A UNIFORM POINCARÉ ESTIMATE FOR QUADRATIC DIFFERENTIALS ON CLOSED SURFACES

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ABSTRACT. We prove a uniform estimate, valid for every closed Riemann surface of genus at least two, that bounds the distance of any quadratic differential to the finite dimensional space of holomorphic quadratic differentials in terms of its antiholomorphic derivative.

1. INTRODUCTION

Given any closed Riemann surface (M, c), c a complex structure, we consider the complex vector space $\mathcal{Q}(M, c)$ of smooth quadratic differentials on (M, c), that is of complex tensors that with respect to a local complex coordinate z take the form

$$\Psi = \psi dz^2, \quad \psi \text{ a smooth function.}$$

Of particular importance is the subspace $\mathcal{H}(M, c)$ of those quadratic differentials that are represented in each complex coordinate chart by a holomorphic function. This space of holomorphic quadratic differentials has finite (complex) dimension $\dim(\mathcal{H}(M, c)) = 0$ for surfaces of genus $\gamma = 0$, $\dim(\mathcal{H}(M, c)) = 1$ if $\gamma = 1$ and

$$\dim(\mathcal{H}(M, c)) = 3(\gamma - 1) \text{ for } \gamma \geq 2$$

by the Riemann-Roch theorem. It canonically represents the tangent space to Teichmüller space $\tau(M)$ at the point $[(M, c)]$, with $\mathcal{H}(M, c)$ equipped with the L^2 inner product,

$$\langle \phi dz^2, \psi dz^2 \rangle_{L^2(M,g)} = \int_M \phi \cdot \bar{\psi} \left| dz^2 \right|^2 dv_g = 4 \int_M \phi \cdot \bar{\psi} \cdot \rho^{-2} \frac{i}{2} dz \wedge d\bar{z},$$

isometric to $T_{[(M,c)]}\tau(M)$ equipped with the Weil-Petersson metric. Here and in the following g stands for the unique (modulo Möbius transformations in the genus zero case) complete metric compatible with c that has constant Gauss curvature $1, 0, -1$ for surfaces of genus $0, 1$ respectively $\gamma \geq 2$ (with unit area in the case $\gamma = 1$) and ρ denotes the conformal factor corresponding to the complex coordinate z, determined by $g = \rho^2 dz d\bar{z}$.

Given any closed Riemann surface of finite genus γ equipped with this canonical choice of metric we now define

$$P_g : \mathcal{Q}(M, c) \to \mathcal{H}(M, c)$$

to be the $L^2(M, g)$-orthogonal projection onto $\mathcal{H}(M, c)$.

Furthermore we denote by $\bar{\partial}\Psi$ the antiholomorphic derivative of a quadratic differential Ψ, that is the tensor given in complex coordinates by

$$\bar{\partial}\Psi = \partial_{\bar{z}}\psi d\bar{z} \otimes dz^2.$$

In this paper we prove an estimate for arbitrary quadratic differentials that is reminiscent of the standard Poincaré inequality for functions

$$(1.1) \qquad \|f - \bar{f}\|_{L^1} \leq C \cdot \|\nabla f\|_{L^1}$$

bounding the distance of an object from its projection onto a finite dimensional subspace, here the constant functions, in terms of a derivative.

However, in stark contrast to the standard Poincaré inequality for functions, our inequality for quadratic differentials is *uniform*, i.e. independent of the geometry of the hyperbolic surface on which we work; it is valid with a constant C depending only on the topology of the surface (i.e. on the genus) and this feature is essential for applications to the Teichmüller harmonic map flow [5] to which we allude briefly below. One further distinction between the normal Poincaré estimate and the new estimate is that we use the $\bar{\partial}$ operator on the right-hand side rather than the full derivative ∇, which makes our estimate also an elliptic estimate, and means that we should make estimates relative to its kernel (i.e. the holomorphic quadratic differentials) rather than the kernel of ∇ (i.e. the constant functions).

Theorem 1.1. *(Main theorem.) Given an arbitrary closed Riemann surface (M, c) of genus at least two, there exists a constant $C < \infty$ depending only on the genus of M such that the following uniform Poincaré estimate holds true. The distance of any quadratic differential $\Psi \in \mathcal{Q}(M, c)$ from its holomorphic part is uniformly bounded in terms of its antiholomorphic derivative in the sense that*

$$(1.2) \qquad \|\Psi - P_g(\Psi)\|_{L^1(M,g)} \leq C \cdot \|\bar{\partial}\Psi\|_{L^1(M,g)}.$$

Here and in the following all norms are computed with respect to the unique hyperbolic metric g compatible with (M, c).

Remark 1.2. While the left-hand side of (1.2) is invariant under a conformal change of the metric, the right hand side is not. It is important here to take the unique hyperbolic conformal metric.

For hyperbolic surfaces contained in a compact region of moduli space, the estimate (1.2) was shown in [5], Lemma 2.1. In that paper, the estimate was used to understand the asymptotics of a new flow which deforms a pair (u, g), where u is a map from a closed surface M to an arbitrary compact Riemannian manifold and g is a constant curvature -1 metric on M, under the gradient flow of the harmonic map energy. Where possible, the flow tries to converge to a branched minimal immersion, and a key to demonstrating this is to argue that the Hopf differential Φ has not only $\bar{\partial}\Phi$ and $P_g(\Phi)$ converging to zero in L^1, but also that Φ itself converges to zero in L^1. The estimate of Theorem 1.1 immediately implies this key fact.

The space of holomorphic quadratic differentials on a surface of genus 0, i.e. on a sphere, is trivial so that in this case the Poincaré estimate takes the following form, and can be proved with standard techniques (cf. [7, Lemma 2.5]).

Proposition 1.3. *There exists a constant $C < \infty$ such that all quadratic differentials $\Psi \in \mathcal{Q}(S^2, g)$ on the sphere satisfy*

$$(1.3) \qquad \|\Psi\|_{L^1(S^2,g)} \leq C\|\bar{\partial}\Psi\|_{L^1(S^2,g)},$$

where g is the metric of constant curvature 1.

Remark 1.4. The moduli space of the sphere, that is the set of equivalence classes of complex structures that agree up to pull-back by an orientation-preserving diffeomorphism, consists of only one point. Since the estimate (1.3) (and also (1.2)) is invariant under the pull-back by diffeomorphisms, Proposition 1.3 essentially just says that a Poincaré estimate is valid for the (unique) complex structure on the sphere.

Similarly, such an estimate is valid for every fixed complex structure of a torus. Contrary to surfaces of higher genus however, this estimate is *not* uniform.

Proposition 1.5. *For any flat unit area torus (T^2, g) there exists $C < \infty$ such that*

$$(1.4) \qquad \|\Psi - P_g(\Psi)\|_{L^1(T^2,g)} \leq C\|\bar{\partial}\Psi\|_{L^1(T^2,g)} \text{ for every } \Psi \in \mathcal{Q}(T^2, g).$$

This estimate is not uniform; given any number $C < \infty$ there exists a torus (T^2, g) that is flat and has unit area but for which (1.4) is violated.

Indeed note that each such torus is isometric to $(\mathbb{C}/\Gamma_{a,b}, g_{eucl})$ for some lattice group $\Gamma_{a,b} = \{n \cdot b + m \cdot (a + \frac{i}{b}), n, m \in \mathbb{Z}\}$ with $a \in \mathbb{R}$ and $b > 0$, and that $\Phi = \phi dz^2$ is a holomorphic quadratic differential if and only if ϕ is a Γ-periodic holomorphic function, i.e. a constant. Thus in this special case the Poincaré estimate for quadratic differentials is equivalent to a refined Poincaré estimate for Γ-periodic functions of

$$\|\phi - \bar{\phi}\|_{L^1(\mathbb{C}/\Gamma)} \leq C \cdot \|\partial_{\bar{z}}\phi\|_{L^1(\mathbb{C}/\Gamma)}.$$

A simple example, say $\phi(x + iy) = \sin(\frac{2\pi}{b}x)$ on $\mathbb{C}/\Gamma_{a=0,b}$ with $b \to \infty$, shows that such an estimate is not uniform.

Theorem 1.1 will be proved by contradiction. If the result were not true, then we would find a sequence of surfaces and quadratic differentials (which without loss of generality would have no holomorphic part at all) which violate (1.2) for larger and larger values of C. The surfaces would have to degenerate by pinching certain necks, because otherwise the result is known from [5]. After normalising so that the L^1 norm is always 1, we then pass to a subsequence to get a noncompact limit surface together with a limit quadratic differential which will be holomorphic (see Lemma 2.1). But a result in [6] (see Lemma 2.2) tells us that the limit quadratic differential inherits the property of having no holomorphic part at all, and thus must be identically zero. The key part of this paper is then to show that the limit inherits the property of having L^1 norm equal to 1, giving a contradiction. The essential point is that we must prove that in this limit, L^1 norm cannot concentrate on degenerating collars and be lost in the limit, and this is articulated by our key Lemma 2.3. Essentially, the only way that L^1 norm of an almost-holomorphic quadratic differential can concentrate on a long collar is if the quadratic differential has a nonvanishing 'principal component' – i.e. its lowest Fourier mode on the collar is not disappearing, and this component looks like a parallel quadratic differential on the collar. However, by assumption our quadratic differentials are orthogonal to all holomorphic quadratic differentials, and in Lemma 2.6 we construct a sequence of holomorphic quadratic differentials which is purely concentrating on the collar, and is becoming parallel. Thus our original sequence cannot concentrate L^1 norm on the collar as desired.

2. Proof of the main result

The basic strategy of the proof of Theorem 1.1 is similar to the one in [5] in that we argue by contradiction and use compactness results in order to pass in the limit to a holomorphic quadratic differential on some limit surface. A key difference is however that in order to obtain the *uniform* version of the Poincaré estimate claimed in Theorem 1.1 we need to be able to deal with degenerating sequences of surfaces, with the local arguments of [5] only applicable for considerations away from the degenerating parts of these surfaces. A crucial part of the proof is thus a discussion, from the point of view of geometric analysis, first of holomorphic quadratic differentials on a sequence of

degenerating hyperbolic surfaces, and then, more generally, of non-holomorphic quadratic differentials with controlled antiholomorphic derivatives.

Contrary to the assertion of the theorem, let us suppose that there exist a sequence of closed hyperbolic surfaces (M_i, c_i, g_i) of fixed genus, and a sequence of (nonholomorphic) quadratic differentials Φ_i on (M_i, c_i) such that

$$\frac{\|P_{g_i}(\Phi_i) - \Phi_i\|_{L^1(M_i, g_i)}}{\|\bar{\partial}\Phi_i\|_{L^1(M_i, g_i)}} \to \infty.$$

Replacing Φ_i by a (multiple of) $P_{g_i}(\Phi_i) - \Phi_i$, using the uniformisation theorem and pulling back by an appropriate family of diffeomorphisms from M to M_i we obtain the following setting:

Assumptions: We assume that there exists a closed surface M of genus $\gamma \geq 2$ such that there is a sequence of complex structures c_i on M and a sequence of quadratic differentials $\Phi_i \in \mathcal{Q}(M, c_i)$ for which the following three assumptions are true:

$$(2.1) \quad P_{g_i}(\Phi_i) = 0 \quad \text{and} \quad \|\Phi_i\|_{L^1(M, g_i)} = 1 \quad \text{and} \quad \|\bar{\partial}\Phi_i\|_{L^1(M, g_i)} \to 0 \text{ as } i \to \infty.$$

Here and in the following g_i stands for the unique complete hyperbolic metric compatible with the complex structure c_i.

Since we know [5] that the Poincaré estimate (1.2) is valid on every compact subset K of moduli space (with a constant C depending a priori on K) the surfaces (M, g_i) must degenerate in moduli space, i.e. the length of the shortest closed geodesic of (M, g_i) must converge to zero as $i \to \infty$.

According to the Deligne-Mumford compactness theorem [1], after passing to a subsequence we may assume that (M, g_i) degenerates to a hyperbolic punctured surface (Σ, h) (i.e. a surface obtained from finitely many closed Riemann surfaces by removing finitely many points, which is equipped with the complete hyperbolic metric that is compatible with the induced complex structure) by collapsing $1 \leq k \leq 3(\gamma - 1)$ geodesics. In practice this means that there exist simple closed geodesics $\{\sigma_i^j\}_{j=1}^k$ on (M, g_i) of length $\ell(\sigma_i^j) \to 0$ as $i \to \infty$ and diffeomorphisms $f_i : \Sigma \to M \setminus \cup_{j=1}^k \sigma_i^j$ such that the metrics and the corresponding complex structures converge

$$f_i^* g_i \to h, \quad f_i^* c_i \to c_\infty \text{ smoothly locally on } \Sigma.$$

Here the limiting surface (Σ, c_∞, h) is a non-compact, possibly disconnected, complete hyperbolic surface with $2k$ punctures corresponding to the collapsing geodesics in the sense that f_i^{-1} extends to a continuous map from M to the compactification of (Σ, h) obtained by filling in k appropriate pairs of punctures with k new points; each geodesic σ_i^j is then mapped by f_i^{-1} to a different one of these (paired) points.

In this situation we then derive a contradiction from the assumptions in (2.1) in three steps; first, and using only local arguments similar to the ones of [5], we obtain that a subsequence of $f_i^* \Phi_i$ converges *locally* to a holomorphic limit Φ_∞; second, we find that Φ_∞ stands orthogonal to the space of integrable holomorphic quadratic differentials on the limit surface, so that the holomorphic quadratic differential Φ_∞ obtained in the first step must be identically zero. Finally, we will show that despite the convergence of the $f_i^* \Phi_i$ being only local, the L^1 norm is preserved globally in the limit $i \to \infty$ and thus that $\|\Phi_\infty\|_{L^1(\Sigma, h)} = 1$ in contradiction to $\Phi_\infty \equiv 0$.

Lemma 2.1. *Let (M, g_i) be a sequence of closed hyperbolic surfaces that degenerates to a hyperbolic punctured surface (Σ, h) as described above. Then for any sequence of quadratic differentials $\Psi_i \in \mathcal{Q}(M, g_i)$ with*

$$\|\Psi_i\|_{L^1(M, g_i)} + \|\bar{\partial}\Psi_i\|_{L^1(M, g_i)} \leq C < \infty$$

there exists a subsequence converging

$$f_i^* \Psi_i \to \Psi_\infty \ \ in \ L_{loc}^1(\Sigma, h)$$

to a quadratic differential $\Psi_\infty \in L^1(\Sigma, h)$. Additionally if $\|\bar{\partial}\Psi_i\|_{L^1(\Sigma,h)} \to 0$ then Ψ_∞ is holomorphic.

Given a sequence Φ_i as in (2.1) we thus find that after passing to a subsequence and pulling-back by diffeomorphisms it converges to a limit Φ_∞ which is an element of the space

$$\mathcal{H}(\Sigma, h) := \{\Psi \text{ a holomorphic quadratic differential on } (\Sigma, h) \text{ with } \|\Psi\|_{L^1(\Sigma,h)} < \infty\}$$

which can be equivalently characterised as the space of holomorphic quadratic differentials with at most a simple pole at each puncture. In the limit $i \to \infty$ the dimension of $\mathcal{H}$ reduces by the number k of collapsing geodesics, i.e. $\dim_{\mathbb{C}}(\mathcal{H}(\Sigma, h)) = 3(\gamma - 1) - k = \dim_{\mathbb{C}}(\mathcal{H}(M, g_i)) - k$, by Riemann-Roch.

As is discussed in [6], Lemma A.11, the definition of $\mathcal{H}(\Sigma, h)$ would be unchanged if we required $\|\Psi\|_{L^\infty(\Sigma,h)} < \infty$ instead of $\|\Psi\|_{L^1(\Sigma,h)} < \infty$. (Note that the volume of (Σ, h) is finite, so the L^∞ norm controls the L^1 norm. On the other hand, controlling the L^1 norm gives sufficient control on the order of any poles for the L^∞ norm also to be controlled.) In particular, all elements in $\mathcal{H}(\Sigma, h)$ lie in L^2, and the space of L^2 quadratic differentials then has a natural projection onto $\mathcal{H}(\Sigma, h)$. Moreover, because every element of $\mathcal{H}(\Sigma, h)$ lies in L^∞, this projection extends naturally to the space of L^1 quadratic differentials.

It thus makes sense to analyse the projection onto $\mathcal{H}(\Sigma, h)$ of any quadratic differential Ψ_∞ obtained as a local $L^1(\Sigma, h)$ limit of a sequence of quadratic differentials with uniformly bounded L^1 norm, and in particular to assert that the limit Φ_∞ of the sequence Φ_i satisfying (2.1) is orthogonal to $\mathcal{H}(\Sigma, h)$. (Together with our knowledge that Φ_∞ is holomorphic, this will imply that $\Phi_\infty \equiv 0$.) In order to prove this, we make use of the following continuity result for the projections onto the spaces $\mathcal{H}(\cdot)$, derived in [6]. A precise definition of the spaces W_i will be given later once we have some more notation.

Lemma 2.2. *Let (M, g_i) be any sequence of closed hyperbolic surfaces that degenerates to a hyperbolic punctured surface (Σ, h) by collapsing k collars. Then there exist subspaces $W_i \subset \mathcal{H}(M, g_i)$ of dimension $3(\gamma - 1) - k$ such that the $L^2(M, g_i)$-orthogonal projections $P_{g_i}^{W_i}$ onto W_i converge to the $L^2(\Sigma, h)$-orthogonal projection $P_h^{\mathcal{H}(\Sigma,h)}$ onto the space of integrable holomorphic quadratic differentials $\mathcal{H}(\Sigma, h)$ in the following sense:*

For any sequence $\Psi_i \in \mathcal{Q}(M, g_i)$ of quadratic differentials on (M, g_i) with $\|\Psi_i\|_{L^1(M,g_i)}$ bounded and with $f_i^* \Psi_i \to \Psi_\infty$ locally in $L^1(\Sigma, h)$, we have

$$f_i^* \left(P_{g_i}^{W_i}(\Psi_i)\right) \to P_h^{\mathcal{H}(\Sigma,h)}(\Psi_\infty) \ \text{smoothly locally on } \Sigma.$$

We remark that a stronger statement holds true for the projections $P_{g_i}^{W_i}$, see Theorem 2.6 of [6], asserting not only local convergence but also convergence of the *global* L^p norms, $1 \leq p \leq \infty$, to the corresponding norm of the limit and thus excluding any concentration of elements of W_i on the degenerating parts of the surface. Conversely, we will later see that elements of $W_i^\perp$ concentrate solely on *degenerating collar regions*.

Returning to our sequence of quadratic differentials Φ_i note that (2.1) implies in particular that $P_{g_i}^{W_i}(\Phi_i) = 0$, so by Lemma 2.2 the limit $\Phi_\infty \in \mathcal{H}(\Sigma, h)$ obtained above must satisfy $P_h^{\mathcal{H}(\Sigma,h)}(\Phi_\infty) = 0$ and thus vanish identically $\Phi_\infty \equiv 0$.

Conversely, we will prove that despite the convergence of $f_i^* \Phi_i$ on Σ being only local, the L^1 norms are preserved globally and thus $\|\Phi_\infty\|_{L^1(\Sigma,h)} = 1$. In this argument, the key

point to be proven is that there can be no concentration of L^1 norm on the degenerating parts of the surface. This follows from the following more general result, controlling almost-holomorphic quadratic differentials in the δ-thin part of the surface (i.e. where the injectivity radius is less than δ) which is central to this paper.

Lemma 2.3. *Let (M, g) be any closed hyperbolic surface. Then there exists a constant $C < \infty$ depending only on the genus of the surface M such that for every quadratic differential $\Psi \in \mathcal{Q}(M, g)$ with*

$$P_g \Psi = 0,$$

and every $\delta > 0$, we have the estimate

$$\|\Psi\|_{L^1(\delta\text{-}thin(M,g))} \leq C \cdot \left(\|\bar{\partial}\Psi\|_{L^1(M,g)} + \delta^{1/2} \|\Psi\|_{L^1(M,g)} \right).$$

Returning to our sequence of quadratic differentials Φ_i satisfying the assumptions in (2.1) we thus find that the L^1 norms on the δ-thick parts $\Sigma_i^\delta := \{p \in \Sigma : \text{inj}_{f_i^* g_i}(p) \geq \delta\}$ of the degenerating surfaces $(\Sigma, f_i^* g_i)$ satisfy

$$(2.2) \qquad \sup_{\delta > 0} \lim_{i \to \infty} \|f_i^* \Phi_i\|_{L^1(\Sigma_i^\delta, f_i^* g_i)} = 1 - \inf_{\delta > 0} \lim_{i \to \infty} \|\Phi_i\|_{L^1(\delta\text{-thin}(M,g_i))} = 1.$$

We remark that the special structure of hyperbolic surfaces, in particular the collar lemma of Keen-Randol [4], leads to the observation that for any $0 < \delta < \text{arsinh}(1)$ the δ-thick part Σ_i^δ of $(\Sigma, f_i^* g_i)$ converges to the δ-thick part Σ^δ of the limiting surface (Σ, h), which is a compact subset of (Σ, h), in the sense that both Σ_i^δ and Σ^δ lie in a fixed (i-independent) compact set and the measure of their symmetric difference converges to zero (see Lemma A.7 in [6]). Combined with the locally uniform convergence of the metrics $f_i^* g_i \to h$ we thus conclude that for any $\delta > 0$

$$(2.3) \qquad \|\Phi_\infty\|_{L^1(\Sigma^\delta, h)} = \lim_{i \to \infty} \|f_i^* \Phi_i\|_{L^1(\Sigma_i^\delta, f_i^* g_i)}$$

so taking the supremum over $\delta > 0$ and using (2.2) we must have $\|\Phi_\infty\|_{L^1(\Sigma, h)} = 1$ in contradiction to the fact that $\Phi_\infty \equiv 0$ which was a consequence of Lemmas 2.1 and 2.2.

Proof of Lemma 2.1. The main tool for the proof of this lemma is the compactness lemma 2.3 of [5] for functions on the euclidean disc D_1 for which the L^1 norm of both the function and its antiholomorphic derivative is bounded.

Let Ψ_i and (M, g_i) be as in Lemma 2.1 and recall that the convergence of the metrics $f_i^* g_i$ also implies convergence of the associated complex structures $f_i^* c_i$. In practice this means that given any compact subset $K \subset \Sigma$ there exists a number $\delta > 0$ and a sequence of atlases covering K which consist of coordinate charts that can be viewed as isometries

$$\phi_i^j : B_{f_i^* g_i}(p^j, \delta) \to (B_{g_H}(0, \delta), g_H),$$

from the balls $B_{f_i^* g_i}(p^j, \delta)$ of radius δ in $(\Sigma, f_i^* g_i)$ to the fixed ball $B_{g_H}(0, \delta)$ of radius δ in the Poincaré hyperbolic disc, and the maps ϕ_i^j converge smoothly to an isometry ϕ_∞^j from $B_h(p^j, \delta) \subset (\Sigma, h)$ to $(B_{g_H}(0, \delta), g_H)$. Here we can assume that for each i, the set K is covered not only by $B_{f_i^* g_i}(p^j, \delta)$ but also by the balls $B_{f_i^* g_i}(p^j, \delta/2)$ with half the radius.

The assumptions of Lemma 2.1 then imply uniform $L^1(B_{g_H}(0, \delta))$ bounds on both the functions ψ_i^j representing $f_i^* \Psi_i$ in these coordinate charts, and their antiholomorphic derivatives. Thus applying Lemma 2.3 of [5] and passing to a subsequence we find that the functions ψ_i^j converge in L^1 on a slightly smaller disc, say on $B_{g_H}(0, \delta/2)$, to a limit ψ_∞^j and that this limit is holomorphic if $\|\partial_{\bar{z}} \psi_i^j\|_{L^1(B_{g_H}(0,\delta))} \to 0$ and thus in particular if $\|\bar{\partial}\Psi_i\|_{L^1(M,g_i)} \to 0$.

Pulling back by the charts ϕ_i^j as well as making use of the convergence of the metrics we then obtain that the quadratic differentials $f_i^* \Psi_i$ converge to a limiting quadratic differential Ψ_∞ in the sense of $L^1(K, h)$ convergence of tensors and that the limit is holomorphic provided the antiholomorphic derivatives converge to zero as described in the lemma. □

The main step of the proof of Theorem 1.1 thus consists in proving Lemma 2.3.

Proof of Lemma 2.3. Let (M, g) be a closed hyperbolic surface of genus γ. We recall that the Keen-Randol collar lemma [4] gives the following explicit description of (M, g) near each simple closed geodesic σ of length $\ell > 0$: there is a neighbourhood $\mathcal{C}$ of σ in (M, g) which is isometric to $\mathcal{C}(\ell)$, where $\mathcal{C}(\ell)$ is the cylinder $(-X(\ell), X(\ell)) \times S^1$ equipped with the metric $\rho^2(s)(ds^2 + d\theta^2)$, with

$$X(\ell) = \frac{2\pi}{\ell} \left(\frac{\pi}{2} - \arctan \left(\sinh \left(\frac{\ell}{2} \right) \right) \right), \quad \text{and} \quad \rho(s) = \frac{\ell}{2\pi \cos(\frac{\ell s}{2\pi})}.$$

We will also use on several occasions that for $z = s + i\theta$,

$$(2.4) \qquad |dz^2| = 2\rho^{-2}, \qquad \text{and} \qquad \|dz^2\|_{L^2(\mathcal{C}(\ell))}^2 = 8\pi \int_{-X}^{X} \rho^{-2}(s) ds \sim \ell^{-3},$$

as $\ell \downarrow 0$.

Further important results in the theory of hyperbolic surfaces, cf. [9], Theorems 4.1.1 and 4.1.6, tell us that the collar regions around geodesics of length $0 < \ell < 2 \operatorname{arsinh}(1)$ are disjoint, the number of closed geodesics σ^j of length less than $2 \operatorname{arsinh}(1)$ is no more than $3(\gamma - 1)$, where γ is the genus of M, and that for any $0 < \delta < \operatorname{arsinh}(1)$ the δ-thin part of any hyperbolic surface consists solely of (subcylinders of) such collars $\mathcal{C}^j$. Furthermore the δ-thin part of each such a collar $\mathcal{C}(\ell)$, $0 < \ell < 2 \operatorname{arsinh}(1)$ is given by the subcylinder $(-X_\delta(\ell), X_\delta(\ell)) \times S^1$ for

$$(2.5) \qquad X_\delta(\ell) = \frac{\pi^2}{\ell} - \frac{2\pi}{\ell} \arcsin \left(\frac{\sinh(\ell/2)}{\sinh(\delta)} \right)$$

if $\delta \in (\ell/2, \operatorname{arsinh}(1))$ respectively by the empty set if $\delta \le \ell/2$. We also remark that using this formula one can easily check the intuitively clear fact that $\rho(X_\delta)$ is of order δ, we write for short $\rho(X_\delta) \sim \delta$, in the sense that there is a constant $C < \infty$ such that for every collar $\mathcal{C}(\ell)$, $0 < \ell < 2 \operatorname{arsinh}(1)$ and every $\delta \in (\ell/2, \operatorname{arsinh}(1))$ we have

$$(2.6) \qquad C^{-1}\delta \le \rho(X_\delta(\ell)) \le C \cdot \delta.$$

In order to prove Lemma 2.3 we need to show that an estimate of the form

$$(2.7) \qquad \|\Psi\|_{L^1(\delta\text{-thin}(\mathcal{C}))} = 2 \int_{-X_\delta(\ell)}^{X_\delta(\ell)} \int_{S^1} |\psi| \, d\theta ds \le C(\delta^{1/2} \|\Psi\|_{L^1(M,g)} + \|\bar{\partial}\Psi\|_{L^1(M,g)})$$

is valid for each such collar $\mathcal{C} \cong \mathcal{C}(\ell)$ and each quadratic diffential $\Psi \in \mathcal{Q}(M, g)$ satisfying $P_g\Psi = 0$. Here and in the following C denotes a constant depending only on the genus of the surface M, which we assume to be fixed.

We prove this claim in two steps. First we show that the assumed orthogonality of Ψ to $\mathcal{H}(M, g)$ implies estimates on mean values on circles of the function $\psi = \psi(s, \theta)$ representing Ψ in the collar coordinates (s, θ).

Lemma 2.4. *Let (M, g) be any closed hyperbolic surface. Then there exists a constant $C < \infty$ depending only on the genus of M such that for any collar region $\mathcal{C} = \mathcal{C}(\ell)$, $\ell > 0$, of (M, g) as described above and any quadratic differential $\Psi = \psi dz^2 \in \mathcal{Q}(M, g)$ satisfying*

$$P_g\Psi = 0,$$

the mean values

$$\alpha(s) := \frac{1}{2\pi} \int_{S^1} \psi(s, \theta) d\theta, \quad s \in (-X(\ell), X(\ell))$$

(where we work with respect to the local complex coordinate $z = s + i\theta$ on the collar) satisfy the estimate

$$(2.8) \qquad \int_{-X(\ell)}^{X(\ell)} |\alpha(s)| \, ds \leq C \cdot \big(\|\bar{\partial}\Psi\|_{L^1(M,g)} + \ell \cdot \|\Psi\|_{L^1(M,g)} \big).$$

In a second step, which will complete the proof of Lemma 2.3, we then estimate the L^1 norm of general quadratic differentials on the thin part of a collar in terms of $\alpha(\cdot)$ and the antiholomorphic derivative.

Lemma 2.5. *Let (M, g) be any closed hyperbolic surface. Then there exists a constant $C < \infty$ depending only on the genus of M such that for any collar region $\mathcal{C}$, any quadratic differential $\Psi \in \mathcal{Q}(M, g)$ and any $\delta > 0$, we have*

$$\|\Psi\|_{L^1(\delta\text{-}thin(\mathcal{C}))} \leq C \cdot \Big(\int_{-X}^{X} |\alpha(s)| \, ds + \|\bar{\partial}\Psi\|_{L^1(M,g)} + \delta^{1/2} \cdot \|\Psi\|_{L^1(M,g)} \Big).$$

Since the δ-thin part of any collar $\mathcal{C}(\ell)$ with $\ell > 2\delta$ is the empty set, and since the claim of Lemma 2.3 is trivially true for large values of δ, combining Lemmas 2.4 and 2.5 immediately gives the claim of the key Lemma 2.3.

Before we prove Lemma 2.4, we remark that the estimate claimed in that lemma remains valid if we weaken the assumption of Ψ being orthogonal to the whole space $\mathcal{H}(M, g)$, and only demand Ψ to be orthogonal to one specific holomorphic quadratic differential (per collar), which is described as follows:

Lemma 2.6. *Let (M, g) be any closed hyperbolic surface. Then there exists a constant $C < \infty$ depending only on the genus of M such that for σ any closed geodesic of length $0 < \ell \leq 2 \operatorname{arsinh}(1)$ and $\mathcal{C}$ its collar neighbourhood described above, there exists a holomorphic quadratic differential Ω with $\|\Omega\|_{L^2(M,g)} = 1$, concentrated on this one collar in the sense that*

$$(2.9) \qquad \|\Omega\|_{L^\infty(M \setminus \mathcal{C}, g)} \leq C\ell^{1/2},$$

and on this collar essentially given as a constant multiple of dz^2 in the sense that

$$(2.10) \qquad \|\Omega - b_0 dz^2\|_{L^\infty(\mathcal{C}, g)} \leq C\ell^{1/2},$$

for a number $b_0 \in \mathbb{C}$ satisfying $\big| 1 - |b_0| \cdot \|dz^2\|_{L^2(\mathcal{C}, g)} \big| \leq C\ell^{1/2}$.

Proof of Lemma 2.6. We prove the lemma by contradiction. Suppose instead that the lemma is false, and thus that there exist a closed surface M, a sequence of metrics g_i on M, and a sequence of collars $\mathcal{C}_i$ on (M, g_i) corresponding to closed geodesics σ_i of length $0 < \ell_i \leq 2 \operatorname{arsinh}(1)$ so that for each i, whenever Ω is a holomorphic quadratic differential on (M, g_i) with $\|\Omega\|_{L^2(M,g_i)} = 1$ then at least one of the two bounds

$$(2.11) \qquad \|\Omega\|_{L^\infty(M \setminus \mathcal{C}_i, g_i)} \leq i \, \ell_i^{1/2},$$

or

(2.12)
$$\|\Omega - b_0 dz^2\|_{L^\infty(\mathcal{C}_i, g_i)} \leq i \, \ell_i^{1/2}, \text{ for some } b_0 \in \mathbb{C} \text{ with } \big| 1 - |b_0| \cdot \|dz^2\|_{L^2(\mathcal{C}_i, g_i)} \big| \leq i\ell_i^{1/2}$$

must be *violated*. To derive a contradiction, we thus construct elements Ω_i which fullfill the two estimates (2.11) and (2.12) for i sufficiently large.

We first remark that standard estimates for holomorphic functions on discs lead to an estimate of the form

$$(2.13) \qquad \|\Phi\|_{L^\infty(\delta\text{-thick}(M,g))} \le C_\delta \|\Phi\|_{L^1(M,g)}$$

valid for all *holomorphic* quadratic differentials Φ on a hyperbolic surface (M, g) and every $\delta > 0$ with C_δ depending only on δ and the genus of the surface, cf. Lemma A.9 in [6]. We conclude that the sequence of surfaces (M, g_i) introduced above must degenerate as $i \to \infty$; indeed, assume that (for a subsequence) the injectivity radius of (M, g_i) is bounded away from zero by some number $\delta > 0$, and thus that the length of any closed geodesic of (M, g_i) is no less than 2δ. Then estimate (2.13) implies that for i sufficiently large (2.11) and (2.12) are both satisfied, say for $b_0 = 0$, for every $\Omega \in \mathcal{H}(M, g_i)$ with $\|\Omega\|_{L^2} = 1$, leading to a contradiction.

The sequence (M, g_i) can thus be analysed with the Deligne-Mumford compactness theorem in the same way as earlier, collapsing k collars $\mathcal{C}_i^j = \mathcal{C}(\ell_i^j)$, $\ell_i^j \to 0$ and yielding a limit (Σ, h).

The Fourier decomposition of holomorphic quadratic differentials Φ on each hyperbolic collar $(\mathcal{C}, g)$

$$(2.14) \qquad \Phi = \left(\sum_{n=-\infty}^{\infty} b_n e^{ns} e^{in\theta} \right) \cdot dz^2, \quad b_n \in \mathbb{C}$$

gives an $L^2(\mathcal{C}, g)$-orthogonal decomposition of each such Φ into its principal part $b_0(\Phi)dz^2$ and its *collar decay* part $\omega^\perp(\Phi)dz^2 := \Phi - b_0(\Phi)dz^2$ which, by Lemma 2.2 and Remark 2.3 of [6] satisfies the key estimate

$$(2.15) \qquad \|\omega^\perp(\Phi)dz^2\|_{L^\infty(\delta\text{-thin}(\mathcal{C},g))} \le C\delta^{-2} e^{-\frac{\pi}{\delta}} \|\Phi\|_{L^1(M,g)}.$$

Following [6], we can then define the subspaces

$$(2.16) \qquad W_i := \{\Theta \in \mathcal{H}(M, g_i) : b_0^j(\Theta)dz^2 = 0 \text{ for every } j \in \{1 \ldots k\}\}$$

of all holomorphic quadratic differentials with principal part equal to zero on each degenerating collar $\mathcal{C}_i^j$, $j = 1 \ldots k$, and it is these subspaces W_i which converge to $\mathcal{H}(\Sigma, h)$ in the sense of Lemma 2.1 (as described in [6]). We furthermore remark that elements of W_i are uniformly controlled by their L^2 norm,

$$(2.17) \qquad \sup_{w \in W_i} \frac{\|w\|_{L^\infty(M,g_i)}}{\|w\|_{L^2(M,g_i)}} \le C < \infty$$

for a constant C independent of i (as follows from Lemma 2.4(i) and Lemma A.8 of [6]).

This implies in particular that the collar $\mathcal{C}_i$, for which (2.11) and (2.12) cannot be satisfied, must degenerate, $\ell_i \to 0$ as $i \to \infty$, and thus that this collar coincides with one of the collapsing collars $\mathcal{C}_i^j$, say $\mathcal{C}_i = \mathcal{C}_i^1$ (for a subsequence).

We will now choose the holomorphic quadratic differentials Ω_i associated with these collars as elements of the $L^2(M, g_i)$-orthogonal complement $W_i^\perp$ of W_i. More precisely, by Lemma 2.4 of [6], we have $\dim(W_i^\perp) = k$ for large enough i, so we can assign to $\mathcal{C}_i = \mathcal{C}_i^1$ the unique element Ω_i of $W_i^\perp$ with $\|\Omega_i\|_{L^2(M,g_i)} = 1$ for which the principal part $b_0^j(\Omega_i)dz^2$ on $\mathcal{C}_i^j$ is equal to zero if $j \neq 1$ but is $b_0(\Omega_i)dz^2$ for some $b_0(\Omega_i) > 0$ if $j = 1$. We then claim that

$$(2.18) \qquad \lambda_i := \|\Omega_i\|_{L^1(M,g_i)} \le C \cdot [\ell_i]^{1/2}$$

and remark that this claim combined with (2.13) and (2.15) directly implies (2.11) and (2.12), thus giving the contradiction that proves Lemma 2.6.

In order to prove the bound (2.18) we now consider the sequence $\widetilde{\Omega}_i := (\lambda_i)^{-1} \cdot \Omega_i$ of holomorphic quadratic differentials normalised to have L^1 norm $\|\widetilde{\Omega}_i\|_{L^1(M,g_i)} = 1$ and prove that the only part of $\widetilde{\Omega}_i$ whose contribution to the L^1 norm does not vanish as $i \to \infty$ is its principal part on $\mathcal{C}_i$.

To start with, Lemma 2.1 allows us to extract a subsequence of $\widetilde{\Omega}_i$ so that $f_i^*\widetilde{\Omega}_i$ converges smoothly locally to a limit $\widetilde{\Omega}_\infty \in \mathcal{H}(\Sigma, h)$, which must indeed be identically zero since by construction $P_{g_i}^{W_i}\widetilde{\Omega}_i = 0$ and thus according to Lemma 2.2 also $P_h^{\mathcal{H}(\Sigma,h)}(\widetilde{\Omega}_\infty) = 0$. We conclude that

$$0 = \|\widetilde{\Omega}_\infty\|_{L^1(\Sigma,h)} = 1 - \inf_{\delta > 0} \lim_{i \to \infty} \|\widetilde{\Omega}_i\|_{L^1(\delta\text{-thin}(M,g_i))}$$

which means that for any $\delta > 0$ all of the L^1 norm of $\widetilde{\Omega}_i$ concentrates in the limit $i \to \infty$ on the δ-thin part of (M, g_i).

We observe that for $\delta > 0$ sufficiently small, the δ-thin part of (M, g_i) is given as the union of the δ-thin parts of the degenerating collars $\mathcal{C}_i^j$, but that estimate (2.15) implies $\|\widetilde{\Omega}_i\|_{L^1(\delta\text{-thin}(\mathcal{C}_i^j))} \le C\delta^{-2}e^{-\frac{\pi}{\delta}} \to 0$ as $\delta \searrow 0$ for each $j \ne 1$.

Meanwhile, (2.15) applied to $\mathcal{C}_i^1 =: \mathcal{C}_i$ shows that the contribution of the collar decay part $\omega^\perp(\widetilde{\Omega}_i)dz^2$ of $\widetilde{\Omega}_i$ on $\mathcal{C}_i$ to the total L^1 norm of $\|\widetilde{\Omega}_i\|_{L^1(M,g_i)} = 1$ vanishes in the limit $i \to \infty$. This means that the only remaining part of $\widetilde{\Omega}_i$, namely the principal part $b_0(\widetilde{\Omega}_i) \cdot dz^2 = (\lambda_i)^{-1}b_0(\Omega_i) \cdot dz^2$ of $\widetilde{\Omega}_i$ on this one collar $\mathcal{C}_i$, must have L^1 norm converging to 1 as $i \to \infty$. Since $\|dz^2\|_{L^1(\mathcal{C}_i)} = 8\pi \cdot X(\ell_i) \le C \cdot [\ell_i]^{-1}$ we thus get an upper bound of

$$\|\Omega_i\|_{L^1(M,g_i)} =: \lambda_i \le C \cdot [\ell_i]^{-1}b_0(\Omega_i)$$

for the L^1 norm of the original holomorphic quadratic differential. But with Ω_i normalised to have $\|\Omega_i\|_{L^2} = 1$ and with the principal and collar decay part being L^2-orthogonal we also know that $\|b_0(\Omega_i) \cdot dz^2\|_{L^2(\mathcal{C}_i,g_i)} \le 1$ which, according to (2.4), means that $b_0(\Omega_i) \le C \cdot [\ell_i]^{3/2}$. The claim (2.18) now follows. $\square$

In summary, from the previous lemma, its proof and the analysis of the spaces W_i carried out in [6], we obtain the following general description of the spaces of holomorphic quadratic differentials on degenerating hyperbolic surfaces.

Corollary 2.7. *Let (M, g_i) be a sequence of hyperbolic surfaces degenerating to a punctured hyperbolic surface (Σ, h) by collapsing k collars $\mathcal{C}_i^j$. Then, for i sufficiently large, the space of holomorphic quadratic differentials $\mathcal{H}(M, g_i)$ splits into*

- *(i) the $3(\gamma - 1) - k$ dimensional subspace W_i defined in (2.16) which converges to the space $\mathcal{H}(\Sigma, h)$ of L^1 holomorphic quadratic differentials on the limit surface as described in Theorem 2.6 of [6], and*
- *(ii) its orthogonal complement $W_i^\perp$, a basis of which is given by holomorphic quadratic differentials $(\Omega_i^j)_{j=1}^k$ concentrating solely on the degenerating collars $\mathcal{C}_i^j$ as described in Lemma 2.6.*

Proof of Lemma 2.4. Let (M, g) be a closed hyperbolic surface and $\mathcal{C} = \mathcal{C}(\ell)$ a collar around a closed geodesic in (M, g). Without loss of generality, we may assume that $\ell \le 2\operatorname{arsinh}(1)$, and can apply Lemma 2.6 to obtain the corresponding holomorphic quadratic differential Ω. To prove (2.8) it is enough to consider collars around geodesics of small length ℓ, in particular small enough so that the number b_0, as in Lemma 2.6 characterising the principal part of Ω on $\mathcal{C}$, satisfies $|b_0| \ge \|dz^2\|_{L^2(\mathcal{C})}^{-1}(1 - C\ell^{1/2}) \ge c\ell^{3/2}$ for some universal $c > 0$, compare (2.4).

Given any quadratic differential $\Psi \in \mathcal{Q}(M, g)$ that is orthogonal to Ω, we combine the relation $\langle \Psi, \Omega \rangle_{L^2(M,g)} = 0$ with this bound on b_0 and with (2.4) to find that

(2.19)
$$\ell^{3/2} \left| \int_{-X}^{X} \int_{S^1} \psi \cdot \rho^{-2} d\theta ds \right| \leq C \cdot \left| \langle \Psi, b_0 dz^2 \rangle_{L^2(\mathcal{C}, g)} \right|$$
$$\leq C \left(\left| \langle \Psi, \Omega - b_0 dz^2 \rangle_{L^2(\mathcal{C},g)} \right| + \left| \langle \Psi, \Omega \rangle_{L^2(M \setminus \mathcal{C}, g)} \right| \right)$$
$$\leq C \left(\| \Omega - b_0 dz^2 \|_{L^\infty(\mathcal{C},g)} + \| \Omega \|_{L^\infty(M \setminus \mathcal{C}, g)} \right) \cdot \| \Psi \|_{L^1(M,g)}$$
$$\leq C \ell^{1/2} \cdot \| \Psi \|_{L^1(M,g)},$$

or equivalently that the mean values $\alpha(s)$ are small on average in the sense that they satisfy

(2.20)
$$\left| \int_{-X(\ell)}^{X(\ell)} \alpha(s) \rho^{-2}(s) ds \right| \leq C \ell^{-1} \| \Psi \|_{L^1(M,g)}.$$

Note that if Ψ were holomorphic, then the function $s \mapsto \alpha(s)$ would be constant and (2.20) would imply $|\alpha| \leq C \ell^2 \| \Psi \|_{L^1}$ and thus in particular the estimate of Lemma 2.4. For general quadratic differentials $\Psi \in \mathcal{Q}(M, g)$ the function $s \mapsto \alpha(s)$ need not be constant but we can still estimate

(2.21)
$$|\alpha(0) - \alpha(s_0)| = \left| \frac{1}{2\pi} \int_0^{s_0} \frac{d}{ds} \left(\int_{\{s\} \times S^1} \psi d\theta \right) ds \right| = \left| \frac{1}{2\pi} \int_0^{s_0} \int_{S^1} (\partial_s \psi + i \partial_\theta \psi) d\theta ds \right|$$
$$\leq \frac{1}{\pi} \int_{[0,s_0] \times S^1} |\partial_{\bar{z}} \psi| d\theta ds$$

for each $s_0 \in (-X, X)$, where we abuse notation by allowing $[0, s_0]$ to denote $[s_0, 0]$ for $s_0 < 0$. Using (2.4), we then write

$$\alpha(0) \cdot \| dz^2 \|_{L^2(\mathcal{C},g)}^2 = 8\pi \int_{-X}^{X} (\alpha(0) - \alpha(s_0)) \rho^{-2}(s_0) ds_0 + 4 \int_{-X}^{X} \int_{S^1} \psi(s, \theta) \rho^{-2}(s) d\theta ds$$

and use (2.4), (2.19) and (2.21) to estimate

(2.22)
$$|\alpha(0)| \leq C \ell^3 \left[\int_0^{X} \left(\rho^{-2}(s_0) \int_{-s_0}^{s_0} \int_{S^1} |\partial_{\bar{z}} \psi| d\theta ds \right) ds_0 + \ell^{-1} \| \Psi \|_{L^1(M,g)} \right]$$
$$\leq C \ell^3 \left(\int_{-X}^{X} \int_{S^1} \rho^{-1} |\partial_{\bar{z}} \psi| d\theta ds \right) \cdot \left(\int_0^{X} \rho^{-1}(s_0) ds_0 \right) + C \ell^2 \| \Psi \|_{L^1(M,g)}$$
$$\leq C \ell \| \bar{\partial} \Psi \|_{L^1(M,g)} + C \ell^2 \| \Psi \|_{L^1(M,g)}.$$

Here we use that

$$\left| \bar{\partial} \Psi \right| = |\partial_{\bar{z}} \psi| \cdot \left| d\bar{z} \otimes dz^2 \right| = 2\sqrt{2} |\partial_{\bar{z}} \psi| \rho^{-3}$$

so that

$$\| \bar{\partial} \Psi \|_{L^1(\mathcal{C},g)} = 2\sqrt{2} \int_{-X}^{X} \int_{S^1} \rho^{-1}(s) |\partial_{\bar{z}} \psi(s, \theta)| d\theta ds.$$

We also used that $\rho(s)$ is monotone in $|s|$ and that $\int_0^{X} \rho^{-1}(s_0) ds_0 = \frac{2\pi}{\ell} \int_0^{X(\ell)} \cos\left(\frac{\ell s}{2\pi} \right) ds \leq \left(\frac{2\pi}{\ell} \right)^2$. Combining (2.22) with (2.21) we thus find that for each $s_0 \in (-X(\ell), X(\ell))$

(2.23)
$$|\alpha(s_0)| \leq C \ell \| \bar{\partial} \Psi \|_{L^1(M,g)} + \frac{1}{\pi} \int_{[0,s_0] \times S^1} |\partial_{\bar{z}} \psi| d\theta ds + C \cdot \ell^2 \| \Psi \|_{L^1(M,g)}.$$

We stress that the second term on the right-hand side of this estimate is *not* the L^1 norm of $\bar{\partial} \Psi$ over $[0, s_0] \times S^1$ but a much smaller integral; indeed the missing factor

$\rho^{-1}(s)$ controls the (euclidean) distance of s to the end of the collar since

$$(2.24) \qquad \rho(s) \cdot (X(\ell) - |s|) \leq \rho(s) \cdot \left(\frac{\pi^2}{\ell} - |s| \right) \leq \sup_{v \in (0, \pi/2)} \frac{v}{\sin(v)} = \frac{\pi}{2}.$$

Integrating (2.23) using Fubini's theorem as well as $X = X(\ell) \leq \frac{\pi^2}{\ell}$ we thus find
$$(2.25)$$
$$\int_{-X}^{X} |\alpha(s_0)| \, ds_0 \leq C \cdot \|\bar{\partial}\Psi\|_{L^1(M,g)} + C \int_{-X}^{X} \int_{S^1} |\partial_{\bar{z}}\psi| \cdot (X - |s|) d\theta ds + C\ell \|\Psi\|_{L^1(M,g)}$$
$$\leq C \|\bar{\partial}\Psi\|_{L^1(M,g)} + C\ell \|\Psi\|_{L^1(M,g)}$$

as claimed in Lemma 2.4. □

The remaining step in the paper is thus:

Proof of Lemma 2.5. We want to estimate the L^1 norm of a general quadratic differentials $\Psi \in \mathcal{Q}(M, g)$ on the δ-thin part of a hyperbolic collar $\mathcal{C}(\ell)$. The basic idea is to extend the function ψ representing $\Psi = \psi dz^2$ on the collar periodically (with period 2π in the i direction) to a function (still denoted by ψ) on the set $(-X(\ell), X(\ell)) \times \mathbb{R} \subset \mathbb{R}^2 \cong \mathbb{C}$ and to derive estimates using the inhomogeneous Cauchy-formula on large domains. Before we proceed with the proof, we remark that the estimate of Lemma 2.5 is trivially true for large values of δ so that we may henceforth assume that $0 < \delta < \delta_0$ and thus also $0 < \ell < 2\delta_0$ for a small number $\delta_0 > 0$ to be chosen later on.

Recall that for any $z_0 \in \mathbb{C}$, any domain $\Omega \subset \mathbb{C}$ containing z_0 and with piecewise C^1 boundary $\partial\Omega$, and any C^1 function ψ the Cauchy-formula gives

$$(2.26) \qquad \psi(z_0) = \frac{1}{2\pi i} \int_{\Omega} \frac{\partial_{\bar{z}}\psi}{z - z_0} dz \wedge d\bar{z} + \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\psi}{z - z_0} dz.$$

Keeping in mind the final goal of getting a bound on $\|\Psi\|_{L^1(\delta\text{-thin}(\mathcal{C}))}$ in terms of $\|\bar{\partial}\Psi\|_{L^1}$, α and a *small* multiple of $\|\Psi\|_{L^1}$ we would like to choose the domains Ω in such a way that the boundary integrals in (2.26) are essentially given in terms of the mean values $\alpha(\cdot)$. Working on *large* rectangles this can be easily achieved for the integrals along lines in the θ direction. Furthermore, applying the Cauchy-formula not just for one such rectangle, but rather taking its mean value over a suitable family of rectangles, also the integrals along lines in the s direction will be essentially controlled in terms of α. We are able to control the first integral in (2.26) in L^1 provided we choose the size of these rectangles dependent on the (large) factor $\rho^{-1}(s_0)$ with which $\partial_{\bar{z}}\psi$ appears in $\|\bar{\partial}\Psi\|_{L^1}$.

Let now $\mathcal{C}(\ell) \cong ((-X(\ell), X(\ell)) \times S^1, \rho^2(ds^2 + d\theta^2))$ be a hyperbolic collar around a closed geodesic of length $0 < \ell < 2\delta_0$. Then for each point $z_0 = (s_0, \theta_0) \in (-X_{\delta_0}, X_{\delta_0}) \times [0, 2\pi]$, representing a point in the δ_0-thin part of the collar, we consider the family of rectangles

$$\Omega_b(z_0) := \{z_0\} + [-\rho^{-1/2}(s_0), \rho^{-1/2}(s_0)] \times [-(\rho^{-1/2}(s_0) + b), \rho^{-1/2}(s_0) + b], b \in [0, 2\pi]$$

and apply the Cauchy-formula to write

$$(2.27) \qquad 2\pi i \psi(z_0) = I_\Omega(z_0, b) + I_V^+(z_0, b) + I_V^-(z_0, b) + I_H^+(z_0, b) + I_H^-(z_0, b)$$

where $I_\Omega(z_0, b) = \int_{\Omega_b(z_0)} \frac{\partial_{\bar{z}}\psi}{z - z_0} dz \wedge d\bar{z}$ while I_H, I_V denote the line integrals along the horizontal respectively vertical paths of $\partial\Omega_b$, that is

$$I_H^\pm(z_0, b) = \mp \int_{h^-(s_0)}^{h^+(s_0)} \frac{\psi(s, \theta_0 \pm (\rho^{-1/2}(s_0) + b))}{s - s_0 \pm i(\rho^{-1/2}(s_0) + b)} ds$$

and

$$I_V^{\pm}(z_0, b) = \pm i \int_{\theta_0 - (\rho^{-1/2}(s_0) + b)}^{\theta_0 + \rho^{-1/2}(s_0) + b} \frac{\psi(h^{\pm}(s_0), \theta)}{\pm \rho^{-1/2}(s_0) + i(\theta - \theta_0)} d\theta.$$

Here and in the following we write for short $h^{\pm}(s_0) := s_0 \pm \rho^{-1/2}(s_0)$ to denote the s limits of the domains of integration.

Remark that to obtain estimates on ψ, be it pointwise or in an L^1 sense, it is sufficient to prove estimates on the mean values with respect to b of all these integrals. While for the terms I_Ω and $I_V^{\pm}$ it is equally simple/difficult to derive bounds on these terms for each individual b, taking such an average over b is crucial in order to bound the integrals $I_H^{\pm}$ along horizontal lines in terms of α which is a mean value in θ and not in s.

With this in mind, we bound
(2.28)
$$\|\Psi\|_{L^1(\delta\text{-thin}(\mathcal{C},g))} = \frac{1}{\pi} \int_{-X_\delta}^{X_\delta} \int_{S^1} \left| \fint_0^{2\pi} \left(I_\Omega + I_V^+ + I_V^- + I_H^+ + I_H^-\right)(z_0, b) db \right| d\theta_0 ds_0$$

$$\leq \frac{1}{\pi} \sup_{b \in [0, 2\pi]} \int_{-X_\delta}^{X_\delta} \int_{S^1} |I_\Omega(z_0, b)| + \left|I_V^+(z_0, b)\right| + \left|I_V^-(z_0, b)\right| d\theta_0 ds_0$$

$$+ \frac{1}{\pi} \int_{-X_\delta}^{X_\delta} \int_{S^1} \left| \fint_0^{2\pi} I_H^+(z_0, b) db \right| + \left| \fint_0^{2\pi} I_H^-(z_0, b) db \right| d\theta_0 ds_0$$

and estimate all terms occurring in this formula individually. As remarked above, we may always assume that $0 < \delta < \delta_0$ for a small fixed number $\delta_0 > 0$, which we chose in particular so that the following remark holds true.

Remark 2.8. For $\delta_0 > 0$ sufficiently small the domains Ω_b have been chosen in such a way that for each collar $\mathcal{C} = \mathcal{C}(\ell)$ with $0 < \ell < 2\delta_0$ and each $0 < \delta \leq \delta_0$:

(i) For every $z_0 \in \delta\text{-thin}(\mathcal{C})$ the points in $\Omega_b(z_0)$ all correspond to points in the 2δ-thin part of $\mathcal{C}$, i.e. $\Omega_b(z_0) \subset (-X_{2\delta}, X_{2\delta}) \times \mathbb{R}$.

(ii) There is a constant $C = C_{\delta_0} < \infty$ depending only on δ_0 such that

$$C^{-1} \rho(s) \leq \rho(s_0) \leq C\rho(s) \text{ for all } z = (s, \theta) \in \Omega_b(z_0), z_0 = (s_0, \theta_0) \in \delta_0\text{-thin}(\mathcal{C}).$$

(iii) The functions $h^{\pm}(s) = s \pm \rho(s)^{-1/2}(s)$ are invertible on $[X_{-\delta_0}, X_{\delta_0}]$, the derivatives of the inverses uniformly bounded and $((h^-)^{-1} - (h^+)^{-1})(s) \leq C_{\delta_0} \rho^{-1/2}(s)$.

The main observation leading to the first statement of the remark is that the expression (2.5) for X_δ combined with the mean value theorem implies that for small values of δ the difference $X_{2\delta} - X_\delta \geq \frac{\pi \cdot \sinh(\ell/2)}{\ell \cdot \sinh(\delta)} \geq c\delta^{-1}$ is much larger than $\rho^{-1/2}(X_\delta) \leq C\delta^{-1/2}$, compare (2.6). The second remark is then a simple consequence of the first and of (2.6). For the final claim, we observe that the derivative of ρ^{-1} is uniformly bounded so that $(\rho^{-1/2})' \leq C\rho^{1/2}$ is small in the δ_0-thin part of the collar that we consider, and thus the derivatives of $h^{\pm}$ are close to one.

Turning back to (2.28), we first estimate the term involving the antiholomorphic derivative (i.e. involving I_Ω). Let $N(s_0) := \left\lceil \frac{\rho^{-1/2}(s_0)}{2\pi} \right\rceil$ and remark that the domain $\Omega_b(z_0)$ can be wrapped around the cylinder $[h^-(s_0), h^+(s_0)] \times S^1$ no more than $2(N(s_0) + 2)$ times.

Using the periodicity of ψ we can thus estimate for each $b \in [0, 2\pi]$
(2.29)

$$
|I_\Omega(z_0, b)| \leq 2 \int_{\Omega_{2\pi}(z_0)} \left| \frac{\partial_{\bar{z}} \psi}{z - z_0} \right| d\theta ds
$$

$$
\leq 2 \int_{z_0 + [-2\pi, 2\pi]^2} \left| \frac{\partial_{\bar{z}} \psi}{z - z_0} \right| d\theta ds + 2 \int_{h^-(s_0)}^{h^+(s_0)} \int_{-2\pi}^{2\pi} \frac{|\partial_{\bar{z}} \psi|}{2\pi} d\theta ds
$$

$$
+ 4 \sum_{k=2}^{N(s_0)+2} \frac{1}{2\pi(k-1)} \cdot \int_{h^-(s_0)}^{h^+(s_0)} \int_{S^1} |\partial_{\bar{z}} \psi| \, d\theta ds
$$

$$
\leq 2 \int_{z_0 + [-2\pi, 2\pi]^2} \left| \frac{\partial_{\bar{z}} \psi}{z - z_0} \right| d\theta ds + C \cdot \int_{h^-(s_0)}^{h^+(s_0)} \int_{S^1} |\partial_{\bar{z}} \psi| \cdot \log(\rho^{-1}(s)) d\theta ds
$$

with the last inequality due to the bound $1 + \sum_{k=1}^{N(s_0)+1} \frac{1}{k} \leq C \cdot \log(N(s_0) + 1) \leq C \log(\rho^{-1}(s))$ being valid for every $s \in [h^-(s_0), h^+(s_0)]$, see Remark 2.8 (ii), if δ_0 is small enough.

Integrating this estimate over $z_0 \in \delta\text{-thin}(\mathcal{C}) = (-X_\delta, X_\delta) \times S^1$ using Fubini's theorem as well as (i) and (iii) of Remark 2.8, we thus obtain that for any $b \in [0, 2\pi]$, we have
(2.30)

$$
\int_{\delta\text{-thin}(\mathcal{C})} |I_\Omega(z_0, b)| \, ds_0 d\theta_0 \leq 2 \int_{-X_\delta - 2\pi}^{X_\delta + 2\pi} \int_{S^1} \left[|\partial_{\bar{z}} \psi| \cdot \int_{z + [-2\pi, 2\pi]^2} \frac{1}{|z - z_0|} ds_0 d\theta_0 \right] d\theta ds
$$

$$
+ C \int_{-X_{2\delta}}^{X_{2\delta}} \int_{S^1} \left[\log(\rho^{-1}) \cdot |\partial_{\bar{z}} \psi| \cdot \left(\int_{(h^+)^{-1}(s)}^{(h^-)^{-1}(s)} 1 \, ds_0 \right) \right] d\theta ds
$$

$$
\leq C \int_{-X_{2\delta}}^{X_{2\delta}} \int_{S^1} |\partial_{\bar{z}} \psi| \cdot \left(\rho^{-1/2} \log(\rho^{-1}) + 1 \right) d\theta ds
$$

$$
\leq C \int_{-X_{2\delta}}^{X_{2\delta}} \int_{S^1} |\partial_{\bar{z}} \psi| \, \rho^{-1} d\theta ds = C \|\bar{\partial} \Psi\|_{L^1(2\delta\text{-thin}(\mathcal{C}))}.
$$

Next we estimate the line integrals $I_V^{\pm}$ over vertical paths. Since we handle both terms in the same way, here we demonstrate the argument only by treating I_V^+.

For any $b \in [0, 2\pi]$ we split $I_V^+(z_0, b)$ into integrals over the line segments $I_k = \{h^+(s_0)\} \times \left(\{\theta_0 + 2\pi \cdot k\} + [0, 2\pi] \right)$, $|k| \leq N(s_0)$, with $N(s_0)$ as above, and a small remainder term that is bounded by $C\rho^{1/2}(s_0) \int_{S^1} |\psi(h^+(s_0), \theta)| \, d\theta$.

The important observation is that $\frac{1}{z - z_0}$ is nearly constant over each such I_k and consequently that the corresponding integrals are essentially given by multiples of the mean values

$$
\alpha(h^+(s_0)) = \fint_{S^1} \psi(h^+(s_0), \theta) d\theta = \fint_{I_k} \psi.
$$

More precisely, we have

(2.31)
$$
\sup_{z \in I_k} \left| \frac{1}{z - z_0} - \frac{1}{\rho^{-1/2}(s_0) + 2\pi i \cdot k} \right| \leq 2\pi \rho(s_0),
$$

so that for each k

(2.32)
$$
\left| \int_{I_k} \frac{\psi(z)}{z - z_0} dz \right| \leq 2\pi \rho^{1/2}(s_0) \cdot |\alpha(h^+(s_0))| + C\rho(s_0) \cdot \int_{S^1} |\psi(h^+(s_0), \theta)| \, d\theta.
$$

Summing up these $2N(s_0) + 1 \leq C\rho^{-1/2}(s_0)$ integrals, we thus find that for each $b \in [0, 2\pi]$

$$
|I_V^+(z_0, b)| \leq C \cdot |\alpha(h^+(s_0))| + C\rho^{1/2}(s_0) \cdot \int_{S^1} |\psi(h^+(s_0), \theta)| \, d\theta.
$$

Integrating these estimates over z_0 in the δ-thin part of the collar and using Remark 2.8 (i) and (iii) as well as (2.6) we thus find that for any $b \in [0, 2\pi]$
(2.33)
$$
\int_{\delta\text{-thin}(\mathcal{C})} \left| I_V^+(z_0, b) \right| ds_0 d\theta_0 \leq C \int_{-X_{2\delta}}^{X_{2\delta}} |\alpha(s)| \, ds + C \left(\sup_{s \in [-X_\delta, X_\delta]} \rho^{1/2}(s) \right) \cdot \|\psi\|_{L^1(2\delta\text{-thin}(\mathcal{C}))}
$$
$$
\leq C \int_{-X}^{X} |\alpha(s)| \, ds + C\delta^{1/2} \|\Psi\|_{L^1(M,g)}.
$$

We finally derive estimates for the integrals $I_H^\pm(z_0, b)$ over the horizontal paths, now not for each individual b but rather for the mean values over $b \in [0, 2\pi]$. We treat the term I_H^+, with I_H^- handled similarly. By Fubini's theorem

$$
\left| \fint_0^{2\pi} I_H^+(z_0, b) db \right| \leq \int_{h^-(s_0)}^{h^+(s_0)} \left| \fint_{\{\theta_0 + \rho^{-1/2}(s_0)\}+[0,2\pi]} \frac{\psi(s, \theta)}{z - z_0} d\theta \right| ds.
$$

Using an estimate similar to (2.31) we now write the interior integrals as $\frac{\alpha(s)}{s - s_0 + i\rho^{-1/2}(s_0)}$ plus a small error term, resulting in
(2.34)
$$
\left| \fint_0^{2\pi} I_H^+(z_0, b) db \right| \leq \rho^{1/2}(s_0) \cdot \int_{h^-(s_0)}^{h^+(s_0)} |\alpha(s)| \, ds + C\rho(s_0) \int_{h^-(s_0)}^{h^+(s_0)} \int_0^{2\pi} |\psi| \, d\theta ds
$$
$$
\leq C \cdot \int_{h^-(s_0)}^{h^+(s_0)} \rho^{1/2}(s) |\alpha(s)| \, ds + C\rho^{1/2}(s_0) \int_{h^-(s_0)}^{h^+(s_0)} \int_0^{2\pi} \rho^{1/2}(s) |\psi| \, d\theta ds,
$$

where we once more use Remark 2.8 (ii) in the last step. Thanks to Remark 2.8, after integration over $z_0 \in \delta\text{-thin}(\mathcal{C})$ this gives
(2.35)
$$
\int_{-X_\delta}^{X_\delta} \int_{S^1} \left| \fint_0^{2\pi} I_H^+(z_0, b) db \right| d\theta_0 ds_0 \leq C \int_{-X_{2\delta}}^{X_{2\delta}} |\alpha(s)| \, \rho^{1/2}(s) \cdot \left((h^-)^{-1} - (h^+)^{-1} \right)(s) ds
$$
$$
+ C\delta^{1/2} \int_{-X_{2\delta}}^{X_{2\delta}} \int_{S^1} |\psi(s, \theta)| \, \rho^{1/2}(s) \cdot ((h^-)^{-1} - (h^+)^{-1})(s) d\theta ds
$$
$$
\leq C \int_{-X}^{X} |\alpha(s)| \, ds + C\delta^{1/2} \|\Psi\|_{L^1(M,g)}.
$$

Inserting these three estimates (2.30), (2.33) and (2.35) into (2.28) immediately gives the claim of Lemma 2.5. $\qquad\square$

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