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Oxidation of Self-Duality to 12 Dimensions and Beyond

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Abstract

Using (partial) curvature flows and the transitive action of subgroups of $\mathrm{O}(d,\mathbb{Z})$ on the indices $\{1,\ldots,d\}$ of the components of the Yang-Mills curvature in an orthonormal basis, we obtain a nested system of equations in successively higher dimensions d, each implying the Yang-Mills equations on d-dimensional Riemannian manifolds possessing special geometric structures. This 'matryoshka' of self-duality equations contains the familiar self-duality equations on Riemannian 4-folds as well as their generalisations on complex Kähler 3-folds and on 7- and 8-dimensional manifolds with G_2 and Spin(7) holonomy. The matryoshka allows enlargement ('oxidation') to a remarkable system in 12 dimensions invariant under Sp(3). There are hints that the underlying geometry is related to the sextonions, a six-dimensional algebra between the quaternions and octonions.

1 Introduction

Many interesting examples of special geometric structures on d-dimensional Riemannian manifolds (M,g) are provided by certain G-invariant covariantly constant (parallel) p-forms $\varphi \in \Lambda^p T^* M$, where $G = \mathrm{Hol}$, the restricted holonomy group of M. If $p < d$, then G is clearly a proper subgroup of $\mathrm{SO}(d)$, since in the generic rotationally invariant case, only the volume form is invariant.

For Riemannian manifolds which are locally neither a product of lower dimensional spaces nor a symmetric space, Berger's list [1] provides the most interesting examples of restricted holonomy groups. These include $\mathrm{U}(n) \subset \mathrm{SO}(2n)$, which leaves the Kähler two-form ω on a $2n$-dimensional Kähler manifold invariant. The $\mathrm{SU}(n)$ Calabi-Yau specialisation has, in addition, an invariant complex n-form, the holomorphic volume form. The group $\mathrm{Sp}(n) \subset \mathrm{SO}(4n)$, $d = 4n$, of $n \times n$ matrices with quaternion elements satisfying $A^\dagger A = 1$, has three invariant Kähler two-forms ω_α, combinable in a two-form, $\omega = \omega_1 i + \omega_2 j + \omega_3 k$, taking values in the imaginary quaternions. These characterise hyper-Kähler geometry. The quaternionic Kähler generalisation has $\mathrm{Hol} = \mathrm{Sp}(n) \cdot \mathrm{Sp}(1) \subset \mathrm{SO}(4n)$, with the three Kähler forms existing only locally. Globally, they define an invariant parallel four-form $\sum \omega_\alpha \wedge \omega_\alpha$. The two exceptional $d = 7$ and 8 geometries with $\mathrm{Hol} = G_2$ and $\mathrm{Spin}(7)$ have, respectively, an invariant three- and four-form. In all these cases, the geometric information can equally well be encoded uniformly in an invariant four-form: the two-forms afford squaring and the three-form in seven dimensions has a Hodge-dual four-form. The Lie group inclusions

$$\mathrm{Sp}(n) \subset \mathrm{SU}(2n) \subset \mathrm{U}(2n) \subset \mathrm{SO}(4n)$$

imply corresponding inclusions of geometries: hyperkähler manifolds are Calabi-Yau manifolds, the latter are Kähler, which in turn are orientable. The two exceptional cases are also part of lower dimensional sets of inclusions:

$$\begin{aligned} \mathrm{U}(2) \subset \mathrm{Sp}(2) \subset \mathrm{SU}(4) \subset{} & \mathrm{Spin}(7) \subset \mathrm{SO}(8) \\ \mathrm{SU}(3) \subset G_2 \subset{} & \mathrm{Spin}(7) \subset \mathrm{SO}(8) \,. \end{aligned}$$

The respective invariant tensors can be obtained by successive reductions of the $4n$-dimensional volume form. For instance, the $\mathrm{Spin}(7)$ invariant four-form in eight dimensions contracted with an arbitrary vector yields the G_2-invariant three-form in the orthogonal seven-dimensional space. Similarly, the latter yields an $\mathrm{SU}(3)$-invariant two-form on projection to the complex three-fold orthogonal to an arbitrary vector.

For Riemannian manifolds (M,g) admitting a G-structure, a principle subbundle of the frame bundle of M, with structure group $G \subset \mathrm{GL}(d,\mathbb{R})$, the tangent space at every point admits an isomorphism with $\mathbb{R}^d$. For every point $p \in M$ there exists a choice of local coordinates with p as the origin in which the Riemannian metric takes the euclidean form $d^2s = g_{ij}dx^i dx^j = \sum_i dx^i dx^i$ and the special geometric structure φ in these coordinates is the constant G-invariant form

$$\varphi = \sum_{(i_1,\ldots,i_p)\in\mathcal{I}^+} dx^{i_1,\ldots,i_p} \,. \tag{1}$$

where $dx^{i_1,\ldots,i_p} := dx^{i_1}\wedge\cdots\wedge dx^{i_p}$ and $\mathcal{I}^+$ is a set of oriented subsets $\{i_1,\ldots,i_p\} \subset \{1,\ldots,d\}$ with $\varphi_{i_1\ldots i_p} = 1$. Differential forms like φ have been called *special democratic forms* [2, 3]. They are 'special' in the sense that they have components $\varphi_{\mu_1\ldots\mu_p}$ equal to $+1,-1$ or 0 in some orthonormal basis, just like the volume form $\mathrm{vol}_d = dx^1\wedge dx^2\wedge\cdots\wedge dx^d =: dx^{12\ldots d}$ on a Euclidean vector space. More precisely, a p-form φ is called *special* if it lies in the $\mathrm{SO}(d,\mathbb{R})$-orbit of

$$\varphi \;=\; \sum_{1\le\mu_1<\ldots<\mu_p\le d} \varphi_{\mu_1\ldots\mu_p}\, dx^{\mu_1\ldots\mu_p} \tag{2}$$

with components $\varphi_{\mu_1\ldots\mu_p} \in \{-1,0,1\}$. There are clearly only a finite number of orbits of special p-forms parametrised by the components $\varphi_{\mu_1\ldots\mu_p} \in \{-1,0,1\}$ under $\mathrm{SO}(d,\mathbb{R})$ or $\mathrm{O}(d,\mathbb{R})$. Distinct sets of components may give rise to special p-forms in the same orbit, because the subgroups $\mathrm{SO}(d,\mathbb{Z}) \subset \mathrm{SO}(d,\mathbb{R})$ or $\mathrm{O}(d,\mathbb{Z}) \subset \mathrm{O}(d,\mathbb{R})$ map the special form φ in equation (1) into a special form parametrised by different components. These groups are isomorphic to the semidirect product of the permutation group S_d acting naturally on $d-1$ or d copies of $\mathbb{Z}_2$, namely $\mathrm{SO}(d,\mathbb{Z}) \cong S_d \ltimes \mathbb{Z}_2^{d-1}$ or $\mathrm{O}(d,\mathbb{Z}) \cong S_d \ltimes \mathbb{Z}_2^d$. Thus, special p-forms which appear to be different may nevertheless be in the same orbit under $\mathrm{SO}(d,\mathbb{R})$ or $\mathrm{O}(d,\mathbb{R})$. The orbit of a special p-form may always be labelled by a choice of a representative (1).

A special p-form φ is called democratic if its set of nonzero components $\{\varphi_{i_1\ldots i_p}\}$ is symmetric under the transitive action of a subgroup of $\mathrm{O}(d,\mathbb{Z})$ on the indices $\{1,\ldots,d\}$. The action of an element $(\sigma,\eta_1,\ldots,\eta_d) \in S_d \ltimes \mathbb{Z}_2^d$, on the components of φ being given by

$$\varphi_{i_1\,\ldots\,i_p} \mapsto \eta_{i_1}\cdots\eta_{i_p}\,\varphi_{\sigma(i_1)\,\ldots\,\sigma(i_p)} \;, \tag{3}$$

where $\eta_i^2 = 1\,,\, i = 1,\ldots,d$. So for a democratic form no choice of indices is privileged. We refer to [2, 3] for further details. It was shown in [2] that knowledge of the above symmetry groups allows an enlargement (oxidation) of the base space; the symmetries may be used to remix the sets of indices $\{(i_1 \dots i_p)\}$ of the nonzero components amongst a larger set of indices $\{1, \dots, D\}$, $D > d$, thus defining special democratic P-forms in D dimensions from special democratic p-forms in d dimensions for successively higher $P \geq p$ and $D \geq d$. In this paper, we consider two such oxidation maps:

a) Oxidation through remixing

This is a map $\varphi \in \Lambda^p\mathbb{R}^d \to \Lambda^p\mathbb{R}^D \ni \Phi$ defining a special democratic p-form Φ in $D > d$ dimensions in terms of the components of a special p-form φ in d-dimensions thus:

$$\varphi = \sum_{(i_1,\dots,i_p)\in\mathfrak{I}^+} dx^{i_1,\dots,i_p} \quad \longmapsto \quad \Phi = \sum_{\sigma\in H\subset S_D} \sum_{(i_1,\dots,i_p)\in\mathfrak{I}^+} dx^{\sigma(i_1)\dots\sigma(i_p)}, \qquad (4)$$

where H is some subgroup of the symmetric group S_D acting on the D indices.

b) Oxidation through heat flow

Alternatively, for $D = d + q$ the nonzero components of a special democratic $P = p + q$-form are given by a map $\varphi \in \Lambda^p\mathbb{R}^d \to \Lambda^{p+q}\mathbb{R}^{d+q} \ni \Phi$ defined by

$$\varphi = \sum_{(i_1,\dots,i_p)\in\mathfrak{I}^+} dx^{i_1,\dots,i_p} \longmapsto \Phi = \sum_{\sigma\in H\subset S_D} \sum_{(i_1,\dots,i_p)\in\mathfrak{I}^+} dx^{\sigma(i_1)\dots\sigma(i_p)\sigma(d+1)\dots\sigma(D)}. \qquad (5)$$

Using these mappings, a nested structure of special forms in successively higher dimensions emerges. This is reminiscent of a matryoshka (матрёшка), a set of nested Russian dolls, traditionally carved in wood, where the inner surface of each doll is basically a copy of the outer surface of the previous doll; but the outer surface can then vary somewhat, depending on the geometry of the bulk.

A remarkable nested stucture of special democratic forms was displayed in [2], which included a U(3)-invariant 2-form in six dimensions, a G_2-invariant 3-form in seven dimensions, and a Spin(7)-invariant 4-form in eight dimensions; corresponding to the embeddings $\mathrm{SU}(3) \subset G_2 \subset \mathrm{Spin}(7)$ mentioned above. It was also shown, that this matryoshka with 3 dolls fits into even larger dolls and interesting properties of a special democratic 6-form in ten dimensions were presented.

Motivated by the discussion in [2] of nested special democratic forms, we shall presently show that there exists a corresponding matryoshka of self-duality equations in successively higher dimensions; each implying the Yang-Mills equations, just as four-dimensional self-duality [4]. Successive sets of equations are 'oxidised' to higher dimensions and 'reduced' to lower dimensions by enhancing or restricting the permutation symmetries on the sets of indices of special geometric tensors. Remarkably, the simplest case of the mapping (5), with $q = D - d = 1$ corresponds to equations for (partial) curvature flows for the vector potentials, hence 'Oxidation through heat flow'. Solutions of the lower dimensional equations then provide initial values for the flow into the extra dimension, the flow to the next doll of the matryoshka. We shall dispay oxidations up to $d = 16$. The representation theory underlying the twelve dimensional system seems to be related to a mathematical curiosity, the algebra of the *sextonions* [5, 6], a six-dimensional algebra between quaternions and octonions. This algebra gives rise to a new row in Freudenthal's magic chart, corresponding to a (non-simple) Lie algebra between $\mathfrak{e}_7$ and $\mathfrak{e}_8$, which has been called $\mathfrak{e}_{7\frac{1}{2}}$ [6].

2 Generalised duality for gauge fields in d > 4

Generalisations of the four-dimensional self-duality equations to higher dimensions were introduced some time ago in [4], where it was shown that restrictions of the Yang-Mills curvature two-form F to an eigenspace of a four-form T, implies the Yang-Mills equations. In a standard orthonormal basis of T^*M these take the form,

$$\frac{1}{2} g^{km} g^{ln} T_{ijkl} F_{mn} = \lambda F_{ij}, \quad i, j, \dots = 1, \dots, d \, . \tag{6}$$

Here T_{mnpq} is a covariantly constant tensor, g^{pr} the inverse metric tensor and $F = dA + A \wedge A$ is the curvature of a connection $D = d + A$ on a Riemannian d-fold (M, g) with values in the Lie algebra of a real gauge group contained in $\mathrm{GL}(n, \mathbb{R})$. These partial-flatness conditions on the curvature are first order equations for the vector potentials A, so they are more amenable to solution than the second order Yang-Mills equations. Indeed, many special solutions are known (see e.g. [8, 9, 10]). The usefulness of the linear curvature constraints (6) follows from the observation [4]:

Theorem 1 *For nonzero eigenvalues λ, the conditions* (6) *imply the Yang-Mills equations $g^{ij} D_i F_{jk} = 0$. Thus, potentials A satisfying these first order equations automatically satisfy the Yang-Mills equations.*

This result follows in virtue of the Bianchi identities $D_{[i}F_{mn]} \equiv 0$. In [4], constant four-forms T in flat euclidean spaces were considered, but it is clear that, more generally [11], it suffices for the consistency condition

$$g^{km}g^{ln}(g^{ip}\nabla_p T_{ijkl})F_{mn} = 0 \tag{7}$$

to hold, which follows if T is co-closed, $g^{ip}\nabla_p T_{ijkl} = 0$. The latter in turn follows if T is parallel (i.e. covariantly constant) with respect to the Levi-Civita connection ∇. In dimensions $d > 4$, the four-form T clearly breaks the d-dimensional rotational invariance of the Yang-Mills equations. Examples of 4-forms and the corresponding partial-flatness conditions (6) invariant under various subgroups of $G \subset \mathrm{SO}(d)$ were studied in [4] for dimensions $4 < d \leq 8$. In particular, interesting examples invariant under $\mathrm{SU}(n)\otimes\mathrm{U}(1))/\mathbb{Z}_2$ and $\mathrm{SU}(n)$ G_2 and Spin(7), in dimensions $d = 2n, 7, 8$ were constructed. The example of $\mathrm{Sp}(n) \otimes \mathrm{Sp}(1)/\mathbb{Z}_2$ was discussed shortly thereafter in [12, 13]. The above groups are precisely the holonomy groups of Calabi-Yau, quaternionic Kähler and exceptional holonomy manifolds, so remarkably, the generalisations of self-duality for most of Berger's special holonomy manifolds [1] were unwittingly constructed before the subject acquired widespread differential geometric interest (e.g. [14, 15, 16, 17, **?**, 11]). On all the above manifolds, there exists a ∇-parallel four-form, so the above-mentioned consistency condition on T is satisfied.

The equations (6) may be expressed in terms of projection operators to the orthogonal eigenspaces (see e.g. [19])

$$\Lambda^I_{ij}F^{ij} = 0\ , \quad I = 1, \dots, \bar{d}_\lambda\,, \tag{8}$$

where the number of equations, $\bar{d}_\lambda$, is the codimension of the eigenspace corresponding to eigenvalue λ. Here, the projector Λ is the analogue of the 't Hooft tensor in four dimensions and we lower (raise) indices using the (inverse) Riemannian metric.

In even dimensions, with $d = 2n$, if the manifold M admits a complex structure J, this provides, at any point p in M, a linear map $J_p\colon\ T_pM \to T_pM$ under which the complexification $T_pM \otimes_{\mathbb{R}} \mathbb{C}$ splits into the eigenspaces $T_p^{(1,0)}M$ and $T_p^{(0,1)}M$, both of which are isomorphic to $\mathbb{C}^n$. This allows the choice of complex coordinates $(z^1, \dots, z^n)$ and $(z^{\bar{1}}, \dots, z^{\bar{n}})$. The complex (1,0)- and (0,1)-forms $\{dz^\alpha\}$ and $\{dz^{\overline{\alpha}}\}$, for $\alpha, \overline{\alpha} = 1, \dots, n$, then provide bases for $T_p^{(1,0)}M$ and $T_p^{(0,1)}M$ respectively. Imposing the reality conditions $dz^{\overline{\alpha}} = \overline{dz^\alpha}$, we may recover $\mathbb{R}^{2n} \simeq \mathbb{C}^n$. The curvature two-form in this basis has components

$F_{\alpha\beta}, F_{\alpha\overline{\beta}}, F_{\overline{\alpha}\overline{\beta}} = \overline{F}_{\alpha\beta}$ and the Riemannian metric locally takes the hermitian form $d^2s = g_{\alpha\overline{\beta}} dz^\alpha dz^{\overline{\beta}} = dz^\alpha dz_\alpha = \sum_\alpha dz^\alpha dz^{\overline{\alpha}}$ and the complex $(n, 0)$ volume form is given by $\Omega = dz^{\alpha_1 \ldots \alpha_n}$. In the complex setting, the equation (6) is a G-invariant equation, where the structure group G is a subgroup of $\mathrm{GL}(n, \mathbb{C}) \subset \mathrm{GL}(2n, \mathbb{R})$. For the particularly important $\lambda = -1$ case, we shall use the following complex variant of Theorem 1.

Theorem 2 *On a Riemannian complex n-fold (M^{2n}, g), with hermitian metric $g = g_{\alpha\overline{\beta}} dz^\alpha dz^{\overline{\beta}}$ and (4,0)-form Φ, the linear curvature constraints,*

$$F_{\alpha\beta} + \tfrac{1}{2} g^{\gamma\overline{\eta}} g^{\delta\overline{\kappa}} \ \Phi_{\alpha\beta\gamma\delta} \ F_{\overline{\eta}\overline{\kappa}} = 0\,, \tag{9}$$

$$g^{\alpha\overline{\beta}} F_{\alpha\overline{\beta}} = 0\,, \tag{10}$$

$$g^{\gamma\overline{\eta}} g^{\delta\overline{\kappa}} \ (g^{\alpha\overline{\rho}} \nabla_{\overline{\rho}} \Phi_{\alpha\beta\gamma\delta}) \ F_{\overline{\eta}\overline{\kappa}} = 0\,, \tag{11}$$

imply the Yang-Mills equations $g^{\alpha\overline{\rho}} D_{\overline{\rho}} F_{\alpha\beta} = g^{\alpha\overline{\rho}} D_{\overline{\rho}} F_{\alpha\overline{\beta}} = 0$.

Proof: Using (9) we have

$$g^{\alpha\overline{\rho}} D_{\overline{\rho}} F_{\alpha\beta} = D^\alpha F_{\alpha\beta} = -\tfrac{1}{2} (\nabla^\alpha \Phi_{\alpha\beta\gamma\delta}) \ F^{\gamma\delta} - \tfrac{1}{2} \Phi_{\alpha\beta\gamma\delta} \ D^\alpha F^{\gamma\delta} = 0, \tag{12}$$

the first term being the left side of (11) and the second vanishes in virtue of the Bianchi identity $D_{\overline{\alpha}} F_{\overline{\gamma}\overline{\delta}} +$ cyclic permutations $= 0$. Similarly, using the Bianchi identity between $D_{\overline{\rho}}, D_{\overline{\beta}}$ and D_a we have, $D_{\overline{\rho}} F_{\alpha\overline{\beta}} = D_{\overline{\beta}} F_{\alpha\overline{\rho}} + D_\alpha F_{\overline{\alpha}\overline{\rho}}$. On contracting with $g^{\alpha\overline{\rho}}$, the second term on the right yields the complex conjugate of the left side of (12) and the first term contains the trace of the (1,1)-part of the curvature, which vanishes by equation (10). □

Already in [4], it was noticed that the lower dimensional cases, including four-dimensional self-duality, the six-dimensional $\mathrm{SU}(3) \otimes \mathrm{U}(1))/\mathbb{Z}_2$-invariant equations and the seven-dimensional G_2-invariant equations, were reductions of the eight-dimensional Spin(7)-invariant set of equations. In the present paper, we show that using the results of [2] these equations also admit a systematic 'oxidation' to higher dimensions starting from the lower dimensional ones.

We consider two types of oxidation. The first is based on the map (4) and uses cyclic permutations to remix the index sets appearing in the lower dimensional equations amongst a larger set of indices. The second oxidation method is based on the heat flow for some appropriate partial curvature. This is related to the $D - d = 1$ case of (5). More specifically, if in $(d-1)$-dimensions, there exist a special set of $d-1$ curvature constraints $f_{ijk} F^{jk} = 0$, $i = 1, \ldots, d-1$, where f is some appropriate tensor, then we can consider the corresponding partial curvature flow

$$\dot{A}_i = f_{ijk} F^{jk} \ , \ i = 1, \ldots, d \, . \tag{13}$$

Identifying the parameter or 'time' of the flow with a d-th independent variable x^d, the left hand side is the $A_d = 0$ 'temporal' gauge form of the curvature components F_{di}, so that the flow equations (13) are in fact linear curvature constraints of the form

$$F_{di} = f_{ijk} F^{jk} \, . \tag{14}$$

Remarkably, in many interesting cases, these constraints may be reformulated in the form (6), thus implying the Yang-Mills equations. The idea of choosing such a temporal gauge to obtain a flow equation is not new. For instance, both Nahm's equations for magnetic monopoles [20] and the generalisations to higher dimensions of Euler's equations for a spinning top [21], arise from the imposition of precisely such a gauge choice on equations of the form (6). The converse idea, that flow equations can be interpreted as gauge covariant equations one dimension higher by gauge un-fixing the component of the gauge potential in the direction of the flow, has been used by Tao [22].

As we shall see, the juxtaposition of the two oxidation methods above yields the advertised matryoshka of self-duality equations, starting from zero curvature in $d = 2$ and including the familiar 4-dimensional self-duality, as well as its generalisations to 6,7 and 8 dimensions mentioned above. Remarkably, the matryoshka affords enlargement to even higher dimensions. We discuss an interesting 12 dimensional extension and display its oxidation to 14 and 16 dimensions.

3 The matryoshka of self-duality equations

Let us begin in two dimensions with the flatness condition $F_{12} = 0$ for the sole component of the curvature two-form. In the complex setting, the curvature only has a (1,1)-part, $F_{z\bar{z}}$, where we use complex coordinates $z = x^1 + ix^2$, $\bar{z} = x^1 - ix^2$. The flatness condition means that the curvature is in the kernel of the volume form. We therefore have,

$$\epsilon_{ij} F^{ij} = 0 \quad \Leftrightarrow \quad F_{12} = 0 \quad \Leftrightarrow \quad F_{z\bar{z}} = 0 \ , \tag{15}$$

Both real and complex forms of the equations are locally rotationally invariant, since their respective invariance algebras $\mathfrak{so}(2)$ and $\mathfrak{u}(1)$ are isomorphic. The rich properties of the solutions of these equations on Riemann surfaces have been investigated by Atiyah and Bott [23].

We oxidise the equation $F_{12} = 0$ to a system in three dimensions by acting on the indices by all permutations generated by the cycle $\sigma = (1\ 2\ 3) \in S_3$, so as to obtain a system of equations invariant under these permutations:

$$\{F_{12} = 0\} \longrightarrow \{F_{\sigma^p(1)\sigma^p(2)} = 0\ ;\ \sigma = (1\ 2\ 3)\ ,\ p = 1, 2\} \tag{16}$$

This of course yields flatness in 3 dimensions; the curvature lies in the kernel of the three-dimensional volume form,

$$\epsilon_{ijk}F^{jk} = 0 \quad \Leftrightarrow \quad F_{12} = F_{23} = F_{31} = 0\ .lad3 \tag{17}$$

Since this is a set of 3 equations for the three vector potentials A_i, $i = 1, 2, 3$, it allows us to write the Yang-Mills curvature flow

$$\frac{\partial}{\partial x^4}A_i(x^i, x^4) = \tfrac{1}{2}\epsilon_{ijk}F^{jk}\,, \quad i = 1, 2, 3\ , \tag{18}$$

with initial (at $x^4 = 0$) flat connection $A_i(x^i, 0)$ satisfying (**??**). This is the gradient flow of the Chern-Simons functional [24]

$$S_{CS} = \int_{M^3} \mathrm{Tr}\ (\tfrac{1}{2}AdA + \tfrac{1}{3}A^3) = \int_{M^3} \mathrm{Tr}\ (\tfrac{1}{2}A_i\partial_jA_k + \tfrac{1}{3}A_iA_jA_k)dx^{ijk}, \tag{19}$$

where $dx^{ijk} = dx^i \wedge dx^j \wedge dx^k$, the volume form. In his canonical quantisation of this theory, Witten [24] considered the 3-fold to be of the form $M^3 = \Sigma \times \mathbb{R}^1$, where the data on the 2-dimensional boundary Σ, a Riemann surface, satisfied the equations (15).

Now applying an x^4-dependent gauge transformation to the vector potentials

$$A_a \mapsto g^{-1}(x^i, x^4)A_ag(x^i, x^4) + g^{-1}(x^i, x^4)\partial_ag(x^i, x^4)\ ,\ a = 1, \ldots, 4, \tag{20}$$

which yields a pure-gauge form for the fourth vector potential, $A_4 = g^{-1}\partial_4 g$. The non-gauge covariant equation (18) now takes the gauge covariant form of the four dimensional SO(4)-invariant anti-self-duality equations

$$F_{ab} + \tfrac{1}{2}\epsilon_{abcd}F^{cd} = 0\,, \quad a, b, c, d = 1, \ldots 4\ , \tag{21}$$

a set of 3 equations for the 4 vector potentials. (The self-duality equations emerge on reversing the x^4-direction of the flow.)

Using a manifestly $\mathfrak{u}(2)$-covariant notation for Yang's complex coordinates (z^{α} , $z^{\overline{\alpha}} := \overline{z^{\alpha}}$, $\alpha, \overline{\alpha} = 1,2$), these equations take the form (c.f. (15)) [25],

$$\begin{aligned} \Omega_{\alpha\beta} F^{\alpha\beta} &= 0 \quad \Leftrightarrow \quad F_{\bar{1}\bar{2}} = 0 && (22) \\ g^{\alpha\overline{\beta}} F_{\alpha\overline{\beta}} &= 0 \quad \Leftrightarrow \quad F_{1\bar{1}} + F_{2\bar{2}} = 0\ . && (23) \end{aligned}$$

This is a system consisting of one complex and one real equation, leaving as the sole non-zero part, the trace-free part of the (1,1)-curvature. The U(2)-invariant metric on $\mathbb{C}^2 \simeq \mathbb{R}^4$ is given by $g_{\alpha\overline{\beta}} dz^{\alpha} dz^{\overline{\beta}} = dz^1 dz^{\bar{1}} + dz^2 dz^{\bar{2}} =: dz^{\alpha} dz^{\overline{\alpha}}$ and the symplectic (2,0) volume form, invariant under SU(2), by $\Omega = \Omega_{\alpha\beta} dz^{\alpha} \wedge dz^{\beta} = dz^1 \wedge dz^2 =: dz^{12}$.

Now, complexifying all the data by dropping all reality conditions (see for instance the discussion in [11]), we obtain the additional equation $F_{\alpha\beta} = 0$, which allows us to choose the holomorphic gauge $A_{\alpha} = 0$. The equation (23) then takes the form of a conservation law [26]

$$g^{\alpha\overline{\beta}} \partial_{\alpha} A_{\overline{\beta}} = \partial^{\overline{\beta}} A_{\overline{\beta}} = 0\ , \quad \alpha, \overline{\beta} = 1, 2\,, \tag{24}$$

which has local solution $A_{\overline{\beta}} = \Omega_{\overline{\beta}\overline{\gamma}} \partial^{\overline{\gamma}} f$, where $\overline{\Omega} = \Omega_{\overline{\alpha}\overline{\beta}} dz^{\overline{\alpha}} \wedge dz^{\overline{\beta}} = dz^{\bar{1}\bar{2}}$ is the symplectic (0,2)-form. The remaining equation in (22) then takes the form of Leznov's wave equation [27]

$$\Box f + \tfrac{1}{2} \Omega^{\alpha\beta} \left[\partial_{\alpha} f\ ,\ \partial_{\beta} f \right] = 0\ , \tag{25}$$

with Laplacian $\Box = g^{\alpha\overline{\beta}} \partial_{\alpha} \partial_{\overline{\beta}} = \partial^{\overline{\alpha}} \partial_{\overline{\alpha}}$. Solutions provide stationary points of the Leznov functional

$$S_L = \int_{M^{\mathbb{C}}} \mathrm{Tr}\ \left(\frac{1}{2} f \Box f + \frac{1}{3} \Omega^{\alpha\beta} f \partial_{\alpha} f \partial_{\beta} f \right)\ . \tag{26}$$

whose variation has the standard heat equation form,

$$\frac{\partial}{\partial t} f = \Box f + \tfrac{1}{2} \Omega^{\alpha\beta} \left[\partial_{\alpha} f\ ,\ \partial_{\beta} f \right]\ . \tag{27}$$

In this case, the left-hand-side side does not allow interpretation as a (gauge-fixed) component of the curvature.

In all the above cases, in dimensions $d = 1, \ldots, 4$, the equations are fully SO(d)-invariant. The special geometric structures characterising these equations are thus precisely the volume forms, which are trivially special democratic forms. The oxidised volume form in d-dimensions $\mathrm{vol}_d = dx^{1\dots d}$ may be obtained from lower dimensional volume forms by taking succesive wedge products with the additional basis one-forms, $\mathrm{vol}_d = \mathrm{vol}_{d-1} \wedge dx^d$.

4 From four to eight dimensions

4.1 Permutation to d=6

To proceed to higher dimensions, we now consider the complex version (22),(23) of the four-dimensional equations. Following the previous mapping from two to three dimensions (16), we now oxidise these equations from $\mathbb{C}^2$ to $\mathbb{C}^3$ by requiring invariance under the cyclic permutations generated by $\sigma = (1\ 2\ 3) \in S_3$, where the indices are now complex;

$$\{F_{\bar{1}\bar{2}} = 0\} \longrightarrow \{F_{\sigma^p(1)\sigma^p(2)} = 0\ ;\ \sigma = (\bar{1}\ \bar{2}\ \bar{3})\ ,\ p = 1, 2\}. \tag{28}$$

This yields the system (c.f. (**??**))

$$\Omega_{\alpha\beta\gamma} F^{\beta\gamma} = 0 \quad \Leftrightarrow \quad \{F_{\bar{1}\bar{2}} = F_{\bar{2}\bar{3}} = F_{\bar{3}\bar{1}} = 0\} \tag{29}$$

$$g^{\alpha\overline{\beta}} F_{\alpha\overline{\beta}} = 0 \quad \Leftrightarrow \quad F_{1\bar{1}} + F_{2\bar{2}} + F_{3\bar{3}} = 0\ ,\quad \alpha, \overline{\alpha} = 1, 2, 3\ , \tag{30}$$

a set of three complex and one real equation. Here $g_{\alpha\overline{\beta}} dz^\alpha dz^{\overline{\beta}}$ is the U(3)-invariant hermitian metric and $\Omega = dz^1 \wedge dz^2 \wedge dz^3 = dz^{123}$, the complex (3,0) volume form. These equations were obtained in [4] as SU(3)⊗U(1))/$\mathbb{Z}_2$-invariant curvature constraints which imply the second order Yang-Mills equations. They later made an appearance in work by Donaldson [28], Uhlenbeck and Yau [18] as the equations for holomorphic connections on three (complex) dimensional Kähler manifolds, g being the Kähler metric.

In the six real coordinates, $x^\alpha := \mathrm{Re}\, z^\alpha$, $x^{\alpha+3} = \mathrm{Im}\, z^\alpha$, $\alpha = 1, 2, 3$, the equations take the form (6), with the special democratic four-form (see [4])

$$T_{(6)} = dx^{1425} + dx^{1436} + dx^{2536}. \tag{31}$$

This is invariant under the group S_3 of permutations of the 3 ordered pairs $(\{1, 4\}, \{2, 3\}, \{4, 5\})$, or, equivalently, the symmetries generated by the permutation $\sigma = (123)(456) \in S_6$. The stabiliser of $T_{(6)}$ in SO(6) is the group $\mathrm{SU}(3) \times \mathrm{U}(1)/\mathbb{Z}_2$ and under this, the space of 2-forms has the following decomposition into eigenspaces of $T_{(6)}$ [4]:

$$\Lambda^2\mathbb{R}^6 = \big(\mathfrak{su}(3)_0\, ,\ \lambda = -1\big) \oplus \big(V_2^3 \oplus \overline{V}^3_{-2}\, ,\ \lambda = 1\big) \oplus \big(\mathbb{R}\omega_0\, ,\ \lambda = 2\big)\, , \tag{32}$$

where $(V_q^n\,,\,\lambda)$ is the n-dimensional irreducible representation of SU(3), the index q denotes the U(1) charge, λ the eigenvalue of $T_{(6)}$ and $\omega_0 = g_{\alpha\overline{\beta}} dz^\alpha \wedge dz^{\overline{\beta}}$ is the invariant metric form associated with g. Two-forms parallel to ω_0 are contained in the $l=2$ eigenspace. Under the action of $T_{(6)}$ the curvature tensor therefore decomposes into $T_{(6)}$-eigenspaces according to

$$F = \big(F_{\gamma\bar{\delta}} - \tfrac{1}{3} g_{\gamma\bar{\delta}} F_0\,,\, \lambda{=}-1\big) \oplus \big(F_{\alpha\beta} \oplus F_{\overline{\alpha}\overline{\beta}}\,,\, \lambda{=}1\big) \oplus \big(F_{\alpha\overline{\alpha}}\,,\, \lambda{=}2\big), \tag{33}$$

where F_0 denotes the trace $g^{\alpha\overline{\beta}} F_{\alpha\overline{\beta}}$. The set of seven equations (29), (30) thus projects the curvature to the 8-dimensional $\mathfrak{su}(3)$ part, the $\lambda = -1$ eigenspace.

Analogously to (23), complexifying the Yang-Mills fields, the equation (30), in the holomorphic gauge $A_\alpha = 0\,,\ \alpha = 1,2,3\,,$ can be locally solved in terms of three prepotentials taking values in the complexification of the gauge group:

$$A_{\overline{\alpha}} = \Omega_{\overline{\alpha}\overline{\beta}\overline{\gamma}} \partial^{\overline{\gamma}} f^{\overline{\beta}}\ . \tag{34}$$

The remaining conditions (29) provide extrema of the Chern-Simons action

$$\begin{aligned} S \ &= \int_{M_{\mathbb{C}}} \mathrm{Tr}\ (\bar{A}\bar{\partial}\bar{A} + \bar{A}^3) \wedge *\overline{\Omega} \\ &= \int_{M_{\mathbb{C}}} \mathrm{Tr}\ (\tfrac{1}{2} A_{\overline{\alpha}} \partial_{\overline{\beta}} A_{\overline{\gamma}} + \tfrac{1}{3} A_{\overline{\alpha}} A_{\overline{\beta}} A_{\overline{\gamma}})\ dz^{\overline{\alpha}\overline{\beta}\overline{\gamma}}\,. \end{aligned} \tag{35}$$

Inserting (34) in (29) yields a wave equation analogous to (25) for the triplet of complex prepotentials f_β,

$$\partial^\beta \partial_{[\alpha} f_{\beta]} + \tfrac{1}{2} \Omega^{\beta\delta\eta} [\partial_\delta f_\eta\ ,\ \partial_{[\alpha} f_{\beta]}] = 0\ . \tag{36}$$

The associated heat flow equation takes the form

$$\frac{\partial}{\partial t} f_\alpha = \partial^\beta \partial_{[\alpha} f_{\beta]} + \tfrac{1}{2} \Omega^{\beta\delta\eta}\ [\partial_\delta f_\eta\ ,\ \partial_{[\alpha} f_{\beta]}]\ . \tag{37}$$

The reduction of (29),(30) to the missing $d=5$ case involves choosing a constant unit vector in $\mathbb{R}^6$ and projecting to the five-dimensional space orthogonal to it. Without loss of generality, we may simply choose one of the basis vectors, say e_6, effectively deleting the variables x^6 and yielding an SO(4)-invariant 4-form $T = dx^{1245}$. The corresponding equations (see [4]) are an embedding of four-dimensional self-duality (21) in five dimensional space. A five dimensional reduction of the Chern-Simons action (35) and corresponding flow equations were discussed some time ago by Nair and Schiff [29].

4.2 Flow to d=7 and d=8

Since the three complex equations (29) have an action (35), we may write down the partial curvature flow, for the three complex potentials A_α, now depending on seven variables $(z^\alpha, z^{\overline{\alpha}}, x^7)$:

$$\frac{\partial}{\partial x^7} A_\alpha = \Omega_{\alpha\beta\gamma} F^{\beta\gamma} \ , \quad \alpha = 1, 2, 3. \tag{38}$$

This being the gradient flow for the functional (35). Now, analogously to the four-dimensional case (20), an x^7-dependent gauge transformation yields the fully gauge covariant form of this partial curvature flow

$$F_{7\alpha} = \Omega_{\alpha\beta\gamma} F^{\beta\gamma} \quad \Leftrightarrow \quad \{F_{71} = F_{\bar{2}\bar{3}} \ , \ F_{72} = F_{\bar{3}\bar{1}} \ , \ F_{73} = F_{\bar{1}\bar{2}}\} \ . \tag{39}$$

Here $\partial/\partial x^7$ denotes the real vector field (the 'time' of the flow) and $\alpha, \beta, \gamma = 1, 2, 3$ are complex indices. The three complex equations (39) together with the real equation,

$$g^{\alpha\overline{\beta}} F_{\alpha\overline{\beta}} = F_{1\bar{1}} + F_{2\bar{2}} + F_{3\bar{3}} = 0 \ , \tag{40}$$

imply the Yang-Mills equations in seven dimensions. Choosing real coordinates $(x^1, \ldots, x^7)$, these equations they take the manifestly G_2-invariant form [4]

$$\psi_{ijk} F^{jk} = 0 \ , \quad i, j, k = 1, \ldots, 7 \, . \tag{41}$$

Here ψ is the G_2-invariant Cayley three form whose components ψ_{ijk} provide structure constants of the algebra of imaginary octonions. Choosing the first six real coordinates as the real and imaginary parts of the complex coordinates as follows, $z^\alpha = x^\alpha + i x^{\alpha+3}$, $\alpha = 1, 2, 3$, we obtain,

$$\psi = dx^{367} + dx^{257} + dx^{147} + dx^{465} + dx^{243} + dx^{135} + dx^{162}. \tag{42}$$

Its four-form dual is given by

$$\varphi := *\psi = dx^{1245} + dx^{1346} + dx^{2356} + dx^{7123} + dx^{1567} + dx^{7246} + dx^{3457}, \tag{43}$$

in terms of which the equations (41) take the form,

$$F_{ij} + \tfrac{1}{2} \varphi_{ijkl} F^{kl} = 0 \ , \quad i, j, k = 1, \ldots, 7 \, , \tag{44}$$

which projects the curvature to the $\lambda = -1$ eigenspace of φ; the eigenspace decomposition of the space of 2-forms being [4]

$$\Lambda^2 \mathbb{R}^7 = (\mathfrak{g}_2 \, , \, \lambda = -1) \oplus (\mathbb{R}^7 \, , \, \lambda = 3). \tag{45}$$

Since the system (41) consists of 7 equations for 7 potentials and has the Chern-Simons type action

$$S_{CS} = \int_{M^7} \mathrm{Tr}\ (AdA + A^3) \wedge *\psi = \int_{M^7} \mathrm{Tr}\ (\tfrac{1}{2}A_i\partial_j A_k + \tfrac{1}{3}A_iA_jA_k)\ \psi^{ijk}\,, \quad (46)$$

we can immediately write down the corresponding partial curvature flow in eight dimensions analogous to (18):

$$\frac{\partial}{\partial x^8}A_i = \tfrac{1}{2}\psi_{ijk}F^{jk}\,, \quad i = 1, \ldots, 7\,. \quad (47)$$

This is the temporal gauge ($A_8 = 0$) form of the Spin(7)-invariant equations in eight dimensions, which were discovered in [4] and shown there to arise as the projection of the curvature form to the $\lambda = -1$ eigenspace of the Spin(7)-invariant 4-form ϕ,

$$F_{ab} + \tfrac{1}{2}\phi_{abcd}F^{cd} = 0\,, \quad a, b, c, d = 1, \ldots 8\ ,, \quad (48)$$

where in terms of the seven dimensional forms ψ , φ in (41) and (44) the four-form ϕ in eight dimensions is given by $\phi = dx^8 \wedge \psi + \varphi$. The decomposition $\Lambda^2\mathbb{R}^8$ into eigenspaces of this 4-form is given by

$$\Lambda^2\mathbb{R}^8 = (\mathfrak{spin}_7\,,\ \lambda = -1) \oplus (\mathbb{R}^7\,,\ \lambda = 3). \quad (49)$$

In complex coordinates, $z^\alpha = x^\alpha + ix^{\alpha+4}$, $\alpha = 1, 2, 3, 4$, the equations (48) take the form (see [4]) incorporating (39),

$$\begin{aligned} F_{\alpha\beta} + \tfrac{1}{2}\Omega_{\alpha\beta\gamma\delta}\ F^{\gamma\delta} = 0 \ &\Leftrightarrow\ \{F_{41} = F_{\bar{2}\bar{3}}\ ,\ F_{42} = F_{\bar{3}\bar{1}}\ ,\ F_{43} = F_{\bar{1}\bar{2}}\} \\ g^{\alpha\overline{\beta}}F_{\alpha\overline{\beta}} = 0 \ &\Leftrightarrow\ F_{1\bar{1}} + F_{2\bar{2}} + F_{3\bar{3}} + F_{4\bar{4}} = 0\ , \end{aligned} \quad (50)$$

where g is the U(4)-invariant hermitian metric on $\mathbb{C}^4 \simeq \mathbb{R}^8$ and $\Omega = dz^{1234}$ is the SU(4)-invariant volume form in $\mathbb{C}^4$. In the complex 'temporal' gauge, $A_4 = 0$, the three complex equations in (50) therefore take the form of a partial curvature flow with complex flow parameter z^4,

$$\frac{\partial}{\partial z^4}A_\alpha \ = \ \tfrac{1}{2}\Omega_{\alpha\beta\gamma}\ F^{\beta\gamma} \quad (51)$$

$$\frac{\partial}{\partial z^4}A_{\bar{4}} \ = \ g_3^{\alpha\overline{\beta}}F_{\alpha\overline{\beta}} \quad (52)$$

where $\Omega_{\alpha\beta\gamma}, g_3^{\alpha\overline{\beta}}$ are the volume form and inverse metric of the complex 3-space orthogonal to the complex vector field $\partial/\partial z^4$. The equation (51) thus gives the complex variation of the Chern-Simons action (35).

All the above duality equations in dimensions up to eight are more or less well-known [4]. Our main result is that the pattern of succesive dimensional oxidation actually continues to higher dimensions. Proceeding further, we see that a particularly interesting 12-dimensional system results.

5 Self-duality in 12 dimensions

Following the method of oxidising the duality equations from $\mathbb{R}^4$ to $\mathbb{R}^6$, we now extend the system (50) in $\mathbb{C}^4$ to $\mathbb{C}^6$ by juxtaposing two additional complex variables z^5, z^6 and then remixing the six complex indices by requiring symmetry under permutations generated by $\sigma = (135)(246) \in S_6$. We thus obtain the equations,

$$g^{\alpha\overline{\beta}}F_{\alpha\overline{\beta}} = F_{1\bar{1}} + F_{2\bar{2}} + F_{3\bar{3}} + F_{4\bar{4}} + F_{5\bar{5}} + F_{6\bar{6}} = 0 \; , \tag{53}$$

together with

$$F_{12} + F_{\bar{3}\bar{4}} + F_{\bar{5}\bar{6}} = 0 \, , \quad F_{34} + F_{\bar{5}\bar{6}} + F_{\bar{1}\bar{2}} = 0 \, , \quad F_{56} + F_{\bar{1}\bar{2}} + F_{\bar{3}\bar{4}} = 0 \tag{54}$$

$$\begin{array}{lll} F_{13} + F_{\bar{4}\bar{2}} = 0 \, , & F_{14} + F_{\bar{2}\bar{3}} = 0 \, , & F_{15} + F_{\bar{6}\bar{2}} = 0 \\ F_{16} + F_{\bar{2}\bar{5}} = 0 \; , & F_{35} + F_{\bar{6}\bar{4}} = 0 \, , & F_{36} + F_{\bar{4}\bar{5}} = 0 \, . \end{array} \tag{55}$$

These equations imply the 12-dimensional Yang-Mills equations! The proof follows from Theorem 2 and the observation that these equations allow expression in the form (9), (10), with the (4,0)-form Φ taking the form

$$\Phi = dz^{1234} + dz^{1256} + dz^{3456} \, . \tag{56}$$

This four-form is thus given by $\Phi = \omega^2$, where ω is the symplectic (2,0)-form

$$\omega = dz^{12} + dz^{34} + dz^{56} \in \Lambda^2\mathbb{C}^6 \, . \tag{57}$$

This is analogous to the $\mathbb{R}^6$ case, except that now everything is complex. The (4,0)-form Φ is manifestly invariant under the action of $\mathrm{Sp}(3) \subset \mathrm{SU}(6) \subset \mathrm{Spin}(12)$.

The three conditions in (54) are equivalent to the four real equations,

$$\begin{aligned} \mathrm{Im}(F_{\bar{1}\bar{2}}) = \mathrm{Im}(F_{\bar{3}\bar{4}}) = \mathrm{Im}(F_{\bar{5}\bar{6}}) &= 0 \; , \\ \mathrm{Re}(\omega^{\overline{\alpha}\overline{\beta}}F_{\overline{\alpha}\overline{\beta}}) = \mathrm{Re}(F_{\bar{1}\bar{2}} + F_{\bar{3}\bar{4}} + F_{\bar{5}\bar{6}}) &= 0 \, , \end{aligned} \tag{58}$$

where the symplectic (0,2)-form $\overline{\omega} = \omega_{\overline{\alpha}\overline{\beta}} dz^{\overline{\alpha}} \wedge dz^{\overline{\beta}} = dz^{\bar{1}\bar{2}} + dz^{\bar{3}\bar{4}} + dz^{\bar{5}\bar{6}}$ and $\omega^{\overline{\alpha}\overline{\beta}}\omega_{\overline{\beta}\overline{\gamma}} = \delta^{\overline{\alpha}}_{\overline{\gamma}}$. The system of equations thus consists of 5 real equations, (53) and (58), together with 6 complex equations (55), a total of 17 real equations.

The entire system (53),(54),(55) in real coordinates for 12-dimensional euclidean space given by $x^i = \mathrm{Re}\, z^i$, $x^{i+6} = \mathrm{Im}\, z^i$, $i = 1, \dots, 6$, takes the following form. Here we denote the indices 10,11,12 by $0, a, b$ respectively.

$$F_{12} + F_{34} + F_{56} + F_{87} + F_{09} + F_{ba} = 0 \tag{59}$$

$$F_{17} + F_{28} + F_{39} + F_{40} + F_{5a} + F_{6b} = 0 \tag{60}$$

$$\begin{aligned}
&F_{13} + F_{42} + F_{97} + F_{80} = 0 \\
&F_{14} + F_{23} + F_{07} + F_{98} = 0 \\
&F_{15} + F_{62} + F_{a7} + F_{8b} = 0 \\
&F_{16} + F_{25} + F_{b7} + F_{a8} = 0 \\
&F_{35} + F_{64} + F_{a9} + F_{0b} = 0 \\
&F_{36} + F_{45} + F_{b9} + F_{a0} = 0 \\
&F_{19} + F_{73} + F_{84} + F_{20} = 0 \\
&F_{10} + F_{74} + F_{92} + F_{38} = 0 \\
&F_{1a} + F_{75} + F_{86} + F_{2b} = 0 \\
&F_{1b} + F_{76} + F_{a2} + F_{58} = 0 \\
&F_{3a} + F_{95} + F_{06} + F_{4b} = 0 \\
&F_{3b} + F_{96} + F_{a4} + F_{50} = 0
\end{aligned} \tag{61}$$

$$\begin{aligned}
&F_{18} + F_{72} = 0 \\
&F_{30} + F_{94} = 0 \\
&F_{5b} + F_{a6} = 0
\end{aligned} \tag{62}$$

These equations have the familiar form (6), with the 4-form $T_{(12)} \in \Lambda^4\mathbb{R}^{12}$ being given by the special democratic form

$$\begin{aligned}
T_{(12)} \quad = \quad & dx^{1234} + dx^{1256} + dx^{1287} + dx^{1209} + dx^{12ba} + dx^{1397} + dx^{1380} \\
& + dx^{1407} + dx^{1498} + dx^{15a7} + dx^{158b} + dx^{16b7} + dx^{16a8} + dx^{2307} \\
& + dx^{2398} + dx^{2479} + dx^{2408} + dx^{25b7} + dx^{25a8} + dx^{267a} + dx^{26b8} \\
& + dx^{3456} + dx^{3487} + dx^{3409} + dx^{34ba} + dx^{35a9} + dx^{35b0} + dx^{36b9} \\
& + dx^{36a0} + dx^{45b9} + dx^{45a0} + dx^{469a} + dx^{46b0} + dx^{5687} + dx^{5609} \\
& + dx^{56ba} + dx^{7890} + dx^{78ab} + dx^{90ab},
\end{aligned} \tag{63}$$

which has a set of 39 non-zero components. The characteristic polynomial of this Sp(3)-invariant four-form, acting on the space of two-forms has been calculated using Maple. The eigenspace decomposition of the space of 2-forms in terms of Sp(3) representations (see e.g. [30, 31]) is given by

$$\begin{aligned}\Lambda^2\mathbb{R}^{12} \;&=\; \big(\mathfrak{sp}_3 \oplus V^{14}(\pi_2) \oplus V^{14}(\pi_2)\,,\, \lambda=-1\big) \;\oplus\; \big(V^{14}(\pi_2)\,,\,\lambda=-3\big)\\ &\quad\oplus\; (\mathbb{C}\omega\,,\,\lambda=-5) \;\oplus\; (\mathbb{R}\omega_0\,,\,\lambda=3)\,.\end{aligned} \tag{64}$$

Here, ω is the symplectic form (57) and ω_0 the metric form $\omega_0 = g_{\alpha\overline{\beta}}dz^\alpha \wedge dz^{\overline{\beta}}$. $V^{14}(\pi_2)$ denotes the 14-dimensional representation with highest weight π_2, the 2nd fundamental weight of $\mathfrak{sp}_3$. The 4-form $T_{(12)}$ is in fact one of six Sp(3)-invariant 4-forms in 12 dimensions. The 17 equations (59)-(61) project the curvature two-form to the 49-dimensional eigenspace with eigenvalue $\lambda = -1$. The other eigenspaces have rather small dimensions compared with $\dim(\Lambda^2\mathbb{R}^{12}) = 66$. We therefore expect the corresponding solutions to be rather trivial. Sp(3), the stabiliser of the 4-form $T_{(12)}$ is a maximal subgroup of SU(6).

The similarity of the equations (53)-(55) to the three and six dimensional sytems in $\mathbb{R}^3$ and $\mathbb{C}^3 \simeq \mathbb{R}^6$ discussed above suggests that this is the counterpart in three dimensional quaternionic space $\mathbb{H}^3 \simeq \mathbb{C}^6$. The imaginary quaternion units satisfy $i^2 = j^2 = k^2 = -1$ and $ij = -ji = k$, together with the relations which result on cyclically permuting (i, j, k). We consider $\mathbb{C}$ to be an $\mathbb{R}$-vector space spanned by $(1, i)$ and $\mathbb{H}$ a $\mathbb{C}$-vector space spanned by $(1, j)$. Scalar multiplication of $z \in \mathbb{C}$ with the quaternionic basis element j satisfies $zj = j\overline{z}$, so quaternions may be written in the form

$$q := z + j\overline{w} = z + wj\,, \quad q \in \mathbb{H}\,,\ z, w \in \mathbb{C}\,. \tag{65}$$

The conjugate quaternion is then given by

$$\overline{q} := \overline{z} - wj = \overline{z} - j\overline{w}\,, \quad q \in \mathbb{H}\,,\ z, w \in \mathbb{C}\ . \tag{66}$$

The conjugate imaginary units are clearly given by $\;\overline{i} = -i\,,\ \overline{j} = -j\,,\ \overline{k} = -k\,.$ Quaternions being noncommutative, conjugation is an involutive antiautomorphism, i.e. $\overline{\overline{q}} = q$ and $\overline{q_1 q_2} = \overline{q}_2\overline{q}_1$. There exist related involutive automorphisms given by conjugation with the quaternion units,

$$\begin{aligned}\mathrm{id}:\ q\ &\mapsto\quad\ \ q = z + wj\,,\\ \alpha:\ q\ &\mapsto\ -iqi = z - wj\,,\\ \beta:\ q\ &\mapsto\ -jqj = \overline{z} + \overline{w}j\,,\\ \gamma:\ q\ &\mapsto\ -kqk = \overline{z} - \overline{w}j\,,\end{aligned} \tag{67}$$

in terms of which the real and imaginary parts of q can be expressed as linear combinations of $q, \alpha(q), \beta(q), \gamma(q)$ (see e.g. [32]).

Now, let M be a three quaternionic-dimensional (i.e. 12 real-dimensional) space. In a local coordinate frame $T_pM \simeq \mathbb{H}^3 \simeq \mathbb{C}^6$. We define three quaternionic coordinates $q^A\,,\ A = 1,2,3$, in terms of pairs of the complex coordinates $z^\alpha := x^\alpha + ix^{\alpha+6}\,,\ \alpha = 1,\dots,6$ used above,

$$
\begin{aligned}
q^1 &:= z^1 + z^2 j \;=\; x^1 + ix^7 + jx^2 + kx^8\,, \\
q^2 &:= z^3 + z^4 j \;=\; x^3 + ix^9 + jx^4 + kx^0\,, \\
q^3 &:= z^5 + z^6 j \;=\; x^5 + ix^a + jx^6 + kx^b
\end{aligned}
\tag{68}
$$

and we denote the conjugate coordinates as $\overline{q^A} = q^{\overline{A}}$.

For any two quaternionic vector fields Q_1, Q_2 the curvature components $F(Q_1, \gamma(Q_2))$ and $F(Q_2, \gamma(Q_1))$ have the same content in terms of real curvature components, since γ is an involutive automorphism. We now denote the basis vectors of the coordinate vector fields on M by $Q_A := \partial/\partial q^A$, their quaternionic conjugates by $Q_{\overline{A}} := \overline{Q_A} = \partial/\partial q^{\overline{A}}$ and their α, β, γ-conjugates by $Q_{\alpha(A)} := \alpha(Q_A)$, etc. The hermitian metric in local quaternionic coordinates is given by $d^2s = g_{A\overline{B}} dq^A dq^{\overline{B}} = dq^1 dq^{\overline{1}} + dq^2 dq^{\overline{2}} + dq^3 dq^{\overline{3}}$.

Proposition 1 *On a three quaternionic dimensional Riemannian manifold, the following 8 quaternionic curvature constraints are equivalent to the system* (59)-(62) *of self-duality equations in 12 dimensions:*

$$
g^{A\overline{B}}\; F(Q_{\overline{B}}\,,\, Q_{\alpha(A)}) \;=\; \sum_{A=1}^{3} F(Q_{\overline{A}}\,,\, Q_{\alpha(A)}) \;=\; 0 \tag{69}
$$

$$
g^{A\overline{B}}\; F(Q_{\overline{B}}\,,\, Q_{\beta(A)}) \;=\; \sum_{A=1}^{3} F(Q_{\overline{A}}\,,\, Q_{\beta(A)}) \;=\; 0 \tag{70}
$$

$$
F(Q_{\overline{1}}\,,\, Q_{\gamma(2)}) \;=\; F(Q_{\overline{2}}\,,\, Q_{\gamma(3)}) \;=\; F(Q_{\overline{3}}\,,\, Q_{\gamma(1)}) \;=\; 0 \tag{71}
$$

$$
F(Q_{\overline{1}}\,,\, Q_{\gamma(1)}) \;=\; F(Q_{\overline{2}}\,,\, Q_{\gamma(2)}) \;=\; F(Q_{\overline{3}}\,,\, Q_{\gamma(3)}) \;=\; 0\,. \tag{72}
$$

Proof: The equivalence to the 17 equations (59)-(62), or equivalently to the complex form (53)-(55) follows from a direct expansion of the quaternionic vector fields in the basis $(1, j)$. $\square$

6 Flowing to 14 dimensions

The similarity between the 3 quaternionic equations in (71), the 3 complex equations in (29) and the 3 real equations in (**??**) immediately suggests that in analogy to the flows (18) and (38), we may write down flows for the three quaternionic partial curvatures in (71) into a futher complex direction, with coordinate z^7. We write, in $M = M^3_{\mathbb{H}} \times \mathbb{C}$ with coordinates (q^1, q^2, q^3, z^7), in analogy with (51) and (52),

$$
\begin{aligned}
\frac{\partial}{\partial z^7} A(Q_1) &= -F(Q_{\overline{2}}\,,\, Q_{\gamma(3)}) \\
\frac{\partial}{\partial z^7} A(Q_2) &= -F(Q_{\overline{3}}\,,\, Q_{\gamma(1)}) \\
\frac{\partial}{\partial z^7} A(Q_3) &= -F(Q_{\overline{1}}\,,\, Q_{\gamma(2)}) \\
\frac{\partial}{\partial z^7} A(Z_{\overline{7}}) &= \sum_{A=1}^{3} F(Q_{\overline{A}}\,,\, Q_{\alpha(A)})
\end{aligned} \tag{73}
$$

together with (70), considered as an equation in 14 dimensions,

$$
\sum_{A=1}^{3} F(Q_{\overline{A}}\,,\, Q_{\beta(A)}) = 0\,. \tag{74}
$$

Writing the quaternionic vector fields $Q_A\,,\ A = 1, \ldots, 3$ in terms of complex vector fields $Z_\alpha\,,\ \alpha = 1, \ldots, 6$ according to the choice in (68) and unravelling the $A(Z_7) = 0$ gauge, we obtain the system

$$
\begin{aligned}
F(Z_7\,,\, Z_1 + Z_2 j) \ + \ F(\overline{Z_3} - j\overline{Z_4}\,,\, \overline{Z_5} - \overline{Z_6} j) &= 0 \\
F(Z_7\,,\, Z_3 + Z_4 j) \ + \ F(\overline{Z_5} - j\overline{Z_6}\,,\, \overline{Z_1} - \overline{Z_2} j) &= 0 \\
F(Z_7\,,\, Z_5 + Z_6 j) \ + \ F(\overline{Z_1} - j\overline{Z_2}\,,\, \overline{Z_3} - \overline{Z_4} j) &= 0 \\
F(Z_7\,,\, \overline{Z_7}) \ - \ \sum_{\alpha=1}^{3} F(\overline{Z_{2\alpha-1}} - j\overline{Z_{2\alpha}}\,,\, Z_{2\alpha-1} - j\overline{Z_{2\alpha}}) &= 0 \\
\sum_{\alpha=1}^{3} F(\overline{Z_{2\alpha-1}} - j\overline{Z_{2\alpha}}\,,\, \overline{Z_{2\alpha-1}} + j Z_{2\alpha}) &= 0
\end{aligned} \tag{75}
$$

Expanding the quaternionic vector fields in the basis $(1, j)$, we obtain equations on $\mathbb{C}^7$, which are contained in the system

$$F_{\alpha\beta} + \tfrac{1}{2}\Phi_{\alpha\beta\gamma\delta}\, F_{\overline{\gamma}\overline{\delta}} = 0 \tag{76}$$

$$g^{\alpha\overline{\beta}} F_{\alpha\overline{\beta}} = 0 \tag{77}$$

with Φ given by the $\mathrm{G}_2^{\mathbb{C}}$-invariant (4,0)-form

$$\Phi = dz^{1234} + dz^{1256} + dz^{3456} + dz^{1375} + dz^{1467} + dz^{2367} + dz^{2457}. \tag{78}$$

By Theorem 2 we therefore have a system of equations which implies the Yang-Mills equations in 14 dimensions.

Unlike the previous analogous cases, the equations (76) are not equivalent to the set (75). The former set contains more equations than the latter. More precisely, (76) includes, for instance, the three equations

$$F_{71} + F_{\bar{3}\bar{5}} + F_{\bar{6}\bar{4}} \;=\; F_{71} + F_{\bar{3}\bar{5}} + F_{64} \;=\; F_{71} + F_{35} + F_{\bar{6}\bar{4}} = 0\,. \tag{79}$$

Under the $\mathrm{G}_2^{\mathbb{C}}$-invariant 4-form Φ, both real and imaginary parts of $F_{\alpha\beta}$ split into their 7- and 14-dimensional irreducible parts. The equations of the form (79) imply that under (76) the real part is projected to the 14-dimensional piece (7 equations) and the imaginary part is zero (21 equations). The real form of the system (76),(77) is given by the set of 29 equations,

$$\begin{aligned}
F_{18} + F_{29} + F_{30} + F_{4a} + F_{5b} + F_{6c} + F_{7d} &= 0\\
F_{12} + F_{34} + F_{56} - F_{89} - F_{0a} - F_{bc} &= 0\\
F_{13} - F_{24} - F_{80} + F_{9a} + F_{bd} + F_{75} &= 0\\
F_{14} + F_{23} - F_{8a} - F_{90} - F_{cd} - F_{76} &= 0\\
F_{15} - F_{26} - F_{8b} + F_{9c} - F_{0d} - F_{73} &= 0\\
F_{16} + F_{25} - F_{8c} - F_{9b} + F_{ad} + F_{74} &= 0\\
F_{17} - F_{35} + F_{46} - F_{8d} + F_{0b} - F_{ac} &= 0\\
F_{72} - F_{36} - F_{45} + F_{9d} + F_{0c} + F_{ab} &= 0
\end{aligned} \tag{80}$$

$$\begin{aligned}
F_{78} - F_{1d} \;&=\; F_{79} - F_{2d} \;=\; F_{70} - F_{3d} \;=\; 0\\
F_{7a} - F_{4d} \;&=\; F_{7b} - F_{5d} \;=\; F_{7c} - F_{6d} \;=\; 0\\
F_{19} - F_{28} \;&=\; F_{10} - F_{38} \;=\; F_{1a} - F_{48} \;=\; 0\\
F_{1b} - F_{58} \;&=\; F_{20} - F_{39} \;=\; F_{2a} - F_{49} \;=\; 0\\
F_{3a} - F_{40} \;&=\; F_{2b} - F_{59} \;=\; F_{3b} - F_{50} \;=\; 0\\
F_{4b} - F_{5a} \;&=\; F_{1c} - F_{68} \;=\; F_{2c} - F_{69} \;=\; 0\\
F_{3c} - F_{60} \;&=\; F_{4c} - F_{6a} \;=\; F_{5c} - F_{6b} \;=\; 0\,.
\end{aligned} \tag{81}$$

These 29 equations correspond to the $\lambda = -1$ eigenspace of the special democratic 4-form given by

$$
\begin{aligned}
T_{(14)} \;=\;\; & dx^{1234} + dx^{1256} + dx^{1298} + dx^{12a0} + dx^{12cb} + dx^{1375} + dx^{13bd} \\
& + dx^{1308} + dx^{139a} + dx^{14a8} + dx^{1409} + dx^{1467} + dx^{14dc} + dx^{15b8} \\
& + dx^{159c} + dx^{15d0} + dx^{16c8} + dx^{16b9} + dx^{1ad6} + dx^{170b} + dx^{1a7c} \\
& + dx^{2367} + dx^{23a8} + dx^{2309} + dx^{23dc} + dx^{2457} + dx^{2480} + dx^{24a9} \\
& + dx^{24db} + dx^{25c8} + dx^{25b9} + dx^{2ad5} + dx^{268b} + dx^{26c9} + dx^{20d6} \\
& + dx^{207c} + dx^{2a7b} + dx^{3456} + dx^{3498} + dx^{34a0} + dx^{34cb} + dx^{35b0} \\
& + dx^{35ac} + dx^{835d} + dx^{93d6} + dx^{36c0} + dx^{36ba} + dx^{83b7} + dx^{937c} \\
& + dx^{94d5} + dx^{45c0} + dx^{45ba} + dx^{460b} + dx^{46ca} + dx^{84d6} + dx^{847c} \\
& + dx^{947b} + dx^{5698} + dx^{56a0} + dx^{56cb} + dx^{9a75} + dx^{8a76} + dx^{9076} \\
& + dx^{890a} + dx^{89bc} + dx^{8057} + dx^{80db} + dx^{8acd} + dx^{90cd} + dx^{9abd} \\
& + dx^{0abc} + dx^{187d} + dx^{297d} + dx^{307d} + dx^{4a7d} + dx^{5b7d} + dx^{6c7d} .
\end{aligned}
$$

Its characteristic polynomial is given by

$$\chi(T_{(14)}) = (\lambda+1)^{62}(\lambda-3)^{14}(\lambda+3)^{7}(\lambda-5)^{7}(\lambda-6) \tag{82}$$

and the above 29 equations correspond to the projection to 62-dimensional $\lambda = -1$ eigenspace.

Deleting all terms containing the 14th index d from the above equations yields the 13-dimensional reduction, corresponding to a flow along a real parameter rather than the complex one chosen in (73). This is also a set of 29 equations, projecting the curvature to the 49-dimensional $\lambda = -1$ eigenspace of the corresponding reduction of the 4-form $T_{(14)}$. The reduced 4-form has characteristic polynomial

$$\chi(T_{(13)}) = (\lambda+1)^{49}(\lambda-3)^{8}(\lambda+3)(\lambda-5)^{2}(\lambda-4)^{6}(\lambda^2+\lambda-4)^{6} . \tag{83}$$

7 Oxidation to 16 dimensions

Analogously to the oxidations (47), (51) and (52) to eight real dimensions, we may oxidise the system (76),(77) in $\mathbb{C}^7$ to one in $\mathbb{C}^8$ by taking $g_{\alpha\overline{\beta}}$ to be the $\mathbb{C}^8$-metric and the (4,0)-form Φ to be given by the $\mathrm{Spin}(7)^{\mathbb{C}}$-invariant,

$$\Phi \;=\; dz^{1234} + dz^{1256} + dz^{1278} + dz^{3456} + dz^{3478} + dz^{5678} + dz^{1368} \tag{84}$$

$$+dz^{1375} + dz^{1467} + dz^{1458} + dz^{2367} + dz^{2457} + dz^{2358} + dz^{2486} . \tag{85}$$

The corresponding system includes the flow equations based on (75),

$$
\begin{aligned}
F(Z_8\,,\,Z_1j-Z_2)\;+\;F(Z_7\,,\,Z_1+Z_2j)\;+\;F(\overline{Z_3}-j\overline{Z_4}\,,\,\overline{Z_5}-\overline{Z_6}j) &= 0\\
F(Z_8\,,\,Z_3j-Z_4)\;+\;F(Z_7\,,\,Z_3+Z_4j)\;+\;F(\overline{Z_5}-j\overline{Z_6}\,,\,\overline{Z_1}-\overline{Z_2}j) &= 0\\
F(Z_8\,,\,Z_5j-Z_6)\;+\;F(Z_7\,,\,Z_5+Z_6j)\;+\;F(\overline{Z_1}-j\overline{Z_2}\,,\,\overline{Z_3}-\overline{Z_4}j) &= 0\\
F(Z_8\,,\,\overline{Z_8})\;+\;F(Z_7\,,\,\overline{Z_7})\;-\;\sum_{\alpha=1}^{3}F(\overline{Z_{2\alpha-1}}-j\overline{Z_{2\alpha}}\,,\,Z_{2\alpha-1}-j\overline{Z_{2\alpha}}) &= 0\\
\sum_{\alpha=1}^{4}F(\overline{Z_{2\alpha-1}}-j\overline{Z_{2\alpha}}\,,\,\overline{Z_{2\alpha-1}}+jZ_{2\alpha}) &= 0
\end{aligned}
\tag{86}
$$

The real form of the full system of equations with (4,0)-form Φ given in (85) is given by

$$
\begin{aligned}
F_{12}+F_{34}+F_{56}+F_{78}-F_{90}-F_{ab}-F_{cd}-F_{ef} &= 0\\
F_{13}-F_{24}-F_{57}+F_{68}-F_{9a}+F_{0b}+F_{ce}-F_{df} &= 0\\
F_{14}+F_{23}+F_{58}+F_{67}-F_{9b}-F_{0a}-F_{cf}-F_{de} &= 0\\
F_{15}-F_{26}+F_{37}-F_{48}-F_{9c}+F_{0d}-F_{ae}+F_{bf} &= 0\\
F_{16}+F_{25}-F_{38}-F_{47}-F_{9d}-F_{0c}+F_{af}+F_{be} &= 0\\
F_{17}-F_{28}-F_{35}+F_{46}-F_{9e}+F_{0f}+F_{ac}-F_{bd} &= 0\\
F_{18}+F_{27}+F_{36}+F_{45}-F_{9f}-F_{0e}-F_{ad}-F_{bc} &= 0\\
F_{19}+F_{20}+F_{3a}+F_{4b}+F_{5c}+F_{6d}+F_{7e}+F_{8f} &= 0\\
F_{79}-F_{1e}\;=\;F_{80}-F_{2f}\;=\;F_{5a}-F_{3c}\;=\;F_{6b}-F_{4d} &= 0\\
F_{70}-F_{2e}\;=\;F_{3d}-F_{6a}\;=\;F_{4c}-F_{5b}\;=\;F_{1f}-F_{89} &= 0\\
F_{69}-F_{1d}\;=\;F_{2c}-F_{50}\;=\;F_{3f}-F_{8a}\;=\;F_{7b}-F_{4e} &= 0\\
F_{1c}-F_{59}\;=\;F_{2d}-F_{60}\;=\;F_{3e}-F_{7a}\;=\;F_{8b}-F_{4f} &= 0\\
F_{1b}-F_{49}\;=\;F_{5f}-F_{8c}\;=\;F_{2a}-F_{30}\;=\;F_{6e}-F_{7d} &= 0\\
F_{6f}-F_{8d}\;=\;F_{5e}-F_{7c}\;=\;F_{40}-F_{2b}\;=\;F_{39}-F_{1a} &= 0\\
F_{7f}-F_{8e}\;=\;F_{5d}-F_{6c}\;=\;F_{10}-F_{29}\;=\;F_{3b}-F_{4a} &= 0\,.
\end{aligned}
\tag{87}
$$

The corresponding 4-form $T_{(16)}\in\Lambda^4\mathbb{R}^{16}$ has characteristic polynomial

$$
\chi(T_{(16)})=(\lambda+1)^{84}(\lambda-3)^{21}(\lambda-7)^{8}(\lambda+5)^{7}\,,
\tag{88}
$$

so the above 36 equations correspond to the vanishing of the imaginary part of $F_{\alpha\beta}$ (28 equations), the 7-dimensional irreducible piece of the real part of $F_{\alpha\beta}$ and the singlet trace condition on the (1,1)-curvature.

Deleting all terms containing f, the 16th index, from the above equations yields 36 equations in 15 dimensions which projects the curvature to the 69-dimensional $\lambda = -1$ eigenspace of the corresponding 4-form $T_{(15)}$, which has characteristic polynomial

$$\chi(T_{(15)}) = (\lambda+1)^{69}(\lambda-6)^{8}(\lambda-3)^{14}(\lambda^2+3\lambda-6)^{7}\,. \tag{89}$$

8 The reductions to $8 < d < 12$

We now briefly comment on some reductions of the above 12-dimensional system to the lower dimensions which were missed out in the discussion above.

d=11

Deleting all terms containing dx^b in (59)-(61) yields a set of 17 equations in 11-dimensions. The correspondingly reduced four-form $T_{(11)} := T_{(12)}|_{dx^b=0}$ has characteristic polynomial

$$\chi(T_{(11)}) = (\lambda+1)^{38}(\lambda-2)^{8}(\lambda-3)^{5}(\lambda-4)^{2}(\lambda^2+\lambda-4). \tag{90}$$

d=10

Reducing the above 11-dimensional 4-form further to the 10-dimensional hypersurface defined, for instance, by $x^6 = 0$ yields a 4-form with characteristic polynomial

$$\chi(T_{(10)}) = (\lambda+1)^{30}(\lambda-1)^{8}(\lambda-3)^{6}(\lambda-4). \tag{91}$$

The $\lambda = -1$ eigenspace corresponds to a set of 15 equations amongst the 45 curvature components. This case is the complex counterpart of the $d = 5$ case discussed at the end of section 4.1. In $\mathbb{C}^5$, these equations take the form (9),(10) with $\alpha, \beta = 1, \ldots, 5$ and the complex (4,0)-form given by the contraction of the (5,0) volume form with a constant unit (0,1)-vector. This (4,0)-form is the SU(4)-invariant volume form in the 4-dimensional complex space orthogonal to this vector. Choosing, this vector, for instance in the direction of the z^5-axis, we obtain $\Phi = dz^{1234}$, yielding the following equations on $\mathbb{C}^5$

$$\begin{aligned}
&F_{1\bar{1}} + F_{2\bar{2}} + F_{3\bar{3}} + F_{4\bar{4}} + F_{5\bar{5}} = 0\\
&F_{12} + F_{\bar{3}\bar{4}} = F_{13} + F_{\bar{4}\bar{2}} = F_{14} + F_{\bar{2}\bar{3}} = 0\\
&F_{15} = F_{25} = F_{35} = F_{45} = 0\,.
\end{aligned} \tag{92}$$

d=9

The most symmetric reduction of (92) to 9-dimensions, making z^5 real, is a trivial embedding of the Spin(7)-invariant set of equations (50) in 9 dimensions.

9 Some open questions

An intruiguing open problem is the relation of the 12-dimensional system to sextonions and to the 'missing row' of the Freudenthal magic square related to $E_{7\frac{1}{2}}$ (see [12, 6]).

In the cases where the duality equations describe (partial) curvature flows, it remains to be seen whether solutions in the bulk can always be seen as arising from solutions on the initial value surface (boundary) of the flow. For instance, to what extent can the known four-dimensional solutions of the self-duality equations (21) be seen as arising from a flow which has a flat 3d connection as its initial value, or do the known solutions of the 8-dimensional Spin(7)-invariant equation [8, 9, 34] arise as a solutions of the flow equation (47) from solutions (e.g. [33]) of the G_2-invariant equation (41) on the initial value seven-fold.

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