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A sharp lower bound on the mean curvature integral with critical power for integral varifolds

Ulrich Menne

This is an announcement of the principal results of [12] using the notation of [11, §1, §2] which is based on Federer [5] and Allard [1]. To describe the results, some additional terminology from [12, 5.1, 5.4, 5.6] is needed.

The space of nonempty closed subsets of a metric space X is topologised by its injection into $\mathbf{R}^X$ associating to each set its distance function, cp. [5, 2.10.21]. Convergence in this topology is termed *locally in Hausdorff distance*.

If $a \in S \subset \mathbf{R}^n$ and S is closed, then S is called *differentiable at* a if and only if $\operatorname{Tan}(S,a)$ is a linear subspace of $\mathbf{R}^n$ and

$$\boldsymbol{\mu}_{1/r} \circ \boldsymbol{\tau}_{-a}[S] \to \operatorname{Tan}(S,a) \quad \text{locally in Hausdorff distance as } r \to 0+.$$

If $a \in S \subset \mathbf{R}^n$ and S is closed, then S is called *twice differentiable at* a if and only if S is differentiable at a and, in case $0 < m = \dim \operatorname{Tan}(S,a) < n$, there exists a homogeneous polynomial function $Q : \operatorname{Tan}(S,a) \to \operatorname{Nor}(S,a)$ of degree 2 such that with $\tau : \operatorname{Tan}(S,a) \times \operatorname{Nor}(S,a) \to \mathbf{R}^n$,

$$\tau(v,w) = v + w \quad \text{for } v \in \operatorname{Tan}(S,a),\ w \in \operatorname{Nor}(S,a),$$
$$\phi_r = r^{-1} \operatorname{Tan}(S,a)_\natural + r^{-2} \operatorname{Nor}(S,a)_\natural \quad \text{for } 0 < r < \infty$$

there holds

$$\phi_r \circ \boldsymbol{\tau}_{-a}[S] \to \tau[Q] \quad \text{locally in Hausdorff distance as } r \to 0+.$$

Note Q is uniquely determined by S and a, hence the *second fundamental form* $\mathbf{b}(S;a)$ and the *mean curvature vector* $\mathbf{h}(S;a)$ of S at a may be defined by $\mathbf{b}(S;a) = D^2Q(0)$ and $\mathbf{h}(S;a) = \operatorname{trace} \mathbf{b}(S;a)$ respectively; here the notion of trace of [5, 1.7.10] is extended in the obvious way.

Suppose m and n are positive integers, $m < n$, $1 \leq p \leq \infty$, V is an m dimensional integral varifold in $\mathbf{R}^n$, $\|\delta V\|$ is a Radon measure, and, if $p > 1$,

$$(H_p) \qquad \begin{aligned} &\delta V(g) = -\textstyle\int g(z) \bullet \mathbf{h}(V;z)\, \mathrm{d}\|V\| z \quad \text{for } g \in \mathscr{D}(\mathbf{R}^n, \mathbf{R}^n), \\ &\mathbf{h}(V;\cdot) \in \mathbf{L}_p(\|V\| \llcorner K, \mathbf{R}^n) \quad \text{whenever } K \text{ is a compact subset of } \mathbf{R}^n. \end{aligned}$$

Instructive examples are constructed in Allard [1, 8.1 (2)], Brakke [3, 6.1], and [9, 1.2]. If $p = m$, then $\mathscr{H}^m \llcorner \operatorname{spt} \|V\| \leq \|V\|$ by Allard [1, 8.3]. If $p > m$ and $p \geq 2$, then there exists a relatively open and dense subset G of $\operatorname{spt} \|V\|$ such that G is an m dimensional submanifold of class 1 of $\mathbf{R}^n$ by Allard [1, 8.1 (1)].

The condition (H_1) is sufficient to establish second order differentiability properties in an approximate sense:

Theorem 1 (cf. [10, 4.8])**.** *If V satisfies (H_1), then there exists a countable collection C of m dimensional submanifolds of class 2 of $\mathbf{R}^n$ with*

$$\|V\|(\mathbf{R}^n \sim \bigcup C) = 0.$$

Moreover, for every member of M of C there holds

$$\mathbf{h}(M; z) = \mathbf{h}(V; z) \quad \text{for } \|V\| \text{ almost all } z \in M.$$

Using different methods, this theorem extends previous results of Schätzle in [13, Theorem 6.1] for the case $n = m + 1$, $p > m$, $p \geq 2$.

If $p = m$, the differentiability properties may be sharpened as follows.

Corollary 2 (cf. [12, 5.11], [10, 4.8])**.** *If V satisfies (H_m) and $S = \operatorname{spt} \|V\|$, then:*

1. *For $\mathscr{H}^m$ almost all $a \in S$ the closed set S is twice differentiable at a with $\dim \operatorname{Tan}(S, a) = m$ and $\mathbf{h}(S; a) = \mathbf{h}(V; a)$.*
2. *For $\|V\|$ almost all a there holds*

$$r^{-m} \int_{\mathbf{B}(a,r)} (|R(z) - R(a) - (\|V\|, m) \operatorname{ap} DR(a)(z-a)|/|z-a|)^2 \, d\|V\| z \to 0$$

as $r \to 0+$, where R maps $w \in S$ such that S is differentiable at w onto $\operatorname{Tan}(S, w)_\natural \in \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$.

To prove the corollary, first, the necessary flatness properties are deduced from the preceding theorem by means of subsolution properties of the distance function associated to a plane. This step utilises ideas from Ecker [4, 1.6, 1.7], Allard [1, 7.5 (6)], and [10, 5.2 (2)]. Second, the differentiability properties are deduced using techniques from [9, §3]. Finally, the relation of $\mathbf{h}(S; \cdot)$ and $\mathbf{h}(V; \cdot)$ is established similarly as in Schätzle [14, Theorem 4.1].

The next theorem for $m = n - 1$ generalises the area formula for the Gauss map from oriented m dimensional submanifolds of class 2 of $\mathbf{R}^n$ to supports of m dimensional integral varifolds satisfying (H_m) with $m \geq 2$.

Theorem 3 (cf. [12, 7.34])**.** *If V satisfies (H_m), $2 \leq m = n - 1$, $S = \operatorname{spt} \|V\|$,*

$$C = (S \times \mathbf{S}^m) \cap \{(a, u) : \mathbf{U}(a - su, s) \cap S = \emptyset \text{ for some } 0 < s < \infty\},$$

and B is an $\mathscr{H}^m$ measurable subset of C, then

$$\int_{\mathbf{S}^m} \mathscr{H}^0 \{a : (a, u) \in B\} \, d\mathscr{H}^m u = \int_S \int_{\mathbf{S}^m \cap \{u : (a,u) \in B\}} |\operatorname{discr}(\mathbf{b}(S; a) \bullet u)| \, d\mathscr{H}^0 u \, d\mathscr{H}^m a,$$

where $\mathbf{b}(S; a) \bullet u : \operatorname{Tan}(S, a) \times \operatorname{Tan}(S, a) \to \mathbf{R}$ denotes the symmetric bilinear function mapping $(v, w) \in \operatorname{Tan}(S, a) \times \operatorname{Tan}(S, a)$ onto $\mathbf{b}(S; a)(v, w) \bullet u \in \mathbf{R}$.

Note $\mathscr{H}^m(S \sim \operatorname{dmn} C) = 0$ by part (1) of Corollary 2.

In view of Theorem 1, the proof of Theorem 3 readily reduces to establishing the following *Lusin property*:

$$\mathscr{H}^m(C[E]) = 0 \quad \text{whenever } E \subset \operatorname{dmn} C \text{ and } \mathscr{H}^m(E) = 0.$$

If $E \subset \{z : \boldsymbol{\Theta}_*^m(\|V\|, z) < \infty\}$, then the key is to establish a suitable version of a weak Harnack estimate for Lipschitzian real valued functions on S. In this respect inspiration is taken from Bombieri and Giusti [2], Hutchinson [7], and Stampacchia [16, §4, §5]. To treat the case $E \subset \{z : \boldsymbol{\Theta}^m(\|V\|, z) = \infty\}$, consider $z \in \mathbf{R}^n$ with $\boldsymbol{\Theta}^m(\|V\|, z) = \infty$. Then the modified monotonicity identity of Kuwert and Schätzle [8, Appendix] (which employs Brakke [3, 5.8]) may be used to estimate barycentres of $\|V\|$ on balls centred at z with suitable radii. In both cases the deduction of the Lusin property from the estimates is carried out analogously to the use of the *Rado-Reichelderfer condition* of Hencl in [6, Theorems 5.1 and 3.5].

Corollary 4 (cf. [12, 7.35]). *If V satisfies (H_m), $2 \leq m = n-1$, and $S = \operatorname{spt} \|V\|$ is nonempty and compact, then*

$$\textstyle\int_{\mathbf{S}^m} |\mathbf{h}(\mathbf{S}^m; z)|^m \, \mathrm{d}\mathscr{H}^m z \leq \int_S |\mathbf{h}(S; z)|^m \, \mathrm{d}\mathscr{H}^m z$$

The weaker estimate resulting from replacing $\mathscr{H}^m$ by $\|V\|$ in the last integral was previously obtained by Kuwert and Schätzle in [8, Appendix] for the case $m = 2$ and certain particular varifolds satisfying (H_p) with $p > m$ by Schulze in [15, Proposition 6.6].

Taking $B = (S \times \mathbf{S}^m) \cap \{(a, u) : (z - a) \bullet u \leq 0 \text{ for } z \in S\}$, Corollary 4 may be deduced from Theorem 3 similarly to Schulze [15, §2].

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Existence of immersed spheres minimizing curvature functionals

Ernst Kuwert
(joint work with Andrea Mondino and Johannes Schygulla)

We consider variational problems for minimizers of Willmore functionals having the topological type of the 2-sphere. Let $[\mathbb{S}^2, \mathbb{R}^3]$ be the space of immersed 2-spheres in $\mathbb{R}^3$. The Willmore functional is given by

$$\mathcal{W}(f) = \frac{1}{4} \int_{\mathbb{S}^2} |\vec{H}|^2 \, d\mu,$$

where $\vec{H}$ is the mean curvature vector and $d\mu$ is the area element. Willmore (1965) proved that $\mathcal{W}(f) \geq 4\pi$ for any closed surface, with equality only for the round spheres. In the talk, we discussed an existence and regularity theorem proved by J. Schygulla in his Ph.D. thesis, see [Schy11]. For embedded surfaces f, one defines the isoperimetric ratio by

$$I(f) = \sqrt{36\pi} \, \frac{V(f)}{A(f)^{3/2}} \in (0, 1],$$

where $A(f)$ is the area and $V(f)$ is the volume enclosed by f.

Theorem 1 (Schygulla [Schy11])**.** *For any $\sigma \in (0, 1]$, there exists a minimizer of the Willmore functional in the class of smooth embeddings $f : \mathbb{S}^2 \to \mathbb{R}^3$ with prescribed isoperimetric ratio $I(f) = \sigma$. As a function of σ, the corresponding minimum $\beta(\sigma)$ is strictly decreasing with*

$$\beta(1) = 4\pi \quad \text{and} \quad \lim_{\sigma \searrow 0} \beta(\sigma) = 8\pi.$$

Moreover, the minimizers converge as $\sigma \searrow 0$ to a round sphere of multiplicity two in the sense of varifolds.

The theorem is partially motivated by a model for cell membranes due to Helfrich (1973). In that model, the energy contains an extra parameter called the spontaneous curvature, and both the area and the enclosed volume are prescribed. The theorem corresponds to the special case of spontaneous curvature zero, where the two conditions reduce to the isoperimetric ratio as single constraint by the scale invariance of the Willmore functional. We refer to [SeBeLip91] for numerical experiments. Under the assumption of axial symmetry, existence of minimizers for any spontaneous curvature have been constructed recently in [ChoVe12].