

Exact higher-spin symmetry in CFT: all correlators in unbroken Vasiliev theory

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Abstract

All correlation functions of conserved currents of the CFT that is dual to unbroken Vasiliev theory are found as invariants of higher-spin symmetry in the bulk of AdS. The conformal and higher-spin symmetry of the correlators as well as the conservation of currents are manifest, which also provides a direct link between the Maldacena-Zhiboedov result and higher-spin symmetries. Our method is in the spirit of AdS/CFT, though we never take any boundary limit or compute any bulk integrals. Boundary-to-bulk propagators are shown to exhibit an algebraic structure, living at the boundary of $\text{SpH}(4)$, semidirect product of $\text{Sp}(4)$ and the Heisenberg group. N-point correlation function is given by a product of N elements.

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1 Introduction

The conformal symmetry has been always important as the ultimate symmetry: of UV and IR fixed points of QFT's, as the symmetry of a dual to a theory of quantum gravity in AdS and for the study of critical phenomena. The Virasoro symmetry combined with the ideas of conformal bootstrap and OPE has already given a harvest of exactly solvable models [1]. These are confined to two space-time dimensions though.

To find exactly solvable models in higher dimensions one calls for an infinite symmetry enveloping the conformal algebra, for the conformal symmetry itself seems to be too weak in $d > 2$. Higher-spin symmetry (HS) could provide a proper replacement for the Virasoro one. However, in $d > 2$ HS symmetry, if unbroken, turns out to be too restrictive for a CFT to have interactions as Maldacena and Zhiboedov have shown recently [2]. The 'minimal models' with exact HS symmetry are free theories. It is tempting to say that nontrivial 'minimal models' in $d > 2$ result from breaking HS symmetries. Indeed, the results of [3] indicate that broken HS symmetry is still quite restrictive.

In this paper we apply HS symmetry to calculate all correlation functions of a CFT that has an exact HS symmetry. The correlators are found for the operators that are dual to HS gauge fields in AdS . We have chosen the simplest HS algebra that underlies the Vasiliev HS theory in AdS_4 . The methods that we shall use in this paper for calculating correlation functions have been spelled out in [4] and follow the general prescription of AdS/CFT, [5–7]. However, at no stage we need to take the boundary limit $z \rightarrow 0$ or deal with integrals over the bulk of AdS . The results are manifestly independent of any coordinate choice in the bulk. The point of view on holography at any z is natural within the Vasiliev approach to HS and has been laid already in [8].

The HS symmetry of the correlators as well as the conservation of the currents are manifest. Therefore, the results of the paper can be also viewed as a complement to the Maldacena-Zhiboedov theorem [2]. To be more precise, [2] classifies all HS algebras together with their unitary irreducible representations, under certain assumptions on the spectrum of conformal

primaries. The latter determines the form of the correlators. We assume the existence of some HS algebra and use it to directly determine all the correlation functions.

In addition, the Maldacena-Zhiboedov result [2] is quite technical in its proof and it would be instructive to have a direct link between HS symmetries and correlation functions. This is necessary in order to study the breaking of HS symmetries. Providing such a link is also a result of the present paper. We show that HS symmetry transformations can be viewed either as bulk symmetries or as boundary symmetries, the latter containing conformal algebra as a subalgebra.

The results can be generalized to any free CFT and it would be interesting to find correlation functions of free $\mathcal{N} = 4$ SYM, reveal the implications of a slightly broken HS symmetry and see if it continues to persist up to the strong coupling.

Higher-spin (HS) theory at present is given by Vasiliev [9–14] in the form of classical equations of motion in the bulk of AdS space that are invariant under diffeomorphisms and HS gauge transformations and whose linearization describes propagation of free fields of all spins $s = 0, 1, 2, 3, \dots$, there is also a truncation to even spins $s = 0, 2, 4, \dots$. The HS fields are gauge fields, being totally-symmetric as tensors,

$$\delta\phi_{\mu_1\dots\mu_s} = \partial_{\mu_1}\xi_{\mu_2\dots\mu_3} + \text{permutations}, \quad (1.1)$$

hence these should be AdS/CFT dual to conserved currents

$$j_{i_1\dots i_s}, \quad \partial^k j_{ki_2\dots i_s} = 0, \quad \Delta = s + 1, \quad s > 0 \quad (1.2)$$

A natural candidate for the CFT dual was conjectured to be a free vector model, [15,16], which has the same spectrum of singlet currents

$$j_{k(s)} = \phi(x)(\overleftarrow{\partial}_k - \overrightarrow{\partial}_k)^s \phi(x), \quad (1.3)$$

where $\phi(x)$ is a vector multiplet of free scalars. Analogous set of currents can be constructed out of free fermion, with the only difference in the conformal weight of $j_0 = \phi^2$, which is 1 or 2, respectively. At free level the AdS/CFT duality is essentially the Flato-Fronsdal theorem [17].

The Flato-Fronsdal theorem states, in its conformal setting, that the tensor square of a free conformal scalar, i.e. the bilocal operator $\phi(x)\phi(y)$, decomposes as a representation of the conformal algebra into a direct sum of conserved currents j_s plus $j_0 = \phi^2(x)$. We can rephrase this as the OPE in terms of the conformal algebra primaries

$$\phi \times \phi = \mathbf{1} + \sum_s j_s \quad (1.4)$$

where $\mathbf{1}$ is the identity operator. A more compact way to state this result is to use the OPE of the fields that are primaries of the HS algebra

$$\phi \times \phi = \mathbf{1} + \mathbb{J} \quad (1.5)$$

where the free conformal scalar ϕ , the identity operator $\mathbf{1}$ and the direct sum $\mathbb{J} = \sum_s j_s$ of the currents are unitary representations of the HS algebra. The latter is reducible under the conformal algebra. Analogous statements are valid for the free fermion theory. In this way HS symmetry allows one to combine an infinite number of the conformal algebra primaries into a finite number of the HS algebra primaries, the job done by the Virasoro algebra in $2d$.

Within the AdS/CFT paradigm Vasiliev higher-spin theory has to be supplemented with certain boundary conditions at the conformal infinity of AdS . Generic boundary conditions

break higher-spin symmetries. For a special choice of boundary behavior the higher-spin symmetry seems to remain unbroken to all orders in perturbation theory [8], implying the boundary theory is free by the Maldacena-Zhiboedov theorem [2], i.e. all correlation functions are given either by free boson or by free fermion.

Despite the powerful result of [2] on the CFT side, the precise mechanism in the bulk is not fully understood yet due to the complexity of the bulk theory. The crucial tests of the conjecture have been performed in [18, 19] where three-point functions have been computed directly from Vasiliev theory by solving bulk equations to the second order with two current sources on the boundary and then taking the boundary limit of the solution. The computations of [18, 19] were quite involved with a simple result in the end, moreover some ambiguities to be resolved have been observed at the intermediate steps.

On the other hand, the formalism of Vasiliev theory suggests introducing observables, [20], the quantities built of HS master fields that are by definition invariant under all HS symmetries provided that they exist in sense of producing a finite result. The simplest such observables are (long)trace operators schematically given by

$$O_n = \text{Tr}(\Phi \star \dots \star \Phi), \quad \delta\Phi = [\Phi, \xi]_\star, \quad (1.6)$$

where it is important that Φ , which is related to HS master field strength, transforms in the adjoint representation of the HS algebra, which is realized as certain \star -product, and the existence of trace operation is also assumed. In this case O_n is invariant under all HS transformations. The existence of such invariants was first pointed out in [21]. If Φ is related to boundary-to-bulk propagators, i.e. it depends on the boundary point \mathbf{x} and bulk point X , $\Phi = \Phi(X, \mathbf{x})$, with the fall-off conditions preserved by HS symmetry, O_n should correspond to the correlation functions of conserved currents

$$\langle j(x_1) \dots j(x_n) \rangle = \sum_{S_n} \text{Tr}(\Phi(X, x_1) \star \dots \star \Phi(X, x_n)). \quad (1.7)$$

Crucial is that the dependence on the bulk point X drops out of (1.7) as the change of X is a particular large HS transformation. This way of extracting correlation functions was proposed by Colombo and Sundell in [4] and confirmed at the level of 2- and 3-point functions. Within the general framework of [4, 22–24] applied in [4] there appear to be several types of divergences that need to be regularized, leading to quite a complicated technique. However, unlike well-defined star-product trace that is used in our paper, the very definition of the trace in [4] requires regularization.

The goal of the present paper is to show that (1.7) is a perfectly finite quantity that can be easily computed using the algebraic structure of Φ that we found. It turns out that the boundary-to-bulk propagators possess certain projective properties within the \star -product. In particular the closed form for all n -point correlation functions of conserved currents can be easily obtained. This calculation is carried out at the free level of HS equations in four dimensions. The nonlinear contribution for appropriate boundary conditions that do not break HS symmetry is believed to leave the result for correlation functions unaffected in accordance with [2, 4, 8, 16].

The other result that lies aside from primary goal of this paper is the group-theoretical approach to certain \star -product elements that arise in the calculation of correlation functions. The boundary-to-bulk propagators turn out to belong to a subalgebra in HS algebra of the elements ($Y \equiv Y^A$ are auxiliary $sp(4) \sim so(3, 2)$ spinors with the Weyl \star -product realization of the HS algebra) that are Gaussians in Y

$$\Phi(f^{AB}, \xi^A, a) = \exp i \left\{ \frac{1}{2} Y f Y + \xi Y + a \right\}, \quad (1.8)$$

where f and ξ depend on X and x . The product of n such elements naturally springs up in our calculations. We show that subalgebra (1.8) can be conveniently parameterized by the $SpH(4)$ group. In other words, each of elements (1.8) and their \star -product belong to $SpH(4)$ – the semidirect product of $Sp(4)$ and the Heisenberg group $(\xi, a) \star (\eta, b) = (\xi + \eta, a + b + \xi\eta)$, where central elements a, b are just numbers and ξ^A and η^A are contracted with the $Sp(4)$ invariant metric. The trace operation projects onto the central part. To be more precise, the quadratic form in $\Phi(f, \xi, a)$ parameterizes the Lie algebra of $SpH(4)$, while the group elements are obtained via the Cayley transform. HS propagators turn out to be even more peculiar as they are not generic elements of type (1.8), lying in the subspace where Cayley transform is not invertible but the \star -product is still well-defined. Finally, n -point correlation function is a centrally projected product of n such propagators (1.8)

The minimal HS prerequisites we need to discuss the constructive approach to correlation functions invariant under unbroken HS symmetry are given in Section 2. In Section 3 we discuss boundary-to-bulk propagators. The observables are introduced in Section 4. The explicit relation between propagators and $SpH(4)$ is presented in Section 5.

The conformal and HS symmetry of the n -point function is manifest. Nevertheless, we consider it important to explicitly expand the result in terms of conformal invariants introduced in [25, 26]. Such rewriting manifests the conformal symmetry of the result but not its HS symmetry and the conservation of the currents is also hard to see. This is done in Section 6 where the examples of 2, 3 and 4-point functions are considered in detail. The main result is the closed form for the n -point function (6.23). Conclusion is given in Section 7.

2 HS prerequisites

The higher-spin equations are formulated in terms of certain master fields. There are two of them, a master HS connection ω , which contains gauge fields themselves with certain their derivatives, and a master field strength B , which contains gauge invariant field strengths together with all their on-mass-shell nontrivial derivatives. The lesson of HS is that when fields are supplemented with certain derivatives thereof to form a master field an extended symmetry acts in a simple way on the master field.

Higher-spin algebra. Crucial for the whole AdS_4 HS story is the isomorphism $so(3, 2) \sim sp(4, \mathbb{R})$ so we will use $sp(4)$ instead. The $sp(4)$ generators $T_{AB} = T_{BA}$, $A, B, \dots = 1\dots 4$ obey

$$[T_{AB}, T_{CD}] = T_{AD}\epsilon_{BC} + 3 \text{ terms}, \quad (2.1)$$

where $\epsilon_{AB} = -\epsilon_{BA}$, $\epsilon_{AB}\epsilon^{AC} = \delta_B^C$ is a symplectic form. Splitting $A = \{\alpha, \dot{\alpha}\}$, etc. along with the choice $\epsilon_{AB} = \text{diag}(\epsilon_{\alpha\beta}, \epsilon_{\dot{\alpha}\dot{\beta}})$, $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}$, $\epsilon_{12} = 1$ *idem.* for $\epsilon_{\dot{\alpha}\dot{\beta}}$, the generators of the bulk Lorentz $so(3, 1) \sim sp_2 \oplus sp_2$ algebra are given by $T_{\alpha\beta}$, $\bar{T}_{\dot{\alpha}\dot{\beta}}$, and $T_{\alpha\dot{\alpha}}$ are the AdS_4 translations.

The higher-spin algebra [21, 27] is a universal enveloping algebra of sp_4 quotiented by the two-sided ideal that is the singleton annihilator, [28, 29]. This is the point where the singleton, i.e. free 3d conformal scalar/fermion enters the story. In the case of $so(3, 2)$ the ideal is resolved by an oscillator realization of sp_4 . Namely, elements of the higher-spin algebra are functions of formally commuting variables Y_A with the Weyl product

$$f(Y) \star g(Y) = \int dU dV f(Y + U)g(Y + V)e^{iV^A U_A} = f(Y) \exp \left\{ i \overleftarrow{\partial}_A \epsilon^{AB} \overrightarrow{\partial}_B \right\} g(Y), \quad (2.2)$$

(ϵ_{AB} is used to raise and lower indices $Y^A = \epsilon^{AB}Y_B$, $Y_A = Y^B\epsilon_{BA}$), which effectively implies

$$[Y_A, Y_B]_\star = 2i\epsilon_{AB}, \quad Y_A \star f(Y) = Y_A f + i \frac{\partial f}{\partial Y^A}, \quad f(Y) \star Y_A = Y_A f - i \frac{\partial f}{\partial Y^A}. \quad (2.3)$$

The sp_4 generators, which form a Lie subalgebra under the \star -commutator, read

$$T_{AB} = -\frac{i}{4}\{Y_A, Y_B\}_\star, \quad [T_{AB}, Y_C] = Y_A\epsilon_{BC} + Y_B\epsilon_{AC}. \quad (2.4)$$

Linearized equations. The full nonlinear Vasiliev equations is a subject of several reviews, [12, 13, 24, 30], see also original works [9–14]. Here we need the linearized equations only, a part of them actually that encode HS field strengths. These are contained in the master field $B(Y|X)$, which is a space-time zero-form. Projection onto the integer spin fields imposes kinematic constraint $B(Y) = B(-Y)$. Various components of $B(Y)$ at the free level are identified as follows,

$$B(Y|X) = \sum B_{A(k)}(X) Y^{A(k)} = \sum_{k,m} B_{\alpha(k),\dot{\alpha}(m)}(X) y^{\alpha(k)} \bar{y}^{\dot{\alpha}(m)}$$

component	meaning
B	the scalar field
$B_{\alpha(k),\dot{\alpha}(k)}, k > 0$	derivatives of the scalar field B , $B_{\alpha\dot{\alpha}} = D_{\alpha\dot{\alpha}}B$, etc.
$B_{\alpha\beta} \quad B_{\dot{\alpha}\dot{\beta}}$	(anti)selfdual parts of the Maxwell spin-one tensor $F_{\mu\nu}$
$B_{\alpha(4)} \quad B_{\dot{\alpha}(4)}$	(anti)selfdual parts of the spin-two Weyl tensor
$B_{\alpha(2s)} \quad B_{\dot{\alpha}(2s)}$	(anti)selfdual parts of the field strength for a spin- s field, which are also called spin- s Weyl tensors
$B_{\alpha(2s+k),\dot{\alpha}(k)} \quad B_{\alpha(k),\dot{\alpha}(2s+k)}, k > 0$	derivatives of the spin- s field strength

The equations for the master fields are first order differential equations that express exterior derivative of all the fields in terms of exterior products of the fields themselves and are called unfolded equations, [31, 32]. The full Vasiliev equations have the unfolded form too. The linearized equations for the master field B read

$$d\Omega + \Omega \star \Omega = 0, \quad (2.5)$$

$$dB + \Omega \star B - B \star \tilde{\Omega} = 0, \quad (2.6)$$

where d is de Rham differential, $\Omega = \frac{1}{2}\Omega_\mu^{AB}dx^\mu T_{AB}$ is a flat sp_4 connection that contains vierbein $h^{\alpha\dot{\alpha}} = \Omega^{\alpha\dot{\alpha}}$ as well as (anti)selfdual parts of the spin-connection, $\Omega^{\alpha\beta}, \bar{\Omega}^{\dot{\alpha}\dot{\beta}}$. $\tilde{\Omega}$ represents the action of an sp_4 automorphism that flips the sign of AdS_4 translations $\tilde{T}_{\alpha\dot{\alpha}} = -T_{\alpha\dot{\alpha}}$. Ω is a vacuum value of the master field ω that contains all HS gauge fields, including graviton as spin-two. It is only the graviton part of Ω that is nonzero. Its purpose is to define AdS_4 background.

Eq. (2.6) does several things: it expresses all components $B_{\alpha(2s+k),\dot{\alpha}(k)}, B_{\alpha(k),\dot{\alpha}(2s+k)}$, with $k > 0$ as rank- k derivatives of Weyl tensors $B_{\alpha(2s)}, B_{\dot{\alpha}(2s)}$; imposes Klein-Gordon equation on the scalar $B(X) = B(0|X)$, $(\square - 2)B(X) = 0$; imposes Bianchi identities and equations of motion, $D^{\beta\dot{\beta}}B_{\alpha(2s-1)\beta} = 0, D^{\beta\dot{\beta}}B_{\dot{\alpha}(2s-1)\dot{\beta}} = 0$, which are equations restricting the form of Weyl tensors and are automatically satisfied once the Weyl tensor is expressed as order- s derivative of the gauge potential.

Reality conditions. \star -product (2.2) defined on Majorana spinors Y_A admits an involution $y_\alpha^\dagger = \bar{y}_{\dot{\alpha}}$, $\bar{y}_{\dot{\alpha}}^\dagger = y_\alpha$ such that

$$(\lambda f + \mu g)^\dagger = \bar{\lambda} f^\dagger + \bar{\mu} g^\dagger, \quad (f \star g)^\dagger = g^\dagger \star f^\dagger. \quad (2.7)$$

The appropriate reality conditions for background one-form Ω and zero-form B read

$$\Omega^\dagger = -\Omega, \quad B^\dagger = \tilde{B}. \quad (2.8)$$

Symmetries and general solution. Eqs. (2.5)-(2.6) are invariant under the global HS symmetries

$$\delta\Omega = d\xi + [\Omega, \xi]_\star = 0, \quad (2.9)$$

$$\delta B = -\xi \star B + B \star \tilde{\xi}, \quad (2.10)$$

where the first equation imposes the invariance of the vacuum $\delta\Omega = 0$. Eq. (2.10) illustrates why it is useful to pack fields together with their derivatives into a master field as the extended symmetry, which is the HS symmetry, gets explicitly algebraic.

Since Ω is flat, (2.9), one can represent it in the pure gauge form

$$\Omega = g^{-1} \star dg, \quad (2.11)$$

where $g = g(Y|X)$ is actually Gaussian $g \sim \exp i(\frac{1}{2} Y^A f_A^B Y_B)$ since Ω occupies only the sp_4 part of the HS algebra and T_{AB} , (2.4), are quadratic in Y . Then one can solve (2.6) and (2.9)

$$\xi(Y|X) = g^{-1} \star \xi(Y|X_0) \star g, \quad (2.12)$$

$$B(Y|X) = g^{-1} \star B(Y|X_0) \star \tilde{g}, \quad (2.13)$$

where X_0 is a point where $g(Y|X_0) = 1$. On the other hand, one can take arbitrary functions $\xi(Y)$ and $B(Y)$ that do not depend on X and obtain the solutions to (2.6) and (2.9) via (2.12) and (2.13). Then the initial data $\xi(Y)$ and $B(Y)$ turn out to be equal to the solutions at X_0 where $g(X_0) = 1$. The Cauchy problem is naturally replaced by a Taylor-like problem as $B(Y)$ parameterizes all nontrivial derivatives at a point.

That the solutions to (2.9) are parameterized by $\xi(Y)$, which is an element of the HS algebra, implies that the HS algebra is the algebra of global symmetries of (2.5)-(2.6).

3 Boundary-to-bulk propagators

Essential ingredient of the following construction is the boundary-to-bulk propagator, $B(X|Y|x, \eta)$, which by definition has two legs, one behaving as a bulk HS master field B , i.e. a generating function of HS field strengths, and the second leg behaving as a generating function for all conserved currents on the boundary. It depends on the bulk coordinate X , which in Poincare coordinates splits as $X = (z, x)$, $x = x^{\alpha\beta}$, where $x^{\alpha\beta} = x^{\beta\alpha}$ is the bi-spinor counterpart to x^i ; on auxiliary bulk variables Y that allows us to pack all the field strengths together with their derivatives into a single master field B ; on boundary coordinate $x \equiv x^{\alpha\beta}$ and on boundary polarization spinor $\eta \equiv \eta^\alpha$ that pretty much as Y is used to pack up all conserved currents

$$j(x, \eta) = \sum_s j_{\alpha(2s)} \eta^\alpha \dots \eta^\alpha. \quad (3.1)$$

Note that $j(x, 0)$ is not a current as it has no indices at all. It is dual to the scalar field of the HS multiplet i.e. $B(0|X)$. Propagator does not depend on any other quantities but described above, it satisfies (2.6) in the bulk and behaves as a conserved current on the boundary [33]

$$\frac{\partial^2}{\partial\eta^\alpha\partial\eta^\beta}\frac{\partial}{\partial x_{\alpha\beta}}B(X, Y|x, \eta) = 0. \quad (3.2)$$

In [18]¹ the propagator was found in Poincare coordinates, $B(X, Y|x, \eta) = B(x, z; y, \bar{y}|x_i, \eta)$. The HS observables and correlation functions do not depend on any particular choice of bulk coordinates as will become clear soon. The expressions below are given for clarity and to illustrate certain general properties of propagators.

Background in Poincare coordinates. To proceed, it is convenient to use AdS_4 Poincare coordinates

$$ds^2 = \frac{1}{z^2}(dz^2 + dx_i dx^i), \quad (3.3)$$

where the 3d boundary coordinates x^i have indices contracted with the flat Minkowski metric η_{ij} . The components of the background connection Ω read

$$\Omega^{\alpha\alpha} = \frac{i}{2z}dx^{\alpha\alpha}, \quad \Omega^{\dot{\alpha}\dot{\alpha}} = -\frac{i}{2z}dx^{\dot{\alpha}\dot{\alpha}}, \quad \Omega^{\alpha\dot{\alpha}} = \frac{1}{2z}(-dx^{\alpha\dot{\alpha}} + i\epsilon^{\alpha\dot{\alpha}}dz), \quad (3.4)$$

where we introduced the mixed epsilon-symbol $\epsilon_{\alpha\dot{\beta}} = -\epsilon_{\dot{\beta}\alpha}$ to single out z -direction. $x^{\alpha\dot{\alpha}}$ should not be confused with the $4d$ coordinates as we adopt the following convention

$$x_{\alpha\dot{\alpha}} = x_\alpha^\beta \epsilon_{\beta\dot{\alpha}}, \quad \epsilon_{\alpha\dot{\beta}} \epsilon^{\beta\dot{\alpha}} = \delta_\alpha^\beta. \quad (3.5)$$

The gauge function $g(Y|x, z)$ that reproduces connection (3.4) has the factorized form

$$g = g_p \star g_z, \quad g_p = e^{\frac{i}{2}P_{\alpha\beta}x^{\alpha\beta}}, \quad g_z = \frac{4\sqrt{z}}{(1+\sqrt{z})^2} e^{\frac{1-\sqrt{z}}{1+\sqrt{z}}\bar{y}_\alpha y^\alpha}, \quad (3.6)$$

where $P_{\alpha\beta} = iy_\alpha^- y_\beta^-$, $y_\alpha^- = \frac{1}{2}(\bar{y}_\alpha - iy_\alpha)$ corresponds to the boundary Poincare translations [8]. The AdS_4 connection Ω then reads

$$\Omega = g_z^{-1} \star (g_p^{-1} \star dg_p) \star g_z + g_z^{-1} \star dg_z, \quad (3.7)$$

where the piece $g_p^{-1} \star dg_p$ is the flat Poincare (boundary) connection in Cartesian coordinates, i.e. $\frac{i}{2}P_{\alpha\beta}dx^{\alpha\beta}$. Note, that $g(Y|x_{\alpha\beta} = 0, z = 1) = 1$.

Bulk-to-boundary propagators. The HS boundary-to-bulk propagator is found to be²

$$B = K \exp i\{-yF\bar{y} + \xi y + \theta\} + K \exp i\{-yF\bar{y} + \bar{\xi}\bar{y} - \theta\} + \left(\begin{matrix} \xi \leftrightarrow -\xi \\ \bar{\xi} \leftrightarrow -\bar{\xi} \end{matrix} \right), \quad (3.8)$$

¹As the conventions of [18] are different from ours and perhaps not fully clear, we redo this HS exercise in Appendix A.

²The boundary-to-bulk propagator is actually a bi-field satisfying the same equations (2.5)-(2.6) on the boundary and in the bulk. Indeed, (2.5)-(2.6) are background independent and instead of taking AdS_4 with a non-degenerate sp_4 connection Ω one may study them over the 3-dimensional boundary, where a natural connection is given by Cartesian coordinates $g_p^{-1} \star dg_p$. Then one can show, see also [8], that (2.5)-(2.6) describe conserved currents. We will not use this fact in the paper, so x and η are just external parameters rather than variables analogous to X and Y .

where $yF\bar{y} \equiv y^\alpha F_{\alpha\dot{\alpha}} \bar{y}_{\dot{\alpha}}$, $\xi y = \xi^\alpha y_\alpha$, θ is an arbitrary constant and K is the $\Delta = 1$ scalar boundary-to-bulk propagator,

$$K = \frac{z}{(x - \mathbf{x})^2 + z^2}. \quad (3.9)$$

$F \equiv F^{\alpha\dot{\alpha}}$ is $\partial^{\alpha\dot{\alpha}} \ln K$ up to some factor

$$F^{\alpha\dot{\alpha}} = - \left(\frac{2z}{(x - \mathbf{x})^2 + z^2} (x - \mathbf{x})^{\alpha\dot{\alpha}} + \frac{(x - \mathbf{x})^2 - z^2}{(x - \mathbf{x})^2 + z^2} i\epsilon^{\alpha\dot{\alpha}} \right). \quad (3.10)$$

A particular spin- s Weyl tensor propagator is encoded in $B_{\alpha(2s)}$ and $B_{\dot{\alpha}(2s)}$ components of (3.8). Bulk polarization spinors ξ and $\bar{\xi}$ represent the boundary polarization spinor η^α parallel transported to the bulk point (x, z) with the parallel transport bispinor $\Pi^{\alpha\beta}$

$$\xi^\alpha = \Pi^{\alpha\beta} \eta_\beta, \quad \Pi^{\alpha\beta} = K \left(\frac{1}{\sqrt{z}} (x - \mathbf{x})^{\alpha\beta} - \sqrt{z} i\epsilon^{\alpha\beta} \right), \quad \bar{\xi}^{\dot{\alpha}} = (\xi^\alpha)^\dagger. \quad (3.11)$$

The symmetrization in (3.8) over ξ , $-\xi$, *idem.* for $\bar{\xi}$, projects onto the bosonic part $B(Y) = B(-Y)$. Parameter θ is introduced for convenience, it is a free parameter in the full Vasiliev equations. At $\theta = 0$ and $\theta = \pi/2$ the theory is parity-invariant, [11,34]. And the Vasiliev theory at $\theta = 0$ and $\theta = \pi/2$ with boundary conditions preserving HS symmetry is conjectured to be dual to the free vector model, bosonic and fermionic, respectively. For other values of θ any boundary conditions break HS symmetry, [8], and there is also a proposal for its dual [35,36]. The fields of the vector model are vectors of some group, e.g. $O(N)$ or $U(N)$. In the case of $O(N)$ all odd-spin singlet currents vanish unless there are other flavor groups. The pure $O(N)$ model should be dual to the minimal bosonic Vasiliev theory, which contains fields of even spins $s = 0, 2, 4, \dots$. $U(N)$ vector model or $O(N)$ model with additional flavors possesses currents of all integer spins and should be dual to the bosonic Vasiliev theory with spectrum $s = 0, 1, 2, 3, \dots$. For brevity we will always refer to vector model without specifying the group.

Invariant properties of propagators. Given some particular coordinates it might not be clear which properties of the propagator are invariant of a particular coordinate choice. Here we wish to collect those that are coordinate independent. First, $F^{\alpha\dot{\alpha}}$ is a projective $Sp(2)$ parameterization of the conformal boundary

$$F^{\alpha\dot{\alpha}}(\lambda x, \lambda z, \lambda \mathbf{x}) = F(x, z, \mathbf{x}), \quad \Pi(\lambda x, \lambda z, \lambda \mathbf{x}) = \lambda^{-\frac{1}{2}} \Pi(x, z, \mathbf{x}). \quad (3.12)$$

A distinguishing property of the HS propagators found in [37] for arbitrary d which has been recently confirmed for $d = 3$ in [38] is that in all cases the solution is based on certain projectors within \star -product algebra. This fact as it seems is closely related to the appearance of δ -like sources in the space-time equations. Their reincarnation results in projectors on the twistor-space side. These can be thought of as analogs of distributions in the \star -product algebra. As it happens to distributions not all of them can be multiplied, see [39] for more detail. Fortunately we will not meet these subtleties.

To be more specific, let us introduce the \star -product projector

$$P_0 = \exp i(-yF\bar{y} + \xi y), \quad P_0 \star P_0 = P_0, \quad (3.13)$$

which requires

$$F^{\alpha\dot{\alpha}} F_{\dot{\alpha}\alpha}^\beta = \epsilon^{\alpha\beta}, \quad F^{\alpha\dot{\alpha}} F_{\alpha\dot{\alpha}}^\beta = \epsilon^{\dot{\alpha}\dot{\beta}}. \quad (3.14)$$

The latter is an invariant property for any choice of bulk coordinates, in particular it holds for (3.10). Note, P_0 is annihilated by two self-commuting oscillators

$$y_\alpha^\pm = F_\alpha^{\dot{\beta}} \bar{y}_{\dot{\beta}} \pm y_\alpha + \xi_\alpha \quad (3.15)$$

as follows

$$y^+ \star P_0 = P_0 \star y^- = 0, \quad P_0 \sim \delta(y^+) \star \delta(y^-), \quad (3.16)$$

which makes it clear e.g., that $P_0 \star \tilde{P}_0$ does not exist. In other terms the \star -product of two propagators from boundary point x and its inverse-reflected point $-x/x^2$ diverges, which as become clear soon is related to the singularity of correlation functions at coincident points.

Anti-holomorphic projector

$$\bar{P}_0 = \exp i(-yF\bar{y} + \bar{\xi}\bar{y}), \quad \bar{P}_0 \star \bar{P}_0 = \bar{P}_0 \quad (3.17)$$

is annihilated by the same y_α^\pm provided that $\xi = -F\bar{\xi}$, which is also an invariant property.

Interplay between bulk and boundary global symmetries. We would like to show that global $Sp(4)$ symmetries can be treated either as symmetries of the bulk theory or as symmetries of the boundary CFT. The difference between the bulk and the boundary is that Y^A variables are explicitly affected by global $Sp(4)$ transformations inducing certain action in the bulk, while boundary coordinate x and polarization spinor η remain untouched.

Let us be given some gauge function $g = g(X|Y)$ that leads to Ω via (2.11) and hence performs certain $Sp(4)$ rotation of Y_A under the adjoint action

$$g^{-1} \star Y_A \star g = \Lambda_A^B Y_B, \quad \Lambda_M^N = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \quad (3.18)$$

Let us look at the large twisted-adjoint rotation (2.13) of the propagator performed by g

$$g^{-1}(X|Y) \star K \exp i \{-yF\bar{y} + \xi y\} \star \tilde{g}(X|Y) = K' \exp i \{-yF'\bar{y} + \xi' y\}, \quad (3.19)$$

$$\begin{aligned} K' &= \frac{K}{\det |A - FC|}, \\ F' &= (A - FC)^{-1}(FD - B), \\ \xi' &= (A - FC)^{-1}\xi. \end{aligned} \quad (3.20)$$

First, propagator preserves its form³. The projector property of the propagator holds upon large twisted-adjoint rotation (2.13). This entails that F' satisfies (3.14). Second, K' , F' and ξ' are exactly K_X , F_X and ξ_X at the point X that is defined by $g(X|Y)$. The latter implies that the two ways of obtaining solution at point X , either by taking the solution itself or first by restricting solution to X_0 where $g(X_0) = 1$ and then performing a large twisted adjoint rotation (2.13), give the same result. This is of course what must have been expected.

One can invert the meaning of (3.20) and treat them as boundary transformations of x and η . Therefore boundary-to-bulk propagator is a representation of $Sp(4)$ in the bulk and on the boundary⁴. Let us mention that similar transformations have already appeared in the context of conformal HS theories [40].

³These transformations are twistor analogs of the ones found recently for HS fields in d -dimensions in [37].

⁴In view of footnote 2, there is a natural action of HS algebra on boundary variables η and x too and the propagator is an equivariant map or intertwining.

It is easy to work out the coordinate dependence of the boundary-to-bulk propagator with the help of (3.20) starting from the base point where $g = 1$

$$B = g^{-1} \star P_0 \star \tilde{g} + c.c. + \left(\frac{\xi \leftrightarrow -\xi}{\tilde{\xi} \leftrightarrow -\tilde{\xi}} \right). \quad (3.21)$$

Indeed, taking for example $f_{\alpha\dot{\alpha}} = i\epsilon_{\alpha\dot{\alpha}}$, a constant ξ^α and using (3.6) one finds (3.8) at $x = 0$.

4 Observables in higher-spin theory

All physical information is encoded in master field $B(Y|X)$ even at the nonlinear level. However, $B(Y|X)$ is not invariant under HS transformations, moreover it transforms in the twisted-adjoint representation of the HS algebra rather than the adjoint one, (2.10). One way of extracting physical data is to construct observables which are invariant under HS symmetries and diffeomorphisms as well. This route was suggested in [22], elaborated further in [4, 23, 24] and is closely related to the action proposal [41, 42] for Vasiliev equations.

That kind of observables do not strictly speaking correspond to any conserved charges, rather to some 'initial data' similar to Cauchy data of classical mechanics. That state of affairs in HS theory is in many ways similar to what happens in pure gravity, which being diffeomorphism invariant admits no stress tensor and conserved charges. However, from the holographic point of view, these observables are actually what one needs to trace the correspondence. Indeed, the gauge invariant correlation functions of the boundary theory should be rewritten in terms of 'initial data' in the bulk.

To construct observables out of $B(Y|X)$ we recall that the \star -product admits uniquely defined supertrace operation⁵ [13]

$$str(F(Y)) = F(0), \quad str(F(Y) \star G(Y) - (-)^{\pi_G \pi_F} G(Y) \star F(Y)) = 0, \quad (4.1)$$

where degree π_F is defined as $F(Y) = (-)^{\pi_F} F(-Y)$. In the bosonic HS theory that we consider the supertrace coincides with the trace as $\pi_B = 0$. Now, having a bosonic field $\Psi(Y)$ that transforms in the adjoint

$$\delta_\xi \Psi = [\Psi, \xi]_\star, \quad (4.2)$$

its corresponding (long)trace operator

$$O_n = \text{Tr}(\Psi \star \dots \star \Psi) \quad (4.3)$$

is a HS gauge invariant quantity. The problem is that master field B transforms in the twisted-adjoint, (2.10). To fix it one observes that the automorphism \tilde{g} turns to an internal one, should we allow δ -functions in the \star -product algebra [43]

$$\delta(y) = \int ds \exp i(sy), \quad (4.4)$$

which behaves nicely under \star -product performing a Fourier transform

$$F(y, \bar{y}) \star \delta(y) = \int ds F(s, \bar{y}) \exp i(sy). \quad (4.5)$$

⁵Similar formulas with the simple trace operation defined below appeared in [4, 23], then the Authors turned to another definition that leads to several types of divergences that need to be regularized.

Sandwiching $F(y, \bar{y})$ with two $\delta(y)$'s one finds

$$\delta(y) \star F(y, \bar{y}) \star \delta(y) = F(-y, \bar{y}) \equiv \tilde{F}(y, \bar{y}). \quad (4.6)$$

The seemingly asymmetric holomorphic form of \tilde{g} is fictitious since $B(-y, \bar{y}) = B(y, -\bar{y})$. Therefore, $\delta(y)$ is a map from the twisted adjoint module of HS algebra (2.10) to the adjoint one

$$\Psi = B \star \delta(y), \quad \delta_\xi \Psi = [\Psi, \xi]. \quad (4.7)$$

Given a set of n adjoint fields $B_i(Y|X) \star \delta(y)$, $i = 1 \dots n$, their mutual HS symmetry invariants are given by operators

$$O_n = \text{str} (B_1 \star \delta \star B_2 \star \delta \star \dots \star B_n \star \delta). \quad (4.8)$$

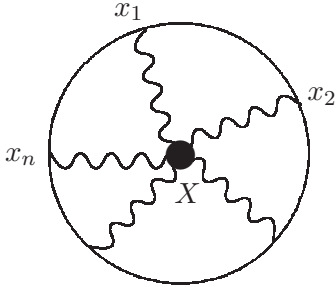


Figure 1: Twistor space Witten diagram for O_n

The observable (4.8) can be also viewed as some n -point interacting vertex. The interaction is governed by the unbroken HS symmetry via the \star -product operation. In our case the set of B_i is given by boundary-to-bulk propagators $B(X, Y|x_i, \eta_i)$ from boundary points x_i to one and the same bulk point X . To make connection with the usual AdS/CFT paradigm, one can identify O_n with an analog of the n -leg Witten diagram. The interaction point is however in the twistor space of Y variables and the interaction vertex is projected onto $Y = 0$ rather than integrated over the twistor space. No integral over the bulk position of the vertex is taken as the interaction point drops out of O_n since a particular large gauge transformation (3.20) allows one to move the interaction point freely over the whole AdS_4 .

One may think that the volume integral splits off. It is worth emphasizing that (4.8) is invariant under all HS transformations (2.10), not only under the $sp(4)$ subalgebra. As we see in order to reproduce the free CFT correlators one needs to take into account only very specific contact interactions in the twistor space⁶. No exchange diagrams is needed.

Let us note that the holomorphic Fourier transform applied to boundary-to-bulk propagator results in δ -function yielding $O_1 =$ derivatives $\delta(0)$. Nevertheless, as we will soon see, all O_n for $n > 1$ are perfectly finite quantities. For O_{2m} it is obvious. Indeed, for even n all δ -function insertions can be pairwise removed using (4.6). If n is odd one is left with a single δ -function inside the trace

$$O_{2m} = \text{str}(B_1 \star \tilde{B}_2 \star \dots \star B_{2m-1} \star \tilde{B}_{2m}), \quad (4.9)$$

$$O_{2m+1} = \text{str}(B_1 \star \tilde{B}_2 \star \dots \star \tilde{B}_{2m} \star B_{2m+1} \star \delta). \quad (4.10)$$

Noticing that

$$\text{str}(F(y, \bar{y}) \star \delta(y)) = \int dy F(y, 0), \quad (4.11)$$

we finally arrive at

$$O_{2m} = (B_1 \star \tilde{B}_2 \star \dots \star B_{2m-1} \star \tilde{B}_{2m}) \Big|_{Y=0}, \quad (4.12)$$

$$O_{2m+1} = \int dy (B_1 \star \tilde{B}_2 \star \dots \star \tilde{B}_{2m} \star B_{2m+1}) \Big|_{\bar{y}=0}. \quad (4.13)$$

Despite not being immediately obvious O_{2m} and O_{2m+1} have the cyclic property for $B_i(Y)$ that satisfy $B_i(Y) = B_i(-Y)$.

⁶This kind of tree level S -matrix can be effectively obtained by substituting $\exp_\star(gB \star \delta)$ instead of B into (2.6) and expanding in the formal coupling g .

General expression. The last step is to replace generic B in O_n with the boundary-to-bulk propagator. The propagator (3.8) has four terms that are generated from the first one by the action of simple discrete groups that will survive in the final expressions. Let us define⁷

$$\rho(\xi y, \bar{\xi} \bar{y}, \theta) = (\bar{\xi} \bar{y}, \xi y, -\theta) \quad \text{and} \quad \pi(\xi, \bar{\xi}) = (-\xi, -\bar{\xi}). \quad (4.14)$$

The propagator (3.8) is rewritten as

$$B = \sum_{\rho \times \pi} \Phi \equiv (1 + \pi)(1 + \rho)\Phi, \quad \Phi(F, \xi, \theta) = K \exp i(-yF\bar{y} + \xi y + \theta). \quad (4.15)$$

Then, the connected n -point correlation function of generating functions of conserved currents is conjectured to be (up to a certain numerical prefactor)

$$\langle j(x_1, \eta_1) \dots j(x_n, \eta_n) \rangle = \sum_{S_n} \sum_{\rho^n \times \pi^n} O_n(\Phi_1, \dots, \Phi_n), \quad (4.16)$$

where Φ_i is a primary term (4.15) of the propagator for the current at the i -th point on the boundary. It depends on x_i, η_i through F , (3.10), and ξ , (3.11). The sum over $\rho^n \times \pi^n$ restores antiholomorphic part of the propagators and projects it onto the bosonic part. The sum over the symmetric group S_n is introduced to account all terms in the trace $\text{Tr}(B \star \tilde{B} \dots B \star \tilde{B})$ except for those at coinciding points, where the full B is $\sum_i B_i$. Equivalently, one may propose (4.16) from the very beginning as the candidate to be an observable with all the requirements being met. In the latter case the sum over S_n is required to make the expression hermitian. In the next section we show that $O_n(\Phi_1, \dots, \Phi_n)$ can be computed explicitly for arbitrary $n > 1$. It is worth emphasizing that no $z \rightarrow 0$ limit is needed. Neither should we deal with any integrals over the bulk of AdS .

5 Algebraic structure of propagators

The boundary-to-bulk propagators are exponents of no higher than quadratic polynomials in Y variables. While each of them has only mixed bilinear term $y\bar{y}$ in the exponent, the mutual \star -product of any two results in some generic bilinear exponent. These Gaussians form a closed subalgebra under the \star -product. So let us consider a generic element

$$\Phi(f, \xi, q) = \exp i \left(\frac{1}{2} f_{AB} Y^A Y^B + \xi^A Y_A + q \right), \quad (5.1)$$

and multiply two such elements within the HS algebra and see what happens.

$Sp(2M) \ltimes$ Heisenberg group and Cayley transform. We may no longer restrict ourselves to $sp(4)$ algebra, as the the following result is valid for $sp(2M)$. First, it is easy to see that

$$\exp i(\xi Y) \star \exp i(\eta Y) = \exp i((\xi + \eta)Y + \xi \eta) \quad (5.2)$$

and hence elements of the form $\Phi(0, \xi, c)$ belong to the Heisenberg group, $\Phi(0, \xi, a) \star \Phi(0, \eta, b) = \Phi(0, \xi + \eta, a + b + \xi \eta)$, [39]. It was shown in [44] that $\Phi(f_1, 0, 0) \star \Phi(f_2, 0, 0)$ computes the $Sp(2M)$ -product provided that $f_{1,2}$ are related to group elements $U_{1,2} \in Sp(2M)$ by Cayley transform,

$$f = (1 - U)(1 + U)^{-1}. \quad (5.3)$$

⁷Recall that θ is a parameter in the Vasiliev equations and it enters the propagator, see discussion after (3.11).

Below we consider the general case of $\Phi(f, \xi, q)$.

Define $SpH(2M)$ as the semidirect product of $Sp(2M)$ and the Heisenberg group, [40, 45], i.e. $SpH(2M)$ consists of triplets $\mathcal{G} = (U_A^B, x_A, c)$, where $U_A^B \in Sp(2M)$ with the following product

$$\mathcal{G}_1 \diamond \mathcal{G}_2 = \left((U_1 U_2)_A^B, x_{1A} + U_{1A}^B x_{2B}, c_1 + c_2 + x_1^A U_{1A}^B x_{2B} \right). \quad (5.4)$$

The $SpH(2M)$ action can be realized by the generalized Cayley transform \mathcal{C} on the \star -product elements of the form $\Phi(f, \xi, q)$, (5.1). For its derivation let us first write down the \star -product of such two different (5.1),

$$\Phi(f_1, \xi_1, 0) \star \Phi(f_2, \xi_2, 0) = \frac{1}{\sqrt{\det |1 + f_1 f_2|}} \Phi(f_{1,2}, \xi_{1,2}, q_{1,2}), \quad (5.5)$$

where

$$f_{1,2AB} = \frac{1}{1 + f_2 f_1} (f_2 + 1) + \frac{1}{1 + f_1 f_2} (f_1 - 1), \quad (5.6)$$

$$\xi_{1,2}^A = \xi_1^B \left(\frac{1}{1 + f_2 f_1} (f_2 + 1) \right)_B^A + \xi_2^B \left(\frac{1}{1 + f_1 f_2} (1 - f_1) \right)_B^A, \quad (5.7)$$

$$q_{1,2} = \frac{1}{2} \left(\frac{1}{1 + f_2 f_1} f_2 \right)_{AB} \xi_1^A \xi_1^B + \frac{1}{2} \left(\frac{1}{1 + f_1 f_2} f_1 \right)_{AB} \xi_2^A \xi_2^B - \left(\frac{1}{1 + f_2 f_1} \right)_{AB} \xi_1^A \xi_2^B. \quad (5.8)$$

The generalized Cayley transform

$$\mathcal{C} : \quad \Phi(f, \xi, q) \rightarrow \mathcal{G}(U, x, c) \quad (5.9)$$

allows one to embed $SpH(2M)$ group into the \star -product algebra, such that

$$r(\mathcal{G}_1) \Phi\left(f(\mathcal{G}_1), \xi(\mathcal{G}_1), q(\mathcal{G}_1)\right) \star r(\mathcal{G}_2) \Phi\left(f(\mathcal{G}_2), \xi(\mathcal{G}_2), q(\mathcal{G}_2)\right) = \quad (5.10)$$

$$r(\mathcal{G}_1 \diamond \mathcal{G}_2) \Phi\left(f(\mathcal{G}_1 \diamond \mathcal{G}_2), \xi(\mathcal{G}_1 \diamond \mathcal{G}_2), q(\mathcal{G}_1 \diamond \mathcal{G}_2)\right). \quad (5.11)$$

Its explicit form reads

$$f_{AB}(\mathcal{G}) = \left(\frac{U - 1}{U + 1} \right)_{AB}, \quad (5.12)$$

$$r(\mathcal{G}) = \frac{2^{M/2}}{\sqrt{\det |1 + U|}}, \quad (5.13)$$

$$\xi_A(\mathcal{G}) = \pm 2 \left(\frac{1}{1 + U} \right)_A^B x_B, \quad (5.14)$$

$$q(\mathcal{G}) = c + \frac{1}{2} \left(\frac{U - 1}{U + 1} \right)_{AB} x^A x^B. \quad (5.15)$$

In principle, the SpH structure of propagators allows one to compute the super-trace that gives HS observables according to the following recipe: (i) map boundary-to-bulk propagators Φ_1, \dots, Φ_n into elements $\mathcal{G}_1, \dots, \mathcal{G}_n$ of SpH by the inverse Cayley transform; (ii) compute the product $\mathcal{G} = \mathcal{G}_1 \diamond \dots \diamond \mathcal{G}_n$; (iii) take the central part of \mathcal{G} and map it back. This method fails, however, when (5.1) is such that $f^2 = I$.

Peculiarity of propagators. As we have already mentioned, boundary-to-bulk propagators (4.15) are not generic elements because the quadratic form f is involutory $f^2 = I$,

$$f = \begin{pmatrix} 0 & -F \\ -F^T & 0 \end{pmatrix}, \quad F \in Sp(2) \implies \det f = 1, \quad f^2 = I. \quad (5.16)$$

Its Cayley transform cannot be inverted to get the group element. The \star -product is still well-defined. Matrices f that are close to being a square root of a unit matrix give group elements that are close to infinity in a sense of having very large matrix elements. General formulae (5.6)-(5.8) being examined on the space of $\Phi(f, \xi, q)$, $f^2 = I$, reduce to

$$f_{1,2} = f_1 \circ f_2 = \frac{1}{f_1 + f_2}(2 + f_2 - f_1), \quad (5.17)$$

$$\xi_{1,2}^A = \frac{1}{2}\xi_1^B(1 + f_1 \circ f_2)_{B^A} + \frac{1}{2}\xi_2^B(1 - f_1 \circ f_2)_{B^A}, \quad (5.18)$$

$$q_{1,2} = \frac{1}{8}\{f_1, f_2\}_{\circ AB}(\xi_1^A \xi_1^B + \xi_2^A \xi_2^B) - \frac{1}{2}(1 + \frac{1}{2}[f_1, f_2]_{\circ})_{AB} \xi_1^A \xi_2^B. \quad (5.19)$$

We can view (5.17)-(5.19) as the extension of the group action to certain cell attached to SpH at infinity. The small-cells that are related by inversion combined with reflection cannot be attached to SpH simultaneously, for $f(-x/x^2) = -f$ and the denominator of (5.17) vanishes.

In order to compute correlation functions one can proceed in two ways at least. Either regularize boundary-to-bulk propagators to shift them to the big cell that is to make them generic, then compute using SpH , (5.4), and remove the regularization at the end. For example, one can duck the issue by redefining $f_i \rightarrow \epsilon f_i$, where ϵ is an arbitrary number $\epsilon \neq 1$. This makes Cayley transform well-defined and allows one to extract trace (4.9) as the limit of the inverse Cayley transform at $\epsilon \rightarrow 1$. In practice, the limit $\epsilon \rightarrow 1$ turns out to be difficult to compute and we found it simpler, to apply the small-cell rules (5.17)-(5.19) directly, leaving aside any regularization problems.

Since $f_{1,2}^2 = I$, eq. (5.17) defines a product on the space of square roots of unity. Its straightforward properties are

$$(f_1 \circ f_2)_{AB} = (f_1 \circ f_2)_{BA}, \quad (5.20)$$

$$f_1 \circ (f_2 \circ f_3) = (f_1 \circ f_2) \circ f_3, \quad (5.21)$$

$$f \circ f = f, \quad (5.22)$$

$$(-f_1) \circ (-f_2) = -f_2 \circ f_1, \quad (5.23)$$

$$f_1 \circ f_2 \circ f_3 = f_1 \circ f_3. \quad (5.24)$$

Associativity (5.21) is induced by the associativity of the \star -product. Last property (5.24) is due to the uniqueness of the \star -product element $F = \exp(\frac{1}{2}(f_- \circ f_+)_{AB} Y^A Y^B)$ which is annihilated by $Y^\pm = (1 \pm f_\pm)Y$ from the right and left for Y^+ and Y^- , respectively. This last 'forgetful' property (5.24) is very important and it will imply that only fields that are adjacent along the n -cycle will effect inside (4.16).

Let us note that despite seemingly $Sp(2)$ -origin of the propagator, i.e. a single Φ contains just $yF\bar{y}$ and not a generic YAY with $A^2 = I$, the product $\Phi_1 \star \dots \star \Phi_n$ falls into $Sp(4)$. Matrices of the particular form (5.16) belong to $Sp(2) \times$ split-complex numbers.

In practice it is convenient to rewrite (5.17)-(5.19) in terms of the following projectors

$$\pi_{ij}^\pm = \frac{1}{2}(1 \pm f_i \circ f_j), \quad \pi_{ij}^\pm \pi_{ij}^\pm = \pi_{ij}^\pm, \quad \pi_{ij}^\pm \pi_{ij}^\mp = 0, \quad (5.25)$$

which have the properties

$$\pi_{ij}^+ \pi_{ik}^+ = \pi_{ik}^+, \quad \pi_{ij}^- \pi_{ik}^+ = 0. \quad (5.26)$$

Using these projectors and (5.20)-(5.24) makes the calculation of the exponent of $\text{Tr}(\Phi_1 \star \dots \star \Phi_n)$ rather straightforward resulting in,

$$\text{Tr}(\Phi_1 \star \dots \star \Phi_n) \sim \exp i \left(\frac{1}{8} \sum_{j=1}^n (f_j \circ f_{j-1} + f_{j+1} \circ f_j)_{AB} \xi_j^A \xi_j^B + \frac{1}{4} \sum_{j=1}^n (1 + f_{j+1} \circ f_j)_{AB} \xi_{j+1}^A \xi_j^B \right), \quad (5.27)$$

where the sum is understood over $1 \dots n \pmod n$. One can see that only adjacent points contribute to the final result (the sum goes along the cycle) which is valid for $sp(2M)$ case as well. For $n = 2$, exponent (5.27) is given by (5.19), which appears to be exactly the same provided that the specific form of $\xi_A = (\xi_\alpha, 0)$, $\bar{\xi}_A = (0, \bar{\xi}_{\dot{\alpha}})$ is taken into account. Calculation of prefactor in (5.27) is much trickier and is left for Appendix B. The prefactor reads

$$\frac{1}{2^{M(n-1)} \sqrt[4]{\prod_{i=1}^n \det |f_i + f_{i+1}|}} \quad (5.28)$$

provided the following condition is met

$$\det |f_i \circ f_j + f_k \circ f_l| = \det |f_j \circ f_i + f_l \circ f_k| \quad (5.29)$$

being the case for propagators. Using (5.23) we see that eq. (5.29) makes the determinant invariant upon simultaneous sign flip $f_i \rightarrow -f_i$ for all i .

When n is odd the additional integration is needed (4.13) in extracting correlation functions. This holomorphic integration yields a prefactor to the determinants (5.28)

$$\frac{1}{\sqrt{\det(f_1 \circ f_n)_{\alpha\beta}}} \quad (5.30)$$

and a contribution to the exponent (5.27) of the form

$$-\frac{i}{2} (f_1 \circ f_n)_{\alpha\beta}^{-1} \xi_{1,n}^\alpha \xi_{1,n}^\beta. \quad (5.31)$$

As we will see, in both even and odd cases the arising structures are simply the conformal ones and the final result does not distinguish even and odd cases very much.

6 N -point functions

Conformal structures. Given a correlator of tensor operators $j(x_a, \eta_a)$ at points x_a , whose tensor structure is encoded in terms of polarization spinors η_a , it may depend on few conformally invariant quantities that involve η 's. This is on top of the ambiguity in functions of conformally invariant ratios $x_{12}x_{34}/(x_{14}x_{24})$.

There are two basic structures [25, 26] that may appear inside correlation functions. One structure depends on two points and two polarization spinors, $P_{ab} = -P_{ba}$, another one depends on three points and one polarization spinor (quadratic in it), $Q_{bc}^a = -Q_{cb}^a$, where indices a, b, \dots enumerate boundary sources.

We will present P_{ab} and Q_{bc}^a together with their 'bulk realization'. To do so⁸ let us define a - b points intertwinings F_{ab}

$$F_{ab}^{\alpha\beta} = F_a^{\alpha\dot{\alpha}} F_b^{\beta}_{\dot{\alpha}}, \quad \det(\epsilon - F_{ab}) = \frac{4x_{ab}^2}{(1+x_a^2)(1+x_b^2)}, \quad (6.1)$$

$$F_{ab\alpha}{}^\beta F_{ab}{}^{\alpha\chi} = \epsilon^{\beta\chi}, \quad F_{ab}{}^\alpha{}_\beta F_{ab}{}^{\chi\beta} = \epsilon^{\alpha\chi}, \quad (6.2)$$

then the conformal structures can be found to have the form

$$P_{ab} = 2\xi_b(\epsilon - F_{ab})^{-1}\xi_a \sim \eta_b x_{ab}^{-1}\eta_a, \quad (6.3)$$

$$Q_{bc}^a = 4\xi_a(\epsilon - F_{ca})^{-1}(\epsilon - F_{bc})(\epsilon - F_{ab})^{-1}\xi_a \sim \eta_a(x_{ab}^{-1} + x_{ca}^{-1})\eta_a. \quad (6.4)$$

These structures can be identified as building blocks of simplest correlators

$$\langle j_{s_1}(x_1, \eta_1) j_{s_2}(x_2, \eta_2) \rangle \sim \frac{1}{x_{12}^2} \delta_{s_1, s_2} (P_{12})^{s_1 + s_2}, \quad (6.5)$$

$$\langle j_{s_1}(x_1, \eta_1) j_0(x_2) j_0(x_3) \rangle \sim \frac{1}{x_{12} x_{23} x_{31}} (Q_{23}^1)^{2s_1}. \quad (6.6)$$

Let us note that one and the same conformal structure can be rewritten in several different ways in the bulk since both ξ and $\bar{\xi}$ originate from the same boundary spinor η . Thus defined conformal structures can be rewritten in terms of \circ -product as follows

$$Q_{bc}^a = \frac{1}{8}\xi_a(f_a \circ (-f_b) + (-f_c) \circ f_a)\xi_a, \quad P_{ab} = -\frac{1}{4}\xi_a(1 + (-f_b) \circ f_a)\xi_b, \quad (6.7)$$

where the origin of minus signs can be understood from alternation in (4.9). These are the building blocks of (5.27). Note, that

$$Q(\xi) = Q(\bar{\xi}), \quad P(\xi_a, \xi_b) = P(\bar{\xi}_a, \xi_b) = -P(\xi_a, \bar{\xi}_b) = -P(\bar{\xi}_a, \bar{\xi}_b). \quad (6.8)$$

2-point functions Let us discuss first the simplest case of two-point functions without any reference to the underlying projector structure of propagators. We first find

$$(12) \equiv \Phi(F_1, \xi_1) \star \Phi(F_2, \xi_2) = \frac{K_1 K_2}{|1 + F_{12}|} \exp i \left(-(F_2 \bar{y} + y + \xi_2)(1 + F_{12})^{-1}(F_1 \bar{y} - y + \xi_1) \right),$$

$$(\bar{1}\bar{2}) \equiv \Phi(F_1, \xi_1) \star \Phi(F_2, \bar{\xi}_2) = \frac{K_1 K_2}{|1 + F_{12}|} \exp i \left(-(F_2 \bar{y} + y)(1 + F_{12})^{-1}(F_1 \bar{y} - y + \xi_1 + F_1 \bar{\xi}) + \bar{\xi}_2 \bar{y} \right)$$

and a similar expressions for $\Phi(F_1, \bar{\xi}_1) \star \Phi(F_2, \bar{\xi}_2)$ and $\Phi(F_1, \bar{\xi}_1) \star \Phi(F_2, \xi_2)$. Next we change $F_2 \rightarrow -F_2$ and $\xi_2 \rightarrow -\xi_2$ and set $Y = 0$ to get $str(\Phi \star \tilde{\Phi})$

$$(12) = (\bar{1}\bar{2}) = \frac{1}{4x_{12}^2} \exp i 2P_{12}, \quad (\bar{1}\bar{2}) = (\bar{1}\bar{2}) = \frac{1}{4x_{12}^2}. \quad (6.9)$$

Finally we need to sum up all contributions and project onto the bosonic part, (4.16), resulting in

$$\langle j(x_1, \eta_1) j(x_2, \eta_2) \rangle = \frac{4}{x_{12}^2} (1 + \cos 2\theta \cos 2P_{12}), \quad (6.10)$$

⁸Since the observables do not depend on coordinate choice in the bulk we have set $x = 0$, $z = 1$ for simplicity reason.

where we recall that $\theta = 0$ and $\theta = \pi/2$ for free boundary bosons and fermions, respectively. Note that $\Delta(j_0) = 2$ in the free fermion vector model and $\langle j_0 j_0 \rangle$ is not reproduced by $\Delta = 1$ propagator.

Let us note that as usual within the AdS/CFT, two-point function can be extracted already from the boundary-to-bulk propagator. Indeed, taking $z \rightarrow 0$ one naively finds for the primary term of (3.8), that the singleton-antisingleton vacuum $\exp y_\alpha \bar{y}^\alpha$ suppresses all other contributions,

$$B \rightarrow z \exp y_\alpha \bar{y}^\alpha + O(z^2). \quad (6.11)$$

As it is explained in [8], equation (2.6), whose solution the propagator is, has a meaningful limit $z \rightarrow 0$ at the conformal boundary, which gives equations for conserved currents. The limiting procedure for the solutions is

$$B \rightarrow z \exp y_\alpha \bar{y}^\alpha T(yz^{\frac{1}{2}}, \bar{y}z^{\frac{1}{2}}). \quad (6.12)$$

Extracting T according to this prescription one finds

$$T = \exp y(x - x)^{-1} \eta, \quad (6.13)$$

which is a correct two-point function of conserved currents, where y plays the role of polarization spinor at point x . Then it is obvious that (3.2) holds true in the boundary limit.

Example: 3-point. We present only the primary term of (4.16), the rest can be generated by the action of the symmetric group S_3 and involutions ρ and π , whose action is described below. Using (5.27), (5.28) and (5.30), (5.31) we obtain

$$(123) = \frac{1}{16 x_{12} x_{23} x_{31}} \exp i \{ (Q_{32}^1 + Q_{13}^2 + Q_{21}^3) + (P_{12} + P_{23} - P_{31}) + 3\theta \}, \quad (6.14)$$

where we also used that $M = 2$ for $Sp(4)$ and the explicit form of f_i (5.16) which gives

$$\sqrt{\det |f_i - f_j|} = 4x_{ij}^2 K_i K_j. \quad (6.15)$$

Example: 4-point. Analogous computation gives

$$(1234) = \frac{1}{64 x_{12} x_{23} x_{34} x_{41}} \exp i \{ (Q_{42}^1 + Q_{13}^2 + Q_{24}^3 + Q_{31}^4) + (P_{12} + P_{23} + P_{34} - P_{41}) + 4\theta \}. \quad (6.16)$$

n -point. The calculation of the n -point function is different for even and odd n due to the difference in observables (4.9) and (4.10). This difference forces one to take into account additional terms (5.30) and (5.31) in odd case. The final result, however, has the unique closed form. To formulate the final result it is convenient to define

$$P_i \equiv P_{i,i+1}(-)^{\delta_{i,n}}, \quad Q_i \equiv Q_{i-1,i+1}^i, \quad Q = \sum_i Q_i, \quad (6.17)$$

where $(-)^{\delta_{j,n}}$ accounts for the sign flip of the last term, c.f. (6.14), (6.16). Then the primary $(12\dots n)$ term is (the sums and the product are understood over $1\dots n \pmod n$),

$$O_n(\Phi_1, \dots, \Phi_n) = (12\dots n) = \frac{1}{2^{2n-2} \prod_i |x_i - x_{i+1}|} \exp i \left\{ \sum_j Q_j + \sum_k P_k + n\theta \right\}. \quad (6.18)$$

Formally $2 - pt$ also fits into this formula.

To sum up to the correlation function according to (4.16) we have to know the action of ρ_i and π_i (4.14). Q -structure is left unaffected by ρ and π , while ρ_i changes the sign of P_{i-1} and θ , and π_i flips the sign of P_i and P_{i-1} , which follows from definitions (6.7) and properties (6.8). The sum over ρ^n gives

$$\sum_{\rho^n} O_n(\Phi_1, \dots, \Phi_n) = \frac{1}{2^{n-2} \prod_i |x_i - x_{i+1}|} \exp i \{Q\} \prod_k \cos(P_k + \theta). \quad (6.19)$$

The sum over π^n affects P_k 's only and can be evaluated using

$$\sum_{\pi^n} \prod_k \cos(P_k + \theta) = 2^n \cos^n \theta \prod_k \cos(P_k) + 2^n \sin^n \theta \prod_k \sin(P_k). \quad (6.20)$$

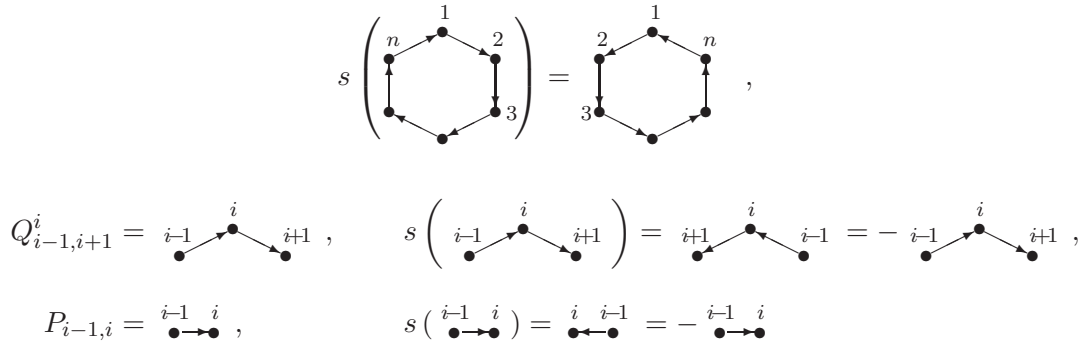
To sum up over S_n it is convenient to introduce the dihedral group, the symmetry group of a regular n -gon, D_n . It contains $2n$ elements, n rotations and n reflections. Elements of D_n map Q , P structures and the prefactor to themselves, while those of S_n/D_n generate new permutations. For example S_4/D_4 produces three terms of the 4-point functions of the scalar operators $j(x_i, 0)$

$$\langle j(x_1, 0) \dots j(x_4, 0) \rangle \sim (x_{12}x_{23}x_{34}x_{41})^{-1} + (x_{13}x_{32}x_{24}x_{41})^{-1} + (x_{12}x_{24}x_{43}x_{31})^{-1}. \quad (6.21)$$

It is easy to see that the rotations r of D_n do not affect Q , (6.17), $r(Q) = Q$ while reflections s flip the sign $s(Q) = -Q$. The same time for $P = \prod_k f_\sigma(P_k)$, where

$$f_n(x) = \begin{cases} \cos x, & n \text{ even;} \\ \sin x, & n \text{ odd,} \end{cases} \quad (6.22)$$

and f_n possesses a well-defined parity σ , $f_\sigma(-x) = (-)^\sigma f_\sigma(x)$, one finds $r(P) = P$ for rotations and $s(P) = (-)^{n\sigma} P$ for reflections. Therefore the sum over D_n affects Q 's producing $f_{n\sigma}(Q)$. It is useful to illustrate the reasoning above with the picture on which each three adjacent points of the n -gon correspond to the Q structure and each two to the P structure. Evidently rotations do not affect P and Q , and reflections produce a sign factor,



Finally, summing up all contributions we get the generating function of connected correlators

$$\langle j(x_1, \eta_1) \dots j(x_n, \eta_n) \rangle = \sum_{S_n} \frac{4}{\prod_i |x_i - x_{i+1}|} \left(\cos(Q) \cos^n \theta \prod_k \cos(P_k) + f_n(Q) \sin^n \theta \prod_k \sin(P_k) \right), \quad (6.23)$$

where $\theta = 0$ ($\theta = \pi/2$) for bosonic (fermionic) vector-model, respectively. The result is real and totally symmetric over n legs as it must be. Note, that sum (6.23) is over S_n and not

S_n/D_n at the price of counting the same terms $2n$ times. For example, one finds a familiar expressions, [4, 19], for the 3-point functions

$$\langle j(x_1, \eta_1) \dots j(x_3, \eta_3) \rangle_{boson} = \frac{4}{x_{12}x_{23}x_{31}} \cos(Q_{32}^1 + Q_{13}^2 + Q_{21}^3) \cos(P_{12}) \cos(P_{23}) \cos(P_{31}) \quad (6.24)$$

$$\langle j(x_1, \eta_1) \dots j(x_3, \eta_3) \rangle_{fermion} = \frac{4}{x_{12}x_{23}x_{31}} \sin(Q_{32}^1 + Q_{13}^2 + Q_{21}^3) \sin(P_{12}) \sin(P_{23}) \sin(P_{31}) \quad (6.25)$$

The 4-point function result is

$$\begin{aligned} \langle j(x_1, \eta_1) \dots j(x_4, \eta_4) \rangle_{boson} = & \frac{4}{x_{12}x_{23}x_{34}x_{41}} \cos(Q_{42}^1 + Q_{13}^2 + Q_{24}^3 + Q_{31}^4) \times \\ & \times \cos P_{12} \cos P_{23} \cos P_{34} \cos P_{41} + (1 \leftrightarrow 4) + (1 \leftrightarrow 2) \end{aligned} \quad (6.26)$$

$$\begin{aligned} \langle j(x_1, \eta_1) \dots j(x_4, \eta_4) \rangle_{fermion} = & \frac{4}{x_{12}x_{23}x_{34}x_{41}} \cos(Q_{42}^1 + Q_{13}^2 + Q_{24}^3 + Q_{31}^4) \times \\ & \times \sin P_{12} \sin P_{23} \sin P_{34} \sin P_{41} + (1 \leftrightarrow 4) + (1 \leftrightarrow 2) \end{aligned} \quad (6.27)$$

and the case $\langle j_2 j_0 j_0 j_0 \rangle_{boson}$ matches (6.9) of [2] provided the disconnected part is excluded. The generating function of correlators we found does not contain j_0 in the case of the free fermion theory, since it has weight 2 and is not covered by the $\Delta = s + 1$ propagator we used.

The correlation functions we found have a factorized form

$$\langle j \dots j \rangle_\theta = \langle j \dots j \rangle_{boson} \cos^n \theta + \langle j \dots j \rangle_{fermion} \sin^n \theta \quad (6.28)$$

Let us note that only for $\theta = 0, \pi/2$ the boundary fall-off of the propagator is preserved by the HS symmetry, [8]. Therefore, our results are meaningful only for $\theta = 0, \pi/2$ and there is no contradiction with the implications of a slightly broken HS symmetry that imply for 3-point functions [3]

$$\langle jjj \rangle_\theta = \langle jjj \rangle_{boson} \cos^2 \theta + \langle jjj \rangle_{fermion} \sin^2 \theta + \langle jjj \rangle_{odd} \cos \theta \sin \theta. \quad (6.29)$$

For generic θ (6.23) is just a generating function for correlation functions of conserved currents.

7 Conclusions

We found all correlation functions of the CFT that is dual to the Vasiliev HS theory in four dimensions with boundary conditions that do not break HS symmetries at all orders in perturbation theory. These are correlation functions of conserved currents of the free $O(N)$ vector model, either bosonic or fermionic one, [2].

CFT	HS
$\langle j \dots j \rangle$	$tr(\Phi \star \dots \star \Phi)$
$[Q, j] = \sum \partial \dots \partial j$	$\delta \Phi = [\Phi, \xi]$
$Q \langle j \dots j \rangle = 0$	$\delta tr(\Phi \star \dots \star \Phi) \equiv 0$

$Tr(\Phi \star \dots \star \Phi)$. The Ward identities are equivalent to the invariance of the trace under the adjoint transformations.

That the result has a simple form (6.23) of exponent of a linear combination of P , Q conformally invariant structures is a consequence of the fact that the propagator is a Gaussian in generating oscillators Y and the \star -product of two Gaussians is a Gaussian again and hence it is no more than quadratic in polarization spinors thus being linear in P 's and Q 's.

The observables O_n can also be viewed as n -point contact Witten diagrams, see discussion after (4.8). Therefore, O_n could be understood as the interaction vertices that are (i) fully determined by unbroken HS symmetry; (ii) sufficient to recover all correlators of the dual CFT.

The exact HS symmetry turns out to be powerful enough to restrict all correlation functions, [2]. It even allows one to find all correlation functions in a closed form as invariants of the HS symmetry. Therefore, HS algebra is a relevant replacement for the Virasoro one when trying to find exactly solvable CFT's in higher-dimensions. The important difference is that all CFT's with exact HS symmetry are free ones.

It opens the avenue to the study of broken HS symmetry, which may still render the model to remain solvable in some sense. Another choice of boundary conditions makes higher-spin symmetries broken by $O(1/N)$ effects. Promisingly, the higher-spin symmetry still restricts the form of correlation functions, [3]. The CFT_3 3-point functions may contain three different structures, (6.29), thus giving a freedom for two relative coefficients. These two turn out to be not independent, leaving only one free parameter which can be identified with the parameter θ in Vasiliev equations. It is interesting to trace explicitly the way the HS symmetry restricts correlation functions of CFT's that slightly break HS symmetries.

It would be instructive to extend the results of this paper to various free CFT's: (i) free bosons in d -dimensions, where boundary-to-bulk propagators for Vasiliev master fields have been recently found in [37]; (ii) free fermions in d ; (iii) free $\mathcal{N} = 4$ SYM, where the projector structure of boundary to bulk propagators could be quite interesting and the implications of HS symmetry when the interactions are turned on remain to be seen; (iv) higher-spin singletons or (anti)-self dual fields in $d = 2n$ dimensions [46].

At present we see three ways of computing correlation functions in the Vasiliev theory:

1. One may use the S -matrix approach solving bulk equations to the n -th order where the solution $B_n(x, z)$

$$B_n(x, z) = \int K_n [B(x_1), \dots, B(x_n)] \quad (7.1)$$

is given by some complicated integral kernel K_n acting on the product of initial data $B(x_i)$ with sources at boundary points x_i . At the end $(n + 1)$ -correlation functions are extracted by taking $z \rightarrow 0$ limit of the solution. This program was performed for 3-point functions in [18, 19].

2. One may compute fair observables [4, 23], i.e. insert solutions $B_n(x, z)$ up to the n -th order into observables $O(B, \dots, B)$, which again calls for a good knowledge of K_n . This completion by nonlinear corrections is a part of the general framework for HS quantum theory of [4, 22–24, 41, 42]

3. One may compute observables (4.9), (4.10) to the leading order, which are fully and explicitly governed by the HS symmetry. This is the simplest of the three and allows one to find all correlation functions (6.23) when higher-spin symmetry is unbroken.

Recent result of [8] indicates that for the boundary conditions conjecturally corresponding to free CFT's the HS symmetry seems to remain unbroken to all orders, which implies, technical detail to be yet clarified though, that No.1 and No.2 are quite long ways to arrive at No.3, which is in accordance with [2]. This is also in accordance with [4], where the first correction to the observables due to HS interactions in the bulk has been found to coincide with the leading term. The correlators found in this paper, (6.23), are point-split and require certain contact terms to be added at coincident points. As was noted in [4] the HS theory may also deliver such contact terms.

The bulk point which is connected to the boundary current and the bulk coordinates themselves can be easily changed by a large twisted-adjoint rotation (2.13). The observables we calculated are manifestly invariant under such transformation. Therefore, once certain general and coordinate-invariant properties of the propagators are understood, there is no need to make any reference to *AdS*. In particular one may try to take the boundary limit $z \rightarrow 0$ for propagators first and then compute the observables 'on the boundary'.

Interestingly, most of the calculations in the present paper do not rely on space-time dimension being four and are formally valid for any $Sp(2M)$, viewing $Sp(2M)$ as a generalized conformal symmetry, [40, 45, 47–51]. On this symplectic way our work generalizes [52], where two and three point correlation functions of $Sp(2M)$ -scalar and vector fields were found.

Another possible application of the elaborated technique can be the calculation of n -point interaction vertices and Neumann coefficients in string field theory along the lines of Moyal formulation [53, 54].

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A Boundary-to-bulk propagators

The right choice of the ansatz for the propagator proceeds from: (i) $B(Y = 0)$ is just a weight $\Delta = 1$ scalar field, propagator for which is well-known; (ii) the derivatives $D_{\alpha\dot{\alpha}}$ of the fields correspond to expansion in translation generator $T_{\alpha\dot{\alpha}}F^{\alpha\dot{\alpha}}$ with yet unknown $F^{\alpha\dot{\alpha}}$; (iii) the boundary polarization spinor η must be transported to the bulk spinor ξ by certain parallel transport bispinor $\xi^\alpha = \Pi^{\alpha\beta}\eta_\beta$, *idem.* for $\bar{\xi}$. This suggests the ansatz to be

$$u = T_{\alpha\dot{\alpha}}F^{\alpha\dot{\alpha}}, \quad v = i\xi^\alpha y_\alpha, \quad \bar{v} = i\bar{\xi}^{\dot{\alpha}}\bar{y}_{\dot{\alpha}}, \quad B = K f(u, v) + K f(u, \bar{v}). \quad (\text{A.1})$$

Eq. (2.6) leads to the following set of equations ($h^{\alpha\dot{\alpha}} = \Omega^{\alpha\dot{\alpha}}$ is a vierbein)

$$d \ln K + \frac{1}{2} F_{\alpha\dot{\alpha}} h^{\alpha\dot{\alpha}} \partial_u \ln f = 0 \quad (DF^{\alpha\dot{\alpha}} \partial_u + 2h^{\alpha\dot{\alpha}} + \frac{1}{2} F^{\alpha}_{\dot{\gamma}} h^{\delta\dot{\gamma}} F_{\delta}^{\dot{\alpha}} \partial_u^2) f = 0, \quad (\text{A.2})$$

$$\left(D\xi^\alpha + \frac{1}{2} F^{\alpha}_{\dot{\gamma}} \xi_\delta h^{\delta\dot{\gamma}} \partial_u \right) \partial_v f = 0, \quad \left(D\xi^{\dot{\alpha}} + \frac{1}{2} F_{\delta}^{\dot{\alpha}} \xi_\gamma h^{\delta\dot{\gamma}} \partial_u \right) \partial_{\bar{v}} f = 0. \quad (\text{A.3})$$

(A.2) determines F up to some factor and gives $f = e^{-2u} h(v)$. The rest of the equations can be easily solved by looking only at the components along dz . In particular it is obvious that the dependence on v can be arbitrary, which just encodes a freedom in normalization of propagators of all spins independently. It is convenient to choose $h(v) = e^v$.

B Determinants

To calculate prefactor (5.28), it is convenient to consider even $n \rightarrow 2n$. The result for odd n is reproduced from even case by setting say $f_2 = f_1$. Indeed, we can set all polarizations to zero, for they do not effect the determinants, and all Φ_i are projectors $\Phi_i \star \Phi_i \sim \Phi_i$.

The determinants arising from \star -product can be rewritten in terms of \circ -product as follows

$$\det \left| \frac{1}{1 + f_1 f_2} \right| = \det \left| \frac{1}{f_1 + f_2} \right| = \frac{1}{2^{4M}} \det |f_1 \circ f_2 + f_2 \circ f_1| \quad (\text{B.1})$$

Applying this identity for $\det |f_1 \circ f_2 + f_3 \circ f_4|$ and using (5.24) we obtain

$$\det |f_1 \circ f_2 + f_3 \circ f_4| = \frac{2^{4M}}{\det |f_1 \circ f_4 + f_3 \circ f_2|}. \quad (\text{B.2})$$

It is convenient to group all terms within the trace into pairs $(\Phi_1 \star \Phi_2) \dots (\Phi_{2n-1} \star \Phi_{2n})$. Taking the trace, for the determinant one obtains

$$X^2 = \prod_{i=1}^{n-1} \frac{1}{\det |f_{2i-1} + f_{2i}|} \prod_{i=1}^{n-1} \frac{\det |f_1 \circ f_{2i+2} + f_{2i+1} \circ f_{2i}|}{2^{4M}} \quad (\text{B.3})$$

Now, property (5.29) peculiar for the propagators states that the determinant is invariant under simultaneous sign flip $f_i \rightarrow -f_i$ for all i . This is equivalent to the statement that the prefactor in $\text{Tr}(\Phi_1 \star \Phi_2) \dots (\Phi_{2n-1} \star \Phi_{2n})$ is equal to the one taken in the reverse order $\text{Tr}(\Phi_{2n} \star \Phi_{2n-1}) \dots (\Phi_2 \star \Phi_1)$. Equating both expression one arrives at the following series of identities

$$\prod_{i=1}^{n-1} \det |f_1 \circ f_{2i+2} + f_{2i+1} \circ f_{2i}| = \prod_{i=1}^{n-1} \det |f_{2n} \circ f_{2i-1} + f_{2i} \circ f_{2i+1}|. \quad (\text{B.4})$$

Particularly, for $n = 2$ we get

$$\det |f_1 \circ f_2 + f_3 \circ f_4| = \det |f_2 \circ f_1 + f_4 \circ f_3|. \quad (\text{B.5})$$

Now, we redefine cyclically all $f_i \rightarrow f_{i+1}$ and multiply both expressions (B.3) with each other. The result is

$$\prod_{i=1}^{2n} \det |f_i + f_{i+1}| X^4 = \prod_{i=1}^{n-1} \frac{\det |f_1 \circ f_{2i+2} + f_{2i+1} \circ f_{2i}|}{2^{4M}} \prod_{i=1}^{n-1} \frac{\det |f_2 \circ f_{2i+3} + f_{2i+2} \circ f_{2i+1}|}{2^{4M}} \quad (\text{B.6})$$

Final step is to apply (B.4) and (B.2) to the first product in the r.h.s. of (B.6)

$$\prod_{i=1}^{2n} \det |f_i + f_{i+1}| X^4 = \prod_{i=1}^{n-1} \frac{\det |f_2 \circ f_{2i+3} + f_{2i+2} \circ f_{2i+1}|}{\det |f_{2n} \circ f_{2i+1} + f_{2i} \circ f_{2i-1}| 2^{4M}} \quad (\text{B.7})$$

Note, that the denominator in (B.7) is that of nominator shifted cyclically by two steps. Therefore both cancel each other yielding the final result

$$X_{2n} = \frac{1}{2^{M(n-1)} \sqrt[4]{\prod_{i=1}^{2n} \det |f_i + f_{i+1}|}} \quad (\text{B.8})$$

Finally, one has to flip the sign of each f_{2i} in accordance with $\tilde{\Phi}$ in (4.9).

References

- [1] A. Belavin, A. M. Polyakov and A. Zamolodchikov, *Infinite Conformal Symmetry in Two-Dimensional Quantum Field Theory*, *Nucl.Phys.* **B241** (1984) 333–380.
- [2] J. Maldacena and A. Zhiboedov, *Constraining Conformal Field Theories with A Higher Spin Symmetry*, **1112.1016**.
- [3] J. Maldacena and A. Zhiboedov, *Constraining conformal field theories with a slightly broken higher spin symmetry*, **1204.3882**.
- [4] N. Colombo and P. Sundell, *Higher Spin Gravity Amplitudes From Zero-form Charges*, **1208.3880**.
- [5] J. M. Maldacena, *The large N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231–252 [[hep-th/9711200](#)].
- [6] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B428** (1998) 105–114 [[hep-th/9802109](#)].
- [7] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253–291 [[hep-th/9802150](#)].
- [8] M. A. Vasiliev, *Holography, Unfolding and Higher-Spin Theory*, **1203.5554**.
- [9] M. A. Vasiliev, *Consistent equation for interacting gauge fields of all spins in $(3+1)$ -dimensions*, *Phys. Lett.* **B243** (1990) 378–382.
- [10] M. A. Vasiliev, *Properties of equations of motion of interacting gauge fields of all spins in $(3+1)$ -dimensions*, *Class. Quant. Grav.* **8** (1991) 1387–1417.
- [11] M. A. Vasiliev, *More on equations of motion for interacting massless fields of all spins in $(3+1)$ -dimensions*, *Phys. Lett.* **B285** (1992) 225–234.
- [12] M. A. Vasiliev, *Higher-spin gauge theories in four, three and two dimensions*, *Int. J. Mod. Phys.* **D5** (1996) 763–797 [[hep-th/9611024](#)].
- [13] M. A. Vasiliev, *Higher spin gauge theories: Star-product and ads space*, [hep-th/9910096](#).
- [14] M. A. Vasiliev, *Nonlinear equations for symmetric massless higher spin fields in $(a)ds(d)$* , *Phys. Lett.* **B567** (2003) 139–151 [[hep-th/0304049](#)].
- [15] E. Sezgin and P. Sundell, *Massless higher spins and holography*, *Nucl.Phys.* **B644** (2002) 303–370 [[hep-th/0205131](#)].
- [16] I. R. Klebanov and A. M. Polyakov, *AdS dual of the critical $O(N)$ vector model*, *Phys. Lett.* **B550** (2002) 213–219 [[hep-th/0210114](#)].
- [17] M. Flato and C. Fronsdal, *One Massless Particle Equals Two Dirac Singletons: Elementary Particles in a Curved Space. 6.*, *Lett.Math.Phys.* **2** (1978) 421–426.
- [18] S. Giombi and X. Yin, *Higher Spin Gauge Theory and Holography: The Three-Point Functions*, *JHEP* **1009** (2010) 115 [[0912.3462](#)].

- [19] S. Giombi and X. Yin, *Higher Spins in AdS and Twistorial Holography*, *JHEP* **1104** (2011) 086 [1004.3736].
- [20] E. Sezgin and P. Sundell, *An Exact solution of 4-D higher-spin gauge theory*, *Nucl.Phys.* **B762** (2007) 1–37 [hep-th/0508158].
- [21] M. A. Vasiliev, *Extended higher spin superalgebras and their realizations in terms of quantum operators*, *Fortsch. Phys.* **36** (1988) 33–62.
- [22] E. Sezgin and P. Sundell, *Geometry and Observables in Vasiliev’s Higher Spin Gravity*, *JHEP* **1207** (2012) 121 [1103.2360].
- [23] N. Colombo and P. Sundell, *Twistor space observables and quasi-amplitudes in 4D higher spin gravity*, *JHEP* **1111** (2011) 042 [1012.0813].
- [24] E. Sezgin and P. Sundell, *Supersymmetric Higher Spin Theories*, 1208.6019.
- [25] M. S. Costa, J. Penedones, D. Poland and S. Rychkov, *Spinning Conformal Correlators*, *JHEP* **1111** (2011) 071 [1107.3554].
- [26] S. Giombi, S. Prakash and X. Yin, *A Note on CFT Correlators in Three Dimensions*, 1104.4317.
- [27] E. S. Fradkin and M. A. Vasiliev, *Candidate to the role of higher spin symmetry*, *Ann. Phys.* **177** (1987) 63.
- [28] S. E. Konstein and M. A. Vasiliev, *Massless representations and admissibility condition for higher spin superalgebras*, *Nucl. Phys.* **B312** (1989) 402.
- [29] S. E. Konstein and M. A. Vasiliev, *Extended higher spin superalgebras and their massless representations*, *Nucl. Phys.* **B331** (1990) 475–499.
- [30] X. Bekaert, S. Cnockaert, C. Iazeolla and M. Vasiliev, *Nonlinear higher spin theories in various dimensions*, hep-th/0503128.
- [31] M. A. Vasiliev, *Equations of motion of interacting massless fields of all spins as a free differential algebra*, *Phys. Lett.* **B209** (1988) 491–497.
- [32] M. A. Vasiliev, *Consistent equations for interacting massless fields of all spins in the first order in curvatures*, *Annals Phys.* **190** (1989) 59–106.
- [33] S. Giombi and X. Yin, *The Higher Spin/Vector Model Duality*, 1208.4036.
- [34] E. Sezgin and P. Sundell, *Holography in 4D (super) higher spin theories and a test via cubic scalar couplings*, *JHEP* **0507** (2005) 044 [hep-th/0305040].
- [35] S. Giombi, S. Minwalla, S. Prakash, S. P. Trivedi, S. R. Wadia *et. al.*, *Chern-Simons Theory with Vector Fermion Matter*, *Eur.Phys.J.* **C72** (2012) 2112 [1110.4386].
- [36] C.-M. Chang, S. Minwalla, T. Sharma and X. Yin, *ABJ Triality: from Higher Spin Fields to Strings*, 1207.4485.
- [37] V. Didenko and E. Skvortsov, *Towards higher-spin holography in ambient space of any dimension*, 1207.6786.

- [38] P. Kraus and E. Perlmutter, *Probing higher spin black holes*, 1209.4937.
- [39] M. Soloviev, *Generalized Weyl correspondence and Moyal multiplier algebras*, *Theor.Math.Phys.* **173** (2012) 13591376.
- [40] O. Gelfond and M. Vasiliev, *$Sp(8)$ invariant higher spin theory, twistors and geometric BRST formulation of unfolded field equations*, *JHEP* **0912** (2009) 021 [0901.2176].
- [41] N. Boulanger and P. Sundell, *An action principle for Vasiliev's four-dimensional higher-spin gravity*, *J.Phys.* **A44** (2011) 495402 [1102.2219].
- [42] N. Boulanger, N. Colombo and P. Sundell, *A minimal BV action for Vasiliev's four-dimensional higher spin gravity*, *JHEP* **1210** (2012) 043 [1205.3339].
- [43] V. Didenko and M. Vasiliev, *Static BPS black hole in 4d higher-spin gauge theory*, *Phys.Lett.* **B682** (2009) 305–315 [0906.3898].
- [44] V. Didenko and M. Vasiliev, *Free field dynamics in the generalized AdS (super)space*, *J.Math.Phys.* **45** (2004) 197–215 [hep-th/0301054].
- [45] O. Gelfond and M. Vasiliev, *Unfolding Versus BRST and Currents in $Sp(2M)$ Invariant Higher-Spin Theory*, 1001.2585.
- [46] X. Bekaert and M. Grigoriev, *Manifestly conformal descriptions and higher symmetries of bosonic singletons*, *SIGMA* **6** (2010) 038 [0907.3195].
- [47] M. Vasiliev, *Relativity, causality, locality, quantization and duality in the $Sp(2M)$ invariant generalized space-time*, hep-th/0111119.
- [48] M. Vasiliev, *Higher spin conserved currents in $Sp(2M)$ symmetric space-time*, *Russ.Phys.J.* **45** (2002) 670–681 [hep-th/0204167].
- [49] O. Gelfond, E. Skvortsov and M. Vasiliev, *Higher spin conformal currents in Minkowski space*, *Theor.Math.Phys.* **154** (2008) 294–302 [hep-th/0601106].
- [50] O. Gelfond and M. Vasiliev, *Higher Spin Fields in Siegel Space, Currents and Theta Functions*, *JHEP* **0903** (2009) 125 [0801.2191].
- [51] O. Gelfond and M. Vasiliev, *Unfolded Equations for Current Interactions of 4d Massless Fields as a Free System in Mixed Dimensions*, 1012.3143.
- [52] M. Vasiliev and V. Zaikin, *On $Sp(2M)$ invariant Green functions*, *Phys.Lett.* **B587** (2004) 225–229 [hep-th/0312244].
- [53] I. Bars, *Map of Witten's $*$ to Moyal's $*$* , *Phys.Lett.* **B517** (2001) 436–444 [hep-th/0106157].
- [54] I. Bars, *MSFT: Moyal star formulation of string field theory*, hep-th/0211238.