

Non-abelian cubic vertices for higher-spin fields in AdS_d

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Abstract

We use the Fradkin-Vasiliev procedure to construct the full set of non-abelian cubic vertices for totally symmetric higher spin gauge fields in AdS_d space. The number of such vertices is given by a certain tensor-product multiplicity. We discuss the one-to-one relation between our result and the list of non-abelian gauge deformations in flat space obtained elsewhere via the cohomological approach. We comment about the uniqueness of Vasiliev's simplest higher-spin algebra in relation with the (non)associativity properties of the gauge algebras that we classified. The gravitational interactions for (partially)-massless (mixed)-symmetry fields are also discussed. We also argue that those mixed-symmetry and/or partially-massless fields that are described by one-form connections within the frame-like approach can have nonabelian interactions among themselves and again the number of nonabelian vertices should be given by tensor product multiplicities.

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1 Introduction

In 1987, Fradkin and Vasiliev solved the problem of cubic interactions for higher spin gauge fields among themselves and with gravity [1, 2]. A key ingredient of their construction was the expansion of all fields around the anti-de Sitter (AdS) background of dimension four instead of the Minkowskian flat background. It proved very convenient to use commuting $sl(2, \mathbb{C})$ Weyl spinors although there is *a priori* nothing fundamental, at that stage, with the dimensionality of the background. The problem can indeed be considered in an AdS_d background of arbitrary dimension $d > 4$, which is the framework of the present paper where we explicitly build and classify all the possible non-abelian couplings between totally-symmetric higher-spin (including spin-2) gauge fields in AdS_d with $d > 4$. By non-abelian cubic vertices, we mean those which non-trivially deform the abelian gauge algebra of the free theory.

For that purpose, we use the Fradkin-Vasiliev procedure whereby the free theory is presented in the frame-like approach, starting from a Lagrangian quadratic in the linear curvature two-forms. The cubic vertices are obtained by substituting non-linear deformations of the curvature two-forms inside the quadratic, free action. The very structure of these non-linear deformations automatically implies that the gauge algebra is non-abelian to the first non-trivial order in deformation. We also adopt the MacDowell-Mansouri-Stelle-West formulation of gravity [3–5] and its higher-spin generalization [6] which makes the AdS_d symmetry manifest through the introduction of an extra field, sometimes called compensator, inside the Lagrangian. For recent works along the same lines, see e.g. [7–12].

Our main result can be stated in a concise way: given three totally-symmetric gauge fields with spins s , s' and s'' , the number of independent non-abelian vertices is given by the tensor product multiplicity

$$\begin{array}{|c|} \hline s-1 \\ \hline \end{array} \otimes \begin{array}{|c|} \hline s'-1 \\ \hline \end{array} \longrightarrow \begin{array}{|c|} \hline s''-1 \\ \hline \end{array}, \quad (1.1)$$

i.e. by all the possible independent ways to contract two rectangular two-row $so(d-1, 2)$ tensors in order to form another two-row rectangular $so(d-1, 2)$ tensor of given length. The lengths of the diagrams involved are related to the spins as indicated above. The gauge fields valued in such irreducible tensor representations of the anti-de Sitter algebra $so(d-1, 2)$ have been proposed for the description of higher-spin fields by Vasiliev in [6]. At the same time, this multiplicity equals the number of non-abelian vertices in Minkowski space, [13]. The vertices we construct are off-shell and not subjected to any transverse/traceless gauge condition. A particular way of contracting indices in (1.1) is given by the Vasiliev higher-spin algebra [14]. This algebra is a unique associative algebra with spectrum of generators (1.1) and all other non-abelian deformations lead to nonassociative algebras that can hardly be consistent at the quartic level.

Among all the possible types of vertices: abelian, non-abelian, Chern-Simons-like etc.⁴, the non-abelian ones contain more information about the full theory, whatever it is. Consistency at the quartic level may however require some abelian cubic vertices to be added, see the discussion in [16] and [17].

The construction that we present here for the classification of the non-abelian cubic vertices in AdS_d uses the $sp(2)$ technology developed by Vasiliev and collaborators [14, 18, 19] and shows some similarity with the cohomological method [20] used in [13] for the classification of the non-abelian algebras in flat space, to first order in deformation. Both the present approach to consistent vertices and the cohomological one have the advantage that they provide a completely algebraic reformulation of the consistent-coupling problem. It is *a priori* not clear that the Fradkin-Vasiliev ansatz leads to the most general non-abelian deformations. As a matter of fact, and in agreement with what was argued in [12], we find that it actually produces the exhaustive list of non-abelian cubic vertices in AdS_d . This follows from the following argument: On the one hand, we have at our disposal [13] the complete classification of non-abelian gauge-algebra deformations, for any given triplet (s, s', s'') of higher-spin gauge fields in flat background. On the other hand we know that to every non-abelian vertex in AdS_d for totally symmetric gauge fields there is a corresponding non-abelian vertex in flat space [16]. Therefore, if one constructs – as we do in this paper – a list of independent non-abelian vertices in AdS_d whose number corresponds to the number of non-abelian vertices in flat space, then one automatically has access to the full list of non-abelian vertices in AdS_d . Indeed, assuming the existence of additional, independent non-abelian vertices in AdS_d , the corresponding flat limit along the lines of [16] – which entails starting from the nontrivial terms in the Lagrangian containing the highest number of partial derivatives, a filtration that can always be done for cubic vertices in AdS_d – would give rise to additional and independent non-abelian vertices in flat space, thereby giving a total number of non-abelian vertices exceeding the upper bound obtained in [13].

Manifestly covariant cubic vertices in flat space of arbitrary dimension have been explicitly written by many authors by now [21–24]⁵. The situation is not exactly the same in AdS_d , see e.g. [29–31] for some very recent endeavours. A noticeable exception is the very general analysis provided in [12], that shows how to classify vertices in AdS_d using the frame-like formalism. In [12] the set of generating functions for non-abelian vertices has also been suggested. Our goal is to elaborate on the algebraic structure of non-abelian cubic vertices. Vertices that explicitly involve the (generalized) Weyl tensors will not be studied here. The triplets of spins (s, s', s'') with $s \leq s' \leq s''$ considered in [12] have to satisfy the triangle inequality $s'' < s + s'$ that coincides with the necessary condition obtained in [13]

⁴See e.g. [12] for some terminology in the present context, and [15] in the general case of a local gauge theory.

⁵The fundamental results on cubic interaction have been obtained by Metsaev within the light-cone approach, [25–27]. For a non-technical review on higher-spin gravity that includes a discussion on cubic vertices, see [17]. See also [28].

for the existence of non-abelian vertices in flat spacetime.

We also discuss gravitational interactions of various (partially)-massless (mixed)-symmetry fields. The gravitational interactions are the simplest ones and we show that these can always be introduced for certain types of gauge fields, though not for all interestingly enough. At the end we give a general argument that the number of nonabelian vertices among various (partially)-massless (mixed)-symmetry fields should again be given by certain tensor product multiplicities.

The plan of the paper is as follows. Section 2 reviews the frame-like formulation of free, totally symmetric higher-spin gauge fields in manifestly $so(d-1, 2)$ -covariant fashion along the lines of [6]. In Section 3 we briefly review the Fradkin-Vasiliev ansatz for cubic, non-abelian vertices in AdS_d , in the frame-like formalism. A more detailed account can be found in [12]. In Section 4 we present the $sp(2)$ -invariant operators from which we construct the full list of non-abelian gauge algebras for candidate cubic vertices. In Section 4 we also show that, among the various gauge algebras that are obtained at the first nontrivial order in interaction, only one can be elevated to an associative, infinite-dimensional higher-spin algebra. This algebra is nothing but the algebra originally found by Eastwood [32], isomorphic to the one used by Vasiliev [14] for his construction of fully nonlinear equations in AdS_d . We then show in Section 5 that all the possible, non-equivalent gauge algebra deformations are indeed realized by consistent cubic vertices, and that their number coincides with the total number of non-abelian gauge algebras in flat spacetime found in [13]. The computation of some coefficients entering the cubic vertices is given in the Appendix. The gravitational interactions for more general types of fields including partially-massless fields and mixed-symmetry fields are considered in Section 6. Finally, Section 7 contains our conclusions.

2 Free fields and the linear action

Nonlinear equations for an infinite tower of totally symmetric gauge fields in arbitrary dimension have been given by Vasiliev in [14]. These equations are background independent, but the gauge algebra contains the AdS_d algebra as maximal finite-dimensional subalgebra, and the simplest exact solution of Vasiliev's equations is empty AdS_d spacetime.

The AdS_d exact solution around which one can linearize the full nonlinear equation is presented in the way used by MacDowell-Mansouri and Stelle-West [3, 5]:

$$R_0^{A,B} = (D_0)^2 = dW_0^{A,B} + W_0^{A,C} \wedge W_{0C}{}^B = 0, \quad (2.1)$$

where $W_0^{A,B} = -W_0^{B,A}$ is the background 1-form connection transforming in the adjoint representation of $so(d-1, 2)$, namely in the antisymmetric rank-2 representation of $so(d-1, 2)$. The differential D_0 is the corresponding covariant derivative around AdS_d . The important ingredient that allows to

combine the vielbein and spin-connection fields of Lorentz-covariant formulation of gravity into the single $so(d-1, 2)$ -connection W_0 is the compensator vector V^A that is constrained to satisfy

$$V^A V^B \eta_{AB} = -\Lambda^{-1}, \quad (2.2)$$

where Λ is the cosmological constant. Assuming one fixes V in such a way that (2.2) is satisfied, the algebra of $so(d-1, 2)$ rotations preserving V is identified with the Lorentz algebra $so(d-1, 1)$. Then one can introduce a one-form frame field

$$E_0^A := D_0 V^A = dV^A + W_0^A{}_B V^B, \quad (2.3)$$

which is assumed to have maximal rank d . From (2.2) we find

$$E_0^A V_A = 0. \quad (2.4)$$

A spin- s massless field in AdS_d spacetime can be described [6] by a one-form $W^{A(s-1), B(s-1)}$ carrying the irreducible representation of the AdS_d isometry algebra $so(d-1, 2)$ described by the traceless two-row rectangular Young diagram of length $s-1$. Then one constructs the linearized higher-spin two-form curvature

$$R_1^{A(s-1), B(s-1)} = D_0 W^{A(s-1), B(s-1)}. \quad (2.5)$$

The curvature (2.5) is gauge invariant with respect to abelian gauge transformations

$$\delta_0 W^{A(s-1), B(s-1)} = D_0 \xi^{A(s-1), B(s-1)}, \quad (2.6)$$

which follows from the fact that $(D_0)^2 = 0$.

To properly describe free massless spin- s field one should impose the following equations of motion [6, 33], called the first on-mass-shell theorem,

$$R_1^{A(s-1), B(s-1)} \approx E_0^M E_0^N C^{A(s-1)}{}_{M, B(s-1)N}, \quad (2.7)$$

where $C^{A(s), B(s)}$ is an irreducible two-row $so(d-1, 2)$ tensor subjected to the extra V -transversal constraint

$$C^{A(s-1)M, B(s)} V_M = 0. \quad (2.8)$$

The zero-form $C^{A(s), B(s)}$ generalizes the Weyl tensor of gravity to the higher-spin case, in the sense that, in the spin-2 case, the Einstein equations linearized around AdS can be written in the form

$$R_1^{A, B} \approx E_0^M E_0^N C^A{}_{M, B}{}^N, \quad (2.9)$$

where $C^{A(2), B(2)}$ only contains the linearized Weyl tensor of gravity when decomposed under $so(d-1, 1)$, as a consequence of the V -transversality condition (2.8).

The quadratic action for the symmetric spin- s gauge field is [6]

$$S_0 = \frac{1}{2} \int d^d x \sum_{p=0}^{s-2} a(s, p) V_{C(2(s-2-p))} G_{MNPQ} R_1^{MB(s-2), NC(s-2-p)D(p)} R_1^P{}_{B(s-2), QC(s-2-p)D(p)}, \quad (2.10)$$

where

$$V_{A(n)} = \overbrace{V_A \dots V_A}^n, \quad G_{M_1 M_2 \dots M_i} = \epsilon_{NM_1 M_2 \dots M_i R_{i+1} \dots R_d} V^N E_0^{R_{i+1}} \dots E_0^{R_d},$$

$$a(s, p) = \alpha_s (-1)^p \Lambda^{-p} \frac{(d-5+2(s-p-2))!! (s-p-1)}{(d-5)!! (s-p-2)!}$$

and α_s is an arbitrary normalization coefficient.

3 Fradkin-Vasiliev procedure

Deformation procedure. Given a quadratic action S_0 (2.10) that is gauge invariant under the gauge transformation (2.6) one looks for a deformation of both the action and gauge transformations by higher-order, field-dependent corrections $S = S_0 + g S_1 + \mathcal{O}(g^2)$, $\delta = \delta_0 + g \delta_1 + \mathcal{O}(g^2)$. The consistency condition reads

$$\delta_0 S_0 + g (\delta_1 S_0 + \delta_0 S_1) + g^2 (\delta_1 S_1 + \delta_0 S_2 + \delta_2 S_0) + \mathcal{O}(g^3) = 0 \quad (3.1)$$

with the first term vanishing because of gauge invariance of S_0 . At the cubic level one looks for a solution of $\delta_1 S_0 + \delta_0 S_1 = 0$. If one succeeds in finding such a cubic part S_1 whose variation under linearized gauge transformations δ_0 vanishes on free mass-shell, then it implies that $\delta_0 S_1$ is proportional to free field equations

$$\delta_0 S_1 = F \left(\frac{\delta S_0}{\delta W}, \xi, W \right), \quad \text{and} \quad F(0, \xi, W) = 0, \quad (3.2)$$

where F is trilinear in its arguments and can be used to extract δ_1 . As always, the cubic action S_1 and the gauge transformations δ_1 are defined modulo field and gauge parameter redefinitions. The problem of extracting δ_1 out of F is purely technical and one does not need to solve it once a nontrivial solution to S_1 is found.

The Fradkin-Vasiliev procedure [1, 2] does not give the general solution to the problem of constructing of cubic action. However, as we will show below, it actually leads to the exhaustive list of non-abelian cubic vertices. To cover all cubic vertices one has to extend the Fradkin-Vasiliev setup with Weyl zero-forms, see [12] for more detail.

Yang–Mills-like transformations. The Fradkin-Vasiliev [1, 2] procedure is based on the idea that one should look for Lagrangians that are quadratic in the curvature two-forms, similarly to what happens in Yang-Mills theory. In other words, in order to generate a cubic action, one replaces the

linearized curvature R_1 by a nonlinear completion R_2 of it inside the quadratic action $S \sim \int R_1 R_1$ from which one starts. Indeed, the action (2.10) is quadratic in the curvatures.

This implies that the one-forms $\{W^k\}$ are valued in some internal algebra whose product we denote by \diamond , with the understanding that the algebra is not necessarily associative. To fix the notation, we have $T_m \diamond T_n = \frac{1}{2} g_{mn}^k T_k$, where the T 's give a basis of the (possibly non-associative) internal algebra \mathcal{A} to which the one-forms belong. As we are going to construct the most general non-abelian cubic vertices coupling symmetric gauge fields around AdS_d , the symbol \diamond does not denote the star product of Vasiliev's theory. As we said, it denotes an arbitrary product that acts on rectangular Young diagrams of $so(d-1,2)$ and can be non-associative. The curvature

$$R = dW + W \diamond W \quad (3.3)$$

is given, in components along the generators T_k , as

$$R^k = dW^k + f_{mn}^k W^m \wedge W^n, \quad f_{mn}^k := g_{[mn]}^k. \quad (3.4)$$

Under the Yang–Mills-like gauge transformation

$$\delta^{YM} W = d\xi + [W, \xi]_\diamond, \quad (3.5)$$

the curvature transforms as

$$\delta^{YM} R = [R, \xi]_\diamond + JAC \quad (3.6)$$

where

$$JAC := [\xi \diamond (W \diamond W) - (\xi \diamond W) \diamond W] - [W \diamond (\xi \diamond W) - (W \diamond \xi) \diamond W] \quad (3.7)$$

$$+ [W \diamond (W \diamond \xi) - (W \diamond W) \diamond \xi] \quad (3.8)$$

is the Jacobiator. It vanishes for an associative algebra. We will not be bothered by the Jacobiator in the following, since it comes at order W^2 and for the problem of cubic vertices we only need the transformation of the curvature to order W . As it will be shown below, to achieve gauge invariance of the cubic vertices, the gauge transformation will receive an extra piece δ_1^{ext} having no simple geometrical interpretation in the current framework. However, as long as we are interested in the cubic vertices and not in the explicit form of $\delta_1^{ext} W$, this issue will not be relevant to us.

Perturbation around AdS_d . We want to include the AdS_d connection as part of the set of one-forms W^k , or in other words, we include the $so(d-1,2)$ generators among the generators T_k of the internal algebra \mathcal{A} . We ask that the one-form gauge fields should be expanded around the AdS_d background solution (2.1)

$$R_0 = dW_0 + W_0 \diamond W_0 = 0, \quad (3.9)$$

namely, we have the weak field decomposition $W = W_0 + W_1$ and impose a constraint on the \diamond -product among the class of two-row Young tableaux with a single column, $\begin{smallmatrix} \square \\ \square \end{smallmatrix}$, namely, that (3.9) should be identical with (2.1). Then, the curvature (3.3) reads, to first order in expansion around AdS_d , as

$$R_1 = dW + W_0 \diamond W + W \diamond W_0 . \quad (3.10)$$

Again, we impose that this formula should reproduce (2.5), which gives an additional restriction on the \diamond -product and implies that the higher spin fields W_1 transform as tensors under the (adjoint) action of $so(d-1, 2) \subset \mathcal{A}$. The linearized gauge transformation (2.6) reads

$$\delta_0 W = d\xi + W_0 \diamond \xi - \xi \diamond W_0 , \quad (3.11)$$

and we rewrite the quadratic action (2.10) in the form

$$S_0^{\{s\}}[W^s] = \int \langle R_1^{\{s\}} , R_1^{\{s\}} \rangle_{W_0} , \quad (3.12)$$

where we added a label $\{s\}$ in order to specify the spin under consideration.

Cubic ansatz. At the next stage we seek a cubic deformation of the quadratic Lagrangian. Following Fradkin and Vasiliev, the idea is to keep the form of the quadratic action (3.12) and replace the linear curvature R_1 with the non-linear $R = R_1 + R_2$, where

$$R_2 = W \diamond W , \quad (3.13)$$

so as to obtain

$$S_0 + S_1 + \mathcal{O}(W^4) = \sum_s \alpha_s \int \langle R^{\{s\}} , R^{\{s\}} \rangle_{W_0} . \quad (3.14)$$

We want to constrain the \diamond -product in such a way that $\delta_1^{YM} S_0 + \delta_0 S_1$ should vanish on the free shell, up to terms of order $\mathcal{O}(W^3 \xi)$, where δ_1^{YM} is the part of (3.5) that is linear in the weak fields:

$$\delta_1^{YM} W = W \diamond \xi - \xi \diamond W . \quad (3.15)$$

Taking into account that $\delta_0 R_2 + \delta_1^{YM} R_1 = [R_1, \xi]_\diamond$ (non-associative terms in (3.6) do not contribute at this order), one can easily compute the variation of the action:

$$\delta_1^{YM} S_0 + \delta_0 S_1 = 2 \sum_s \alpha_s \int \langle R_1^{\{s\}} , [R_1, \xi]_\diamond^{\{s\}} \rangle_{W_0} + \mathcal{O}(W^3 \xi) , \quad (3.16)$$

where $[R_1, \xi]_\diamond^{\{s\}}$ denotes the restriction of $[R_1, \xi]_\diamond$ to the spin- s sector. According to the central result recalled in (2.7), this variation on free shell gives

$$\delta_1^{YM} S_0 + \delta_0 S_1 \approx 2 \sum_s \alpha_s \int \langle (E_0 E_0 C)^{\{s\}} , ([E_0 E_0 C, \xi]_\diamond)^{\{s\}} \rangle_{W_0} + \mathcal{O}(W^3 \xi) , \quad (3.17)$$

where (E_0E_0C) is the *r.h.s.* of (2.7).

By arguments similar to those at the beginning of this section, if one succeeds in adjusting the free coefficients α_s in such a way that the gauge variation (3.17) is zero on free shell and up to terms cubic in the fields, then there exists a certain completion δ_1^{ext} of δ_1^{YM} that yields the full gauge invariance of the action $S_0 + S_1$. We recall that δ_1 was split into a Yang–Mills-like part δ_1^{YS} plus the rest δ_1^{ext} , where the latter cannot be presented in a simple, geometric, form within the current approach. Whatever δ_1^{ext} is, the vanishing of the right-hand side of (3.17) up to terms cubic in the fields is sufficient to prove that the action $S_0 + S_1$ is gauge invariant under a certain δ_1 transformation containing the non-abelian part δ_1^{YM} . Let us add the comment that, by construction, $\delta_1^{ext}W$ is linear in R_1 and in the gauge parameters, and therefore does not contribute to the non-abelian nature of the gauge transformation at the first nontrivial order where we work; only δ_1^{YM} does.

The Fradkin-Vasiliev procedure amounts to solving

$$0 = \sum_{k,m,n} \alpha_k \int \langle (E_0E_0C)^{\{k\}}, f_{mn}^k (E_0E_0C)^{\{m\}} \xi^{\{n\}} \rangle_{W_0} =: \sum_{k,m,n} I_{mn}^k \quad (3.18)$$

for the free coefficients α_s and for the structure constants f_{mn}^k . Let us consider the terms in (3.18) involving only fields and gauge parameters of three fixed spins k , m and n . Obviously, such terms are independent from the others and to solve (3.18) they should cancel among each other. The Fradkin–Vasiliev condition, in the fixed sector we consider, therefore reads

$$I_{m,n}^k + I_{n,k}^m + I_{k,m}^n = 0. \quad (3.19)$$

Regrouping terms pairwise, it implies that one should have

$$\alpha_k f_{nm}^k \int \langle (E_0E_0C)^{\{k\}}, \xi^{\{n\}} (E_0E_0C)^{\{m\}} \rangle_{W_0} = \alpha_m f_{kn}^m \int \langle (E_0E_0C)^{\{m\}}, (E_0E_0C)^{\{k\}} \xi^{\{n\}} \rangle_{W_0}, \quad (3.20)$$

where there is no sum over the Latin indices k, m and n . Our aim in this paper is therefore to find the most general solution of the above equation.

This leads us to the following two problems:

1. Find the full set of independent f_{mn}^k coefficients. This is done in the next Section 4;
2. For each independent product rule found in item 1, solve (3.20). This is done in Section 5.

4 Non-abelian deformations

Let us recall that the one-form gauge fields W^s entering the formulation of free higher-spin theory around AdS_d transform as $so(d-1, 2)$ tensors characterized by a Young diagram made of two rows of

equal lengths $(s-1)$. In the spin- s sector one therefore has the following correspondence

$$W^s \longleftrightarrow W^{A(s-1), B(s-1)}. \quad (4.1)$$

It is convenient to follow the notation of [6, 14] and introduce a set of $2(d+1)$ bosonic oscillators Y_α^A , $\alpha = 1, 2$ that are used to realize $\mathfrak{sp}(2)$ generators $K_{\alpha\beta}$ by

$$K_{\alpha\beta} := \frac{i}{2} \left(Y_\alpha^A \frac{\partial}{\partial Y^{\beta A}} + Y_\beta^A \frac{\partial}{\partial Y^{\alpha A}} \right), \quad (4.2)$$

so that indeed

$$[K_{\alpha\beta}, K_{\gamma\delta}] = \epsilon_{\gamma(\alpha} K_{\beta)\delta} + \epsilon_{\delta(\alpha} K_{\beta)\gamma}, \quad (4.3)$$

where one raises and lowers indices with the $\mathfrak{sp}(2)$ -invariant symbol $\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}$ according to the rule $Y^\alpha = \epsilon^{\alpha\beta} Y_\beta$, $Y_\alpha = Y^\beta \epsilon_{\beta\alpha}$ where $\epsilon^{12} = 1 = \epsilon_{12}$. One then represents the spin- s gauge field by

$$W^s := \frac{1}{(s-1)!(s-1)!} W^{A(s-1), B(s-1)} Y_A^1 \dots Y_A^1 Y_B^2 \dots Y_B^2 \quad (4.4)$$

so that the $\mathfrak{sp}(2)$ -singlet conditions

$$[K_{\alpha\beta}, W^s] = 0 \quad (4.5)$$

impose that the coefficients $W^{A(s-1), B(s-1)}$ are two-row irreducible tensors of $\mathfrak{gl}(d+1)$, see e.g. [12] for more details and references.

Given two $\mathfrak{sp}(2)$ -singlet fields $W^n(Y)$ and $W^m(Z)$ – we hereby double the set of Y_α^A oscillators by introducing the oscillators Z_α^A that play exactly the same role, there is a natural operator that contracts a pair of indices:

$$\tau_{YZ}^{\alpha\beta} := \frac{\partial^2}{\partial Y_\alpha^A \partial Z_{A\beta}}.$$

From $W^n(Y)$ and $W^m(Z)$ one can produce another $\mathfrak{sp}(2)$ singlet by acting on the product $W^n(Y) W^m(Z)$ with some $\mathfrak{sp}(2)$ -invariant operator built out of $\tau_{YZ}^{\alpha\beta}$ and then setting $Z_\alpha^A = Y_\alpha^A$. As an $\mathfrak{sp}(2)$ module, $\tau_{\alpha\beta}$ decomposes into $\bullet \oplus \square$, so that the problem is to find all the $\mathfrak{sp}(2)$ -invariants of $\bullet \oplus \square$. There are two generating $\mathfrak{sp}(2)$ -invariants:

$$s_{YZ} := \tau_{YZ}^{\alpha\beta} \epsilon_{\alpha\beta} \equiv \frac{\partial^2}{\partial Y_1^A \partial Z_{2A}} - \frac{\partial^2}{\partial Y_2^A \partial Z_{1A}}, \quad (4.6)$$

$$p_{YZ} := \det(\tau_{YZ}^{\alpha\beta}) \equiv \frac{\partial^2}{\partial Y_1^A \partial Z_{1A}} \frac{\partial^2}{\partial Y_2^B \partial Z_{2B}} - \frac{\partial^2}{\partial Y_1^A \partial Z_{2A}} \frac{\partial^2}{\partial Y_2^B \partial Z_{1B}}. \quad (4.7)$$

The Vasiliev higher-spin algebra [14] is defined as a certain quotient of the Weyl algebra or of the universal enveloping algebra $\mathcal{U}(so(d-1, 2))$, where the Weyl algebra is realized by the star product algebra

$$W^n(Y) \star W^m(Z) = \exp\left(\frac{1}{2} s_{YZ}\right) W^n(Y) W^m(Z) \Big|_{Z=Y} \quad (4.8)$$

modulo the ideal generated by the traces. Fortunately for us, the long tail of terms projecting out the ideal does not contribute to the Fradkin-Vasiliev condition (3.20) since any η^{AB} -proportional term vanishes in the variation of the action when put on the free mass-shell.

One may try to define some other $\mathfrak{sp}(2)$ -invariant product rules via

$$W^n(Y) \diamond W^m(Z) = \sum_k \chi_{n,m}^k(s_{YZ}, p_{YZ}) W^n(Y) W^m(Z) \Big|_{Z=Y}, \quad (4.9)$$

with $\chi_{n,m}^k(\bullet, \bullet)$ being a polynomial function in its two arguments, which can depend on k, m and n . The list of all the possible inequivalent functions $\chi_{n,m}^k$ gives all the inequivalent ways to contract two two-row $so(d-1, 2)$ -Young diagrams with lengths $\check{n} := n-1$ and \check{m} in order to produce a similar Young diagram with length \check{k} . The corresponding composition rules (4.9) are not associative if they contain at least one p operator. Indeed, by the universal property, the only associative algebra on the vector space of two-row rectangular $so(d-1, 2)$ Young tableau is given by $\mathcal{A} \cong \frac{\mathcal{U}(so(d-1, 2))}{I_{singl.}}$, where $I_{singl.}$ is the ideal that annihilates the scalar Dirac singleton, see e.g. [34] and references therein. The corresponding associative product (4.8) is generated by the s contraction only. Crucial in this line of reasoning is the fact that the higher-spin tensors generating the algebra under consideration are required to transform under the adjoint action of $so(d-1, 2)$, which in physical terms means that the corresponding higher-spin gauge fields couple to gravity in the way explained below (3.10).

Leaving aside all the possible constraints that will be imposed on the cubic vertices when investigating gauge invariance of the action $S = S_0 + g S_1 + g^2 S_2$ at order $\mathcal{O}(g^2)$, let us find all the possible independent $\mathfrak{sp}(2)$ -invariant contractions of $\mathfrak{sp}(2)$ singlets given by two-row rectangular diagrams of some particular lengths \check{n} and \check{m} with $\check{n} \leq \check{m}$. As it was explained above, in the general case (of arbitrarily long Young diagrams $f^{\check{n}}(Y)$ and $g^{\check{m}}(Z)$ with degree of homogeneity in Y_α^A and Z_α^A being $2\check{n}$ and $2\check{m}$, respectively), all the independent polynomials in s_{YZ} and p_{YZ} produce independent contractions. On the other hand, it is obvious that finite Young diagrams cannot be contracted in an infinite number of independent ways. Moreover, it is clear that contractions with sufficiently large powers of s and p annihilate any given Young diagrams, each being a monomial of finite degree in Y or Z , like $f^{\check{n}}(Y)$ and $g^{\check{m}}(Z)$. So, our goal is to study the independent contractions for finite Young diagrams. This problem can be solved by representation theory methods, where it amounts to taking tensor product of two representations associated with W^n and W^m and decomposing the result into irreducible two-row Young tableaux parts. This being said, we will make a more direct analysis that gives an explicit realization of all the independent contractions in terms of polynomials in the operators s and p .

Given two $\mathfrak{sp}(2)$ -singlets $f^{\check{n}}(Y)$ and $g^{\check{m}}(Z)$ of degree \check{n} and \check{m} in Y and Z respectively, first note

⁶In the following we will often use the notation s and p in place of s_{YZ} and p_{YZ} when no confusion can arise.

that the action of a single p_{YZ} operator on $f(Y)g(Z)$ contracts twice as many indices as s_{YZ} does, see (4.7). The total number of contracted indices in one of the two Young tableaux will be called the degree of contraction and denoted by k . So, for the contraction $p^\alpha s^\beta$, the degree of contraction is $k = 2\alpha + \beta$. Obviously, only contractions of the same degree may be linearly dependent. The next thing to note is that the significant difference between s^2 and p is that p contracts the same number of indices in the first and in the second row of, say, the first Young tableau. In contrast to p , s^2 contains terms that contract two indices in only the first or the second row of $f^{\tilde{n}}(Y)$.

In general, let us consider the operator $\mathcal{O}_1^{(\alpha,\beta)} = p^\alpha s^\beta$. The maximal number of indices it contracts in the first row of $f^{\tilde{n}}(Y)$ is $M(\mathcal{O}_1^{(\alpha,\beta)}) = \alpha + \beta$. Let us note that $M(\mathcal{O})$, *i.e.* the number of indices contracted in the first row of $f^{\tilde{n}}(Y)$ by an operator \mathcal{O} , is a quantity that cannot be changed by using Young symmetry properties of $f^{\tilde{n}}(Y)$. Now we consider the operator $\mathcal{O}_2^{(\alpha-1,\beta+2)} = p^{\alpha-1} s^{\beta+2}$ of the same degree as $\mathcal{O}_1^{(\alpha,\beta)}$. The maximal number M_2 of indices contracted in the first row now is $\alpha + \beta + 1$. From the fact that $\mathcal{O}_2^{(\alpha-1,\beta+2)}$ contains the terms where $\alpha + \beta + 1$ indices are contracted in the first row and $\mathcal{O}_1^{(\alpha,\beta)}$ does not, it follows that $\mathcal{O}_1^{(\alpha,\beta)}$ and $\mathcal{O}_2^{(\alpha-1,\beta+2)}$ produce linearly independent contractions. Following this logic, one can show that all the contractions $p^\alpha s^\beta$ of fixed degree $k = 2\alpha + \beta$ are independent when $k \leq \tilde{n}$. We call this *Case I* in what follows.

On the other hand, in *Case II* when $k > \tilde{n}$, the above logic is not applicable because some operators \mathcal{O} such as s^k are such that $M(\mathcal{O}) > \tilde{n}$, namely they have the maximal number of contractions in the first row of $f^{\tilde{n}}(Y)$ exceeding the total number of indices available. Still, the operators with $M(\mathcal{O}) \leq \tilde{n}$ are linearly independent, following the logic explained above. More precisely, all the operators $p^\alpha s^\beta$ of fixed degree $k = 2\alpha + \beta$ and having $M = \alpha + \beta$ are independent for $\alpha + \beta \leq \tilde{n}$. Let us call them *definitely-independent Case II* operators. What is less easy to see is that all the remaining operators (*i.e.* those that have $M > \alpha + \beta$) of the same degree can be given as linear combinations of those having $M = \alpha + \beta \leq \tilde{n}$. One can prove this proposition from the associativity of the tensor product, as follows.

Let us consider the operators belonging to *Case I* and let us compute how many of them can contract two Young tableaux of respective lengths \tilde{n} and \tilde{m} (with $\tilde{n} \leq \tilde{m}$) and produce a resulting Young tableau of length $\check{\ell}$. The first obvious relation is

$$\tilde{n} + \tilde{m} - k = \check{\ell}. \quad (4.10)$$

Now we fix \tilde{n} , \tilde{m} , $\check{\ell}$, and consequently k . The independent contractions belonging to *Case I* (so that $k \leq \tilde{n}$) are such that $\tilde{m} \leq \check{\ell}$. So, the operators in *Case I* can be alternatively be specified by

$$\tilde{n} \leq \tilde{m} \leq \check{\ell}. \quad (4.11)$$

By definition of *Case I*, all the operators $p^\alpha s^\beta$ in this case are independent, so the total number of independent operators equals to the number of partitions of k as $k = 2\alpha + \beta$ with non-negative α and

β . It is easy to see that the number of such partitions is

$$N_I = \left[\frac{k}{2} \right] + 1 = \left[\frac{\check{n} + \check{m} - \check{\ell}}{2} \right] + 1. \quad (4.12)$$

This gives the desired multiplicity of contractions of $f^{\check{n}}$ and $g^{\check{m}}$ that produce a Young tableau $h^{\check{\ell}}$ with $\check{\ell} \geq \check{m}$.

Obviously, once the multiplicities (4.12) for *Case I*, (4.11), are known, and from the associativity of the tensor product, one can derive the multiplicities of the contractions of $f^{\check{n}}$ and $g^{\check{m}}$ that give rise to $h^{\check{\ell}}$ with $\check{\ell} < \check{m}$. This is nothing but *Case II* since $\check{\ell} < \check{m}$ is equivalent to $\check{k} > \check{n}$, cf. (4.10). From (4.12) one can find that the multiplicity in this case is

$$N_{II} = \left[\frac{\check{n} + \check{\ell} - \check{m}}{2} \right] + 1. \quad (4.13)$$

Now we want to show that this multiplicity is the number of definitely-independent operators as was explained above, which will therefore prove that the remaining operators are just linear combinations of the definitely-independent ones, thereby proving our proposition.

So, we compute the number of definitely-independent operators $p^\alpha s^\beta$ with the fixed degree $k = 2\alpha + \beta$ in *Case II* and having $\alpha + \beta \leq \check{n}$. This multiplicity is the number of partitions of k in the form $k = 2\alpha + \beta$ such that $\alpha + \beta \leq \check{n}$ and where both α and β are non-negative integers. It is not hard to show that this gives exactly

$$N = \check{n} - \left[\frac{k}{2} \right] = \left[\check{n} - \frac{k}{2} \right] + 1 = \left[\frac{\check{n} + \check{\ell} - \check{m}}{2} \right] + 1,$$

as anticipated. To conclude, the definitely-independent contractions indeed provide a basis of operators in *Case II*.

To summarize, both possibilities $\check{m} \leq \check{\ell}$ and $\check{m} > \check{\ell}$ have been considered, and the bases of all the possible contractions have been given.

5 Trace-associativity or invariant-normed algebra condition

With the basis for independent contractions known, we can find the solution to Fradkin-Vasiliev condition (3.20) derived in Section 3. The on-shell curvatures are V -transversal because of (2.7) and (2.8), thereby only the last term in the sum (2.10) of the quadratic action remains non-zero on-shell,

$$S_0 \approx \frac{1}{2} \int d^d x a(k, k-2) G_{MNPQ} R^{MB(k-2), ND(k-2)} R^P{}_{B(k-2), Q}{}_{D(k-2)}, \quad (5.1)$$

and therefore nontrivial cubic interactions are obtained by substituting $R = R_1 + R_2$ in the above expression, instead of using the full action (2.10).

The first on-mass-shell theorem can be rewritten as

$$R_0^{A(k-1),B(k-1)} \approx E_0^M E_0^N C^{A(k-1),B(k-1)};_{MN} \quad \text{or} \quad R_0^{\{k\}} \approx E_0^M E_0^N C^{\{k\}};_{MN}.$$

Here $C^{\{k\}};_{MN}$ has two groups of indices: (i) $2(k-1)$ indices having a symmetry of two-row rectangular Young tableau denoted implicitly by $\{k\}$, and (ii) two indices M and N antisymmetrized by contraction with two frame fields. It is important to split the indices of C into two groups, because only the indices $\{k\}$ are sensitive to gauge transformations – see also [11]. Indeed, from

$$\delta_1^{YM} R^{\{k\}} = -\xi^{\{n\}} \diamond R^{\{m\}} + R^{\{n\}} \diamond \xi^{\{m\}} + \mathcal{O}(W^2),$$

it follows that

$$\delta_1^{YM} (E_0^M E_0^N C^{\{k\}};_{MN}) = E_0^M E_0^N (-\xi^{\{n\}} \diamond C^{\{m\}};_{MN} + C^{\{n\}};_{MN} \diamond \xi^{\{m\}}). \quad (5.2)$$

Therefore

$$\delta_1^{YM} S_0 + \delta_0 S_1 \approx 2 \int d^d x a(k, k-2) G_{MNPQ} R^{MB(k-2),ND(k-2)} \delta_1^{YM} R^P{}_{B(k-2),D(k-2)} + \mathcal{O}(W^3 \xi) \quad (5.3)$$

can be rewritten as

$$\begin{aligned} \delta_1^{YM} S_0 + \delta_0 S_1 \approx & \int d^d x a(k, k-2) G_{MNPQ} E^R E^S C_{RS; MN; B(k-2), D(k-2)} \times \\ & E_0^T E_0^U (-\xi^{\{n\}} \diamond C^{\{m\}};_{TU} + C^{\{n\}};_{TU} \diamond \xi^{\{m\}})^{PQ};_{B(k-2), D(k-2)} + \mathcal{O}(W^3 \xi). \end{aligned} \quad (5.4)$$

One can write Lorentz indices everywhere instead of (anti)-de Sitter indices because of (i) the on-mass-shell theorem states that curvatures are V -transversal on-shell; and (ii) since the symbol G_{MNPQ} defined in (2) contains an explicit contraction of the totally antisymmetric $so(d-1, 2)$ tensor with a compensator V , so that all the remaining indices of the antisymmetric tensor run only over the V -transversal, or Lorentz, directions. Using the identity [6]

$$E_0^C G_{A_1 \dots A_k} = \frac{1}{(d-k+1)} \sum_{i=1}^{i=k} (-)^{i+k} \delta_{A_i}^C G_{A_1 \dots \hat{A}_i \dots A_k}$$

one can show that

$$G_{MNPQ} E_0^R E_0^S E_0^T E_0^U \propto \delta_{[MNPQ]}^{[RSTU]} G. \quad (5.5)$$

We do not specify the precise coefficient because it only depends on the dimension of the space and cancels in the following computations.

Let us focus on the first term in the bracket of (5.4). According to (5.5) it can be rewritten as

$$\int d^d x G a(k, k-2) \delta_{[MNPQ]}^{[RSTU]} C_{RS; MN; B(k-2), D(k-2)} (-\xi^{\{n\}} \diamond C^{\{m\}};_{TU})^{PQ};_{B(k-2), D(k-2)}. \quad (5.6)$$

Due to the tracelessness of the Weyl tensor, the indices R and S can be contracted with P and Q only, so we can rewrite (5.6) as

$$\int d^d x G a(k, k-2) C^{MN;B(k-1),D(k-1)} (-\xi^{\{n\}} \diamond C^{\{m\};MN})_{B(k-1),D(k-1)} . \quad (5.7)$$

Regrouping terms as in (3.20) one finds that the Fradkin-Vasiliev condition is equivalent to

$$\begin{aligned} a(k, k-2) C^{MN;B(k-1),D(k-1)} (\xi^{\{n\}} \diamond C^{\{m\};MN})_{B(k-1),D(k-1)} = \\ a(m, m-2) C^{MN;B(m-1),D(m-1)} (C^{\{k\};MN} \diamond \xi^{\{n\}})_{B(m-1),D(m-1)} . \end{aligned} \quad (5.8)$$

Suppose that the particular \diamond -product between $\xi^{\{n\}}$ and $C^{\{m\};MN}$ is realized as $k_{n,m}^k p^\alpha s^\beta$ and produces a Young tableau that belongs to the spin- k sector, which implies

$$\check{k} = \check{n} + \check{m} - 2\alpha - \beta . \quad (5.9)$$

In the appendix it is shown in (8.11) that

$$\begin{aligned} C^{MN;B(k-1),D(k-1)} (\xi^{\{n\}} p^\alpha s^\beta C^{\{m\};MN})_{B(k-1),D(k-1)} = \\ \frac{(\alpha+\beta+1)}{(\alpha'+\beta+1)} C^{MN;B(m-1),D(m-1)} (C^{\{k\};MN} p^{\alpha'} s^\beta \xi^{\{n\}})_{B(m-1),D(m-1)} , \end{aligned} \quad (5.10)$$

where $\alpha' = \check{n} - \alpha - \beta$. Therefore, in order to solve (5.8) the \diamond -product between $C^{\{k\}}$ and $\xi^{\{n\}}$ should have the form $k_{k,n}^m p^{\alpha'} s^\beta$ and

$$a(k, k-2) k_{n,m}^k = \frac{(\alpha+\beta+1)}{(\alpha'+\beta+1)} a(m, m-2) k_{k,n}^m .$$

In terms of the spins m , n , k and the free parameter β , it gives

$$a(k, k-2) k_{n,m}^k = \frac{(n+m-k+\beta+1)}{(n+k-m+\beta+1)} a(m, m-2) k_{k,n}^m . \quad (5.11)$$

This equation explicitly displays the implication of Fradkin-Vasiliev condition on the free coefficients a and k . Obviously, if the particular \diamond contraction between $\xi^{\{n\}}$ and $C^{\{m\};MN}$ is realized as $k_{n,m}^k p^\alpha s^\beta$, then one can always find $k_{k,n}^m$ so as to satisfy (5.11), which means that every such contraction can be promoted to a consistent higher-spin cubic vertex. Having classified all the independent contractions in Section 4, we thereby classified all the independent higher-spin cubic vertices.

Finally, it is easy to see that the number of independent contractions given in Section 4 coincides with the number of possible non-abelian algebra deformations obtained in [13]⁷, thereby proving that our list of independent non-abelian vertices in AdS_d is exhaustive.

⁷We recall that, for a triplet of spins with $s \leq s' \leq s''$, the non-abelian deformations of the gauge algebra can give rise to vertices with a number of derivatives k ranging from $k^{min} = s'' + s' - s$ to $k_o^{max} = 2s' - 1$ for odd $\mathbf{s} := s + s' + s''$ or to $k_e^{max} = 2s' - 2$ for even \mathbf{s} . Therefore the multiplicity of non-abelian vertices is $N_o = \frac{s+s'-s''+1}{2} \equiv \left\lfloor \frac{s+s'-s''}{2} \right\rfloor + 1$ for odd \mathbf{s} and $N_e = \frac{s+s'-s''}{2}$ for even \mathbf{s} , which exactly matches the multiplicity formula found in Section 4.

6 (Mixed)-symmetry (partially)-massless fields

In this section we discuss how to construct gravitational interactions in anti-de Sitter space for gauge fields of various types⁸ and discuss briefly general non-abelian interactions. The simplest example is provided by a spin- s partially-massless field of depth- t . Partially-massless fields [41] have the following higher-derivative transformation law

$$\delta\phi_{\mu_1\dots\mu_s} = D_{\mu_1}\dots D_{\mu_t}\xi_{\mu_{t+1}\dots\mu_s} + \dots, \quad (6.1)$$

where the parameter $t \in \{1, \dots, s\}$ is called the depth and ... stands for the terms with less derivatives. As shown in [42] a spin- s partially-massless field of depth- t can be described by a one-form connection that takes values in the irreducible tensor representation of $so(d-1, 2)$ defined by a two-row Young diagram

$$\delta W^{A(s-1), B(s-t)} = D_0 \xi^{A(s-1), B(s-t)}, \quad \begin{array}{|c|} \hline s-1 \\ \hline s-t \\ \hline \end{array}. \quad (6.2)$$

Massless fields arise at $t = 1$. The equations of motion are similar to (2.7)

$$R^{A(s-1), B(s-t)} = D_0 W^{A(s-1), B(s-t)}, \quad R^{A(s-1), B(s-t)} = E_0^M E_0^N C^{A(s-1)}_{M, B(s-t)N}, \quad (6.3)$$

where the Weyl tensor for partially-massless field has the symmetry of $\begin{array}{|c|} \hline s \\ \hline s-t+1 \\ \hline \end{array}$ and it is V -transverse.

As before we write the most general quadratic corrections to the field strength of the spin-2 field $W^{U,U}$ and to that of the partially-massless field $W^{A(s-1), B(s-t)}$

$$\begin{aligned} R^{U,U} &= D_\Omega W^{U,U} + g_1 W^{A(s-2)U, B(s-t)} \wedge W_{A(s-2), B(s-t)}^U + \\ &\quad + g_2 W^{A(s-1), B(s-t-1)U} \wedge W_{A(s-1), B(s-t-1)}^U, \\ R^{A(s-1), B(s-t)} &= D_\Omega W^{A(s-1), B(s-t)} + W^A_{,M} \wedge W^{MA(s-2), B(s-t)} + W^B_{,M} \wedge W^{A(s-1), MB(s-t-1)}. \end{aligned}$$

Note that there are two independent contributions to $R^{U,U}$. The quadratic correction to $R^{A(s-1), B(s-t)}$ is just an $so(d-1, 2)$ -covariant derivative. The quadratic actions for the graviton and partially-massless field read

$$\begin{aligned} S^{\{2\}} &= \alpha_2 \int R^{U,U} \wedge R^{V,V} G_{UUVV} \\ S^{\{pm\}} &= \sum \alpha_{q,m}^{s,t} \int R^{UA(s-m-2)C(m), UC(q)B(s-q-2)} \wedge R^V_{A(s-m-2)}{}^{C(m), VC(q)}_{B(s-q-2)} V_{2q+2m} G_{UUVV} \end{aligned}$$

where $a_{q,m}^{s,t}$ are certain coefficients fixed up to an overall factor [42], which we identify with $\alpha_{0,0}^{s,t}$.

⁸For some results on interactions of mixed-symmetry fields on flat background see [27, 36–39], as for anti-de Sitter space a few results are available [7, 9–11]. Interactions of partially-massless fields has been studied recently in [40]

Using the general formulae (3.5), (3.6) and (3.20), one requires the gauge invariance of the cubic terms on the free mass-shell, resulting in the condition $\delta S^{\{2\}} + \delta S^{\{pm\}} = 0$, where

$$\begin{aligned} \delta S^{\{2\}} &= 4g_1\alpha_2 \int \overbrace{C_{uu,vv} \xi_{a(s-2)}^{v, b(t)} C^{a(s-2)uv, b(t)u}}^{\mathbf{A}} + 4g_2\alpha_2 \int \overbrace{C_{uu,vv} \xi_{a(s-1), b(t-1)}^v C^{a(s-1)u, b(t-1)uv}}^{\mathbf{B}}, \\ \delta S^{\{pm\}} &= \alpha_{0,0}^{s,t}(-2s)\mathbf{A} + \alpha_{0,0}^{s,t}\left(-\frac{2st}{s-1}\right)\mathbf{B}. \end{aligned}$$

Obviously, the condition $\delta S^{\{2\}} + \delta S^{\{pm\}} = 0$ admits a unique solution. The ratio g_1/g_2 is a fixed number. Therefore the freedom in g_1, g_2 does not lead to two different types of gravitational interactions.

Let us now comment of the general case of gravitational interactions of mixed-symmetry and/or partially-massless fields described by one-form connections $W^{\mathbf{Y}}$ with values in any irreducible tensor representation of $so(d-1, 2)$ specified by a Young diagram \mathbf{Y} with rows of lengths s_1, s_2, \dots, s_n , $\mathbf{Y} = \mathbb{Y}(s_1, \dots, s_n)$. The dictionary between $W^{\mathbf{Y}}$ and the metric-like formalism was given in [43–46]. The case of one-forms $W^{\mathbf{Y}}$ does not cover the variety of all possible types of mixed-symmetry and partially-massless fields. In order to take into consideration all gauge fields possible one has to include gauge connections $W^{\mathbf{Y}}$ that are forms of higher degree too. However, only one-form connections $W^{\mathbf{Y}}$ can give rise to a Lie algebra and only one-forms can source gravity in the Fradkin-Vasiliev framework as in this case one can write $W^{\mathbf{Y}} \wedge W^{\mathbf{Y}}$ contribution to the spin-2 field strength $R^{U,U}$ as we did above. The most general ansatz reads

$$R^{U,U} = D_\Omega W^{U,U} + \sum_{\mathbf{Y}/\square} g_i W^{;U} \wedge W^{;U}, \quad (6.4)$$

$$R^{\mathbf{Y}} = D_\Omega W^{\mathbf{Y}} + \sum_i W^{B, M} \wedge W^{A(s_1), \dots, MB(s_i-1), \dots}, \quad (6.5)$$

where in the first line the sum is over all possible ways to cut one cell from \mathbf{Y} such that the result is a valid Young diagram. The number of such ways is equal to the number of blocks of \mathbf{Y} . If there are no rows in \mathbf{Y} that have equal length, then the sum is over all rows and in the i -th summand one isolates one index in the i -th row, denotes it by U and contracts the rest of the indices pairwise. The deformation of $R^{\mathbf{Y}}$ is just a covariant derivative with respect to dynamical spin-2 connection $W^{U,U}$.

The linear equations of motion for $W^{\mathbf{Y}}$ read [43, 45–48]

$$R^{\mathbf{Y}} = E_0^M E_0^N \Pi_{MN} (C^{\mathbf{X}})^{\mathbf{Y}}, \quad (6.6)$$

where the generalized Weyl tensor $C^{\mathbf{X}}$ is an irreducible $so(d-1, 2)$ -tensor having the symmetry of $\mathbf{X} = \mathbb{Y}(s_1+1, s_2+1, s_3, \dots, s_n)$ and the projector Π_{MN} isolates two indices of C and projects onto \mathbf{Y} . The Weyl tensor for generic mixed-symmetry field is not fully-transverse and satisfies more complicated V -dependent constraints, [43, 45–48], which implies that C contains more than one Lorentz component in general. This is not the case for totally-symmetric (partially)-massless fields.

In order for the gravitational interactions of $W^{\mathbf{Y}}$ to exist in the first nontrivial order one has to prove that there is enough free coefficients to impose the invariance of the cubic vertex on the free mass-shell, (3.2). We will give an argument that this is indeed true despite the fact that the quadratic actions are not known in full generality. In general to construct a Lagrangian the connection $W^{\mathbf{Y}}$ has to be supplemented with certain additional fields, see e.g. [49–51] for specific examples. Fortunately, to check the gauge invariance of the cubic vertex we only need to know the on-shell action, *i.e.* the terms in the action to which the generalized Weyl tensor contribute,

$$(S^{\{2\}} + S^{\{\mathbf{Y}\}})\Big|_{on-shell} = \alpha_2 \int R^{U,U} \wedge R^{V,V} G_{UUVV} + \sum_{\mathbf{Y}/\boxplus} \alpha_n \int R^{;UU} \wedge R^{;VV} G_{UUVV}, \quad (6.7)$$

where the sum is over all possible ways to isolate two anti-symmetric indices in tensor with the symmetry of \mathbf{Y} , these are to be contracted with G_{UUVV} , the rest are contracted pairwise. These leading terms can be extracted from the results of [43, 52, 53]. That the Weyl tensor is not fully V -transverse imposes severe restrictions on such terms. Indeed, one would naively add to (6.7) the terms where in addition to a pair of anti-symmetrized indices one isolates a group of symmetric indices to be contracted with V . These additional V contractions may be nonzero as the Weyl tensor is not fully V -transverse. Taking then the variation of (6.7), one finds

$$\delta(S^{\{2\}} + S^{\{\mathbf{Y}\}})\Big|_{on-shell} \sim \alpha_2 \int [R, \xi]^{UU} \wedge R^{V,V} G_{UUVV} + \sum_{\mathbf{Y}/\boxplus} \alpha_i \int [R, \xi]^{;UU} \wedge R^{;VV} G_{UUVV}, \quad (6.8)$$

where $[R, \xi]$ can be read off from (6.4)-(6.5) according to general formulae of Section 3. One observes that $\xi^{A,B}$ contributes only to $\delta S^{\{\mathbf{Y}\}}$ and not to $\delta S^{\{2\}}$. Therefore, $\xi^{A,B}$ -variation must vanish on its own. Indeed, that there is no \boxplus in the symmetric tensor product $Sym(\mathbf{X} \otimes \mathbf{X})$ for any \mathbf{X} implies that any singlet built of $\xi^{A,B}$ and two Weyl tensors $C^{\mathbf{X}}$ is identically zero. Now we have to cancel the $\xi^{\mathbf{Y}}$ -part of the variation. Note that $\delta S^{\{2\}}$ has no V explicitly besides in G_{UUVV} since neither the deformation (6.4) nor the spin-2 action contain V . The latter implies that $\delta S^{\{\mathbf{Y}\}}$ and hence the on-shell part of $S^{\{\mathbf{Y}\}}$ must not have any explicit V -contractions. This justifies the form of (6.7). Then, using the symmetric basis for Young diagrams it is easy to see that the sums in (6.4) and (6.5) produce pairwise identical terms in $\delta S^{\{2\}}$ and $\delta S^{\{\mathbf{Y}\}}$. In particular all the terms in the sum of (6.7) vanish except for the one where two anti-symmetrized indices UU belong to the first two rows of \mathbf{Y} . Equivalently, using the freedom of adding total derivatives of the form $\int D_0(RRVG)$, [43, 44, 52, 53], one can reduce the number of terms in the sum of (6.7) to a single term described above. Again all the ratios g_i/g_j are certain fixed numbers and hence the gravitational interactions are essentially unique.

Let us make some comments about general non-abelian interactions of (mixed)-symmetry and/or partially-massless fields. We restrict ourselves to those gauge fields in the metric-like approach that are described by one-form connections $W^{\mathbf{Y}}$ within the frame-like approach. The condition for the

variation to vanish amounts to

$$(A|B \diamond C) - (A \diamond B|C) = 0, \quad (6.9)$$

where A, B, C correspond to two Weyl tensors and one gauge parameter; \diamond stands for some particular way of contracting indices; $(x|y)$ takes the singlet part. Given some $A \diamond B$ one can always adjust the contraction $B \diamond C$ such that (6.9) is true. As we argued above, see also [11], already the gravitational interactions restrict the freedom of adding topological terms $\int D_0(RRVG)$ in such a way that the Weyl tensor has no V -contractions in the on-shell action. The appearance of the Weyl tensor contracted with a number of compensators V would invalidate the arguments above. Therefore we see that each independent way of contracting indices among two connections $W^{\mathbf{Y}_1}$ and $W^{\mathbf{Y}_2}$ gives rise to a consistent cubic vertex, which is non-abelian by definition.

Similitude with Yang–Mills and invariant-normed algebra. The parallel between the above discussion and the spin-1 case is obvious, and we have seen that it is always possible to contract the indices of rectangular two-row Young tableaux in such a way that the resulting cubic action is consistent at that order. This becomes clear if one highlights the similitude of the Fradkin-Vasiliev construction with the Yang-Mills one. The Fradkin-Vasiliev procedure is precisely inspired by the Yang-Mills, geometric treatment of gauge systems. Consider, as a starting point, a positive sum of n Maxwell’s actions for a set of one-form gauge fields $\{A^a\}_{a=1,\dots,n}$

$$S_0[A^a] = \int_{M_4} \langle F_1, F_1 \rangle \equiv \int_{M_4} k_{ab} F_1^a \wedge *F_1^b, \quad F_1^a := dA^a, \quad (6.10)$$

where k_{ab} is diagonalized to $k_{ab} = c_a \delta_{ab}$ with $c_a > 0$ for the sake of unitarity. In order to introduce cubic interactions one performs the substitution

$$F_1^a \longrightarrow F^a := F_1^a + g f^a_{bc} A^a A^b \quad (6.11)$$

inside S_0 while disregarding quartic terms, as we did with the Fradkin-Vasiliev procedure. By definition of F^a and because A^a are one-forms, one has

$$f^a_{bc} = -f^a_{cb}, \quad (6.12)$$

which defines an internal anti-commutative algebra \mathcal{A} with basis elements $\{e_a\}$ and product law \diamond given by

$$e_a \diamond e_b = f^c_{ab} e_c = -e_b \diamond e_a. \quad (6.13)$$

As is well-known and easy to see – a cohomological derivation can be found in [35], the resulting deformed action $S_0 + S_1$ is consistent to order $\mathcal{O}(g)$ provided one has the following antisymmetry condition

$$f_{abc} := k_{ad} f^d_{bc} = f_{[abc]}. \quad (6.14)$$

In turn, this means that \mathcal{A} is an invariant-normed (sometimes called graded-symmetric) algebra, namely

$$\forall x, y, z \in \mathcal{A}, \quad (x \diamond y, z) = (x, y \diamond z), \quad (6.15)$$

where the norm is defined by

$$(x, y) = k_{ab} x^a y^b, \quad x = x^a e_a, \quad y = y^a e_a. \quad (6.16)$$

Given some constants f^a_{bc} that satisfy $f^a_{bc} = -f^a_{cb}$, it is always possible to find f_{abc} that are completely antisymmetric, thereby producing a consistent cubic vertex.

The story repeats itself in the higher-spin context where the internal index a is replaced with a rectangular two-row tensor representation of $so(d-1, 2)$. The fact that the Yang-Mills index a now has an inner structure in the higher-spin case implies that there is a multiplicity of choices for the \diamond -products or equivalently for the constants f^a_{bc} 's – and where one may need to add a color index on every higher-spin gauge fields in order to ensure the antisymmetry of $f^a_{bc} = -f^a_{cb}$; this is the case for example when the \diamond -product is given by pure p contractions in the sector of odd spins. The determination of these multiplicities was done in Section 4 or could be obtained from group theory.

As in the spin-1 Yang-Mills case, the invariant-norm condition $(x \diamond y, z) = (x, y \diamond z)$ can also be achieved in the higher-spin case, for every independent choice of \diamond -product.

What will severely constrain the \diamond -product is the Jacobi condition that arises at second order in the coupling constant g ,

$$f^a_{b[c} f^b_{de]} = 0. \quad (6.17)$$

In the spin-1 case, it implies that f^a_{bc} define the structure constants of a semi-simple Lie algebra.

7 Conclusions

In this paper we have classified and explicitly built all the possible non-abelian cubic vertices among totally symmetric gauge fields in AdS_d . The universal property of the universal enveloping algebra guarantees that there exists only one gauge algebra that can lead to an associative higher-spin algebra, and that the latter precisely coincides with the algebra used by Vasiliev in [14] for the construction of his nonlinear equations. When pushing the analysis of vertices to the next, quartic order, one typically finds that the internal algebra with (graded)-antisymmetric structure constant should obey the Jacobi identity, which is automatically satisfied if the commutator arises from the underlying associative structure, see e.g. the discussion and the results reviewed in [54]. It is likely that the

only cubic vertex that has a chance to be promoted to the next order is the one associated with the so-called “s-contraction” rule of Section 4, where the latter is the germ for the associative algebra used in [14] via the Moyal-Weyl star-product formula (4.8). There is still a loophole in that there can exist a higher-spin algebra, which is essentially a Lie algebra. For example, a Poisson contraction of the Vasiliev algebra, i.e. the one where $\exp \hbar s$, (4.8), is expanded to the leading order in a formal non-commutativity parameter \hbar would seem a good candidate. However, the Poisson contraction is inconsistent even at the cubic level, as was pointed in [1] for the $4d$ case and the statement is valid for any d . The technique developed in this paper can be used to examine the question of uniqueness of higher-spin algebra in full generality and we leave it for a future publication.

We view the determination of cubic vertices as one way to gain insight into the structure and uniqueness of the full theory proposed in [14, 55, 56]. In this sense, our results strongly confirm the belief that Vasiliev’s construction is the unique way to obtain fully nonlinear and consistent interactions among higher-spin gauge fields. In the spirit of the Noether procedure for consistent interactions this implies that Vasiliev’s theory can be viewed as the gauging of the rigid star-product algebra $hu(1|2 : [d - 1, 2])$, and that this is the only way to construct a fully nonlinear theory starting in perturbation around a fixed (here AdS_d) background.

We showed that the (partially)-massless (mixed)-symmetry gauge fields that are described by one-form connections $W^{\mathbf{Y}}$ valued in irreducible representations of $so(d - 1, 2)$ can interact with gravity. This gives a nontrivial indication that within the metric-like approach one will face certain difficulties in trying to make interact with gravity those gauge fields that are described by gauge connections of higher degree within the frame-like approach we use. It seems that the frame-like approach contains more information about interactions even at the linear level. Another example of this phenomenon was observed in [57], where a simple argument prevents constructing Lagrangians for certain types of fermionic fields, which is highly nontrivial to see in the metric-like approach [58, 59]. The gravitational interactions for fields that are described by forms of higher degree in the frame-like approach are severely constrained, see e.g. [60] and references therein. The gauge transformations for the p -form gauge fields can only be deformed à la Chaplin-Manton [61] or Freedman-Townsend [62], so that the gauge algebra in the p -form sector remains abelian although the gauge transformations are modified non-trivially, sometimes even non-polynomially.

We also argued that those mixed-symmetry and/or partially-massless fields that are described by one-form connections within the frame-like approach can have nonabelian interactions among themselves and again the number of nonabelian vertices should be given by tensor product multiplicities. Within the metric-like approach such gauge fields have the gauge parameter whose Young diagram is obtained by removing cells from the first row of the Young diagram of the field potential. For the rest of gauge fields, which are all nonunitary in AdS, [63, 64], within the metric-like approach one still can

write a lot of terms for the most general ansatz for the cubic vertex, but we expect that the gauge invariance will result in a trivial solution only.

The technique used in the paper can be generalized to various cases of (partially)-massless fields [42] and (mixed)-symmetry fields [43, 45, 46, 46, 53].

Note added

During the final stage when the file was being prepared for submission to the arXives, the paper [65] appeared where cubic vertices for (partially-)massless fields are constructed, following a different procedure. The tools presented there allow the construction of all possible types of vertices. The nature of the gauge algebras associated with the vertices is not clear, though, except in the Born-Infeld cases for obvious reasons. After identifying which of the vertices in [65] are non-abelian, it would be interesting to see if their number is indeed given by certain tensor product multiplicities as we showed in the present paper. Some simple examples show that this is the case.

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8 Appendix

Here we introduce some notations and prove certain identities required for solving the Fradkin-Vasiliev condition for a general cubic vertex.

Let us first introduce a projection f operation which antisymmetrizes two indices that belong to different rows of a Young diagram

$$f(W) = W^{M_1 N_1; A(m-1), B(m-1)} = \frac{1}{2} \left(W^{A(m-1) M_1, B(m-1) N_1} - W^{A(m-1) N_1, B(m-1) M_1} \right).$$

This operation is relevant to p contraction

$$\begin{aligned} W^n_p W^m &= W^{A(n-1) M, B(n-1) N} \left(W^{A(m-1)}_{M, B(m-1)}{}_N - W^{A(m-1)}_{N, B(m-1)}{}_M \right) = \\ &= 2W^{A(n-1) M, B(n-1) N} W_{MN}; A(m-1), B(m-1)} = 2W^{MN; A(n-1), B(n-1)} W_{MN}; A(m-1), B(m-1)}. \end{aligned}$$

An iterative application of an f -projector α times gives

$$f^\alpha(W^{A(m),B(m)}) = W^{M_1N_1,M_2N_2,\dots,M_\alpha N_\alpha;A(\gamma),B(\gamma)}. \quad (8.1)$$

where $\gamma = m - \alpha$. It is straightforward to check that the right hand side of (8.1) possesses symmetry of $\mathbf{Y}(\alpha, \alpha)$ in the antisymmetric basis in the first group of 2α indices in the same time having symmetry of $\mathbf{Y}(\gamma, \gamma)$ in the symmetric basis in the remaining indices.

By iterative application of

$$W^{M_1N_1,\dots,M_{\alpha-1}N_{\alpha-1},AB;A(\gamma),B(\gamma)} = \frac{1}{2} \cdot \frac{\gamma + 2}{\gamma + 1} \cdot W^{M_1N_1,\dots,M_{\alpha-1}N_{\alpha-1},A(\gamma+1),B(\gamma+1)}$$

one can find

$$W^{\overbrace{AB,\dots,AB,AB}^\alpha;A(\gamma),B(\gamma)} = \frac{1}{2^\alpha} \cdot \frac{\alpha + \gamma + 1}{\gamma + 1} \cdot W^{A(\alpha+\gamma),B(\alpha+\gamma)}. \quad (8.2)$$

Another useful representation appears when one symmetrizes only M and N indices among each other in (8.1) resulting in

$$W^{M_1N_1,M_2N_2,\dots,M_\alpha N_\alpha;A(\gamma),B(\gamma)} \rightarrow W^{\overbrace{MN,MN,\dots,MN}^\alpha;A(\gamma),B(\gamma)}. \quad (8.3)$$

This tensor has a symmetry of two row rectangular Young diagram in symmetric convention in both groups of indices. Obviously the same symmetry can be reached in a different way

$$f^\gamma(W^{M(\alpha+\gamma),N(\alpha+\gamma)}) = W^{A_1B_1,A_2B_2,\dots,A_\gamma B_\gamma;M(\alpha),N(\alpha)} \rightarrow W^{\overbrace{AB,AB,\dots,AB}^\gamma;M(\alpha),N(\alpha)}, \quad (8.4)$$

which implies that right hand sides of (8.3) and (8.4) are proportional, that is

$$W^{\overbrace{MN,MN,\dots,MN}^\alpha;A(\gamma),B(\gamma)} = X(\alpha, \gamma) W^{\overbrace{AB,AB,\dots,AB}^\gamma;M(\alpha),N(\alpha)} \quad (8.5)$$

with some $X(\alpha, \gamma)$. To find X we symmetrize M with A and N with B in both sides of (8.5), which, according to (8.2) results in

$$\frac{1}{2^\alpha} \cdot \frac{\alpha + \gamma + 1}{\gamma + 1} \cdot W^{A(\alpha+\gamma),B(\alpha+\gamma)} = X(\alpha, \gamma) \frac{1}{2^\gamma} \cdot \frac{\alpha + \gamma + 1}{\alpha + 1} \cdot W^{A(\alpha+\gamma),B(\alpha+\gamma)}, \quad (8.6)$$

which in turn implies

$$X(\alpha, \gamma) = 2^{\gamma-\alpha} \frac{\alpha + 1}{\gamma + 1}. \quad (8.7)$$

So, we introduce a notation

$$\begin{aligned} W^{M(\alpha),N(\alpha);A(\gamma),B(\gamma)} &= W^{A(\gamma),B(\gamma);M(\alpha),N(\alpha)} = \\ &= \frac{2^\alpha}{\alpha + 1} W^{\overbrace{MN,MN,\dots,MN}^\alpha;A(\gamma),B(\gamma)} = \frac{2^\gamma}{\gamma + 1} W^{\overbrace{AB,AB,\dots,AB}^\gamma;M(\alpha),N(\alpha)}. \end{aligned} \quad (8.8)$$

One can proceed in the same manner by breaking each small sub-Young diagram into even smaller pieces using the same formulas.

Computation Our goal is to find out how to relate two terms of (5.8). Terms

$$C^{MN;B(k-1),D(k-1)}(-\xi^{\{n\}} \diamond C^{\{m\}};_{MN})_{B(k-1),D(k-1)} \quad (8.9)$$

and

$$C^{MN;B(m-1),D(m-1)}(C^{\{k\}};_{MN} \diamond \xi^{\{n\}})_{B(m-1),D(m-1)} \quad (8.10)$$

are proportional and our aim is to find the proportionality coefficient. The M and N indices are not involved in \diamond -product. They are used just to contract two Weyl tensors in the same way in both expressions. So we can omit them and treat the Weyl tensors as having effectively two indices less each.

As a warm up exercise let us find the proportionality coefficient for the case when \diamond is represented by the p contraction only in some power α . Let us also introduce $\check{k} = k - 1$, $\check{n} = n - 1$ and $\check{m} = m - 1$. In this terms $2\alpha = \check{n} + \check{m} - \check{k}$.

$$\begin{aligned} & C^{MN;B(k-1),D(k-1)}(\xi^{\{n\}} p^\alpha C^{\{m\}};_{MN})_{B(k-1),D(k-1)} \rightarrow \\ & C^{B(k-1),D(k-1)}(\xi^{\{n\}} p^\alpha C^{\{m\}})_{B(k-1),D(k-1)} = \\ & = 2^\alpha C_{A(\check{k}),B(\check{k})} \xi^{CD,\dots,CD;A(\check{n}-\alpha),B(\check{n}-\alpha)} C_{CD,\dots,CD}^{A(\check{m}-\alpha),B(\check{m}-\alpha)} = \\ & = 2^\alpha C_{A(\check{m}-\alpha)U(\check{n}-\alpha),B(\check{m}-\alpha)V(\check{n}-\alpha)} \xi_{AB,\dots,AB}^{U(\check{n}-\alpha),V(\check{n}-\alpha)} C^{A(\check{m}),B(\check{m})} = \\ & = (\alpha + 1) C_{A(\check{m}-\alpha)U(\check{n}-\alpha),B(\check{m}-\alpha)V(\check{n}-\alpha)} \xi_{A(\alpha),B(\alpha)}^{U(\check{n}-\alpha),V(\check{n}-\alpha)} C^{A(\check{m}),B(\check{m})} = \\ & = \frac{2^{\check{n}-\alpha+1}(\alpha + 1)}{(\check{n} - \alpha + 1)} C_{A(\check{m}-\alpha)U(\check{n}-\alpha),B(\check{m}-\alpha)V(\check{n}-\alpha)} \xi_{A(\alpha),B(\alpha)}^{UV,\dots,UV} C^{A(\check{m}),B(\check{m})} = \\ & = \frac{2^{\check{n}-\alpha+1}(\alpha + 1)}{(\check{n} - \alpha + 1)} C_{UV,\dots,UV;A(\check{m}-\alpha),B(\check{m}-\alpha)} \xi_{A(\alpha),B(\alpha)}^{UV,\dots,UV} C^{A(\check{m}),B(\check{m})} = \\ & = \frac{\alpha + 1}{\check{n} - \alpha + 1} (C^{\{k\}} p^{\check{n}-\alpha} \xi^{\{n\}})_{B(m-1),D(m-1)} C^{B(m-1),D(m-1)} \rightarrow \\ & \frac{\alpha + 1}{\alpha' + 1} C^{MN;B(m-1),D(m-1)}(C^{\{k\}};_{MN} p^{\alpha'} \xi^{\{n\}})_{B(m-1),D(m-1)}, \end{aligned}$$

where $\alpha' = \check{n} - \alpha$.

Analogously one can show that

$$\begin{aligned} & C^{MN;B(k-1),D(k-1)}(\xi^{\{n\}} p^\alpha s^\beta C^{\{m\}};_{MN})_{B(k-1),D(k-1)} = \\ & \frac{\alpha + \beta + 1}{\alpha' + \beta + 1} C^{MN;B(m-1),D(m-1)}(C^{\{k\}};_{MN} p^{\alpha'} s^\beta \xi^{\{n\}})_{B(m-1),D(m-1)}, \end{aligned} \quad (8.11)$$

where $\check{k} = \check{n} + \check{m} - 2\alpha - \beta$ and $\alpha' = \check{n} - \alpha - \beta$. To show this, let us note that s^β is

$$\begin{aligned} \xi s^\beta C &= \xi(Y) \left(\frac{\partial^2}{\partial Y_1^A \partial Z_{2A}} - \frac{\partial^2}{\partial Y_2^A \partial Z_{1A}} \right)^\beta C(Z) = \\ \xi(Y) \sum_{i=0}^{\beta} \frac{(-)^i \beta!}{i!(\beta-i)!} \left(\frac{\partial^2}{\partial Y_1^A \partial Z_{2A}} \right)^{\beta-i} \left(\frac{\partial^2}{\partial Y_2^A \partial Z_{1A}} \right)^i C(Z). \end{aligned} \quad (8.12)$$

Each term of the expansion (8.12) has non-zero projection to the space of tensors with a symmetry encoded by the rectangular Young diagram \mathbf{Y}_r as well as other projections encoded by non-rectangular Young diagrams \mathbf{Y}_{nr} . Since in (8.11) $\xi s^\beta C$ appears only contracted with other tensor valued in \mathbf{Y}_r , each term of the expansion (8.12) contributes only with its \mathbf{Y}_r -shaped part. This allows us to keep track of only the first term in (8.12), while the others give some fixed proportional contributions. The following computation relates the first term of the left hand side of (8.11) and the last term of the right hand side of (8.11)

$$\begin{aligned}
& C^{MN;B(k-1),D(k-1)}(\xi^{\{n\}} p^\alpha s^\beta C^{\{m\}}; MN)_{B(k-1),D(k-1)} \Big|_{1st} \rightarrow \\
& 2^\alpha C_{A(\tilde{n}+\tilde{m}-2\alpha-\beta),B(\tilde{n}+\tilde{m}-2\alpha-\beta)} \xi^{CD\dots CD;A(\tilde{n}-\alpha-\beta)M(\beta),B(\tilde{n}-\alpha)} C_{CD\dots CD;A(\tilde{m}-\alpha),B(\tilde{m}-\alpha-\beta)}_{M(\beta)} = \\
& \quad 2^\alpha C_{A(\tilde{n}+\tilde{m}-2\alpha-2\beta)K(\beta),B(\tilde{n}+\tilde{m}-2\alpha-2\beta)L(\beta)} \times \\
& \quad \xi^{CD\dots CD;A(\tilde{n}-\alpha-\beta)M(\beta),B(\tilde{n}-\alpha-\beta)L(\beta)} C_{CD\dots CD;A(\tilde{m}-\alpha-\beta)K(\beta),B(\tilde{m}-\alpha-\beta)}_{M(\beta)} = \\
& \quad (\alpha+1) C_{A(\tilde{m}-\alpha-\beta)U(\tilde{n}-\alpha-\beta)K(\beta),B(\tilde{m}-\alpha-\beta)V(\tilde{n}-\alpha-\beta)L(\beta)} \times \\
& \quad \xi_{A(\alpha),B(\alpha)}^{U(\tilde{n}-\alpha-\beta)M(\beta),V(\tilde{n}-\alpha-\beta)L(\beta)} C_{A(\tilde{m}-\beta)K(\beta),B(\tilde{m}-\beta)}_{M(\beta)} = \\
& \quad (\alpha+1) C_{A(\tilde{m}-\alpha-\beta)U(\tilde{n}-\alpha-\beta)K(\beta),B(\tilde{m}-\alpha-\beta)V(\tilde{n}-\alpha-\beta)L(\beta)} \times \\
& \quad \frac{(\beta+1)(\tilde{n}-\alpha-\beta+1)}{(\tilde{n}-\alpha+1)} \xi_{A(\alpha),B(\alpha)}^{M(\beta)L(\beta);U(\tilde{n}-\alpha-\beta),V(\tilde{n}-\alpha-\beta)} C_{A(\tilde{m}-\beta)K(\beta),B(\tilde{m}-\beta)}_{M(\beta)} = \\
& \quad (-1)^\alpha (\alpha+1) \frac{(\beta+1)(\tilde{n}-\alpha-\beta+1)}{(\tilde{n}-\alpha+1)} C_{A(\tilde{m}-\alpha-\beta)U(\tilde{n}-\alpha-\beta)K(\beta),B(\tilde{m}-\alpha-\beta)V(\tilde{n}-\alpha-\beta)L(\beta)} \times \\
& \quad \xi_{B(\alpha),A(\alpha)}^{M(\beta)L(\beta);U(\tilde{n}-\alpha-\beta),V(\tilde{n}-\alpha-\beta)} C_{A(\tilde{m}-\beta)K(\beta),B(\tilde{m}-\beta)}_{M(\beta)} = \\
& \quad (-1)^\alpha (\alpha+1) \frac{(\beta+1)(\tilde{n}-\alpha-\beta+1)}{(\tilde{n}-\alpha+1)} \frac{\alpha+\beta+1}{(\alpha+1)(\beta+1)} C_{A(\tilde{m}-\alpha-\beta)U(\tilde{n}-\alpha-\beta)K(\beta),B(\tilde{m}-\alpha-\beta)V(\tilde{n}-\alpha-\beta)L(\beta)} \times \\
& \quad \xi_{B(\alpha)}^{M(\beta),A(\alpha)}^{L(\beta);U(\tilde{n}-\alpha-\beta),V(\tilde{n}-\alpha-\beta)} C_{A(\tilde{m}-\beta)K(\beta),B(\tilde{m}-\beta)}_{M(\beta)} = \\
& \quad (-1)^\alpha \frac{(\tilde{n}-\alpha-\beta+1)(\alpha+\beta+1)}{(\tilde{n}-\alpha+1)} \frac{2^{\tilde{n}-\alpha-\beta}}{(\tilde{n}-\alpha-\beta+1)} C_{UV\dots UV;A(\tilde{m}-\alpha-\beta)K(\beta),B(\tilde{m}-\alpha-\beta)L(\beta)} \times \\
& \quad \xi_{B(\alpha)}^{M(\beta),A(\alpha)}^{L(\beta);UV\dots UV} C_{A(\tilde{m}-\beta)K(\beta),B(\tilde{m}-\beta)}_{M(\beta)} = \\
& \quad (-1)^\alpha (-1)^{\alpha+\beta} 2^{\tilde{n}-\alpha-\beta} \frac{\alpha+\beta+1}{\tilde{n}-\alpha+1} C_{UV\dots UV;A(\tilde{m}-\alpha-\beta)K(\beta),B(\tilde{m}-\alpha-\beta)L(\beta)} \times \\
& \quad \xi_{A(\alpha)}^{L(\beta),B(\alpha)}^{M(\beta);UV\dots UV} C_{A(\tilde{m}-\beta)K(\beta),B(\tilde{m}-\beta)}_{M(\beta)} = \\
& \quad \frac{\alpha+\beta+1}{\alpha'+\beta+1} C^{MN;B(m-1),D(m-1)}(C^{\{k\}}; MNP^{\alpha'} s^\beta \xi^{\{n\}})_{B(m-1),D(m-1)} \Big|_{(\beta+1)th}.
\end{aligned}$$

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