

# Construction of oscillatory singularities

## Abstract

One way to understand more about spacetime singularities is to construct solutions of the Einstein equations containing singularities with prescribed properties. The heuristic ideas of the BKL picture suggest that oscillatory singularities should be very common and give a detailed picture of how these could look. The more straightforward case of singularities without oscillations is reviewed and existing results on that subject are surveyed. Then recent theorems proving the existence of spatially homogeneous solutions with oscillatory singularities of a specific type are presented. The proofs of these involve applications of some ideas concerning heteroclinic chains and their stability. Some necessary background from the theory of dynamical systems is explained. Finally some directions in which this research might be generalized in the future are pointed out.

## 1 Introduction

One of the characteristic features of general relativity is its prediction of spacetime singularities. These can already be observed in explicit solutions of the Einstein equations such as the Schwarzschild solution and the Friedman-Lemaitre-Robertson-Walker (FLRW) solutions. These explicit solutions have a high degree of symmetry. The question arose early whether the known singularities might be artefacts of symmetry. The work of Lifshitz and Khalatnikov [13] supported this idea. These authors made what they believed to be a general ansatz for the form of the geometry near the singularity and found that it could not accommodate the full number of free functions expected. They concluded that generic solutions of the Einstein equations do not develop singularities. The result was heuristic in nature.

The conclusions of [13] were shown to be invalid by the singularity theorems of Penrose and Hawking [5]. In contrast to the arguments in [13] the results of the singularity theorems were based on mathematical proofs. It was shown that singularities occur for open sets of solutions of the Einstein equations. More specifically, using the point of view of the initial value problem, they occur for the solutions arising from open sets of initial data. This means that the occurrence of singularities is stable in a certain precise sense. A positive feature of these theorems is that the hypotheses are relatively weak. On the other hand the conclusions are also relatively weak. What is proved is that the spacetimes

to which the theorems apply are geodesically incomplete. Very little is said about the nature of the singularities. There is no information on whether the singularities are accompanied by large energy densities or tidal forces, as might be expected on the basis of physical intuition.

Later, Belinskii, Khalatnikov and Lifshitz (abbreviated in what follows by BKL) presented a more detailed picture of spacetime singularities [3]. Their arguments are heuristic. They are similar to the work of [13] with the important difference that this time the ansatz used is more complicated, including oscillatory behaviour. This allows the full number of free functions to be included. Some of the main assertions belonging to the BKL picture are:

- the solutions of the partial differential equations are approximated by solutions of ordinary differential equations near the singularity
- the approach to the singularity is oscillatory
- solutions of the Einstein equations with matter are approximated by solutions of the vacuum equations near the singularity

The solutions of the ordinary differential equations in the first point correspond to spatially homogeneous solutions of the Einstein equations. In the original work one of the most general classes of spatially homogeneous solutions, the Bianchi type IX solutions, played a central role. These solutions were also studied independently at about the same time by Misner [14] who called them the Mixmaster model. On the basis of heuristic and numerical work the conclusion was reached that these solutions show a highly oscillatory behaviour near the singularity. This statement is correct but, as will be discussed below, it took a long time to prove it. Here the connection to the second point above can be seen. Moreover, the oscillations are observed in the vacuum case and this makes contact with the third point. From these remarks it follows that if the BKL picture is correct then spatially homogeneous solutions of the vacuum equations are very important in understanding singularities in solutions of the Einstein equations without symmetries and with quite general matter. Note, however, that after forty years it is still not known whether the BKL picture is correct. At least there have been many studies of solutions with symmetries which give good agreement with the conclusions obtained by specializing the general BKL picture to symmetric cases.

Since the general case is very complicated to treat it makes sense to start looking for a better understanding of the question of the validity of the BKL picture by concentrating on the spatially homogeneous case. This is especially true due to the fact that the wider significance of the homogeneous case is intimately related to the main claims of the BKL picture. This leads to the study of certain systems of ordinary differential equations. A concept from the theory of ODE which turns out to be of particular relevance in this context is that of heteroclinic chains. They will be discussed in Sect. 4. In special cases the dynamics near the singularity is convergent rather than oscillatory and in this

easier case there do exist results for inhomogeneous spacetimes. Since these are relevant for the conceptual approach used in the case of oscillatory asymptotics they will be discussed in Sect. 3. In order to even define the concepts ‘monotone’ and ‘oscillatory’ being used it is necessary to introduce some basic notation and terminology and this is the subject of Sect. 2.

## 2 Notation and terminology

Let  $M$  be a four-dimensional manifold and  $g_{\alpha\beta}$  a Lorentzian metric on  $M$ . Let  $S_t$  be a foliation by spacelike hypersurfaces whose leaves are parametrized by  $t$ . Let  $g_{ab}(t)$  and  $k_{ab}(t)$  be the induced metric and the second fundamental form of the leaves of the foliation. Define the Hubble parameter as  $H = -\frac{1}{3}g^{ab}k_{ab}$ . It is possible to do a 3+1 decomposition of the Einstein equations for the metric  $g_{\alpha\beta}$  based on the foliation  $S_t$  which brings in a lapse function and a shift vector. This can of course be done in the presence of matter and also in a closely analogous way in other dimensions.

Let  $\lambda_i$  be the eigenvalues of  $k_{ab}$  with respect to  $g_{ab}$ , i.e. the solutions of  $k_{ab}X^b = \lambda_i g_{ab}X^b$  for some vector  $X^b$ . For a well-behaved foliation approaching the singularity towards the past  $H > 0$  in a neighbourhood of the singularity. Hence the quantities  $p_i = -\lambda_i/3H$ , the generalized Kasner exponents, are well-defined functions of  $t$  and  $x$ , where  $x$  denotes a point on one of the leaves of the foliation. It follows from the definition that  $\sum_i p_i = 1$  so that only two of the  $p_i$  are independent. The way in which different leaves are identified with each other depends on the choice of the shift vector. Suppose that the limit  $t \rightarrow 0$  corresponds to the approach to a singularity. Fix a point  $x_0$  and consider the functions  $p_i(t, x_0)$ . A function  $p_i$  is said to be oscillatory near the singularity if  $\liminf_{t \rightarrow 0} p_i(t) < \limsup_{t \rightarrow 0} p_i(t)$ . It is said to be convergent near the singularity if  $\liminf_{t \rightarrow 0} p_i(t) = \limsup_{t \rightarrow 0} p_i(t)$ , in other words if  $p_i$  tends to a limit as  $t \rightarrow 0$ . Similar definitions can be made for other geometric quantities. Informally, a singularity is said to be oscillatory or convergent if some important geometric quantities have the corresponding properties. It is important that the quantities concerned are dimensionless, since otherwise they could be expected to diverge as the singularity is approached.

These definitions depend a lot on the choice of 3+1 decomposition. Things are simpler in the case of spatially homogeneous spacetimes. There it is natural to choose the foliation to consist of level hypersurfaces of a Gaussian time coordinate based on a hypersurface of homogeneity. Then geometric quantities such as  $p_i$  depend only on  $t$ .

## 3 The convergent case

The second and third points listed above as parts of the BKL picture are only supposed to hold in this form with certain restrictions on the type of matter which is coupled to the Einstein equations. Assuming the validity of the first

point this is directly related to the behaviour of homogeneous models with that type of matter. Two related types of matter which are known to be exceptional from this point of view are the scalar field and the stiff fluid. Here we will concentrate on the second. Consider a perfect fluid with linear equation of state  $p = (\gamma - 1)\rho$  for a constant  $\gamma$ . An extreme case of this from the point of view of ordinary physics is  $\gamma = 2$ . This is the stiff fluid where the velocity of sound of the fluid  $\sqrt{\gamma - 1}$  is equal to the velocity of light. (We use geometrical units.) It is also possible to consider ultrastiff fluids with  $\gamma > 2$ . These very exotic matter models have been considered in certain scenarios for the early universe. According to the BKL picture the generalized Kasner exponents in a general spatially homogeneous solution of the Einstein-Euler equations with a linear equation of state are oscillatory near the singularity in the case  $1 \leq \gamma < 2$  and convergent for  $2 \leq \gamma$ . (The case  $\gamma < 1$  will not be considered in what follows.) In fact for  $\gamma > 2$  the  $p_i$  all converge to the limit  $\frac{1}{3}$ . These limiting values agree with the values of these quantities in the FLRW solutions and correspond to isotropization. In the case  $\gamma = 2$ , on the other hand, the  $p_i$  may converge to different limits in different solutions and at different spatial points in a single inhomogeneous solution.

What kind of mathematical results could count as a rigorous version of these expectations? It would be desirable to prove that an open neighbourhood with respect to some reasonable topology of the FLRW data leads to solutions which have the predicted asymptotics. This may be called the forwards result since it goes from the data to the asymptotics. Unfortunately this kind of result is hard to obtain. What is sometimes easier is to get a backwards result which goes from the asymptotics to the solution. The solutions with the predicted asymptotics can be parametrized by certain free functions. (Here we ignore complications arising from the Einstein constraints.) Call these functions the asymptotic data. The idea is then to ask whether for asymptotic data belonging to a large class there exists a solution with the correct asymptotics and exactly those free functions. A theorem of this kind was proved for stiff fluids in [1] and for ultrastiff fluids in [6]. In this work the notion of ‘large class’ was defined to mean containing as many free functions as the general solution. It was only proved in the case that these functions are analytic ( $C^\omega$ ). Until very recently the forwards problem was unsolved, even in the stiff case, but a theorem of this kind has been announced by Rodnianski and Speck [20].

For comparison it is of interest to quote what is known in a case with symmetry, that of the Gowdy solutions. Note that for the Gowdy solutions the BKL picture does not predict oscillations and indeed none are found. In this case the backwards problem was solved in [7] for the analytic case and in [16] for the smooth case. The forwards problem for Gowdy was solved in [19]. The aim of this section was to exhibit the backwards problem as a route to obtaining rigorous results and to show that at least in some cases it can open the way to obtaining results of the type which are most desirable.

Just as some matter models such as the stiff fluid can suppress oscillations others can stimulate them. For instance a Maxwell field can produce BKL type oscillations within Bianchi types where there are no oscillations in the vacuum

case. An example is discussed in Sect. 6. The BKL picture can be applied in higher dimensions and gives different results in that case. It says, for instance, that in the case of the Einstein vacuum equations generic oscillations disappear when the spacetime dimension is at least eleven but are present in all lower dimensions. The higher dimensional models come up in the context of string theory and there it is typical that there are other matter fields present. There is the dilaton which, as a scalar field, tends to suppress oscillations and the  $p$ -forms which, as generalizations of the Maxwell field, encourage them. Many of the conclusions of the heuristic analysis which lead to the conclusion that the singularity is convergent can be made rigorous using techniques generalizing those of [1]. For more information on this the reader is referred to [4]. To conclude it should be noted that there is not a single rigorous result which treats solutions which are both inhomogeneous and oscillatory.

## 4 Heteroclinic chains

Consider a system of  $k$  ordinary differential equations. This can be thought of as being defined by a vector field defined on an open subset of  $\mathbb{R}^k$ , a geometrical formulation which corresponds to the point of view of dynamical systems. A stationary (i.e. time-independent) solution of the ODE system corresponds to a fixed point of the vector field. A solution of the ODE system which is time-dependent corresponds to an integral curve of the vector field, an orbit of the dynamical system. An orbit which tends to a stationary solution  $p$  as  $t \rightarrow -\infty$  and a stationary point  $q \neq p$  as  $t \rightarrow +\infty$  is called a heteroclinic orbit. An orbit which tends to  $p$  as  $t \rightarrow -\infty$  and as  $t \rightarrow +\infty$ , but is not itself stationary, is called a homoclinic orbit.

Suppose now that  $p_i$  is a sequence of points, finite or infinite, such that for each  $i$  there is a heteroclinic orbit from  $p_i$  to  $p_{i+1}$ . This configuration is called a heteroclinic chain. If the chain is finite and comes back to its starting point then it is called a heteroclinic cycle. It turns out that homoclinic orbits and heteroclinic cycles are not robust in the sense that a sufficiently general small perturbation of the vector field will destroy them. It is therefore perhaps surprising that they are found in models for many phenomena in nature. Given that all physical measurements only have finite precision it seems difficult to be able to rule out small perturbations in mathematical models for real phenomena. The answer to this question appears to be that there is some absolute element in the system whose presence is not subject to uncertainty and which therefore cannot be perturbed. In the case of spacetime singularities it is the singularity itself which appears to play this role. These considerations are admittedly rather vague and it would be desirable to understand issues of this type more precisely. In any case, heteroclinic cycles do seem to be widespread in models for the dynamics of solutions of the Einstein equations near their singularities.

Coming back to more general dynamical systems, it is of interest to have criteria for the stability properties of heteroclinic chains. In other words, the aim is to find out under what conditions solutions which start close to a heteroclinic

chain approach it as  $t \rightarrow \infty$ . The case of most interest is not that where a solution approaches one of the stationary points belonging to the heteroclinic chain (call them vertices) but the case where the solution follows successive heteroclinic orbits within the chain. When the chain is a cycle this means that the solutions exhibit oscillatory behaviour, repeatedly approaching the vertices of the cycle. In the next section it is shown that this is exactly what happens in the context of the BKL picture.

A solution which is converging to a heteroclinic chain spends most of its time near the vertices. Since the vector field vanishes at the vertices it is small near them and the solution moves slowly there. This suggests that the behaviour of the solution while it is near the vertices could have strong influence on the stability of the cycle. A way of studying the local flow near a vertex is to linearize there and examine the eigenvalues of the linearization. It turns out that this can give valuable information about the stability of the cycle.

## 5 Bianchi models

A spatially homogeneous solution of the Einstein equations is one which admits a symmetry group whose orbits are spacelike hypersurfaces. The only case where the group cannot be assumed to be three-dimensional is the class of Kantowski-Sachs spacetimes. They are not discussed further in this paper. All the rest are the Bianchi models. In that case it can be assumed without loss of generality that the spatial manifold is simply connected and then it can be identified with the Lie group itself. The reason for this is that the dynamics on the universal covering manifold is the same as that on the original manifold. Thus in Bianchi models it can be assumed that  $M = I \times G$  where  $I$  is an interval and  $G$  is a simply connected three-dimensional Lie group. These Lie groups are in one to one correspondence with their Lie algebras and the three-dimensional Lie algebras were classified by Bianchi into types I-IX. The metric can be written in the form

$$- dt^2 + \sum g_{ij}(t)\theta^i \otimes \theta^j \tag{1}$$

where  $\theta^i$  is a basis of left invariant one-forms on  $G$ . Let  $e_i$  be the dual basis of vector fields. The evolution equations depend on the Lie group chosen through the structure constants defined by  $[e_i, e_j] = c_{ij}^k e_k$ . It is customary to distinguish between Class A models where  $c_{kj}^k = 0$  and Class B models which are the rest.

There are many ways in which the evolution equations for Bianchi models can be written as a system of ordinary differential equations. Consider for simplicity the vacuum models of Class A. In that case it can be shown that there is a basis of the Lie algebra with the property that diagonal initial data give rise to diagonal solutions. This is still true when the matter is described by a perfect fluid but it does not hold for general matter. A form of the equations for Class A models which has turned out to be particularly useful for proving theorems is the Wainwright-Hsu system [21]. The basic variables are called  $\Sigma_+, \Sigma_-, N_1, N_2, N_3$ . The first two are certain linear combinations of the general-

ized Kasner exponents. The quantity  $N_1$  is given in the Bianchi type IX case by  $\frac{1}{H} \sqrt{\frac{g_{11}}{g_{22}g_{33}}}$  and the other two  $N_i$  are related to this by cyclic permutations. The Gaussian time coordinate  $t$  is replaced by a coordinate  $\tau$  which satisfies  $\frac{d\tau}{dt} = H$ . In this time coordinate the singularity is approached for  $\tau \rightarrow -\infty$ . Two important related features of this system is that the variables are dimensionless and that their evolution equations form a closed system, not depending on  $H$ . They must satisfy one algebraic condition which is the Hamiltonian constraint in the 3+1 formalism. There results a four-dimensional dynamical system. Stationary solutions of this system correspond to self-similar spacetimes. When a solution of the Wainwright-Hsu system is given the quantities  $H$  and  $t$  can be determined by integration.

The different Bianchi types are represented by subsets of the state space of the Wainwright-Hsu system where the  $N_i$  have certain combinations of signs (positive, negative or zero). The simplest Bianchi type is Bianchi type I, the Abelian Lie algebra. The vacuum solutions of this type are the Kasner solutions. They are self-similar and form a circle in the state space, the Kasner circle  $\mathcal{K}$ . Let  $\mathcal{T}$  be the equilateral triangle circumscribing the Kasner circle which is symmetric under reversal of  $\Sigma_-$ . It is tangential to  $\mathcal{K}$  at three points  $T_i$ , the Taub points. They correspond to flat spacetimes. Each solution of Bianchi type II is a heteroclinic orbit joining two Kasner solutions. Its projection to the  $(\Sigma_+, \Sigma_-)$ -plane is a straight line which passes through a corner of  $\mathcal{T}$ . Concatenating Bianchi type II solutions gives rise to many heteroclinic chains. Given a point of  $\mathcal{K}$  which is not one of the Taub points consider the straight line joining it to the closest corner of the triangle  $\mathcal{T}$ . The part of this straight line inside  $\mathcal{K}$  is the projection of an orbit of type II and it intersects  $\mathcal{K}$  in exactly one other point. In this way it is possible to define a continuous map from  $\mathcal{K}$  to itself, the BKL map. (The map is defined to be the identity at the Taub points.) The two most general Bianchi types of Class A are type VIII and type IX, corresponding to the Lie algebras  $sl(2, \mathbb{R})$  and  $su(2)$ . As already mentioned BKL concentrated on type IX. Type VIII shows many of the same features but may be even more complicated.

When specialized to the Bianchi Class A case the BKL picture suggests that as the singularity is approached in a Bianchi type IX solution the dynamics should be approximated by a heteroclinic chain of Bianchi type II solutions, successive vertices of which are generated by the BKL map. In particular the solution is oscillatory in the sense that the  $\alpha$ -limit set of a solution of this kind should consist of more than one point. (The  $\alpha$ -limit set consists of those points  $x$  such there is a sequence of times tending to  $-\infty$  along which the value of the solution converges to  $x$ .) In fact there should be three non-collinear points in the  $\alpha$ -limit set. This implies that both  $\Sigma_+$  and  $\Sigma_-$  are oscillatory in the approach to the singularity. This statement about the  $\alpha$ -limit set was proved in fundamental work of Ringström [17]. He showed that there are at least three non-Taub points in the  $\alpha$ -limit set of a generic Bianchi IX solution. (He also described the exceptions explicitly.) In addition Ringström showed that the entire  $\alpha$ -limit set of a Bianchi type IX solution is contained in the union of the

points of type I and II [18]. This means that in some sense the type IX solution is approximated by solutions of types I and II. The question of whether the corresponding statement holds for solutions of Bianchi type VIII is still open.

After the results just discussed it still took a long time before theorems about convergence to heteroclinic cycles were published. There is a heteroclinic cycle of Bianchi II solutions which comes back to its starting point after three steps. Let us call this particularly simple example ‘the triangle’. It turns out that there is a codimension one manifold with the property that any Bianchi type IX solution which starts on this manifold converges to the triangle [11] in the past time direction. This result extends to a much larger class of heteroclinic chains generated by iterating the BKL map. The essential condition is that the vertices of the chain should remain outside an open neighborhood of the Taub points. The manifold constructed in [11], which may be referred to as the unstable manifold of the triangle (unstable towards the future and hence stable towards the past), is only proved to be Lipschitz continuous. An alternative approach to this problem was given in [2]. It has the advantage that the stable manifold is shown to be continuously differentiable. On the other hand it cannot treat all the heteroclinic chains covered by the results of [11]. In particular, it does not cover heteroclinic cycles such as the triangle. The reason for this restriction is the need to avoid resonances, certain linear relations between the eigenvalues with integer coefficients. When this extra condition is satisfied it can be shown, using a theorem of Takens, that the flow near any vertex of the chain is equivalent to the linearized flow by a diffeomorphism. Some investigations of the case of chains which may approach the Taub points have been carried out in [15].

## 6 Construction of solutions converging to the triangle

The construction of solutions converging to heteroclinic chains of Bianchi type II solutions in the approach to the initial singularity will be illustrated by the case of the triangle. As indicated in Sect. 4 the stability of heteroclinic chains is related to the eigenvalues of the linearization about the vertices. Since a vertex of the triangle lies on the Kasner circle which consists of stationary points the linearization automatically has a zero eigenvalue. Of the three other eigenvalues one is negative, call it  $-\mu$ , and two are positive, call them  $\lambda_1$  and  $\lambda_2$ . The two positive eigenvalues are distinct and we adopt the convention that  $\lambda_1 < \lambda_2$ . The eigendirection corresponding to  $\lambda_2$  is tangent to the triangle. The proofs of [11] are dependent on the fact that  $\mu$  is smaller than both positive eigenvalues.

A solution which converges to the triangle repeatedly passes the vertices. Consider a point on the triangle which is close to a vertex and on an orbit approaching it. Let  $S$  be a manifold passing through the point which is transverse to the heteroclinic orbit. Call this an incoming section. Similarly we can consider an outgoing section close to the vertex and passing through the orbit leaving it. A solution which starts on the incoming section sufficiently close

to the heteroclinic cycle also intersects the outgoing section. Taking the first point of intersection defines a local mapping from the incoming section to the outgoing section, the local passage. There is a similar local mapping from the outgoing section of one vertex to the ingoing section of the next in the cycle. This is called an excursion. Composing three passages and three excursions gives a mapping from the ingoing section of a vertex to itself. It is important for the proof of [11] that this mapping is Lipschitz and that by restricting the domain of the mapping to a small enough neighbourhood of the triangle the Lipschitz constant can be made as small as desired. The norm used to define the Lipschitz property is that determined by the flat metric  $dx^2 + dy^2 + dz^2$ , where  $(x, y, z)$  are some regular coordinates on the section. The mapping from a small local section to itself is a contraction. The excursions are expanding mappings but the expansion factor is bounded. The passages are contractions which can be made arbitrarily strong and which can therefore dominate the effect of the excursions. Once the contraction has been obtained the manifold being sought can be constructed in a similar way to the stable manifold of a stationary solution.

A system similar to the Wainwright-Hsu system can be obtained for solutions of Bianchi type  $VI_0$  with a magnetic field [10]. It also includes certain solutions of Bianchi types I and II with magnetic fields and vacuum solutions of types I, II and  $VI_0$ . It uses variables  $\Sigma_+$  and  $\Sigma_-$  which have the same geometrical interpretation as in the vacuum case. The Kasner circle can be considered as a subset of the state space for the magnetic system. It has been proved that solutions of this system are oscillatory [22]. There are two families of heteroclinic orbits defined by vacuum solutions of type II. The third family in the vacuum case is replaced by heteroclinic orbits defined by Bianchi type I solutions with magnetic fields. The projections of these heteroclinic orbits onto the  $(\Sigma_+, \Sigma_-)$ -plane are the same straight lines as are obtained from heteroclinic orbits in the vacuum case. Thus the exactly the same heteroclinic chains are present. However their stability properties might be different. Now the stability of the triangle will be considered in the case with magnetic field, following [12]. The result is very similar to that in the vacuum case - there is a one-dimensional unstable manifold - but the proof is a lot more subtle. The reason that the method of proof of [11] does not apply directly is that the eigenvalue configuration is different. Compared to the vacuum case one of the eigenvalues is halved. Then it can happen that the negative eigenvalue is larger in modulus than one of the positive eigenvalues.

A problem which results from the different eigenvalue configuration is that the return map is no longer Lipschitz. More precisely it no longer has this property with respect to the Euclidean metric  $dx^2 + dy^2 + dz^2$ . It does, however, have the desired Lipschitz property with respect to the singular metric  $\frac{x^2+y^2}{x^2}dx^2 + \frac{x^2+y^2}{y^2}dy^2 + dz^2$  and this observation allows the proof of [11] to be generalized to the case with magnetic field. It is important to know that the properties of eigenvalues and eigenvectors alone are not enough to make this proof work. It is also necessary to use the existence of certain invariant mani-

folds which follows from the geometric background of the problem. At this point it is necessary to remember that we are not just dealing with a heteroclinic cycle in an arbitrary dynamical system but with one in a system with very special properties.

It turns out that the difficulties just discussed can be avoided by a clever but elementary device which comes down to replacing the variable representing the magnetic field by its square. Although this provides a very simple way of studying the heteroclinic chains in the Bianchi VI<sub>0</sub> model with magnetic field it cannot be expected that this kind of trick will apply to more general matter models. By contrast the new method is potentially much more generally applicable. There is one case where it is already known to give new results, as will now be explained. In addition to the results obtained on vacuum models results on models with a perfect fluid with linear equation of state  $p = (\gamma - 1)\rho$  were obtained in [11]. It turns out, however, that the techniques of [11] only work under an assumption on  $\gamma$  which has no physical interpretation. In the case of the triangle the condition is  $\gamma < \frac{5-\sqrt{5}}{2} \sim 1.38$ . For other chains other inequalities are obtained. These arise because the linearization has another positive eigenvalue coming from the fluid and it must be ensured that this eigenvalue is greater than  $\mu$ . With the method of [12] this restriction can be replaced by the inequality  $\gamma < 2$ , saying that the speed of sound in the fluid is smaller than the speed of light.

## 7 Future challenges

This section discusses some directions in which the known results on the construction of oscillatory singularities might be extended in the future. There are dynamical systems describing Bianchi models of types I and II with magnetic fields which are not just special cases of those included in the system describing models of type VI<sub>0</sub>. The reason for this is that the Maxwell constraints become less restrictive in the more special Bianchi types. It is possible to choose the basis so that the magnetic field only has one non-vanishing component but the price to be paid is that the metric becomes non-diagonal in that basis. In the dynamical systems describing solutions of types I and II the already familiar heteroclinic chains are still found. There are, however, additional heteroclinic orbits corresponding to the off-diagonal elements of the metric. This means that, in contrast to the models analysed up to now, the stable manifold of a point on the Kasner circle may be of dimension greater than one. This means that when it is desired to continue a heteroclinic chain towards the past there is not a unique choice any more. New ideas are required to solve this type of problem.

Similar difficulties are met in Bianchi models of Class B. There it is believed on the basis of heuristic considerations that there is precisely one type, VI<sub>− $\frac{1}{3}$</sub> , which shows oscillatory behaviour similar to that found in types VIII and IX. There are no rigorous results on Class B comparable to the results of Ringström on models of Class A. In this case too the stable manifold of a Kasner solution

may have dimension greater than one. It can also not be expected that invariant manifolds of the type exploited in [12] will exist.

Perhaps the most exciting challenge in this field is to construct inhomogeneous spacetimes with oscillatory singularities. The simplest class of inhomogeneous vacuum spacetimes where oscillations are expected are the  $T^2$  models. These have a two-dimensional isometry group acting on spacelike hypersurfaces and so are effectively inhomogeneous in just one space dimension. They include the Gowdy spacetimes as a subset but it is believed that generic  $T^2$ -symmetric vacuum spacetimes have a much more complicated oscillatory behaviour near the singularity. The models of Bianchi type  $VI_{-\frac{1}{9}}$  are locally isometric to  $T^2$  models but not locally isometric to Gowdy models. Thus understanding more about Bianchi models of Class B appears a very natural first step towards a better understanding of the inhomogeneous case.

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