*

$$\int_{[1,a] \times S^1} \psi_2(z)(K_{1,h}(z, z, t)e^{2\varphi(z)} - K_{1,g}(z, z, t)) dA_g(z) \sim t^{-1} \sum_{j=0}^{\infty} \hat{a}_j t^j.$$

The coefficients  $\hat{a}_j$  depend on the parameter  $a$ . There is a remainder term that also depends on  $a$  as  $O(e^{-\frac{c}{a^4 t}})$ , for a positive constant  $c$ .

*Proof* In order to deal with the integral  $I_1(t)$ , we first recall what  $K_{1,h}$  and  $K_{1,g}$  are. Recall that  $h$  was extended to the horn  $\tilde{Z}$  and that  $K_{1,h}(z, w, t)$  denotes the heat kernel for  $\Delta_h$  on  $\tilde{Z}$ . The idea of this proof is to use the local asymptotic expansion of the corresponding heat kernels and find a uniform bound on the remainder term.

The universal covering of  $\tilde{Z}$  is  $\hat{Z} = \mathbb{R}^+ \times \mathbb{R}$  with projection  $\pi : \hat{Z} \rightarrow \tilde{Z}$  and group of deck transformations  $\Gamma = \mathbb{Z}$ . The metric  $h$  on  $\tilde{Z}$  induces a metric  $\hat{h}$  on  $\hat{Z}$ , that has the same curvature properties as  $h$ . In addition,  $\hat{h} = e^{2\hat{\varphi}} \hat{g}_0$ , where  $\hat{g}_0$  is the lift of  $g_0$  to  $\hat{Z}$  and is precisely the hyperbolic metric on  $\mathbb{H}$ , and the function  $\hat{\varphi}$  is a lift of  $\varphi$  ( $\tilde{\varphi}$  the extension of  $\varphi$  to  $\tilde{Z}$ ),  $\hat{\varphi} = \tilde{\varphi} \circ \pi$ . It follows that  $\hat{h}$  and  $\hat{g}_0$  are quasi-isometric. Therefore, by Proposition 2.1 in [15], the injectivity radius of  $\hat{h}$  is bounded from below by a positive constant independent of the point. In this way,  $(\hat{Z}, \hat{h})$  has bounded geometry. Let  $k_h$  denote the heat kernel of  $\Delta_{\hat{h}}$  in  $\hat{Z}$ . It satisfies the following estimate:

$$k_h(\tilde{z}, \tilde{w}, t) \leq Ct^{-1} e^{-\frac{c d^2(\tilde{z}, \tilde{w})}{t}}, \tag{4.6}$$

where  $\tilde{z}, \tilde{w} \in \hat{Z}$  and  $0 < t \leq 1$ , [8]. It is not difficult to verify that

$$K_{1,h}(z, w, t) = \sum_{m \in \mathbb{Z}} k_h(\tilde{z}, \tilde{w} + m, t), \tag{4.7}$$

where  $\pi(\tilde{z}) = z, \pi(\tilde{w}) = w$ .

The construction above can be performed for the kernel  $K_{1,g}$  as well. Then the integral  $I_1(t)$  becomes:

$$\int_1^a \int_0^1 \tilde{\psi}_2(\tilde{z}) \left( \sum_{m \in \mathbb{Z}} k_h(\tilde{z}, \tilde{z} + m, t) e^{2\hat{\varphi}(\tilde{z}+m)} - \sum_{l \in \mathbb{Z}} k_g(\tilde{z}, \tilde{z} + l, t) \right) dA_{\hat{g}}(\tilde{z}),$$

because  $F = \mathbb{R}^+ \times [0, 1]$  is a fundamental domain for  $\Gamma$  and the domain corresponding to  $Z'_a$  in  $F$  is  $[1, a] \times [0, 1]$ ; and  $\tilde{\psi}_2$  is the natural extension and lift of  $\psi_2$  to  $\mathbb{H}$ . Thus

$$\begin{aligned} I_1(t) &= \int_1^a \int_0^1 \tilde{\psi}_2(\tilde{z}) (k_h(\tilde{z}, \tilde{z}, t) e^{2\hat{\varphi}(\tilde{z})} - k_g(\tilde{z}, \tilde{z}, t)) dA_{\hat{g}}(\tilde{z}) \\ &\quad + \int_1^a \int_0^1 \tilde{\psi}_2(\tilde{z}) \sum_{m \neq 0} (k_h(\tilde{z}, \tilde{z} + m, t) e^{2\hat{\varphi}(\tilde{z}+m)} - k_g(\tilde{z}, \tilde{z} + m, t)) dA_{\hat{g}}(\tilde{z}). \end{aligned} \tag{4.8}$$

We will start by estimating the second term on the right-hand side of (4.8). Note that  $\hat{\varphi} = \tilde{\varphi} \circ \pi$  implies that the function  $e^{2\hat{\varphi}}$  is bounded. This, the fact that the metrics  $\hat{h}$  and  $\hat{g}$  are quasi-isometric and the estimate on the heat kernel  $k_h$  imply that

$$\sum_{m \neq 0} k_h(\tilde{z}, \tilde{z} + m, t) e^{2\hat{\varphi}(\tilde{z}+m)} \ll t^{-1} \sum_{m \neq 0} \exp\left(-\frac{c_1 d_{\hat{g}}^2(\tilde{z}, \tilde{z} + m)}{t}\right). \tag{4.9}$$

The explicit expression of the hyperbolic distance in the upper half plane gives

$$d_{\hat{g}}((\tilde{x}, \tilde{y}), (\tilde{x} + m, \tilde{y})) = \cosh^{-1} \left( 1 + \frac{m^2}{2\tilde{y}^2} \right).$$

If  $s \geq 1$ ,  $\cosh^{-1}(s) = \log(s + \sqrt{s^2 - 1})$ ; this implies

$$d_{\hat{g}}((\tilde{x}, \tilde{y}), (\tilde{x} + m, \tilde{y})) = \log \left( 1 + \frac{m^2}{2\tilde{y}^2} + \frac{|m|}{\tilde{y}} \sqrt{\frac{m^2}{4\tilde{y}^2} + 1} \right) \geq \log \left( 1 + \frac{m^2}{2\tilde{y}^2} \right).$$

For  $\tilde{y} = y \in [1, a]$ ,  $\log(1 + \frac{m^2}{2y^2}) \geq \log(1 + \frac{1}{2a^2})$ . Thus

$$e^{-\frac{c_1 d_{\hat{g}}^2(\tilde{z}, \tilde{z}+m)}{t}} \leq e^{-\frac{c_1 \log(1+1/2a^2)^2}{2t}} e^{-\frac{c_1 \log(1+m^2/2\tilde{y}^2)^2}{2t}}.$$

In addition,  $0 \leq s \leq 1$  satisfies  $\log(1 + s) \geq s/2$ . Applying this to  $s = (2a^2)^{-1}$  gives

$$\sum_{m \neq 0} e^{-\frac{c_1 d_{\hat{g}}^2(\tilde{z}, \tilde{z}+m)}{t}} \leq e^{-\frac{c_1}{2^5 a^4 t}} \sum_{m \neq 0} e^{-\frac{c_1 \log(1+\frac{m^2}{2\tilde{y}^2})^2}{2t}} \leq e^{-\frac{c_2}{a^4 t}} \sum_{m \neq 0} e^{-\frac{c_1 \log(1+\frac{m^2}{2a^2})^2}{2t}}, \tag{4.10}$$

with  $c_2$  a positive constant. In order to estimate the series, we compare it with an integral using the fact that  $\exp\left(-\frac{c_1 \log(1+\frac{m^2}{2a^2})^2}{2t}\right)$  is a decreasing function of  $m$ . We proceed in the following way:

$$\begin{aligned} \sum_{m \neq 0} e^{-\frac{c_1 \log(1+\frac{m^2}{2a^2})^2}{2t}} &\ll \int_1^{\infty} e^{-\frac{c_1 \log(1+\frac{u^2}{2a^2})^2}{2t}} du \\ &\leq \int_1^{\sqrt{2}a} e^{-\frac{c_1 \log(1+\frac{u^2}{2a^2})^2}{2t}} du + \int_{\sqrt{2}a}^{\infty} e^{-\frac{2c_1 \log(\frac{u}{\sqrt{2}a})^2}{t}} du \\ &\ll (\sqrt{2}a - 1) + a \int_0^{\infty} e^{-\frac{2c_1 v^2}{t}} e^v dv \ll a(1 + \sqrt{t}e^{ct}) \ll a, \end{aligned} \tag{4.11}$$

where for the integral on the right-hand side, we used the change of variables  $v = \log(\frac{u}{\sqrt{2}a})$ ; and in the middle step, we used that for  $x \geq 1$ ,  $(\log(x^2 + 1))^2 \geq (\log(x))^2$ . Now we can use (4.9) and the bounds above to estimate the second term on the right-hand side of equation (4.8):

$$\begin{aligned} &\int_1^a \int_0^1 |\tilde{\psi}_2(\tilde{z})| \sum_{m \neq 0} (k_h(\tilde{z}, \tilde{z} + m, t) e^{2\hat{\varphi}(\tilde{z}+m)} - k_g(\tilde{z}, \tilde{z} + m, t)) |dA_{\hat{g}}(\tilde{z}) \\ &\ll t^{-1} \int_1^a \int_0^1 |\tilde{\psi}_2(\tilde{z})| \sum_{m \neq 0} e^{-\frac{c_1 d_{\hat{g}}^2(\tilde{z}, \tilde{z}+m)}{t}} |dA_{\hat{g}}(\tilde{z}) \\ &\ll t^{-1} e^{-\frac{c_2}{a^4 t}} \int_1^a \sum_{m \neq 0} e^{-\frac{c_1 \log(1+\frac{m^2}{2a^2})^2}{2t}} \frac{dy}{y^2} \ll t^{-1} a e^{-\frac{c_2}{a^4 t}}. \end{aligned} \tag{4.12}$$

Let us remark that in Eq. (4.12), the right-hand side is a  $O(e^{-c/a^4 t})$  as  $t \rightarrow 0$  with  $c > 0$ .

Now, let us denote the first term on the right-hand side of Eq. (4.8) by  $\tilde{I}_1(t)$ . The heat kernels  $k_h(\tilde{z}, \tilde{z}, t)$  and  $k_g(\tilde{z}, \tilde{z}, t)$  have a uniform local asymptotic expansion as  $t \rightarrow 0$  of the usual form:

$$k_*(\tilde{z}, \tilde{z}, t) = t^{-1} \sum_{k=0}^N a_k(\hat{*}, \tilde{z}) t^k + \mathcal{R}_N(\hat{*}, \tilde{z}, t), \text{ for any } N \geq 0, \tag{4.13}$$

where  $* = g, h$ . For the remainder terms, there is a constant  $C > 0$  such that

$$|\mathcal{R}_N(\hat{h}, \tilde{z}, t)| \leq Ct^N \text{ and } |\mathcal{R}_N(\hat{g}, \tilde{z}, t)| \leq Ct^N \tag{4.14}$$

independent of  $\tilde{z}$ . Replacing the corresponding expansion in  $\tilde{I}_1(t)$ , we obtain

$$\begin{aligned} \tilde{I}_1(t) &= \int_1^a \int_0^1 \tilde{\psi}_2(\tilde{z}) t^{-1} \left( \sum_{k=0}^N a_k(\hat{h}, \tilde{z}) e^{2\hat{\varphi}(\tilde{z})} - a_k(\hat{g}, \tilde{z}) \right) t^k dA_{\hat{g}}(\tilde{z}) \\ &\quad + \int_1^a \int_0^1 (\mathcal{R}_N(\hat{h}, \tilde{z}, t) e^{2\hat{\varphi}(\tilde{z})} - \mathcal{R}_N(\hat{g}, \tilde{z}, t)) dA_{\hat{g}}(\tilde{z}). \end{aligned} \tag{4.15}$$

Note that each integral converges separately since the integrands are bounded and the domain has finite area. So, strictly speaking, we do not need to consider relative objects in this part. However, when we take  $a = t^{-1/5}$  and we take the limit as  $t \rightarrow 0$ , the need of considering the relative integral becomes clear.

We estimate the integrals of the remainder terms using Eq. (4.14):

$$\begin{aligned} &\left| \int_1^a \int_0^1 \tilde{\psi}_2(\tilde{z}) (\mathcal{R}_N(\hat{h}, \tilde{z}, t) e^{2\hat{\varphi}(\tilde{z})} - \mathcal{R}_N(\hat{g}, \tilde{z}, t)) dA_{\hat{g}}(\tilde{z}) \right| \\ &\leq \int_1^a \int_0^1 (|\mathcal{R}_N(\hat{h}, \tilde{z}, t) e^{2\hat{\varphi}(\tilde{z})}| + |\mathcal{R}_N(\hat{g}, \tilde{z}, t)|) dA_{\hat{g}}(\tilde{z}) \ll t^N \int_1^\infty \frac{dy}{y^2} \ll t^N, \end{aligned} \tag{4.16}$$

for  $0 < t \leq 1$ . Note that this estimation is independent of  $a$ . This finishes the proof of Proposition 4.4.  $\square$

**Proposition 4.5** *Let  $\varphi|_Z(z)$ ,  $\Delta_g \varphi|_Z(z)$ , and  $|\nabla_g \varphi|_g|_Z(z)$  with  $z = (y, x)$ , be  $O(y^{-k})$  as  $y \rightarrow \infty$ , with  $k \geq 1$ . Then, for  $0 < t \leq 1$ , we have*

$$|I_2(t)| = |\text{Tr}(M_{\chi_{Z_a}} M_{\psi_2} (T^{-1} e^{-t\Delta_{1,h}} T - e^{-t\Delta_{1,g}}))| \ll a^{-k+1/2} t^{-3/2}. \tag{4.17}$$

*Proof* To prove Proposition 4.5, we want to apply Duhamel’s principle on the cusp  $Z$ . However, the heat operators involved in the trace correspond to Laplacians in the horn  $\tilde{Z}$ . Therefore, in order to make the computations easier, we first replace them by the heat operators  $e^{-t\Delta_{Z,h}}$  and  $e^{-t\Delta_{Z,g}}$  corresponding to the extensions of the Laplacians on the cusps with respect to Dirichlet boundary conditions. Then, we apply Duhamel’s principle to  $e^{-t\Delta_{Z,h}}$  and  $e^{-t\Delta_{Z,g}}$ . We have to take into account more terms, but we avoid the problem of the singularity at  $y = 0$ . Using Eqs. (2.13) and (2.15) to replace the respective kernels, we obtain

$$\begin{aligned} \text{Tr}(M_{\chi_{Z_a}} M_{\psi_2} (T^{-1} e^{-t\Delta_{1,h}} T - e^{-t\Delta_{1,g}})) &= \text{Tr}(M_{\chi_{Z_a}} M_{\psi_2} (T^{-1} e^{-t\Delta_{Z,h}} T - e^{-t\Delta_{Z,g}})) \\ &\quad - \int_M \chi_{Z_a}(z) \psi_2(z) (p_{h,D}(z, z, t) e^{2\varphi(z)} - p_{1,D}(z, z, t)) dA_g(z). \end{aligned}$$



From Eq. (2.14) and  $\text{supp}(\psi_2) = Z_{5/4}$ , it follows that

$$\begin{aligned} & \left| \int_M \psi_2(z) (p_{h,D}(z, z, t) e^{2\varphi(z)} - p_{1,D}(z, z, t)) dA_g(z) \right| \\ & \ll \int_{Z_{\frac{5}{4}}} t^{-1} y \left( e^{-\frac{cd_h(z, \partial Z)}{t}} + e^{-\frac{c'd_g(z, \partial Z)}{t}} \right) dA_g(z) \ll \int_{\frac{5}{4}}^{\infty} t^{-1} y e^{-\frac{c_1 \log(y)^2}{t}} \frac{dy}{y^2} \\ & \leq t^{-1} e^{-\frac{c_1 \log(5/4)^2}{2t}} \int_{\frac{5}{4}}^{\infty} y^{-1} e^{-\frac{c_1 \log(y)^2}{2t}} dy \ll e^{-\frac{c_1 \log(5/4)^2}{4t}}. \end{aligned}$$

Let us now continue with the estimation of the trace of the operator:

$$M_{\chi_{Z_a}} M_{\psi_2} (T^{-1} e^{-t\Delta_{Z,h}} T - e^{-t\Delta_{Z,g}}).$$

The kernel of  $T^{-1} e^{-t\Delta_{Z,h}} T - e^{-t\Delta_{Z,g}}$  is given by

$$e^{\varphi(z)} K_{Z,h}(z, w, t) e^{\varphi(w)} - K_{Z,g}(z, w, t),$$

and for  $z = w$  it takes the form  $K_{Z,h}(z, z, t) e^{2\varphi(z)} - K_{Z,g}(z, z, t)$ . From the usual form of Duhamel’s principle, we infer

$$\begin{aligned} & K_{Z,h}(z, w, t) e^{2\varphi(w)} - K_{Z,g}(z, w, t) \\ & = \int_0^t \int_M K_{Z,h}(z, z', s) e^{2\varphi(z')} (\Delta_{Z,g} - \Delta_{Z,h}) K_{Z,g}(z', w, t - s) dA_g(z') ds. \end{aligned}$$

Then taking  $z = w$  in the equation above and using the transformation of the Laplacian, we obtain

$$\begin{aligned} & \text{Tr} (M_{\chi_{Z_a}} M_{\psi_2} (T^{-1} e^{-t\Delta_{Z,h}} T - e^{-t\Delta_{Z,g}})) \\ & = \int_{Z_a} \psi_2(z) \int_0^t \int_Z \left\{ K_{Z,h}(z, z', s) e^{2\varphi(z')} (1 - e^{-2\varphi(z')}) \right. \\ & \quad \left. \Delta_{Z,g} K_{Z,g}(z', z, t - s) \right\} dA_g(z') ds dA_g(z). \end{aligned}$$

Recall that  $\text{supp}(\psi_2) = Z_{5/4}$ , let us first assume that  $a > 5/4$ , so  $4a/5 > 1$ . Split the integral as the sum of the following terms:

- (1)  $J_1 = \int_0^t \int_{Z_a} \int_{[1, \frac{4a}{5}] \times S^1} \dots dA_g(z') dA_g(z) ds.$
- (2)  $J_2 = \int_0^{t/2} \int_{Z_a} \int_{Z_{\frac{4a}{5}}} \dots dA_g(z') dA_g(z) ds.$
- (3)  $J_3 = \int_{t/2}^t \int_{Z_a} \int_{Z_{\frac{4a}{5}}} \dots dA_g(z') dA_g(z) ds.$

In this part, we only describe the main lines of the proof. The proof of the estimation of each integral is given in Appendix B2. The methods are very similar to the ones used to prove Theorem 3.1.

Let  $k \geq 1$  and suppose that  $\varphi(y, x) = O(y^{-k})$  as  $y \rightarrow \infty$ . Then so are  $\psi = 1 - e^{-2\varphi}$  and  $\tilde{\psi} = e^{2\varphi} - 1$ . Thus, for  $J_1$ , we have

$$J_1 = \int_0^t \int_{Z_a} \int_{[1, \frac{4a}{5}] \times S^1} \psi_2(z)(K_{1,h}(z, z', s) + p_{h,D}(z, z', s))e^{2\varphi(z')} \psi(z') \Delta_{Z,g}(K_{1,g}(z', z, t-s) + p_{1,D}(z', z, t-s)) dA_g(z') dA_g(z) ds. \tag{4.18}$$

On this region  $a \leq y < \infty$  and  $1 \leq y' \leq \frac{4a}{5}$ . Thus,  $1 < \frac{5}{4} \leq \frac{y}{y'}$ , so  $\log(y/y')$  is bounded away from 0. Using the estimates of the heat kernels, we obtain

$$|J_1| \ll a e^{-\frac{c'}{t}},$$

for some constants  $c' > 0$ .

For  $J_2$ , let us use that the variable  $z' \in Z_{\frac{4a}{5}}$  to multiply the inside integral by the characteristic function  $\chi_{Z_{\frac{4a}{5}}}(z')$ . Then,

$$J_2 = \int_0^{t/2} \int_{Z_a} \int_{Z_{\frac{4a}{5}}} \psi_2(z) K_{Z,h}(z, z', s) e^{2\varphi(z')} \chi_{Z_{\frac{4a}{5}}}(z') \psi(z') \Delta_{Z,g} K_{Z,g}(z', z, t-s) dA_g(z') dA_g(z) ds.$$

Writing this integral in terms of traces of the corresponding operators, we infer

$$\begin{aligned} |J_2| &= \left| \int_0^{t/2} \text{Tr}(M_{\psi_2} e^{-s\Delta_{Z,h}} M_{e^{2\varphi}} M_{\chi_{Z_{\frac{4a}{5}}}} M_{\psi} \Delta_{Z,g} e^{-(t-s)\Delta_{Z,g}}) ds \right| \\ &\ll \int_0^{t/2} \|M_{\chi_{Z_{\frac{4a}{5}}}} M_{\psi} \Delta_{Z,g} e^{-(t-s)\Delta_{Z,g}}\|_1 ds \\ &= \int_{t/2}^t \|M_{\chi_{Z_{\frac{4a}{5}}}} M_{\psi} \Delta_{Z,g} e^{-s\Delta_{Z,g}}\|_1 ds. \end{aligned}$$

To obtain a bound, we use a similar method as in Section 3.1. Let  $\phi$  be the auxiliary function defined by Eq.(3.2) with  $\beta = 1/2$ . Then the trace norm of the operator  $M_{\chi_{Z_{\frac{4a}{5}}}} M_{\psi} \Delta_{Z,g} e^{-s\Delta_{Z,g}}$  satisfies:

$$\|M_{\chi_{Z_{\frac{4a}{5}}}} M_{\psi} \Delta_{Z,g} e^{-s\Delta_{Z,g}}\|_1 \leq \|M_{\chi_{Z_{\frac{4a}{5}}}} M_{\psi} \Delta_{Z,g} e^{-s/2\Delta_{Z,g}} M_{\phi}^{-1}\|_2 \|M_{\phi} e^{-s/2\Delta_{Z,g}}\|_2.$$

The terms on the right-hand side can be estimated in a similar way as before to obtain

$$\begin{aligned} \|M_{\chi_{Z_{\frac{4a}{5}}}} M_{\psi} \Delta_{Z,g} e^{-s/2\Delta_{Z,g}} M_{\phi}^{-1}\|_2 &\ll s^{-7/4} (a^{-k} + a^{-k+1/2}), \\ \|M_{\phi} e^{-s/2\Delta_{Z,g}}\|_2 &\ll s^{-3/4}. \end{aligned}$$

It follows that

$$|J_2| \ll \int_{t/2}^t s^{-7/4} (a^{-k} + a^{-k+1/2}) \cdot s^{-3/4} ds \ll a^{-k+1/2} t^{-3/2}.$$

Now, for  $J_3$ , we proceed in a similar way as for  $J_2$  to obtain

$$|J_3| \ll \int_{t/2}^t a^{-k+1/2} s^{-7/4} s^{-3/4} ds \ll a^{-k+1/2} t^{-3/2},$$

see Appendix B for all the details. From all the equations above, we obtain

$$|\text{Tr}(M_{\psi_2}(T^{-1}e^{-t\Delta_{Z,h}}T - e^{-t\Delta_{Z,g}}))| \ll a^{-k+1/2}t^{-3/2} + ae^{-c'/t} \ll a^{-k+1/2}t^{-3/2},$$

for  $0 < t < 1$ . □

**Theorem 4.6** *Let  $\nu \geq 1$ . Write  $z \in Z$  as  $z = (y, x)$ . Let  $\varphi|_Z(z)$ ,  $\Delta_g\varphi|_Z(z)$ , and  $|\nabla_g\varphi|_g|_Z(z)$  be  $O(y^{-k})$  as  $y \rightarrow \infty$  with  $k \geq 5\nu + 8$ . In addition, if  $\nu \geq 3$  we require for  $2 \leq \ell \leq \nu$  that  $|\nabla^\ell\varphi|_g|_Z(z) = O(y^{-k})$  with  $k \geq 5(\nu - 2) - 1$ . Then under these conditions, there is an expansion of the relative heat trace of the form:*

$$\text{Tr}(T^{-1}e^{-t\Delta_h}T - e^{-t\Delta_g}) = t^{-1} \sum_{i=0}^{\nu} a_i t^i + O(t^\nu), \text{ as } t \rightarrow 0. \tag{4.19}$$

*Proof* The argument of the proof started above. To complete the proof, we need to put together the proofs of Propositions 4.4 and 4.5 in a consistent manner. First of all, we need to make all our estimates independent of  $a$ . In particular, the estimate of Eq. (4.12). This particular estimate is going to determine our result. In Eq. (4.12), the right-hand side is estimated by  $t^{-1}ae^{-\frac{c_2}{4t}}$ . Taking  $a = t^{-1/5}$ , we get  $a^4t = t^{1/5}$ . Therefore, Eq. (4.12) becomes:

$$\int_1^{t^{-1/5}} \int_0^1 |\tilde{\psi}_2(\tilde{z}) \sum_{m \neq 0} (k_h(\tilde{z}, \tilde{z} + m, t)e^{2\hat{\varphi}(\tilde{z}+m)} - k_g(\tilde{z}, \tilde{z} + m, t))| dA_{\hat{g}}(\tilde{z}) \ll e^{-\frac{c_2}{2t^{1/5}}}.$$

The next step is to make sure that the asymptotic expansion in Eq. (4.15) is kept when we pass to the limit as  $t \rightarrow 0$ . Before we continue with the asymptotics of  $I_1(t)$ , let us consider again the estimate of  $I_2(t)$  and replace  $a = t^{-1/5}$  in Eq. (4.17). In order to have

$$|\text{Tr}(M_{\psi_2}(T^{-1}e^{-t\Delta_{Z,h}}T - e^{-t\Delta_{Z,g}}))| \ll (t^{-1/5})^{-k+1/2}t^{-3/2} \ll t^\nu \tag{4.20}$$

with  $\nu \geq 1/2$  we need that  $\frac{k}{5} - \frac{1}{10} - \frac{3}{2} \geq \nu$ . Thus,  $k$  should satisfy  $k \geq 5\nu + 8$ . This condition applies to the conformal factor and its derivatives up to second order.

Now, let us go back to the asymptotics of  $I_1(t)$ . Let  $\nu \geq 1$ . Replacing  $a = t^{-1/5}$  in Eq. (4.15),  $\tilde{I}_1(t)$  becomes:

$$\begin{aligned} \tilde{I}_1(t) &= t^{-1} \int_1^{t^{-1/5}} \int_0^1 \tilde{\psi}_2(\tilde{z}) \sum_{j=0}^{\nu} t^j (e^{2\hat{\varphi}} a_j(\hat{h}, \tilde{z}) - a_j(\hat{g}, \tilde{z})) dA_{\hat{g}}(\tilde{z}) \\ &\quad + \int_1^{t^{-1/5}} \int_0^1 \tilde{\psi}_2(\tilde{z}) (e^{2\hat{\varphi}} \mathcal{R}_\nu(\hat{h}, \tilde{z}, t) - \mathcal{R}_\nu(\hat{g}, \tilde{z}, t)) dA_{\hat{g}}(\tilde{z}). \end{aligned} \tag{4.21}$$

The integral of the remainder terms was estimated in Eq. (4.16), independently of  $t$  and  $a$ . In what follows, we set  $\tilde{\psi}_2 = 1$  and drop the hat in  $\hat{\varphi}$ . To deal with the convergence of the integrals in the first term on the right-hand side in Eq. (4.21) we fix  $j$  and split each integral as follows:

$$\int_1^{t^{-1/5}} \int_0^1 (e^{2\varphi} a_j(\hat{h}) - a_j(\hat{g})) dA_{\hat{g}} = \int_1^\infty \int_0^1 (e^{2\varphi} a_j(\hat{h}) - a_j(\hat{g})) dA_{\hat{g}} - \int_{t^{-1/5}}^\infty \int_0^1 (e^{2\varphi} a_j(\hat{h}) - a_j(\hat{g})) dA_{\hat{g}}$$

Our goal is to prove that for each  $j$  the integral over  $[1, \infty) \times [0, 1]$  converges and that the integral over  $[t^{-1/5}, \infty) \times [0, 1]$  can be suitably estimated.

First of all, note that the region of integration  $[1, \infty) \times [0, 1]$  has finite area respect to both metrics  $\hat{g}$  and  $\hat{h}$ . Since  $\hat{g}$  is the hyperbolic metric on  $\mathbb{H}^2$ , the functions  $a_k(\hat{g}, \tilde{z})$  are bounded, therefore integrable. Let us describe the general picture. Our goal is to prove the following equation:

$$\begin{aligned} \tilde{I}_1(t) &= t^{-1} \sum_{j=0}^{\nu} t^j \int_1^\infty \int_0^1 (e^{2\varphi} a_j(\hat{h}, \tilde{z}) - a_j(\hat{g}, \tilde{z})) dA_{\hat{g}}(\tilde{z}) \\ &\quad - t^{-1} \sum_{j=0}^{\nu} t^j \int_{t^{-1/5}}^\infty \int_0^1 (e^{2\varphi} a_j(\hat{h}, \tilde{z}) - a_j(\hat{g}, \tilde{z})) dA_{\hat{g}}(\tilde{z}) + O(t^\nu) \\ &= \sum_{j=0}^{\nu} (t^{-1} t^j \tilde{a}_j + O(t^\nu)) + O(t^\nu) = \sum_{j=0}^{\nu} t^{-1} t^j \tilde{a}_j + O(t^\nu), \end{aligned} \tag{4.22}$$

where the coefficients  $\tilde{a}_j$  are given by

$$\tilde{a}_j = \int_1^\infty \int_0^1 (e^{2\varphi} a_j(\hat{h}, \tilde{z}) - a_j(\hat{g}, \tilde{z})) dA_{\hat{g}}(\tilde{z}).$$

For each  $j$  with  $0 \leq j \leq \nu$ , we find conditions on the decay of  $\varphi$ , on the number of derivatives that should decay, and on the order of that decay such that the corresponding integral converges or is suitably estimated. At the end, we impose the strongest condition on  $\varphi$  and its derivatives coming from all the terms together.

At each level  $j$  (the subindex of the heat invariant), we assume that  $\varphi$  and its derivatives (we will see each time how many derivatives we need) decay as  $y^{-k}$  at infinity, then we find  $k$  in terms of  $\nu$  and  $j$ .

Let us proceed with the analysis of the heat invariants. We analyze the convergence and estimation of the integrals simultaneously.

For  $a_0$ , we have

$$\int_1^\infty \int_0^1 (e^{2\varphi} - 1) dA_{\hat{g}} = A_{\hat{h}}([1, \infty) \times [0, 1]) - 1$$

and

$$t^{-1} \int_{t^{-1/5}}^{\infty} \int_0^1 |e^{2\varphi} - 1| dA_{\hat{g}} \ll \int_{t^{-1/5}}^{\infty} y^{-k} \frac{dy}{y^{-2}} = t^{-1} \frac{1}{k+1} t^{\frac{k+1}{5}}$$

In order to have  $t^{-1} t^{\frac{k+1}{5}} \leq t^\nu$ , we need  $\varphi$  to decay as  $k \geq 5\nu + 4$ .

For  $a_1$ , the integrals are

$$\begin{aligned} \int_1^{\infty} \int_0^1 (e^{2\varphi} R_{\hat{h}} - R_{\hat{g}}) dA_{\hat{g}} &= \int_1^{\infty} \int_0^1 ((\Delta_{\hat{g}}\varphi + R_{\hat{g}}) - R_{\hat{g}}) dA_{\hat{g}} \\ &= \int_1^{\infty} \int_0^1 \Delta_{\hat{g}}\varphi dA_{\hat{g}} \ll 1 \end{aligned}$$

and

$$\int_{t^{-1/5}}^{\infty} \int_0^1 |e^{2\varphi} \Delta_{\hat{g}}\varphi| dA_{\hat{g}} \ll \int_{t^{-1/5}}^{\infty} y^{-k} \frac{dy}{y^{-2}} = \frac{1}{k+1} t^{\frac{k+1}{5}}.$$

Here, we need  $\Delta_{\hat{g}}\varphi$  to decay as  $k \geq 5\nu - 1$ .

The second heat invariant  $a_2$  is given in [17] as  $a_2 = \frac{\pi}{60} \int_M R^2 dA$ . In our case, we obtain

$$\begin{aligned} \int_1^{\infty} \int_0^1 (e^{2\varphi} R_{\hat{h}}^2 - R_{\hat{g}}^2) dA_{\hat{g}} &= \int_1^{\infty} \int_0^1 e^{-2\varphi} (\Delta_{\hat{g}}\varphi + R_{\hat{g}})^2 - R_{\hat{g}}^2 dA_{\hat{g}} \\ &= \int_1^{\infty} \int_0^1 e^{-2\varphi} (\Delta_{\hat{g}}\varphi)^2 + e^{-2\varphi} (\Delta_{\hat{g}}\varphi) R_{\hat{g}} dA_{\hat{g}} \ll 1. \end{aligned}$$

For integral over  $[t^{-1/5}, \infty) \times [0, 1]$ , we have

$$t \int_{t^{-1/5}}^{\infty} \int_0^1 |e^{-2\varphi} (\Delta_{\hat{g}}\varphi)^2 + e^{-2\varphi} (\Delta_{\hat{g}}\varphi) R_{\hat{g}}| dA_{\hat{g}} \ll \frac{t^{\frac{2k+1}{5}+1}}{2k+1} + \frac{t^{\frac{k+1}{5}+1}}{k+1}$$

The left-hand side is bounded by  $t^\nu$  if  $\frac{k+1}{5} + 1 \geq \nu$ , i.e if  $k \geq 5\nu - 6$ . In this case  $\nu \geq 2$ , and we need two derivatives.

Now, let us go one step forward and consider the third heat invariant as it is given in [19]:

$$a_3 = \frac{1}{4\pi} \int_M -9|\nabla R|^2 + 4R^3 dA$$

Before we proceed, let us perform some computations:

$$\begin{aligned} \nabla_{\hat{h}} R_{\hat{h}} &= -2e^{-2\varphi}(\Delta_{\hat{g}}\varphi - 1)(\nabla_{\hat{h}}\varphi) + e^{-2\varphi}\nabla_{\hat{h}}(\Delta_{\hat{g}}\varphi) \\ |\nabla_{\hat{h}} R_{\hat{h}}|_{\hat{h}}^2 &= 4e^{-4\varphi}(\Delta_{\hat{g}}\varphi - 1)^2|\nabla_{\hat{h}}\varphi|_{\hat{h}}^2 - 4e^{-4\varphi}(\Delta_{\hat{g}}\varphi - 1)\langle\nabla_{\hat{h}}\varphi, \nabla_{\hat{h}}(\Delta_{\hat{g}}\varphi)\rangle_{\hat{h}} \\ &\quad + e^{-4\varphi}|\nabla_{\hat{h}}(\Delta_{\hat{g}}\varphi)|_{\hat{h}}^2 \\ R_{\hat{h}}^3 &= e^{-6\varphi}(\Delta_{\hat{g}}\varphi + R_{\hat{g}})^3 = e^{-6\varphi}((\Delta_{\hat{g}}\varphi)^3 - 3(\Delta_{\hat{g}}\varphi)^2 + 3(\Delta_{\hat{g}}\varphi) - 1) \end{aligned}$$

Plugging the expressions above in the integrals under consideration, we obtain:

$$\begin{aligned} &\int_1^\infty \int_0^1 e^{2\varphi} \left( -9|\nabla_{\hat{h}} R_{\hat{h}}|_{\hat{h}}^2 + 4R_{\hat{h}}^3 \right) - \left( -9|\nabla R_{\hat{g}}|^2 + 4R_{\hat{g}}^3 \right) dA_{\hat{g}} \\ &= 4 \int_1^\infty \int_0^1 e^{-4\varphi} \left( (\Delta_{\hat{g}}\varphi)^3 - 3(\Delta_{\hat{g}}\varphi)^2 + 3(\Delta_{\hat{g}}\varphi) \right) + (1 - e^{-4\varphi}) dA_{\hat{g}} \\ &\quad - 9 \int_1^\infty \int_0^1 e^{-4\varphi} \left\{ 4(\Delta_{\hat{g}}\varphi - 1)^2|\nabla\varphi|^2 - 4(\Delta_{\hat{g}}\varphi - 1)\langle\nabla\varphi, \nabla(\Delta_{\hat{g}}\varphi)\rangle + |\nabla(\Delta_{\hat{g}}\varphi)|^2 \right\} dA_{\hat{g}} \end{aligned}$$

Since  $|\nabla_{\hat{h}}\varphi|_{\hat{h}}^2 = e^{-2\varphi}|\nabla_{\hat{g}}\varphi|_{\hat{g}}^2$  and we drop the subindice when we consider the metric  $\hat{g}$ . In the first integral of the last equality, all functions decay at infinity. For convergence of the second integral, it is enough to require boundedness of the integrand, i.e.,  $|\nabla(\Delta\varphi)| \ll 1$ .

Now, we estimate the integrals on the region  $[t^{-\frac{1}{5}}, \infty) \times [0, 1]$ . As above, let us assume that  $|\nabla_{\hat{g}}^\ell\varphi| = O(y^{-k})$ , for  $0 \leq \ell \leq 3$  then  $|\Delta_{\hat{g}}\varphi - 1| \ll 1$ ,  $e^{-4\varphi} - 1 = O(y^{-k})$ , and

$$\begin{aligned} &t^2 \int_{t^{-1/5}}^\infty \int_0^1 |e^{2\varphi} \left( -9|\nabla R_{\hat{h}}|^2 + 4R_{\hat{h}}^3 \right) - \left( -9|\nabla R_{\hat{g}}|^2 + 4R_{\hat{g}}^3 \right)| dA_{\hat{g}} \\ &\ll t^2 \int_{t^{-1/5}}^\infty \int_0^1 (|\Delta_{\hat{g}}\varphi|^3 + |\Delta_{\hat{g}}\varphi|^2 + |\Delta_{\hat{g}}\varphi| + |1 - e^{-4\varphi}| \\ &\quad + |\nabla\varphi|^2 + |\nabla\varphi||\nabla(\Delta_{\hat{g}}\varphi)| + |\nabla(\Delta_{\hat{g}}\varphi)|^2) dA_{\hat{g}} \\ &\ll t^2 \int_{t^{-1/5}}^\infty (y^{-3k} + y^{-2k} + y^{-k}) \frac{dy}{y^2} \\ &= \frac{t^{\frac{3k+1}{5}+2}}{3k+1} + \frac{t^{\frac{2k+1}{5}+2}}{2k+1} + \frac{t^{\frac{k+1}{5}+2}}{k+1}. \end{aligned}$$

In the same way as in the previous case, we need that  $\frac{k+1}{5} + 2 \geq \nu$ . This is achieved if  $k \geq 5\nu - 11$ , ( $\nu \geq 3$ ).

General formulas for the coefficients in the expansion of the heat kernel are very complicated and only known explicitly for few of them. However, it is known that the functions  $a_k(\hat{h}, \tilde{\tau})$  are polynomials of degree  $2k$  in the scalar curvature  $(2R_{\hat{h}})$  and half powers of the Laplacian. The leading coefficients of this polynomials are described in [17] and in a more explicit form by Branson et al. in [3]. We refer to Lemma 1.3 and Eq. (1.4) in [3].

$$a_j(\Delta) = \int_M (j(j-1)c_j)|\nabla^{j-2}R|^2 + \text{polynomial}(R, \nabla R, \dots, \nabla^{j-3}R),$$

for  $j \geq 3$ . These are the heat coefficients for a closed Riemann surface (in [3]  $R$  denotes the scalar curvature). Applying this to our case, we require at least  $|\nabla_{\hat{h}}^{j-2}R_{\hat{h}}|$  to be bounded for  $0 \leq \ell \leq j - 2$ . In terms of the conformal factor, this condition translates to  $|\nabla^\ell \varphi| \ll 1$  for  $2 \leq \ell \leq j$ . Under these requirements, the integrals defining the coefficients  $\tilde{a}_j$  converge.

Now let us estimate the integral over  $[t^{-1/5}, \infty) \times [0, 1]$ , assuming that  $|\nabla^\ell \varphi| = O(y^{-k})$  for  $2 \leq \ell \leq j$ :

$$\begin{aligned} & \int_{t^{-1/5}}^\infty \int_0^1 (j(j-1)c_j) \left( e^{2\varphi} |\nabla_{\hat{h}}^{j-2}R_{\hat{h}}|^2 - |\nabla_{\hat{g}}^{j-2}R_{\hat{g}}|^2 \right) \\ & + e^{2\varphi} \text{polynomial} \left( R_{\hat{h}}, \nabla_{\hat{h}}R_{\hat{h}}, \dots, \nabla_{\hat{h}}^{j-3}R_{\hat{h}} \right) \\ & - \text{polynomial} \left( R_{\hat{g}}, \nabla_{\hat{g}}R_{\hat{g}}, \dots, \nabla_{\hat{g}}^{j-3}R_{\hat{g}} \right) dA_{\hat{g}}(\tilde{z}) \end{aligned} \tag{4.23}$$

If  $j \geq 3$ ,  $\nabla_{\hat{g}}^{j-2}R_{\hat{g}} = 0$ , therefore the leading term is of the form  $|\nabla_{\hat{h}}^{j-2}R_{\hat{h}}|^2 = O(y^{-2k})$ . Now, let us consider the terms involved in the polynomial. For that, we assume that the polynomial is of the form:

$$p_{2j}(x_1, \dots, x_r) = \sum a_{i_1 \dots i_r} x_1^{i_1} \dots x_r^{i_r},$$

then we have terms of the form:

$$e^{2\varphi} a_{i_1 \dots i_r} R_{\hat{h}}^{i_1} (\nabla_{\hat{h}}R_{\hat{h}})^{i_2} \dots (\nabla_{\hat{h}}^{j-3}R_{\hat{h}})^{i_r} - a_{i_1 \dots i_r} R_{\hat{g}}^{i_1} (\nabla_{\hat{g}}R_{\hat{g}})^{i_2} \dots (\nabla_{\hat{g}}^{j-3}R_{\hat{g}})^{i_r}$$

If  $i_j \neq 0$  for some  $j > 1$ , the second term vanishes. So we are left only with

$$e^{2\varphi} a_{i_1 \dots i_r} R_{\hat{h}}^{i_1} (\nabla_{\hat{h}}R_{\hat{h}})^{i_2} \dots (\nabla_{\hat{h}}^{j-3}R_{\hat{h}})^{i_r}$$

that involve at least one derivative of  $R_{\hat{h}}$ :  $\nabla_{\hat{h}}^\ell R_{\hat{h}} = (\nabla_{\hat{h}}^\ell(\Delta\varphi + R_{\hat{g}}))^{i_j} = (\nabla_{\hat{h}}^\ell(\Delta\varphi))^{i_j} = O(y^{-k \cdot i_j})$ .

If  $i_j = 0$  for all  $j > 1$ , we have terms of the form:

$$\begin{aligned} a_{i_1 \dots i_r} \left( e^{2\varphi} R_{\hat{h}}^{i_1} - R_{\hat{g}}^{i_1} \right) &= a_{i_1 \dots i_r} \left( e^{2(1-i_1)\varphi} (\Delta\varphi + R_{\hat{g}})^{i_1} - R_{\hat{g}}^{i_1} \right) \\ &= a_{i_1 \dots i_r} \left( e^{2(1-i_1)\varphi} \sum_{\ell=0}^{i_1} \binom{i_1}{\ell} (\Delta\varphi)^\ell (R_{\hat{g}})^{i_1-\ell} - R_{\hat{g}}^{i_1} \right) \end{aligned}$$

$$|a_{i_1 \dots i_r} (e^{2\varphi} R_{\hat{h}}^{i_1} - R_{\hat{g}}^{i_1})| \ll \left( \sum_{\ell=1}^{i_1} y^{-k\ell} \right) + (e^{2(1-i_1)\varphi} - 1) R_{\hat{g}}^{i_1},$$

and recall that  $1 - e^{-2\ell\varphi} = O(y^{-k})$ . Therefore,

$$t^{j-1} \int_{t^{-1/5}}^\infty \int_0^1 \left( e^{2\varphi} a_j(\hat{h}) - a_j(\hat{g}) \right) dA_{\hat{g}} \ll t^{j-1} t^{\frac{k+1}{5}},$$

the last term is bounded by  $t^\nu$  if  $k \geq 5\nu - 5(j - 1) - 1$ . We have finished the proof of Eq. (4.22).

It is interesting to see how, as we want to have more terms in the expansion, although more derivatives need to be considered, the conditions on their decay become weaker. However, this fact does not have any implication on our purposes of defining relative determinants. We could try to further refine the requirements to minimize conditions on  $\varphi$  but that will imply a deeper analysis of the heat invariants that is beyond the purpose of this article.  $\square$

**Corollary 4.7** *If the conformal factor  $\varphi$  and all its derivatives decay at infinity to infinite order, then there is a complete asymptotic expansion of the relative heat trace as  $t \rightarrow 0$ :*

$$\text{Tr} \left( T^{-1} e^{-t\Delta_h} T - e^{-t\Delta_g} \right) = t^{-1} \sum_{j=0}^{\infty} a_j t^j.$$

**Corollary 4.8** *Let  $h = e^{2\varphi} g$  with  $\varphi|_Z(z)$ ,  $\Delta_g \varphi|_Z(z)$ , and  $|\nabla_g \varphi|_g|_Z(z)$  be  $O(y^{-k})$  as  $y \rightarrow \infty$  with  $k \geq 11$ . Then the relative heat trace has an expansion of the form:*

$$\text{Tr} \left( T^{-1} e^{-t\Delta_h} T - e^{-t\Delta_g} \right) = a_0 t^{-1} + a_1 + O(\sqrt{t}) \text{ as } t \rightarrow 0. \tag{4.24}$$

We will see in Sect. 5.1 that this condition is sufficient to define the relative determinant.

*Proof* The condition  $k \geq 11$  comes from taking  $\nu = 1/2$  in Eq. (4.20). In the part corresponding to  $\tilde{I}_1(t)$ , we take  $\nu = 1$ . The heat invariants  $a_0$  and  $a_1$  require  $\varphi$  to decay at least as  $k = 9$  and  $\Delta\varphi$  to decay as  $k = 4$ . The strongest condition is then determined by  $I_2$ .  $\square$

To compute the coefficients in the expansion (4.19), we use that the coefficients in the local expansion of the heat kernels are given by universal functions. Taking  $\nu = 2$ , we have that

$$\begin{aligned} \text{Tr} \left( T^{-1} e^{-t\Delta_h} T - e^{-t\Delta_g} \right) &= \frac{t^{-1}}{4\pi} (A_h - A_g) + t \frac{\pi}{60} \left( \int_M R_h^2(z) dA_h(z) - \int_M R_g^2(z) dA_g(z) \right) \\ &+ O(t^2), \text{ as } t \rightarrow 0, \end{aligned} \tag{4.25}$$

where the constant term vanishes due to Gauss–Bonnet’s theorem. Equation (4.24) becomes:

$$\text{Tr} \left( T^{-1} e^{-t\Delta_h} T - e^{-t\Delta_g} \right) = \frac{t^{-1}}{4\pi} (A_h - A_g) + O(\sqrt{t}), \text{ as } t \rightarrow 0, \tag{4.26}$$

### 4.2 Asymptotics of other relative heat traces

Let us consider again surfaces with several cusps. Let  $(M, g)$  be a swc of genus  $p$  and with  $m$  cusps. Assume that  $M$  can be decomposed as  $M = M_0 \cup Z_{a_1} \cup \dots \cup Z_{a_m}$ , where  $a_i \geq 1$  for  $1 \leq i \leq m$ . Let  $\bar{\Delta}_{a,0}$  be the direct sum  $\bigoplus_{j=1}^m \Delta_{a_j,0}$  of the Dirichlet Laplacians  $\Delta_{a_j,0}$  defined in Definition 2.1.

Proposition 6.4 in [12] establishes that the operator  $e^{-t\Delta_g} - e^{-t\bar{\Delta}_{a,0}}$  is trace class and its trace has the following asymptotic expansion as  $t \rightarrow 0$ :

$$\begin{aligned} \text{Tr} \left( e^{-t\Delta_g} - e^{-t\bar{\Delta}_{a,0}} \right) &= \frac{A_g}{4\pi} t^{-1} + \left( \frac{\gamma m}{2} + \sum_{j=1}^m \log(a_j) \right) \frac{1}{\sqrt{4\pi t}} \\ &+ \frac{m \log(t)}{2 \sqrt{4\pi t}} + \frac{\chi(M)}{6} + \frac{m}{4} + O(\sqrt{t}), \end{aligned} \tag{4.27}$$



where  $\gamma$  is the Euler constant. A close examination of the proof of Eq.(4.27) in [12] shows that the term  $\sum_{j=1}^m \frac{\log(a_j)}{\sqrt{4\pi t}}$  can be replaced by  $e^{-t/4} \sum_{j=1}^m \frac{\log(a_j)}{\sqrt{4\pi t}}$ .

In particular, we can consider the relative determinant of the pair  $(\Delta_g, \bar{\Delta}_{1,0})$ . To that purpose we consider the trace  $\text{Tr}(e^{-t\Delta_g} - e^{-t\bar{\Delta}_{1,0}})$ , where the trace is taken in an extended  $L^2$  space that is given by

$$L^2(M, dA_g) \oplus \oplus_{j=1}^m L^2([1, a_j], y^{-2}dy) = L^2(M_0, dA_g) \oplus \oplus_{j=1}^m (L^2_0(Z_{a_j}) \oplus L^2([1, \infty), y^{-2}dy)). \tag{4.28}$$

Thus, using Proposition 3.5 and Eqs. (3.6) and (4.27) we obtain the following asymptotic expansion as  $t \rightarrow 0$ :

$$\text{Tr}(e^{-t\Delta_g} - e^{-t\bar{\Delta}_{1,0}}) = \frac{A_g}{4\pi} t^{-1} + \frac{\gamma m}{2} \frac{1}{\sqrt{4\pi t}} + \frac{m \log(t)}{2 \sqrt{4\pi t}} + \frac{\chi(M)}{6} + \frac{m}{4} + O(\sqrt{t}). \tag{4.29}$$

Together with Eq.(4.26), this gives

$$\begin{aligned} \text{Tr}(T^{-1}e^{-t\Delta_h}T - e^{-t\bar{\Delta}_{1,0}}) &= \frac{A_h}{4\pi} t^{-1} + \frac{\gamma m}{2} \frac{1}{\sqrt{4\pi t}} + \frac{m \log(t)}{2 \sqrt{4\pi t}} \\ &+ \frac{\chi(M)}{6} + \frac{m}{4} + O(\sqrt{t}), \end{aligned} \tag{4.30}$$

where the transformation  $T$  is the identity in the space  $\oplus_{j=1}^m L^2([1, a_j], y^{-2}dy)$ .

### 5 Relative determinants on surfaces with asymptotically cusp ends

#### 5.1 Definition

The relative determinant on a surface with hyperbolic cusps was already considered by Müller in [14]. Therefore, we restrict our attention to the definition and properties of the relative determinant on asymptotically hyperbolic surfaces. Let  $(M, g)$  be a swc and let  $h = e^{2\varphi}g$ . In order to define the relative determinant of the pairs  $(\Delta_h, \Delta_g)$ , and  $(\Delta_h, \Delta_{1,0})$ , we need to verify that the conditions given in Sect. 2.1 are satisfied. Let  $k \geq 1$ , let us define the following set of functions:

$$\mathcal{F}_k := \{\psi \in C^\infty(M) \mid \psi(z), |\nabla_g \psi|(z) \text{ and } \Delta_g \psi(z) \text{ are } O(i(z)^{-k}) \text{ as } y = i(z) \rightarrow \infty\}.$$

Sections 3.2 and 4 establish that the first and second conditions are fulfilled provided that  $\varphi \in \mathcal{F}_1$  and  $\varphi \in \mathcal{F}_{11}$ , respectively.

The third condition in Sect. 2.1 is about the behavior of the relative heat trace for big values  $t$ . The trace class property together with the fact that  $\sigma_{ac}(\Delta_{1,0}) = [1/4, \infty)$  and Lemma 2.22 in [14] give the existence of a constant  $C_1 > 0$  such that

$$\text{Tr}(T^{-1}e^{-t\Delta_h}T - e^{-t\Delta_{1,0}}) = 1 + O(e^{-C_1t}), \quad \text{as } t \rightarrow \infty, \tag{5.1}$$

where the value 1 on the right-hand side comes from  $\dim \ker \Delta_h - \dim \ker \Delta_{1,0}$  and the trace is taken in  $L^2(M, dA_g)$ . This condition is satisfied even when  $\varphi \in \mathcal{F}_1$ .

Let us prove that the condition  $\varphi \in \mathcal{F}_{11}$  suffices to define the relative determinant of  $(\Delta_h, \Delta_{1,0})$ . The relative zeta function  $\zeta(s; \Delta_h, \Delta_{1,0})$  converges on  $\text{Re}(s) > 1$ . It follows from the asymptotic expansions (4.30) and (5.1) that the function  $\zeta(s; \Delta_h, \Delta_{1,0})$  has a meromorphic continuation to the complex plane, that it is regular at  $s = 0$ . This continuation is denoted again by  $\zeta$ . The proof of the existence of the continuation and regularity at  $s = 0$  is classical in the literature. However, we include it here to remark that it is enough to have a truncated asymptotic expansion.

For the sake of simplicity, let us take  $m = 1$  and let us fix the notation in Eq. (4.30) above:

$$a_0 = \frac{A_h}{4\pi}, \quad a_{10} = \frac{\gamma}{4\sqrt{\pi}}, \quad a_{11} = \frac{1}{4\sqrt{\pi}}, \quad a_2 = \frac{\chi(M)}{6} + \frac{1}{4}.$$

Now, let us write  $\zeta(s; \Delta_h, \Delta_{1,0})$  as  $\zeta_1(s) + \zeta_2(s)$  with

$$\begin{aligned} \zeta_1(s) &:= \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} (\text{Tr}(T^{-1}e^{-t\Delta_h}T - e^{-t\Delta_{1,0}}) - 1) dt \quad \text{and} \\ \zeta_2(s) &:= \frac{1}{\Gamma(s)} \int_1^\infty t^{s-1} (\text{Tr}(T^{-1}e^{-t\Delta_h}T - e^{-t\Delta_{1,0}}) - 1) dt. \end{aligned}$$

Equation (5.1) implies that  $\zeta_2(s)$  is analytic at  $s = 0$ . As for  $\zeta_1(s)$  and  $\text{Re}(s) > 1$ , we have that:

$$\begin{aligned} \zeta_1(s) &= \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} (a_0t^{-1} + (a_{10} + a_{11} \log t)t^{-1/2} + a_2 - 1 + \vartheta(t))dt \\ &= \frac{1}{\Gamma(s)} \left( \frac{a_0}{s-1} + \frac{a_{10}}{s-1/2} - \frac{a_{11}}{(s-1/2)^2} + \frac{a_2-1}{s} + \vartheta_1(s) \right), \end{aligned}$$

where  $\vartheta(t) = O(\sqrt{t})$  and  $\vartheta_1(s)$  is a function that is analytic at  $s = 0$ .

Therefore, we can define the (regularized) relative determinant of  $(\Delta_h, \Delta_{1,0})$  as in Sect. 2.1:

$$\det(\Delta_h, \Delta_{1,0}) = \exp \left( -\frac{d}{ds} \zeta(s; \Delta_h, \Delta_{1,0}) \Big|_{s=0} \right).$$

Note that we only need to require that the function  $\varphi$  and its derivatives up to order two, have a decay of order 11 at infinity. The definition of  $\det(\Delta_h, \Delta_g)$  is done in the same way.

### 5.2 Polyakov’s formula for the relative determinant. Extremals

In [16], the authors proved that on compact surfaces, with and without boundary and under suitable restrictions, the regularized determinant of the Laplace operator has an extremum. In this section, we discuss the generalization of the extremal property of determinants given by OPS to certain cases of surfaces with asymptotically cusp ends. The main tool to study extremal properties of determinants is Polyakov’s formula that relates the determinant of a given metric to the determinant of a conformal perturbation of it. The formula obtained

here for relative determinants is the same as the one for regularized determinants on compact surfaces given in [16]. The proofs of the variational formula and of Polyakov’s formula follow the main lines of the corresponding proofs in [16] but we focus in the technical details that allow us to perform each step in the main proof.

### 5.2.1 Polyakov’s formula

In this section, we first consider  $\varphi, \psi \in \mathcal{F}_k$  with  $k \geq 11$  and  $u \in \mathbb{R}$ , let us define the family of metrics:

$$h_u := e^{2(\varphi+u\psi)} g = e^{2u\psi} h.$$

The corresponding Laplace operators and area elements are given by the equations:

$$\Delta_u := \Delta_{h_u} = e^{-2u\psi} \Delta_h, \quad dA_u := dA_{h_u} = e^{2u\psi} dA_h.$$

Let us consider the family of unitary maps given by

$$T_u : L^2(M, dA_u) \rightarrow L^2(M, dA_h), \quad f \mapsto f e^{u\psi},$$

and the following functional:

$$F : \mathcal{F}_k \rightarrow \mathbb{C}, \quad \psi \mapsto F_s(\varphi + u\psi) := \zeta(s; \Delta_u, \Delta_{1,0}),$$

$$\zeta(s; \Delta_u, \Delta_{1,0}) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} (\text{Tr}(T_u e^{-t\Delta_u} T_u^{-1} - T e^{-t\Delta_{1,0}} T^{-1}) - 1) dt,$$

where the trace is taken in  $L^2(M, dA_h)$ . The variation of  $\zeta$  at  $\varphi$  in the direction of  $\psi$  is defined as

$$\frac{\delta \zeta}{\delta \psi}(s; \Delta_h, \Delta_{1,0}) := \left. \frac{\partial}{\partial u} F_s(\varphi + u\psi) \right|_{u=0}.$$

In order to proceed with the computation of the derivative in the equation above, we need the following lemma:

**Lemma 5.1**

$$\left. \frac{d}{du} \text{Tr} (T_u e^{-t\Delta_u} T_u^{-1} - T e^{-t\Delta_{1,0}} T^{-1}) \right|_{u=0} = -t \text{Tr}(\dot{\Delta}_h e^{-t\Delta_h}),$$

where  $\dot{\Delta}_h \equiv \left. \frac{\partial}{\partial u} \Delta_u \right|_{u=0} = -2\psi \Delta_h$ .

*Proof* Let  $H_u = T_u \Delta_u T_u^{-1}$ . Then  $H_u$  is a family of self-adjoint operators acting on  $L^2(M, dA_h)$ . Note that  $e^{-tH_u} = T_u e^{-t\Delta_u} T_u^{-1}$ . It is also clear that

$$\frac{d}{du} \text{Tr}(T_u e^{-t\Delta_u} T_u^{-1} - T e^{-t\Delta_{1,0}} T^{-1}) = \text{Tr} \left( \frac{d}{du} e^{-tH_u} \right).$$

Let  $u_1, u_2 > 0$ , with  $u_1 > u_2$ . Let us apply Duhamel’s principle in terms of the operators:

$$e^{-tH_{u_1}} - e^{-tH_{u_2}} = \int_0^t -e^{-sH_{u_1}} H_{u_1} e^{-(t-s)H_{u_2}} + e^{-sH_{u_1}} H_{u_2} e^{-(t-s)H_{u_2}} ds.$$

Dividing by  $u_1 - u_2$  the previous equation and letting  $u_2 \rightarrow u_1$ , we obtain

$$\frac{d}{du} e^{-tH_u} \Big|_{u=u_1} = - \int_0^t e^{-sH_{u_1}} \left( \frac{d}{du} H_u \Big|_{u=u_1} \right) e^{-(t-s)H_{u_1}} ds.$$

Therefore, we get

$$\frac{d}{du} \text{Tr} (T_u e^{-t\Delta_u} T_u^{-1} - T e^{-t\Delta_{1,0}} T^{-1}) = -t \text{Tr} (\dot{H}_u e^{-tH_u}). \tag{5.2}$$

Let us compute the derivative  $\dot{H}_u$ :

$$\frac{d}{du} H_u = \psi T_u \Delta_u T_u^{-1} + T_u \left( \frac{d}{du} \Delta_u \right) T_u^{-1} - T_u \Delta_u \psi T_u^{-1}.$$

Thus, we get

$$\text{Tr} (\dot{H}_u e^{-tH_u}) = \text{Tr} (\psi \Delta_u e^{-t\Delta_u}) + \text{Tr} (\dot{\Delta}_u e^{-t\Delta_u}) - \text{Tr} (\Delta_u \psi e^{-t\Delta_u}).$$

From the rate of decay assumed for  $\psi$  and  $\Delta_g \psi$ , we have that the operators  $\psi e^{-t\Delta_u}$  and  $\Delta_u \psi e^{-t\Delta_u}$  are trace class. Using in addition that  $e^{-t\Delta_u} \Delta_u$  is bounded for all  $t > 0$ , we obtain

$$\text{Tr} (\Delta_u \psi e^{-t\Delta_u}) = \text{Tr} (e^{-\frac{t}{2}\Delta_u} \Delta_u \psi e^{-\frac{t}{2}\Delta_u}) = \text{Tr} (\psi e^{-t\Delta_u} \Delta_u) = \text{Tr} (\psi \Delta_u e^{-t\Delta_u}).$$

In this way, we get

$$\text{Tr} (\dot{H}_u e^{-tH_u}) = \text{Tr} (\dot{\Delta}_u e^{-t\Delta_u}) = -2 \text{Tr} (\psi \Delta_u e^{-t\Delta_u}).$$

Taking  $u = 0$  in the previous equation together with Eq. (5.2) implies the statement of the lemma. □

We are ready to compute the variation of the relative zeta function:

$$\begin{aligned} & \frac{\delta \zeta}{\delta \psi} (s; \Delta_h, \Delta_{1,0}) \\ &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{d}{du} (\text{Tr}(T_u e^{-t\Delta_u} T_u^{-1} - T e^{-t\Delta_{1,0}} T^{-1}) - 1) \Big|_{u=0} dt \\ &= \frac{-1}{\Gamma(s)} \int_0^\infty t^s \text{Tr}(-2\psi \Delta_h e^{-t\Delta_h}) dt = \frac{-2}{\Gamma(s)} \int_0^\infty t^s \frac{\partial}{\partial t} \text{Tr}(\psi e^{-t\Delta_h}) dt. \end{aligned}$$

Since

$$\frac{\partial}{\partial t} \psi e^{-t\Delta_h} = \frac{\partial}{\partial t} \psi (e^{-t\Delta_h} - P_{\ker(\Delta_h)}),$$

we have that

$$\frac{\delta \zeta}{\delta \psi} (s; \Delta_h, \Delta_{1,0}) = \frac{-2}{\Gamma(s)} \int_0^\infty t^s \frac{\partial}{\partial t} \text{Tr}(\psi (e^{-t\Delta_h} - P_{\ker(\Delta_h)})) dt. \tag{5.3}$$

In the classical proof of the variational formula of the spectral zeta function, the next step is to do integration by parts in Eq. (5.3). Before we do that, we have to verify the good decay

of  $\text{Tr}(\psi(e^{-t\Delta_h} - P_{\ker(\Delta_h)}))$  for big and small values of  $t$ . In addition, we need to make sure that we can obtain an expansion of the trace, for small values of  $t$ , whose remainder term can be integrated. We accomplish that in the following two lemmas:

**Lemma 5.2** *There exists a constant  $c > 0$  such that*

$$\text{Tr}(\psi(e^{-t\Delta_h} - P_{\ker(\Delta_h)})) = O(e^{-ct}), \text{ as } t \rightarrow \infty.$$

*Proof* Let  $t > 1$  and let us write

$$\psi(e^{-t\Delta_h} - P_{\ker(\Delta_h)}) = \psi e^{-\frac{1}{2}\Delta_h} (e^{-(t-\frac{1}{2})\Delta_h} - P_{\ker(\Delta_h)}),$$

where we used that  $e^{-\frac{1}{2}\Delta_h} P_{\ker(\Delta_h)} = P_{\ker(\Delta_h)}$ . By Corollary 3.3, we have that  $\psi e^{-\frac{1}{2}\Delta_h}$  is trace class. On the other hand, for  $f \in L^2(M, dA_h)$ , the spectral theorem implies that

$$e^{-t\Delta_h} f - P_{\ker(\Delta_h)} f = e^{-t(\Delta_h - P_{\ker(\Delta_h)})} f.$$

Note that  $\sigma_{\text{ess}}(\Delta_h) = [1/4, \infty)$  implies that 0 is an isolated eigenvalue of  $\Delta_h$  and  $\sigma(\Delta_h - P_{\ker(\Delta_h)}) \subseteq [c_1, \infty)$  for some  $c_1 \in (0, 1/4]$ . Thus

$$\|e^{-t(\Delta_h - P_{\ker(\Delta_h)})}\|_{L^2(M, h)} \leq e^{-c_1 t}$$

for any  $t > 0$ . If  $t > 1, t - \frac{1}{2} > 0$ ; therefore, the trace satisfies the desired estimate:

$$\begin{aligned} |\text{Tr}(\psi(e^{-t\Delta_h} - P_{\ker(\Delta_h)}))| &\leq \|\psi e^{-\frac{1}{2}\Delta_h} (e^{-(t-\frac{1}{2})\Delta_h} - P_{\ker(\Delta_h)})\|_1 \\ &\leq \|\psi e^{-\frac{1}{2}\Delta_h}\|_1 \|e^{-(t-\frac{1}{2})(\Delta_h - P_{\ker(\Delta_h)})}\|_{L^2(M, h)} \ll e^{-c_1 t}. \end{aligned}$$

This proves Lemma 5.2. □

**Lemma 5.3** *For  $0 < t \leq 1$ , the trace of the operator  $\psi(e^{-t\Delta_h} - P_{\ker(\Delta_h)})$  has the following expansion:*

$$\text{Tr}(\psi(e^{-t\Delta_h} - P_{\ker(\Delta_h)})) = \int_M \psi(z) \left( \frac{1}{4\pi t} + \frac{R_h(z)}{12\pi} - \frac{1}{A_h} \right) dA_h + O(t)$$

as  $t \rightarrow 0$ .

*Proof* In order to prove Lemma 5.3, we use a method similar to the one used in Sect. 4.1 to prove the existence of the expansion of the relative heat trace  $\text{Tr}(e^{-t\Delta_h} - e^{-t\Delta_g})$  for small  $t$ . We start by considering the parametrix kernel  $Q_h(z, z', t)$  defined by Eq. (4.2):

$$Q_h(z, w, t) = \varphi_1(z)K_{W, h}(z, w, t)\psi_1(w) + \varphi_2(z)K_{1, h}(z, w, t)\psi_2(w),$$

where the functions  $\varphi_i$  and  $\psi_i, i = 1, 2$ , are defined in Sect. 4.1. From Lemma 4.3, we can restrict our attention to  $\int_M \psi(z)(Q_h(z, z, t) - \frac{1}{A_h})dA_h(z)$  and split the integral as the sum of the following two terms:

$$\begin{aligned} L_1(t) &= \int_{M_2} \psi(z)\psi_1(z) \left( K_{W, h}(z, z, t) - \frac{1}{A_h} \right) dA_h(z) \\ L_2(t) &= \int_{Z_{\frac{5}{4}}} \psi(z)\psi_2(z) \left( K_{1, h}(z, z, t) - \frac{1}{A_h} \right) dA_h(z). \end{aligned}$$

Using the asymptotic expansion of the kernel  $K_{W,h}(z, z, t)$ , we obtain

$$L_1(t) = \int_{M_2} \psi(z)\psi_1(z) \left( \frac{1}{4\pi t} + \frac{R_h(z)}{12\pi} - \frac{1}{A_h} + \mathcal{R}_1(z, t) \right) dA_h(z). \tag{5.4}$$

For  $L_2(t)$ , we use the same construction and notation as in the proof of Proposition 4.4. Now, let  $a > 5/4$  and let us split the integral  $L_2(t)$  as the sum  $L_2 = \tilde{J}_1(t) + \tilde{J}_2(t) + \tilde{J}_3(t)$ , where the  $\tilde{J}_i, i = 1, 2, 3$ , are given by

$$\begin{aligned} \tilde{J}_1(t) &= \int_{\frac{5}{4}}^{\infty} \int_0^1 \hat{\psi}(\tilde{z})\hat{\psi}_2(\tilde{z}) \left( k_h(\tilde{z}, \tilde{z}, t) - \frac{1}{A_h} \right) dA_{\hat{h}}(\tilde{z}), \\ \tilde{J}_2(t) &= \int_{\frac{5}{4}}^a \int_0^1 \hat{\psi}(\tilde{z})\hat{\psi}_2(\tilde{z}) \sum_{m \neq 0} k_h(\tilde{z}, \tilde{z} + m, t) dA_{\hat{h}}(\tilde{z}), \\ \tilde{J}_3(t) &= \int_a^{\infty} \int_0^1 \hat{\psi}(\tilde{z})\hat{\psi}_2(\tilde{z}) \sum_{m \neq 0} k_h(\tilde{z}, \tilde{z} + m, t) dA_{\hat{h}}(\tilde{z}). \end{aligned}$$

For  $\tilde{J}_1$ , we use the local asymptotic expansion of the heat kernel  $k_h(\tilde{z}, \tilde{z}, t)$ , whose remainder term is uniformly bounded, see [7]:

$$\tilde{J}_1(t) = \int_{\frac{5}{4}}^{\infty} \int_0^1 \hat{\psi}(\tilde{z})\hat{\psi}_2(\tilde{z}) \left( \frac{1}{4\pi t} + \frac{R_{\hat{h}}(\tilde{z})}{12\pi} - \frac{1}{A_h} + \mathcal{R}_{1,1}(\tilde{z}, t) \right) dA_{\hat{h}}(\tilde{z}) \tag{5.5}$$

For  $\tilde{J}_2(t)$ , in the same way as in the proof of Proposition 4.4, we can estimate the series as in Eq. (4.10). Then we estimate the integral in the same way as in Eqs. (4.11) and (4.12):

$$\begin{aligned} \tilde{J}_2(t) &\ll \int_{\frac{5}{4}}^a y^{-11} e^{-\frac{c_2}{a^4 t}} \sum_{m \neq 0} e^{-\frac{c_1 \log(1 + \frac{m^2}{2a^2})^2}{2t}} \frac{dy}{y^2} \\ &\ll e^{-\frac{c_2}{a^4 t}} \int_{\frac{5}{4}}^a y^{-11} \int_1^{\infty} e^{-\frac{c_1 \log(1 + \frac{u^2}{2a^2})^2}{2t}} du \frac{dy}{y^2} \ll a e^{-\frac{c_2}{a^4 t}}. \end{aligned} \tag{5.6}$$

The integral  $\tilde{J}_3$  can be bounded as

$$\begin{aligned} \tilde{J}_3(t) &\leq \int_{Z_a} \psi(z)\psi_2(z) K_{1,h}(z, z, t) dA_h(z) \\ &\ll t^{-1} \int_a^{\infty} y^{-12} dy \ll t^{-1} a^{-11}. \end{aligned} \tag{5.7}$$

Taking  $a = t^{-1/5}$  in the same way as we did in the proof of Theorem 4.6 and putting Eqs. (5.4), (5.5), (5.6), and (5.7) together, we obtain:

$$\begin{aligned} & \text{Tr}(\psi(e^{-t\Delta_h} - P_{\ker(\Delta_h)})) \\ &= \int_{M_2} \psi(z)\psi_1(z) \left( \frac{1}{4\pi t} + \frac{R_h(z)}{12\pi} - \frac{1}{A_h} + \mathcal{R}_1(z, t) \right) dA_h(z) \\ &+ \int_{\frac{5}{4}}^{\infty} \int_0^1 \hat{\psi}(\tilde{z})\hat{\psi}_2(\tilde{z}) \left( \frac{1}{4\pi t} + \frac{R_{\hat{h}}(\tilde{z})}{12\pi} - \frac{1}{A_h} + \mathcal{R}_{1,1}(\tilde{z}, t) \right) dA_{\hat{h}}(\tilde{z}) + O(t), \end{aligned}$$

where  $O(t)$  is clearly independent of  $z$ . Now we know that  $|\mathcal{R}_1(z, t)| \ll t$  and  $|\mathcal{R}_{1,1}(\tilde{z}, t)| \ll t$  uniformly in  $z$ . Therefore, we can make the following estimate:

$$\int_{M_2} \psi(z)\psi_1(z)\mathcal{R}_1(z, t)dA_h(z) + \int_{\frac{5}{4}}^{\infty} \int_0^1 \hat{\psi}(\tilde{z})\hat{\psi}_2(\tilde{z})\mathcal{R}_{1,1}(\tilde{z}, t)dA_{\hat{h}}(\tilde{z}) \ll t.$$

This finishes the proof of Lemma 5.3. □

The rest of the proof now follows the same lines as in [16]. Let us mention the main steps of it. Going back to the variation of the relative zeta function, we may now apply integration by parts in Eq. (5.3) to obtain for  $\text{Re}(s) > 0$ :

$$\frac{\delta\zeta}{\delta\psi}(s; \Delta_h, \Delta_{1,0}) = \frac{2s}{\Gamma(s)} \int_0^{\infty} t^{s-1} \text{Tr}(\psi(e^{-t\Delta_h} - P_{\ker(\Delta_h)}))dt.$$

We split this integral as

$$\begin{aligned} \frac{\delta\zeta}{\delta\psi}(s; \Delta_h, \Delta_{1,0}) &= \frac{2s}{\Gamma(s)} \left( \int_0^1 t^{s-1} \text{Tr}(\psi(e^{-t\Delta_h} - P_{\ker(\Delta_h)}))dt \right. \\ &\quad \left. + \int_1^{\infty} t^{s-1} \text{Tr}(\psi(e^{-t\Delta_h} - P_{\ker(\Delta_h)}))dt \right). \end{aligned} \tag{5.8}$$

From Lemma 5.2, the integral in second term on the right-hand side of Eq. (5.8) is an entire function of  $s$ . Since  $\Gamma(s)^{-1} \sim s$ , it follows that

$$\frac{d}{ds} \frac{2s}{\Gamma(s)} \int_1^{\infty} t^{s-1} \text{Tr}(\psi(e^{-t\Delta_h} - P_{\ker(\Delta_h)}))dt \Big|_{s=0} = 0$$

Using Lemma 5.3, the first term on the right-hand side of (5.8), becomes:

$$\begin{aligned} & \frac{2s}{\Gamma(s)} \int_0^1 t^{s-1} \text{Tr}(\psi(e^{-t\Delta_h} - P_{\ker(\Delta_h)}))dt \\ &= \frac{2s}{\Gamma(s)} \left\{ \frac{1}{s} \int_M \psi(z) \left( \frac{R_h(z)}{12\pi} - \frac{1}{A_h} \right) dA_h + \text{analytic in } s \text{ near } 0 \right\}. \end{aligned}$$

The next step is to take the derivative with respect to  $s$  at  $s = 0$ . Using  $\frac{1}{\Gamma(s)} = s + O(s^2)$ , we have

$$\begin{aligned} & \frac{d}{ds} \frac{2s}{\Gamma(s)} \int_0^1 t^{s-1} \text{Tr}(\psi(e^{-t\Delta_h} - P_{\ker(\Delta_h)})) dt \Big|_{s=0} \\ &= \int_M 2\psi(z) \left( \frac{R_h(z)}{12\pi} - \frac{1}{A_h} \right) dA_h. \end{aligned}$$

Thus,

$$\begin{aligned} \frac{\delta}{\delta\psi} \log \det(\Delta_h, \Delta_{1,0}) &= -\frac{\delta}{\delta\psi} \frac{d}{ds} \zeta(s; \Delta_h, \Delta_0) \Big|_{s=0} \\ &= -\frac{1}{6\pi} \int_M \psi(\Delta_g \varphi + R_g) dA_g + \frac{\delta}{\delta\psi} \log A_h. \end{aligned} \tag{5.9}$$

Finally, it is very easy to show that any  $\psi$  in the domain of  $F$  satisfies:

$$\begin{aligned} \frac{1}{2} \frac{\partial}{\partial u} \int_M |\nabla_g(\varphi + u\psi)|^2 dA_g \Big|_{u=0} &= \langle \psi, \Delta_g \varphi \rangle, \\ \frac{\partial}{\partial u} \int_M R_g(\varphi + u\psi) dA_g \Big|_{u=0} &= \int_M R_g \psi dA_g, \end{aligned}$$

Integrating (5.9), we obtain

$$\log \det(\Delta_h, \Delta_{1,0}) = -\frac{1}{12\pi} \int_M |\nabla_g \varphi|^2 dA_g - \frac{1}{6\pi} \int_M R_g \varphi dA_g + \log A_h + C.$$

Notice that if  $\varphi = 0$ ,  $\Delta_h = \Delta_g$ . Therefore, the last equation implies  $C = \log \det(\Delta_g, \Delta_{1,0})$ . In this way, we have proved Polyakov’s formula:

**Theorem 5.4** *Let  $(M, g)$  be a surface with cusps and let  $h = e^{2\varphi}g$  be a conformal transformation of  $g$  with  $\varphi \in \mathcal{F}_{11}$ . For the corresponding relative determinants, we have the following formula:*

$$\begin{aligned} \log \det(\Delta_h, \Delta_{1,0}) &= -\frac{1}{12\pi} \int_M |\nabla_g \varphi|^2 dA_g - \frac{1}{6\pi} \int_M R_g \varphi dA_g \\ &\quad + \log A_h + \log \det(\Delta_g, \Delta_{1,0}). \end{aligned} \tag{5.10}$$

### 5.2.2 Extremal properties of the relative determinant

Given Polyakov’s formula for the relative determinant, the study of the extremal properties of it is exactly the same as in OPS [16] for the case when  $\chi(M) < 0$ . We assume now that  $\chi(M) < 0$ . Let us recall the analysis in [16] as we adapt it to our case. On  $\mathcal{F}_{11}$  consider the following functional:

$$\Phi(\varphi) = \frac{1}{2} \int_M |\nabla_g \varphi|^2 dA_g + \int_M R_g \varphi dA_g - \pi \chi(M) \log \left( \int_M e^{2\varphi} dA_g \right). \tag{5.11}$$



It is straightforward that  $\Phi$  is translation invariant and that minimizing  $\Phi$  is the same as maximizing  $\log \det(\Delta_h, \Delta_{1,0})$  for metrics of constant area. Since we are considering  $\chi(M) < 0$ , we have that  $\Phi$  is convex. In the same way as in [16], we have that

$$\Phi(\varphi) = -6\pi \log \det(\Delta_h, \Delta_{1,0}) + \pi(6 - \chi(M)) \log(A_h).$$

Let us drop the constraint  $A_h = 1$ . Then, if  $\varphi$  is a minimizer of  $\Phi$  the equation  $\frac{\delta\Phi}{\delta\psi}(\varphi) = 0$  holds for all  $\psi \in \mathcal{F}_{11}$ . This implies that

$$R_h = e^{-2\varphi}(\Delta_g\varphi + R_g) = \frac{2\pi\chi(M)}{\int_M e^{2\varphi} dA_g},$$

i.e.,  $R_h$  should be constant. If  $A_h = 2\pi(2p + m - 2)$ , it follows that  $R_h = -1$ , where  $p$  is the genus of  $M$  and  $m$  is the number of cusps.

On the other hand, if  $R_h = \text{constant}$  we have that

$$\begin{aligned} \frac{\delta\Phi}{\delta\psi}(\varphi) &= \int_M e^{2\varphi}\psi R_h dA_g - \frac{\pi\chi(M)}{A_h} \int_M 2\psi e^{2\varphi} dA_g \\ &= \int_M \frac{e^{2\varphi}\psi}{A_h} (R_h A_h - 2\pi\chi(M)) dA_g = 0, \end{aligned}$$

because of Gauss–Bonnet theorem. Thus, the critical points of  $\Phi$  are the metrics of constant curvature. The convexity of  $\Phi$  assures that the critical points are minima.

Our problem is to find a maximizer of the relative determinant among metrics inside the following conformal class:

$$\text{Conf}_{1,11}(g) = \{h|h = e^{2\psi}g, \text{ with } \psi \in \mathcal{F}_{11} \text{ and } A_h = 2\pi(2p + m - 2)\}.$$

If the initial metric  $g$  on  $M$  is a metric of negative constant curvature  $g = \tau$  with  $R_\tau = -1$ , and we take the conformal class  $\text{Conf}_{1,11}(\tau)$ ,  $\tau$  itself is the maximizer of the relative determinant and  $\tau \in \text{Conf}_{1,11}(\tau)$ . The maximizer trivially exists inside the conformal class. However, if the starting metric  $g$  on  $M$  is a metric that is hyperbolic only in the cusps, the differential equation for the curvature on the cusps is

$$-e^{2\varphi} = \Delta_g\varphi - 1.$$

This implies that in the cusps the function  $\varphi$  should decay at infinity as  $y^{-1}$ . In this case, the function  $\varphi$  is outside the conformal class under consideration. Therefore, in order to have a maximizer of the relative determinant inside the conformal class, we need to be able to define the relative determinant for Laplacians whose metrics have conformal factors  $e^{2\varphi}$  with  $\varphi$  having a decay as  $y^{-1}$  at infinity.

As it was mentioned in the introduction, in [1], Albin et al. consider renormalized determinants of Laplacians on more general surfaces that also include swac. In that case, the authors use Vaillant’s results in [20] to have an asymptotic expansion of the renormalized trace. The conditions on the conformal factor imposed by Vaillant are different to ours, but conformal factors that decay as  $y^{-1}$  at infinity are included. Then Ricci flow is used to prove existence of the maximizer.

We could use the fact that if an operator is trace class, its trace coincides with its renormalized trace. Thus, we could use Vaillant’s result to define our relative determinant in terms of the renormalized determinant of the Laplacian and of the one of our model operator. However, in the Ricci flow proof in [1], two different rescalings take place. When we consider relative

determinants, re-scaling implies to modify the model operator as well. This is an interesting open problem.

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### Appendix A

In this appendix, we give the proof of Lemma 2.4. We prove the estimate of  $K_{1,h}$ . The estimate of  $K_h$  then follows by a standard gluing parametrix construction. We use the notation introduced in Sect. 4.1 and Proposition 4.4. Let us recall Eq. (4.7):

$$K_{1,h}(z, w, t) = \sum_{m \in \mathbb{Z}} k_h(\tilde{z}, \tilde{w} + m, t),$$

where  $\pi(\tilde{z}) = z, \pi(\tilde{w}) = w$ , and  $\tilde{z} = (x_1, y_1)$  and  $\tilde{w} = (x_2, y_2)$  can be chosen so that  $0 \leq x_i \leq 1$ .

We know that  $d_h(z, w) = \inf_{m \in \mathbb{Z}} d_{\tilde{h}}(\tilde{z}, \tilde{w} + m) \leq d_{\tilde{h}}(\tilde{z}, \tilde{w} + m)$  for all  $m \in \mathbb{Z}$ . Then using the estimate in Eq. (4.6) with constant  $c_1 > 0$  corresponding to the metric  $h$ , we obtain

$$\begin{aligned} K_{1,h}(z, w, t) &\ll t^{-1} \sum_{m \in \mathbb{Z}} \exp\left(-\frac{c_1 d_{\tilde{h}}^2(\tilde{z}, \tilde{w} + m)}{t}\right) \\ &\leq t^{-1} e^{-\frac{c_1 d_{\tilde{h}}^2(z,w)}{2r}} \sum_{m \in \mathbb{Z}} e^{-\frac{c_1 d_{\tilde{h}}^2(\tilde{z}, \tilde{w} + m)}{2r}} \\ &\leq t^{-1} e^{-\frac{c_1 d_{\tilde{h}}^2(z,w)}{2r}} \left( e^{-\frac{c_2 d_{\tilde{g}}^2(\tilde{z}, \tilde{w})}{2r}} + \sum_{m \neq 0} e^{-\frac{c_2 d_{\tilde{g}}^2(\tilde{z}, \tilde{w} + m)}{2r}} \right) \end{aligned}$$

Now we use the formula for the hyperbolic distance to estimate it; for  $m \neq 0$ , we have

$$\begin{aligned} d_{\tilde{g}}((x_1, y_1), (x_2 + m, y_2)) &= \cosh^{-1} \left( 1 + \frac{(x_1 - x_2 - m)^2 + (y_1 - y_2)^2}{2y_1 y_2} \right) \\ &\geq \log \left( 1 + \frac{(x_1 - x_2 - m)^2}{2y_1 y_2} \right) \geq \log \left( 1 + \frac{(|m| - 1)^2}{2y_1 y_2} \right) \end{aligned}$$

since  $-1 \leq x_1 - x_2 \leq 1$  and  $(|m| - 1)^2 \leq (x_1 - x_2 - m)^2 \leq (|m| + 1)^2$ , if  $|m| \neq 0$ . We proceed now to estimate the series in the same way as in (4.11), but we do not need to restrict the values of  $y_1$  and  $y_2$  to  $[1, a]$  any more. We keep the value  $y_1 y_2$  in the estimates instead of using the bound  $a^2$

$$\begin{aligned} \sum_{|m| \geq 2} e^{-\frac{c_2 d_g^2(\bar{z}, \bar{w}+m)}{t}} &\leq \sum_{|m| \geq 1} e^{-\frac{c_2 \log(1 + \frac{m^2}{2y_1 y_2})^2}{2t}} \\ &\ll \int_1^\infty e^{-\frac{c_2 \log(1 + \frac{u^2}{2y_1 y_2})^2}{2t}} du \\ &\ll y_1^{1/2} y_2^{1/2} (1 + \sqrt{t} e^{ct}) \leq C(\tau) y_1^{1/2} y_2^{1/2} \end{aligned}$$

for some constant  $C(\tau)$  that depends on  $\tau, 0 < t \leq \tau$ . Putting all the terms together, we obtain

$$\begin{aligned} K_{1,h}(z, w, t) &\ll t^{-1} e^{-\frac{c_1 d_h^2(z,w)}{2t}} \left( 2 + e^{-\frac{c_2 d_g^2(\bar{z}, \bar{w})}{2t}} + \sum_{m \neq 0} e^{-\frac{c_1 \log(1 + \frac{m^2}{2y_1 y_2})^2}{2t}} \right) \\ &\ll t^{-1} y_1^{1/2} y_2^{1/2} e^{-\frac{c_1 d_h^2(z,w)}{2t}}. \end{aligned}$$

For the derivatives of the heat kernel, we apply the results by Cheng et al. in [8, Theorems 6, 7], to  $(\mathbb{H}, \hat{h})$  that has bounded geometry. The first two derivatives of the heat kernel  $K_{1,h}$  can be estimated in the same way as we did for the heat kernel. As the authors point out in [8], the constant in each estimate will depend on the curvature of  $M$  and its covariant derivatives.

### Appendix B

#### B1 Observation

In the proof of Theorem 3.1, we repeatedly make use of the following elementary facts:

(1) For any  $a > 0$ , and  $b, n, m \in \mathbb{R}$ , we have that

$$\int_n^m e^{-ax^2 - bx} dx = \frac{e^{b^2/4a}}{\sqrt{a}} \int_{\sqrt{a}(n + \frac{b}{2a})}^{\sqrt{a}(m + \frac{b}{2a})} e^{-v^2} dv \leq \frac{\sqrt{\pi} e^{b^2/4a}}{\sqrt{a}}.$$

(2) For any  $c > 0, 0 < t \leq T, k, \ell \geq 0$  with  $k + \ell > 2$  we have

$$\int_1^\infty \int_1^\infty y^{-k} y'^{-\ell} e^{-\frac{c}{t} \log(y/y')^2} dy dy' \leq \sqrt{t} e^{(1-k)^2 t/c}. \tag{5.12}$$

(3) Let  $\varphi \in C^\infty(M), \psi = e^{-2\varphi} - 1$  and  $\tilde{\psi} = e^{2\varphi} - 1$ . If  $\varphi|_Z(y, x), \Delta_g \varphi|_Z(y, x)$  and  $|\nabla_g \varphi|_g|_Z(y, x)$  are  $O(y^{-k})$  as  $y \rightarrow \infty$ , then so are  $\psi|_Z(y, x), \Delta_g \psi|_Z(y, x), |\nabla_g \psi|_g|_Z(y, x)$  and the analogous functions corresponding to  $\tilde{\psi}$ .

(4) For  $a, b, c > 0$ , the function  $f(t) = t^{-a} e^{-ct^{-b}}$  is bounded on  $(0, \infty)$  and  $\lim_{t \rightarrow 0} f(t) = 0$ .

B2 Proof of the bounds of the integrals  $J_1, J_2$  and  $J_3$  in Proposition 4.5

Let us start with  $J_1$  that is given by Eq. (4.18):

$$J_1 = \int_0^t \int_{Z_a} \int_{[1, \frac{4a}{5}] \times S^1} \psi_2(z)(K_{1,h}(z, z', s) + p_{h,D}(z, z', s))e^{2\varphi(z')} \\ \psi(z')\Delta_{Z,g}(K_{1,g}(z', z, t-s) + p_{1,D}(z', z, t-s)) dA_g(z') dA_g(z) ds.$$

Note that on this region  $a \leq y < \infty$  and  $1 \leq y' \leq \frac{4a}{5}$ ,  $\log(y/y')$  is bounded away from zero. Using the estimates of the heat kernels and their derivatives, we obtain

$$|J_1| \ll \int_0^t \int_a^\infty \int_1^{\frac{4a}{5}} s^{-1}(t-s)^{-2}y \left( e^{-\frac{c \log(y/y')^2}{s}} + e^{-\frac{c \log(y)^2}{s}} e^{-\frac{c \log(y')^2}{s}} \right) \\ y'^{-k+1} \left( e^{-\frac{c \log(y/y')^2}{t-s}} + e^{-\frac{c \log(y)^2}{t-s}} e^{-\frac{c \log(y')^2}{t-s}} \right) \frac{dy'}{y'^2} \frac{dy}{y^2} ds \\ \ll at^{-2} \int_0^{t/2} \int_a^\infty s^{-1}y^{-1} \left( e^{-\frac{c \log(5y/4a)^2}{s}} + e^{-\frac{c \log(y)^2}{s}} \right) dy ds \\ + at^{-1} \int_{t/2}^t \int_a^\infty (t-s)^{-2}y^{-1} \left( e^{-\frac{c \log(5y/4a)^2}{t-s}} + e^{-\frac{c \log(y)^2}{t-s}} \right) dy ds.$$

Since  $y \geq a > \frac{5}{4}$ , we have an estimate in  $s$ :

$$e^{-\frac{c \log(5y/4a)^2}{s}} + e^{-\frac{c \log(y)^2}{s}} \leq e^{-\frac{c \log(5/4)^2}{2s}} \left( e^{-\frac{c \log(5y/4a)^2}{2s}} + e^{-\frac{c \log(y)^2}{2s}} \right)$$

and  $\int_a^\infty y^{-1} e^{-\frac{c \log(5y/4a)^2}{2s}} dy = \int_{\frac{5}{4}}^\infty v^{-1} e^{-\frac{c \log(v)^2}{2s}} dv \ll \sqrt{s}$ . We get a similar estimate for  $t-s$ , and together these give

$$|J_1| \ll at^{-2} \int_0^{t/2} s^{-1} e^{-\frac{c \log(5/4)^2}{2s}} \int_{\frac{5}{4}}^\infty y^{-1} e^{-\frac{c \log(y)^2}{2s}} dy ds \\ + at^{-1} \int_{t/2}^t (t-s)^{-2} e^{-\frac{c \log(5/4)^2}{2(t-s)}} \int_{\frac{5}{4}}^\infty y^{-1} e^{-\frac{c \log(y)^2}{2(t-s)}} dy ds \\ \ll at^{-2} \int_0^{t/2} s^{-1/2} e^{-\frac{c \log(5/4)^2}{2s}} ds + at^{-1} \int_{t/2}^t (t-s)^{-3/2} e^{-\frac{c \log(5/4)^2}{2(t-s)}} ds \\ \ll at^{-2} e^{-\frac{c \log(5/4)^2}{4t}} \int_0^{t/2} ds + at^{-1} e^{-\frac{c \log(5/4)^2}{2t}} \int_{t/2}^t ds \ll a(t^{-1} + 1)e^{c_1/t} \ll ae^{-\frac{c'}{t}},$$

for some constants  $c_1, c' > 0$ , where we also used part (4) of Observation B.1.

For  $J_2$ , we had reduced the problem to the following estimate:

$$|J_2| \ll \int_{t/2}^t \|M_{\chi Z \frac{4a}{5}} M_\psi \Delta_{Z,g} e^{-(s/2)\Delta_{Z,g}} M_\phi^{-1}\|_2 \|M_\phi e^{-(s/2)\Delta_{Z,g}}\|_2 ds.$$

Now we proceed to estimate each of the HS norms appearing as integrand on the right-hand side as follows:

$$\begin{aligned} & \|M_{\chi Z \frac{4a}{5}} M_\psi \Delta_{Z,g} e^{-(s/2)\Delta_{Z,g}} M_\phi^{-1}\|_2^2 \\ &= \int_{Z \frac{4a}{5}} \int_Z |\psi(z) \Delta_{Z,g} K_{Z,g}(z, z', s/2) \phi(z')^{-1}|^2 dA_g(z') dA_g(z) \\ &\ll \int_{\frac{4a}{5}}^\infty \int_1^\infty y^{-2k} y y' s^{-4} \left( e^{-\frac{4c}{s}(\log(y/y'))^2} + e^{-\frac{4c}{s}(\log(y'y))^2} \right) y' \frac{dy'}{y'^2} \frac{dy}{y^2} \\ &= s^{-4} \int_{\frac{4a}{5}}^\infty \int_1^\infty y^{-2k-1} e^{-\frac{4c}{s}(\log(y'/y))^2} dy' dy \\ &\quad + s^{-4} \int_{\frac{4a}{5}}^\infty \int_1^\infty y^{-2k-1} e^{-\frac{4c}{s}(\log(y'))^2} dy' dy. \end{aligned}$$

The first integral in the last equality above can be estimated by fixing  $y$  and making the change of variables  $v = \log(y'/y)$ ,  $y' = ye^v$ ,  $dy' = ye^v dv$ :

$$\begin{aligned} s^{-4} \int_{\frac{4a}{5}}^\infty \int_{-\log(y)}^\infty y^{-2k} e^v e^{-\frac{4c}{s}v^2} dv dy &\ll s^{-4} e^{\frac{s}{4c}} \sqrt{s} \int_{\frac{4a}{5}}^\infty y^{-2k} \int_{-\infty}^\infty e^{-v^2} dv dy \\ &\ll s^{-7/2} a^{-2k+1} e^{\frac{s}{4c}}. \end{aligned}$$

As for the second integral, we obtain in a similar way:

$$s^{-4} \int_{\frac{4a}{5}}^\infty \int_1^\infty y^{-2k-1} e^{-\frac{4c}{s}(\log(y'))^2} dy' dy \ll s^{-7/2} e^{\frac{s}{4c}} a^{-2k}.$$

Thus,

$$\|M_{\chi Z \frac{4a}{5}} M_\psi \Delta_{Z,g} e^{-s/2\Delta_{Z,g}} M_\phi^{-1}\|_2 \ll s^{-7/4} \left( a^{-k} + a^{-k+1/2} \right).$$

For the operator  $M_\phi e^{-(s/2)\Delta_{Z,g}}$ , using Eq. (3.4) we have

$$\begin{aligned} & \|M_\phi e^{-(s/2)\Delta_{Z,g}}\|_2^2 \\ & \ll \int_1^\infty \int_1^\infty s^{-2} y^{-1} y y' \left( e^{-\frac{2c}{s}(\log(y/y'))^2} + e^{-\frac{2c}{s}(\log(yy'))^2} \right)^2 \frac{dy'}{y'^2} \frac{dy}{y^2} \\ & \ll \int_1^\infty \int_1^\infty s^{-2} y'^{-1} y^{-2} \left( e^{-\frac{4c}{s}(\log(y/y'))^2} + e^{-\frac{4c}{s}(\log(yy'))^2} \right) dy' dy \\ & \ll s^{-2} \sqrt{s} e^{s/4c} + s^{-2} \int_1^\infty y'^{-1} e^{-\frac{4c}{s}(\log(y'))^2} dy' \ll s^{-3/2} (1 + e^{s/4c}). \end{aligned}$$

Since  $s \leq t \leq 1$  we have that  $\|M_\phi e^{-(s/2)\Delta_{Z,g}}\|_2 \ll s^{-3/4}$ . It follows that

$$|J_2| \ll \int_{t/2}^t s^{-7/4} (a^{-k} + a^{-k+1/2}) \cdot s^{-3/4} ds \ll a^{-k+1/2} t^{-3/2}.$$

Now, for  $J_3$ , we have

$$\begin{aligned} J_3 &= \int_{t/2}^t \int_{Z_a} \int_{Z_{\frac{4a}{5}}} \psi_2(z) K_{Z,h}(z, z', s) e^{2\varphi(z')} \chi_{Z_{\frac{4a}{5}}}(z') \\ & \quad (\Delta_{Z,g} - \Delta_{Z,h})_{z'} K_{Z,g}(z', z, t-s) dA_g(z') dA_g(z) ds. \end{aligned}$$

Remember that  $\Delta_{Z,g} - \Delta_{Z,h} = (e^{2\varphi(z')} - 1)\Delta_{Z,h} = \tilde{\psi}(z')\Delta_{Z,h}$ , so the previous equation becomes:

$$\begin{aligned} J_3 &= \int_{t/2}^t \int_{Z_a} \int_{Z_{\frac{4a}{5}}} \{\psi_2(z) K_{Z,h}(z, z', s) \chi_{Z_{\frac{4a}{5}}}(z') \tilde{\psi}(z') \\ & \quad (\Delta_{Z,h} K_{Z,g}(z', z, t-s)) e^{-2\varphi(z)}\} dA_h(z') dA_h(z) ds \\ &= \int_{t/2}^t \int_{Z_a} \int_{Z_{\frac{4a}{5}}} \{\psi_2(z) (\Delta_{Z,h} K_{Z,h}(z, z', s) \tilde{\psi}(z')) \chi_{Z_{\frac{4a}{5}}}(z') \\ & \quad K_{Z,g}(z', z, t-s) e^{-2\varphi(z)}\} dA_h(z') dA_h(z) ds \\ &= \int_{t/2}^t \int_{Z_a} \int_{Z_{\frac{4a}{5}}} \{\psi_2(z) e^{-2\varphi(z)} K_{Z,g}(z, z', t-s) \chi_{Z_{\frac{4a}{5}}}(z') \\ & \quad (\Delta_{Z,h} \tilde{\psi}(z') K_{Z,h}(z', z, s))\} dA_h(z') dA_h(z) ds. \end{aligned}$$

Writing this in terms of the corresponding operators, we obtain

$$J_3 = \int_{t/2}^t \text{Tr}(M_{\psi/2} M_{e^{-2\varphi}} e^{-(t-s)\Delta_{Z,s}} M_{\chi Z \frac{4a}{5}} \Delta_{Z,h} M_{\tilde{\psi}} e^{-s\Delta_{Z,h}}) ds,$$

$$|J_3| \leq \int_{t/2}^t \|M_{\chi Z \frac{4a}{5}} \Delta_{Z,h} M_{\tilde{\psi}} e^{-s\Delta_{Z,h}}\|_1 ds.$$

We are now working in  $L^2(M, dA_h)$  therefore to simplify notation, we do not write the subindex  $h$  in the trace and the HS norms. In the same way as above, we have

$$\|M_{\chi Z \frac{4a}{5}} \Delta_{Z,h} M_{\tilde{\psi}} e^{-s\Delta_{Z,h}}\|_1 \leq \|M_{\chi Z \frac{4a}{5}} \Delta_{Z,h} M_{\tilde{\psi}} e^{-s\Delta_{Z,h}/2} M_{\phi^{-1}}\|_2 \|M_{\phi} e^{-s\Delta_{Z,h}/2}\|_2$$

The kernel of the operator  $M_{\chi Z \frac{4a}{5}} \Delta_{Z,h} M_{\tilde{\psi}} e^{-s\Delta_{Z,h}/2} M_{\phi^{-1}}$  is

$$\chi Z \frac{4a}{5}(z')(\Delta_{Z,h}(\tilde{\psi}(z')K_{Z,h}(z', z, s))\phi(z))^{-1}.$$

Using the decay assumptions on  $\varphi$  and its derivatives, we have that

$$|\Delta_{Z,h}(\tilde{\psi}K_{Z,h})|^2 \ll |\tilde{\psi}\Delta_{Z,h}K_{Z,h}|^2 + |K_{Z,h}\Delta_{Z,h}\tilde{\psi}|^2 + 2|\langle \nabla\tilde{\psi}, \nabla K_{Z,h} \rangle|^2$$

$$\ll y'^{-2k+1}y(s^{-4} + s^{-2} + s^{-3}) \left( e^{-\frac{c}{s}(\log(y/y'))^2} + e^{-\frac{c}{s}(\log(yy'))^2} \right)^2.$$

Since for  $0 < s < 1$  we have that  $s^{-4} + s^{-2} + s^{-3} \ll s^{-4}$ , we can estimate the HS norm by

$$\|M_{\chi Z \frac{4a}{5}} \Delta_{Z,h} M_{\tilde{\psi}} e^{-s\Delta_{Z,h}/2} M_{\phi^{-1}}\|_2^2$$

$$= \int_Z \int_Z |\chi Z \frac{4a}{5}(z')\tilde{\psi}(z')\Delta_{h,z'}K_h(z', z, s/2)\phi(z))^{-1}|^2 dA_h(z')dA_h(z)$$

$$\ll s^{-4} \int_1^\infty \int_{\frac{4a}{5}}^\infty y^2 y'^{-2k+1} \left( e^{-\frac{2c}{s}(\log(y/y'))^2} + e^{-\frac{2c}{s}(\log(yy'))^2} \right)^2 \frac{dy'}{y'^2} \frac{dy}{y^2}$$

$$\ll s^{-4} \int_{\frac{4a}{5}}^\infty \int_1^\infty \left( y'^{-2k-1} e^{-\frac{4c}{s}(\log(y/y'))^2} + y'^{-2k-1} e^{-\frac{4c}{s}(\log(yy'))^2} \right) dy' dy$$

$$\ll (a^{-2k+1} + a^{-2k})s^{-7/2}e^{s/4c} \ll a^{-2k+1}s^{-7/2}.$$

We finally obtain

$$\|M_{\phi^{-1}} e^{-(s/2)\Delta_{Z,h}} \tilde{\psi} \Delta_h\|_2 \leq a^{-k+1/2} s^{-7/4}.$$

For the operator  $e^{-(s/2)\Delta_{Z,h}} M_{\phi}$ , the proof goes in the same way as for the operator  $M_{\phi} e^{-(s/2)\Delta_{Z,s}}$ . At the end, we obtain

$$\|e^{-s\Delta_{Z,h}} M_{\phi}\|_2 = \left( \int_Z \int_Z |K_{Z,h}(z, z', s/2)\phi(z')|^2 dA_h(z')dA_h(z) \right)^{1/2} \ll s^{-3/4}.$$

In this way,

$$|J_3| \ll \int_{t/2}^t a^{-k+1/2} s^{-7/4} s^{-3/4} ds \ll a^{-k+1/2} t^{-3/2}.$$

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