

Exact higher-spin symmetry in CFT: Free fermion correlators from Vasiliev theory

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N -point correlation functions of conserved currents and weight-two scalar operators of the three-dimensional free fermion vector model are found as invariants of the higher-spin symmetry in four-dimensional AdS. These are the correlators of the unbroken Vasiliev higher-spin theory. The results extend the recent work arXiv:1210.7963 and are complementary to arXiv:1301.3123 where the correlators were computed entirely on the boundary.

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I. INTRODUCTION

Higher-spin theories constructed by Vasiliev [1–6] have attracted much attention [7–14] as simple models of AdS/CFT [15–17], with the CFT duals being vector models, some of which are of practical value, e.g., the critical $O(N)$ model [18–20].

The higher spin (HS) symmetry is an infinite-dimensional extension of the conformal symmetry and is strong enough to fix the form of all correlation functions [21]. However, the Maldacena-Zhiboedov theorem [21], which can be thought of as an extension of the Coleman-Mandula no-go theorem, tells us that under some mild assumptions such as unitarity, locality¹ and operator product expansion (OPE) it is impossible to have interacting CFTs that allow infinitely many conserved HS charges. In other words, if the bulk theory admits such a boundary behavior that leaves HS symmetry unbroken, the corresponding dual CFT is free. On the other hand, the extension of the Maldacena-Zhiboedov result [22] shows that broken HS symmetry still restricts correlation functions and thus can be effectively used as a source of nontrivial integrable models once the mechanism of breaking is understood. As a starting point it is required to understand how the exact HS symmetry can be used to efficiently determine the form of correlation functions.

Recently by using the HS symmetry as a higher-dimensional replacement of the Virasoro algebra, the correlation functions of all orders of conserved currents in the three-dimensional CFTs that have exact HS symmetry have been found in [23]. By the Maldacena-Zhiboedov theorem these are free theories, either free boson or free fermion. The main goal of [23] was to give an explicit formula for all correlators relying on symmetry requirements only.

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¹Let us note that in context of HS theories the notion of local CFTs should be treated with great care as the bulk theories are nonlocal.

The constructive formula for the n -point correlation function proposed in [23] reads

$$\langle j(x_1, \eta_1) \cdots j(x_n, \eta_n) \rangle = \sum_{S_n} \text{Tr}(\Psi(x, x_1, \eta_1) \star \cdots \star \Psi(x, x_n, \eta_n)), \quad (1)$$

and it is analogous to the definition of long-trace operators, with the difference being that the trace is taken in the infinite-dimensional HS algebra rather than in $SU(N)$. Let us now explain the constituents of (1) in detail. On the CFT side we focus on symmetric and traceless tensors, $j_{a_1 \cdots a_s}$, which are in addition conserved $\partial^m j_{ma_2 \cdots a_s} = 0$ and thus are primary fields to be referred to as currents. It is convenient to pack all currents $j_{a_1 \cdots a_s}$ into a generating function,

$$j(x, \eta) = \sum_s j_{a_1 \cdots a_s} \eta^{a_1} \cdots \eta^{a_s} = \sum_s j_{\alpha_1 \cdots \alpha_s} \eta^\alpha \cdots \eta^\alpha, \quad (2)$$

where η^a 's are null polarization vectors, $\eta^a \eta_a = 0$. In $3d$ the spinor language has a great advantage and instead of lightlike vector η^a we introduced two-component spinor η^α . Then, given the left-hand side of (1), one can extract the n -point correlation function of some particular currents of spins s_1, \dots, s_n as the order $2s_1, \dots, 2s_n$ Taylor coefficient of $\eta_1^\alpha, \dots, \eta_n^\alpha$. The definition of the right-hand side of (1) requires a certain AdS/CFT inspired technique that was laid down in [11, 24, 25]. The key object $\Psi(x, x_i, \eta_i)$ is the Fourier transform with respect to some auxiliary variable of the boundary-to-bulk propagator for the master field strength of the Vasiliev HS theory in AdS₄. It depends on the AdS₄ coordinate x ; on the boundary point x_i where the current $j(x_i, \eta_i)$ is inserted; on the polarization η_i^α that encodes the index structure; on the auxiliary variable Y that generates HS algebra and was left implicit.

There are a few important facts about $\Psi(x, x_i, \eta_i)$. Firstly, it behaves like a (set of) conserved current with respect to x_i and η_i . Secondly, it transforms in the adjoint of the HS algebra,

$$\delta \Psi = [\Psi, \xi]_\star, \quad (3)$$

where \star denotes the product with respect to Y in the HS algebra. It is also assumed that the HS algebra admits a trace, which has the right property to make (1) invariant under all HS transformations (3). In particular, since the conformal algebra is itself a subalgebra of the HS algebra, (1) is conformally invariant and behaves as a conserved current in each slot. From the bulk point of view, a large adjoint transformation $\Psi \rightarrow g^{-1}(x) \star \Psi \star g(x)$, where $g(x)$ occupies only the $SO(3, 2)$ subgroup, allows one to move x freely in the AdS. Therefore, the dependence on the bulk point x drops out of (1). Lastly, the sum over the symmetric group makes the result symmetric in its arguments. From the bulk point of view the sum is necessary to have the trace (1) real, that is the symmetrization is driven by appropriate reality conditions for master fields.

One can also view (1) as originating from a Witten-like diagram (see Fig. 1) for a vertex

$$V_n = \text{Tr}(\Psi \star \cdots \star \Psi). \tag{4}$$

Since the trace does not depend on the interaction point x in the bulk the integral over AdS drops out. The sum of such diagrams seems to be what the Vasiliev theory reduces to with the boundary conditions that do not break HS symmetry.

Let us note that the formula (1) is quite general and can be applied to any CFT with HS symmetry, e.g. free scalar and boson in d dimensions, free limit of super Yang-Mills, generalized free fields and perhaps to the duals of $3d$ HS theories, etc.

According to [19,20] the currents $j_{a_1 \cdots a_s}$ should be originated either from free scalar ϕ or from free fermion ψ . These are the conformal primaries appearing in the OPE $\phi \times \phi$ or $\psi \times \psi$. The only difference between free boson and free fermion at the level of currents $j_{a_1 \cdots a_s}$ is that

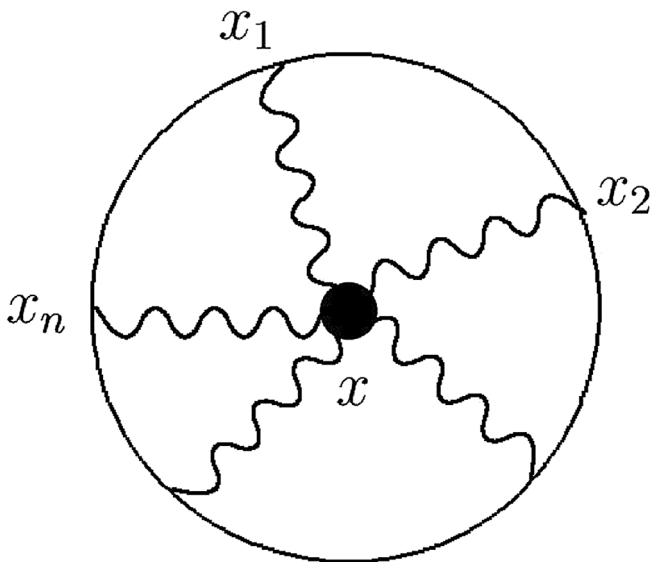


FIG. 1. Witten diagram in twistor space.

the first member of the family j_0 , $j_0 = \phi^2$ or $j_0 = \psi^2$, which is not a genuine current, may have different conformal weights, $\Delta = 1$ or $\Delta = 2$ depending on whether it is made of a boson or a fermion, respectively. In [23] only the case with $\Delta = s + 1$ boundary conditions was considered, while the case of the $\Delta = 2$ operator, which we will refer to as \tilde{j}_0 , was not included. Therefore, the part of the correlators of the free fermion model were not reproduced. This is the gap we would like to fill in the present paper, so we would like to compute various correlators of the form

$$\langle \tilde{j}_0 \cdots \tilde{j}_0 j(x_1, \eta_1) \cdots j(x_n, \eta_n) \rangle. \tag{5}$$

The results of the paper shows that (1) works well, giving all correlators of the operators that are dual to the higher-spin multiplet in the bulk of AdS_4 . Our results are closely related to the recent paper [26] by Gelfond and Vasiliev. In [26] the operator product algebra of free boson and free fermion was found explicitly and then used to compute all the correlators. The advantage of having the operator algebra at hand is that the overall prefactors, which are left undetermined in our approach, can be fixed in terms of N (N is the number of free fields in the multiplet). Complementary to [26] our method relies on the AdS/CFT and provides a link between the computations entirely in the bulk and on the boundary. We expect that our basic formula (1) is the prolongation of [26] to the bulk of AdS. At the same time our work and [23] are similar to [26] in a sense that both approaches are in fact $sp(2M)$ covariant with $M = 2$ for the case of interest AdS_4/CFT_3 since $so(3, 2) \sim sp(4)$. Particularly, using this fact the authors of [26] were able to straightforwardly generalize their results to reproduce correlation functions of $4d$ conserved currents as being realized via embedding into $sp(8)$. These formally coincide² with those of [23] as the basic building blocks, the conformally invariant structures, remain the same as well as the general $sp(2M)$ formula of [23] used to derive the correlation functions.

The outline of the paper is as follows. For the reader's convenience, we summarize our key results in the next section and then present our derivation in the following sections. In particular, we derive the generating functions and correlation functions in Sec. III. The conclusions are in Sec. IV. Certain technicalities are collected in the Appendix.

In this paper we use the same strategy and techniques as developed in [23], where all the results for $\Delta = s + 1$ have been obtained. For the reader's convenience, in our derivation we will quote the main results of [23] and put them side by side with the new results obtained involving \tilde{j}_0 .

²We mean that the dependence on the tensor conformally invariant structures is the same, but the prefactors $|x_i - x_j|^{-\delta}$ come with powers δ that depend on d of course.

TABLE I. Conformal structures. Note R is not independent from P , Q or S . Coordinates on the boundary are parametrized by symmetric bispinors $\mathbf{x} \equiv (\mathbf{x}^{\alpha\beta}) = (\mathbf{x}^{\beta\alpha})$ and we have chosen to suppress the spinor indices. The subscript on \mathbf{x}_i refers to the i th point at which some operator is inserted, η_i is the boundary polarization, $\mathbf{x}_{ij} = \mathbf{x}_i - \mathbf{x}_j$ and $x_{ij} = \sqrt{-\det \mathbf{x}_{ij}}$ is the distance between the i th point and the j th point. Parity is determined under the transformation $\mathbf{x} \rightarrow -\mathbf{x}$ and $\eta \rightarrow i\eta$, with positive being even and negative being odd.

Conformal structures	Explicit coordinate representation	Number of points	Parity
P_{ij}	$\eta_i \mathbf{x}_{ij}^{-1} \eta_j$	2	Even
Q_{jk}^i	$\eta_i [\mathbf{x}_{ij}^{-1} - \mathbf{x}_{ik}^{-1}] \eta_j$	3	Even
S_{ik}^j	$\frac{\eta_i \mathbf{x}_{ij} \mathbf{x}_{jk} \eta_k}{x_{ij} x_{jk} x_{ki}}$	3	Odd
R_{ij}	$\eta_i \mathbf{x}_{i(i+1)}^{-1} \cdots \mathbf{x}_{(j-1)j}^{-1} \eta_j$	$j - i + 1$	$(-)^{j-i+1}$

II. MAIN RESULTS: EXAMPLES

In the Vasiliev higher-spin theory there is a free parameter³ θ that allows one to interpolate smoothly between the duals. Our results correspond to the free boson at $\theta = 0$ and free fermion at $\theta = \pi/2$. We prefer to keep θ everywhere. The correlators we consider are connected correlators and they are defined up to an overall factor, which cannot be fixed in (1).

Before going to the examples, note for tensor operators it is convenient to use conformally invariant structures, i.e. P , Q and S , as introduced in [27,28]. However, we find that starting from two \tilde{j}_0 insertions it is no longer convenient to make use of the S structure, especially in correlators of orders higher than three. Instead, we find it more convenient to introduce a new object R . Some basic properties of these structures are listed in Table I. In our calculation, we will need the following (they only differ from those in Table I by numerical factors):

$$\begin{aligned}
Q^i &= -\frac{1}{8} \eta_i [\mathbf{x}_{i(i+1)}^{-1} - \mathbf{x}_{i(i-1)}^{-1}] \eta_i, \\
P_{i(i+1)} &= \frac{1}{4} \eta_i \mathbf{x}_{i(i+1)}^{-1} \eta_{i+1}, \quad 1 \leq i \leq n-1, \\
P_{01} &= P_{n(n+1)} = P_{n1} = -\frac{1}{4} \eta_n \mathbf{x}_{n1}^{-1} \eta_1, \\
R_{jk} &= -\frac{c}{2(2i)^{k-j}} \eta_j \mathbf{x}_{j(j+1)}^{-1} \cdots \mathbf{x}_{(k-1)k}^{-1} \eta_k, \\
R_{0j} &= R_{nj}, \quad R_{j(n+1)} = R_{j0}, \quad \forall j,
\end{aligned} \tag{6}$$

where n is the order of correlation functions. The meaning of the above definitions is that P , Q , S and R structures involve a pair or a triple of the points adjacent along the cycle $1, 2, \dots, n$ for the n -point function. Note that

³There are infinitely many free constants in the parity-violating Vasiliev theory, θ being the first of them. We expect that the effect of the rest of the constants is just to renormalize θ .

$$R_{jk} \sim \eta_j \mathbf{x}_{j(j+1)}^{-1} \cdots \mathbf{x}_{(n-1)n}^{-1} \mathbf{x}_{n1}^{-1} \cdots \mathbf{x}_{(k-1)k}^{-1} \eta_k \tag{7}$$

if $j > k$ (after the replacement $R_{0j} \rightarrow R_{nj}$ or $R_{j(n+1)} \rightarrow R_{j0}$, if any). Finally, $c = 1$ if R_{jk} does not contain \mathbf{x}_{n1}^{-1} and $c = -1$ otherwise.

In the examples below j_s refers to the insertion of $j_{a_1 \dots a_s}$, i.e. $\Delta = s + 1$ operator. In particular the generating function corresponding to j_s contains $(\Delta = 1)$ -operator $j_0 = \phi^2$. The insertion of the $(\Delta = 2)$ operator ψ^2 is denoted by \tilde{j}_0 . Of course, all correlators involving j_0 and \tilde{j}_0 simultaneously must vanish.

A. Two-point functions

Two point functions are fixed by conformal symmetry up to a number:

$$\begin{aligned}
\langle j_s j_s \rangle &\propto \frac{1}{x_{12}^2} (\cos^2 \theta \cos^2 P_{12} + \sin^2 \theta \sin^2 P_{12}), \\
\langle \tilde{j}_0 \tilde{j}_0 \rangle &\propto \frac{1}{x_{12}^4}.
\end{aligned} \tag{8}$$

B. Three-point functions

Three point functions are known to be fixed up to a number of constants:

$$\begin{aligned}
\langle j_s j_s j_s \rangle &\propto \frac{1}{x_{12} x_{23} x_{31}} [\cos^3 \theta \cos(Q^1 + Q^2 + Q^3) \\
&\quad \times \cos P_{12} \cos P_{23} \cos P_{31} + i \sin^3 \theta \\
&\quad \times \sin(Q^1 + Q^2 + Q^3) \sin P_{12} \sin P_{23} \sin P_{31}], \\
\langle \tilde{j}_0 j_s j_s \rangle &\propto \sin^2 \theta \frac{\cos(Q^2 + Q^3)}{x_{12} x_{23} x_{31}} R_{02} \sin P_{23}, \\
\langle \tilde{j}_0 \tilde{j}_0 j_s \rangle &\propto \sin \theta \frac{\sin Q^3}{x_{12} x_{23} x_{31}} R_{03}, \quad \langle \tilde{j}_0 \tilde{j}_0 \tilde{j}_0 \rangle = 0.
\end{aligned} \tag{9}$$

The first formula, borrowed from [11,23] correctly reproduces three-point functions of free boson (first part) and three-point functions of the conserved currents, j_s , $s \geq 1$ of free fermion (second part). These two contributions were previously obtained by solving Vasiliev equations to the second order in [8]. The second formula coincides with the one from [28], which was claimed to have been obtained from the Vasiliev theory too. Let us note the appearance of the odd conformally invariant structure⁴ S , which is a particular case of our R structure: $R_{02} \propto \xi_3 \mathbf{x}_{31}^{-1} \mathbf{x}_{12}^{-1} \xi_2 \sim S^1$. The third correlator is in fact fixed up to an overall coefficient by the conformal symmetry and in this degenerate case the R structure coincides with Q : $R_{03} \propto \xi_3 \mathbf{x}_{31}^{-1} \mathbf{x}_{12}^{-1} \mathbf{x}_{23}^{-1} \xi_3 \sim Q^3$. The last correlator, which vanishes identically irrespectively of θ , is correct both for the free fermion theory and the critical boson.

⁴ S as defined in [28] contains a factor of P , which we removed from our definition.

The latter seems to be accidental as our considerations based on the exact HS symmetry do not apply to the critical boson. All the three-point functions above can be found also in [26].

C. Four-point functions

The four-point functions were missing in the literature before [23] apart from a very specific correlators used in [21],

$$\begin{aligned}
\langle j_s j_s j_s j_s \rangle &\propto \sum_{\mathbb{S}_4} \frac{\cos(Q^1 + Q^2 + Q^3 + Q^4)}{x_{12} x_{23} x_{34} x_{41}} (\cos^4 \theta \cos P_{12} \cos P_{23} \cos P_{34} \cos P_{41} + \sin^4 \theta \sin P_{12} \sin P_{23} \sin P_{34} \sin P_{41}), \\
\langle \tilde{j}_0 j_s j_s j_s \rangle &\propto \sin^3 \theta \sum_{\mathbb{S}_4} \frac{\sin(Q^2 + Q^3 + Q^4)}{x_{12} x_{23} x_{34} x_{41}} R_{02} \sin P_{23} \sin P_{34}, \\
\langle \tilde{j}_0 \tilde{j}_0 j_s j_s \rangle &\propto \sin^2 \theta \sum_{\mathbb{S}_4} \frac{\cos(Q^3 + Q^4)}{x_{12} x_{23} x_{34} x_{41}} [i R_{03} P_{34} + R_{02} R_{24} (\xi_2 \rightarrow \xi_3, \mathbf{x}_2 \leftrightarrow \mathbf{x}_3)], \\
\langle \tilde{j}_0 \tilde{j}_0 \tilde{j}_0 j_s \rangle &\propto \sin \theta \sum_{\mathbb{S}_4} \frac{\sin Q^4}{x_{12} x_{23} x_{34} x_{41}} R_{04}, \\
\langle \tilde{j}_0 \tilde{j}_0 \tilde{j}_0 \tilde{j}_0 \rangle &\propto \sum_{\mathbb{S}_4} \frac{\text{tr}(\mathbf{x}_{12}^{-1} \mathbf{x}_{23}^{-1} \mathbf{x}_{34}^{-1} \mathbf{x}_{41}^{-1})}{x_{12} x_{23} x_{34} x_{41}}.
\end{aligned} \tag{10}$$

The first formula is borrowed from [23]. Despite containing two-pieces, first of the free boson and second of the free fermion, the free fermion piece vanishes when at least one of the polarizations η is set to zero in accordance with the fact that the $\Delta = s + 1$ propagator cannot account for the $\Delta = 2$ operator \tilde{j}_0 . The correlators with different number of insertions of \tilde{j}_0 are given afterwards. In particular, $\langle \tilde{j}_0 \tilde{j}_0 \tilde{j}_0 \tilde{j}_0 \rangle$ depends nontrivially on the two conformally invariant ratios and this dependence is reproduced by the enumerator.

The general case of n -point correlation functions requires more technicalities and is discussed below.

III. GENERATING FUNCTIONS AND CORRELATION FUNCTIONS

In this section, we will calculate (1) specified to the $4d$ Vasiliev theory. A few basic definitions are given below:

- (i) The $4d$ HS algebra is the Weyl algebra with $sp(4)$ vectors Y^A as generating elements obeying

$$[Y_A, Y_B] = 2i\epsilon_{AB}, \tag{11}$$

where ϵ_{AB} is the $sp(4)$ invariant metric, which is used to raise and lower indices $Y^A = \epsilon^{AB} Y_B$, $Y_A = Y^B \epsilon_{BA}$. It is convenient to use the Weyl \star -product realization. Then, the elements of the HS algebra are functions of formally commuting variables Y_A with the product

$$\begin{aligned}
f(Y) \star g(Y) &= \int dU dV f(Y + U) g(Y + V) e^{iV^A U_A} \\
&= f(Y) \exp \{i \tilde{\partial}_A \epsilon^{AB} \tilde{\partial}_B\} g(Y).
\end{aligned} \tag{12}$$

In the calculation, we will suppress spinor indices in most places.⁵

- (ii) The adjoint HS field Ψ in (1) is a Fourier transform of the boundary-to-bulk propagator for the master field strength B [1–5]. Namely, $\Psi = B \star \delta$ and $\delta(y) = \int \frac{d^d p}{2\pi} e^{i p y}$. Here $Y^A = (y^\alpha, \bar{y}^{\dot{\alpha}})$, i.e. $Y = (y, \bar{y})$.
- (iii) We need the boundary-to-bulk propagator of B with $\Delta = s + 1$ boundary conditions, which describes bosonic fields of all spins in the bulk, and $\Delta = 2$ propagator for the scalar component of the HS multiplet. The propagators, [7,8], see also [23], read

$$\begin{aligned}
\Delta &= s + 1, \\
B_i &= B(x, x_i, \eta_i) \\
&= K_i e^{-iy F_i \bar{y}} (e^{-iy \xi_i + i\theta} + e^{iy \xi_i + i\theta} \\
&\quad + e^{-iy \bar{\xi}_i - i\theta} + e^{iy \bar{\xi}_i - i\theta}),
\end{aligned} \tag{13}$$

$$\Delta = 2, \quad B'_i = B'(x, x_i) = K_i^2 (1 - iy F_i \bar{y}) e^{-iy F_i \bar{y}}, \tag{14}$$

where the spinors ξ and $\bar{\xi}$ are the bulk polarization spinors, these are obtained by the parallel transport of the boundary polarization η_i to the bulk. $F \equiv F^{\alpha\dot{\alpha}}$ is the wave vector from the bulk point x towards the

⁵For this purpose, we follow the convention that implicit spinor indices will always be contracted from the upper-left to the lower-right direction, e.g. $YMY \equiv Y^A M_A^B Y_B$. To be consistent with this, the implicit index positions on a matrix are always $M = M_{\bullet}^{\bullet}$ and a generalized notion of “transpose,” $\check{M}_{\bullet}^{\circ} = \epsilon^{\circ\check{\diamond}} (M^T)^*_{\check{\diamond}\bullet}$, is introduced so that indices on a transposed matrix are also at the correct positions. Note the relation $\epsilon^{AB} = \text{diag}(\epsilon^{\alpha\beta}, \epsilon^{\dot{\alpha}\dot{\beta}})$.

boundary point x_i . K_i is the Witten $\Delta = 1$ propagator for the scalar field. More details given in Poincare coordinates can be found in the Appendix.

A. Strategy

Given the above definitions, we will take the following steps to calculate (1):

- (1) Suppose that there are n currents on the boundary, we have

$$B = \sum_{i=1}^n B_i \quad \text{or} \quad B'_i. \quad (15)$$

Our strategy to compute $V_n = \text{Tr}((B \star \delta)^n)$ is to calculate first $Z_n = \text{Tr}(B_1 \star \delta \star \cdots \star B_n \star \delta)$ and

- then to apply the permutation group \mathbb{S}_n on Z_n to obtain V_n .⁶
 (2) To calculate Z_n , it will be convenient to start with

$$\begin{aligned} \Phi_i &= K_i e^{-iyF_i\bar{y} - iy\xi_i - i\bar{y}\bar{\xi}_i + i\Theta_i} = K_i e^{-\frac{i}{2}Yf_iY - iY\Xi_i + i\Theta_i}, \\ f_i &= \begin{pmatrix} \check{F}_i \\ F_i \end{pmatrix}, \end{aligned} \quad (16)$$

where $\Xi^A = \{\xi^\alpha, \bar{\xi}^{\dot{\alpha}}\}$, Θ_i is a constant and we have suppressed the spinor indices. Then B_i can be obtained by applying the following projections, successively:

$$\begin{aligned} \hat{\rho}_0: \Phi_i &\rightarrow \Phi_{i0} = K_i e^{-iyF_i\bar{y} - iy\xi_i + i\theta}, \\ \hat{\rho}: \Phi_{i0} &\rightarrow \Phi'_{i0} = K_i e^{-iyF_i\bar{y}} (e^{-iy\xi_i + i\theta} + e^{iy\xi_i + i\theta}), \\ \hat{\pi}: \Phi'_{i0} &\rightarrow B_i = K_i e^{-iyF_i\bar{y}} (e^{-iy\xi_i + i\theta} + e^{iy\xi_i + i\theta} + e^{-i\bar{y}\bar{\xi}_i - i\theta} + e^{i\bar{y}\bar{\xi}_i - i\theta}). \end{aligned} \quad (17)$$

For B'_i , we have

$$B'_i = K_i (1 - iyF_i\bar{y}) \Phi_i|_{\Xi_i = \Theta_i = 0} = K_i \left(1 - \frac{i}{2} Yf_i Y\right) \Phi_i|_{\Xi_i = \Theta_i = 0} = K_i \left(1 - \frac{i}{2} \partial_{\Xi} f \partial_{\Xi}\right)_i \Phi_i|_{\Xi_i = \Theta_i = 0}. \quad (18)$$

$$\partial_{\Xi} f \partial_{\Xi} = \partial_{\Xi_A} f_A^B \partial_{\Xi^B}, \quad \partial_{\Xi_A} (Y\Xi) = Y^A, \quad \partial_{\Xi^A} (Y\Xi) = -Y_A. \quad (19)$$

- (3) Since both B_i and B'_i can be obtained from Φ_i either by projection or by an operator that is irrelevant for the star product, we can firstly calculate $(\tilde{\Phi} = \delta \star \Phi \star \delta)$

$$Z_n = \text{Tr}(\Phi_1 \star \delta \star \cdots \star \Phi_n \star \delta) = \begin{cases} \Phi_1 \star \tilde{\Phi}_2 \star \cdots \star \Phi_{n-1} \star \tilde{\Phi}_n|_{Y=0} & : n \text{ even,} \\ \int \frac{dy}{2\pi} \Phi_1 \star \tilde{\Phi}_2 \star \cdots \star \Phi_{n-2} \star \tilde{\Phi}_{n-1} \star \Phi_n|_{\bar{y}=0} & : n \text{ odd,} \end{cases} \quad (20)$$

and then apply the above operations to recover Z_n .

In the following subsections, we work backwards along the steps outlined here.

B. The building block of generating functions

In this subsection, we firstly calculate Z_n in (20). Let us note that our computations are $sp(2M)$ covariant although we need the specialization to $sp(4)$ only.

Given (16), we find that

⁶We will not consider contact terms in this calculation. So in fact we are calculating

$$V_n = \text{Tr}((B \star \delta)^n) - (\text{contact terms}).$$

$$\begin{aligned}
 \Phi_1 \star \Phi_2 &= \frac{K_1 K_2}{\sqrt{|1 + f_1 f_2|}} e^{-\frac{i}{2}Y(f_1 \circ f_2)Y - iY(\Xi_1 \circ \Xi_2) - \frac{i}{2}(\Theta_1 \circ \Theta_2)}, \\
 f_1 \circ f_2 &= (2 + f_2 - f_1)(f_1 + f_2)^{-1}, \\
 \Xi_1 \circ \Xi_2 &= \frac{1}{2}(1 + f_1 \circ f_2)\Xi_1 + \frac{1}{2}(1 - f_1 \circ f_2)\Xi_2, \\
 \Theta_1 \circ \Theta_2 &= -\frac{1}{8}\Xi_1(f_1 \circ f_2 + f_2 \circ f_1)\Xi_1 - \frac{1}{4}\Xi_1(1 + f_2 \circ f_1)\Xi_2 - \frac{1}{8}\Xi_2(f_1 \circ f_2 + f_2 \circ f_1)\Xi_2 \\
 &\quad + \frac{1}{4}\Xi_2(1 + f_1 \circ f_2)\Xi_1 + \Theta_1 + \Theta_2.
 \end{aligned} \tag{21}$$

In the special case $\tilde{f} = f$ and $f^2 = 1$, one can find the following useful properties:

$$\begin{aligned}
 (f_1 + \dots + f_n)^2 &= \sqrt{|f_1 + \dots + f_n|}, \\
 (f_1 \circ f_2)^\vee &= f_1 \circ f_2, \\
 f_1 \circ (f_2 \circ f_3) &= (f_1 \circ f_2) \circ f_3 = f_1 \circ f_2 \circ f_3 = f_1 \circ f_3, \Rightarrow f_1 \circ \dots \circ f_n = f_1 \circ f_n, \\
 (f_1 \circ f_2)(f_1 \circ f_3) &= 1 + f_1 \circ f_2 - f_1 \circ f_3, \\
 (f_1 \circ f_3)(f_2 \circ f_3) &= 1 - f_1 \circ f_3 + f_2 \circ f_3, \\
 \Xi_1 \circ (\Xi_2 \circ \Xi_3) &= (\Xi_1 \circ \Xi_2) \circ \Xi_3 = \Xi_1 \circ \Xi_2 \circ \Xi_3 = \Xi_1 \circ \Xi_3, \Rightarrow \Xi_1 \circ \dots \circ \Xi_n = \Xi_1 \circ \Xi_n, \\
 \Theta_1 \circ (\Theta_2 \circ \Theta_3) &= (\Theta_1 \circ \Theta_2) \circ \Theta_3 = \Theta_1 \circ \Theta_2 \circ \Theta_3,
 \end{aligned} \tag{22}$$

where most relations were already known in [23]. Note, in particular, that the \circ product for f_i and Ξ_i are ‘‘forgetful.’’ Using such properties, it is easy to write down the general result,

$$\begin{aligned}
 \Phi_1 \star \Phi_2 \cdots \star \Phi_n &= N_n \exp \left\{ -\frac{i}{2}Y(f_1 \circ f_n)Y - iY(\Xi_1 \circ \Xi_n) \right. \\
 &\quad \left. + i(\Theta_1 \circ \dots \circ \Theta_n) \right\}, \\
 N_n &= 2^{2-n} \prod_{i=1}^n \frac{K_{i+1}}{|1 + f_i f_{i+1}|^{1/4}}, \\
 \Theta_1 \circ \dots \circ \Theta_n &= -\frac{1}{8} \sum_{i=1}^n [\Xi_i(f_{i+1} \circ f_i + f_i \circ f_{i-1})\Xi_i \\
 &\quad + 2\Xi_i(1 + f_{i+1} \circ f_i)\Xi_{i+1}] + \sum_{i=1}^n \Theta_i,
 \end{aligned} \tag{23}$$

where for any given n , we have defined $K_{n+1} = K_1$, $f_{n+1} = f_1$, $f_0 = f_n$ and $\Xi_{n+1} = -\Xi_1$.

Note (20) also contains $\tilde{\Phi}_i$'s. This can be easily taken into account by noticing that, to go from Φ_i to $\tilde{\Phi}_i$, one only needs to do the following replacement in (23):

$$\begin{aligned}
 f_i \rightarrow \tilde{f}_i &= I' f_i I' = -f_i, & \Xi_i \rightarrow \tilde{\Xi}_i &= I' \Xi_i, \\
 I' &= \begin{pmatrix} -1 & \\ & 1 \end{pmatrix} \otimes \mathbf{1}_2,
 \end{aligned} \tag{24}$$

The constant Θ_i is the same for Φ_i and $\tilde{\Phi}_i$. Plugging these results into (20) and (23), one can find that

$$\begin{aligned}
 Z_n &= 2^{2-2n} \prod_{i=1}^n \frac{e^{i[\hat{Q}^i + \hat{P}_{i(i+1)} + \Theta_i]}}{x_{i(i+1)}}, & \hat{Q}^i &= \hat{Q}_{(i+1)(i-1)}^i, \\
 \hat{Q}_{jk}^i &= \Xi_i \mathcal{Q}_{jk}^i \Xi_i, & \mathcal{Q}_{jk}^i &= -\frac{1}{8}(\tilde{f}_j \circ f_i + f_i \circ \tilde{f}_k), \\
 \hat{P}_{ij} &= \Xi_i \mathcal{P}_{ij} \Xi_j, & \mathcal{P}_{ij} &= -\frac{1}{4}(1 + \tilde{f}_j \circ f_i)I'.
 \end{aligned} \tag{25}$$

For later convenience, let us introduce $\Xi_0 = \Xi_n$. Let us also absorb the minus sign in Ξ_{n+1} into $\mathcal{P}_{n(n+1)} = \mathcal{P}_{01} = \mathcal{P}_{n1}$, so that

$$\Xi_{n+1} = \Xi_1, \quad \mathcal{P}_{n1} = \frac{1}{4}(1 + \tilde{f}_1 \circ f_n)I'. \tag{26}$$

Now suppose that the j th through k th nodes are all $\Delta = 2$ scalars, then from (18)

$$\begin{aligned}
 Z_n^{j,k} &= \prod_{i=j}^k K_i \left(1 - \frac{i}{2} \partial_{\Xi} f \partial_{\Xi} \right)_i Z_n |_{\Xi_j = \dots = \Xi_k = 0}, \\
 &= \hat{R}_{(j-1)(k+1)} Z_n |_{\Xi_j = \dots = \Xi_k = 0}, \\
 \hat{R}_{(j-1)(k+1)} &= \Xi_{j-1} \mathcal{R}_{(j-1)(k+1)} \Xi_{k+1}, \\
 \mathcal{R}_{(j-1)(k+1)} &= i^{k-j+2} \mathcal{P}_{(j-1)j} \hat{f}_j \mathcal{P}_{j(j+1)} \cdots \hat{f}_k \mathcal{P}_{k(k+1)},
 \end{aligned} \tag{27}$$

where \hat{f}_i is defined in (A5). In deriving this result, we have assumed that not all points are $\Delta = 2$ scalars. For the case when all n points are $\Delta = 2$ scalars, we note

$$Z_n^{2,n} = \Xi_1 \mathcal{R}_{1(n+1)} \Xi_1 e^{\hat{Q}} 2^{2-2n} \prod_{i=1}^n \frac{e^{i\Theta_i}}{x_{i(i+1)}}, \Rightarrow Z_n^{1,n} = K_1 \left(1 - \frac{i}{2} \partial_{\Xi} f \partial_{\Xi} \right)_1 Z_n^{2,n} |_{\Xi_1=0} = \text{tr}[\hat{f}_1 \mathcal{R}_{1(n+1)}] 2^{2-2n} \prod_{i=1}^n \frac{e^{i\Theta_i}}{x_{i(i+1)}}. \quad (28)$$

C. Projections and correlation functions

Given Z_n and $Z_n^{j,k}$ we can now impose the projections defined in (17).

Firstly note $\xi = -F\bar{\xi}$, with which one can obtain

$$\begin{aligned} Q^i &\equiv \hat{Q}^i(\xi_i) = \hat{Q}^i(\bar{\xi}_i), & P_{ij} &\equiv \hat{P}_{ij}(\xi_i, \xi_j) = \hat{P}_{ij}(\bar{\xi}_i, \xi_j) = -\hat{P}_{ij}(\xi_i, \bar{\xi}_j) = -\hat{P}_{ij}(\bar{\xi}_i, \bar{\xi}_j), \\ R_{jk} &\equiv \hat{R}_{jk}(\xi_i, \xi_j) = \hat{R}_{jk}(\bar{\xi}_i, \xi_j) = -\hat{R}_{jk}(\xi_i, \bar{\xi}_j) = -\hat{R}_{jk}(\bar{\xi}_i, \bar{\xi}_j), \end{aligned} \quad (29)$$

where $\hat{Q}^i(\xi_i) = \hat{Q}^i|_{\bar{\xi}_i=0}$, $\hat{P}_{ij}(\xi_i, \xi_j) = \hat{P}_{ij}|_{\bar{\xi}_i=\bar{\xi}_j=0}$ and so on. Now using (17), we find

$$\begin{aligned} \hat{\rho}_0 Z_n &= 2^{2-2n} \prod_{i=1}^n \frac{e^{i[Q^i + P_{i(i+1)} + \theta]}}{x_{i(i+1)}}, \\ \hat{\rho} \hat{\rho}_0 Z_n &= 2^{2-n} \prod_{i=1}^n \frac{e^{i(Q^i + \theta)}}{x_{i(i+1)}} \left[\prod_i^n \cos P_{i(i+1)} + i^n \prod_i^n \sin P_{i(i+1)} \right], \\ Z_n &= \hat{\pi} \hat{\rho} \hat{\rho}_0 Z_n = 4 \prod_{i=1}^n \frac{e^{iQ^i}}{x_{i(i+1)}} \left[\cos^n \theta \prod_i^n \cos P_{i(i+1)} + \sin^n \theta \prod_i^n \sin P_{i(i+1)} \right], \\ \hat{\rho}_0 Z_n^{j,k} &= 2^{2-2n} R_{(j-1)(k+1)} \prod_{i=1}^n \frac{e^{i[Q^i + P_{i(i+1)} + i\theta]}}{x_{i(i+1)}} \Big|_{\Xi_j = \dots = \Xi_k = 0}, \\ \hat{\rho} \hat{\rho}_0 Z_n^{j,k} &= 2^{1-n-k+j} i^{n-(k-j+2)} R_{(j-1)(k+1)} \prod_{i=1}^n \frac{e^{i(Q^i + \theta)}}{x_{i(i+1)}} \prod_{i'}^l \sin P_{i'(i'+1)}, \\ Z_n^{j,k} &= \hat{\pi} \hat{\rho} \hat{\rho}_0 Z_n^{j,k} = -\frac{i}{4^{k-j}} (-\sin \theta)^{n-(k-j+1)} R_{(j-1)(k+1)} \prod_{i=1}^n \frac{e^{iQ^i}}{x_{i(i+1)}} \prod_{i'}^l \sin P_{i'(i'+1)}, \end{aligned} \quad (30)$$

where $\prod_{i'}^l$ goes over all points for which $P_{i'(i'+1)} \neq 0$ and one has to set $\Xi_j = \dots = \Xi_k = 0$ for $Z_n^{j,k}$.

Our last step is to obtain V_n . This can be achieved by applying the permutation group, \mathbb{S}_n , on Z_n or $Z_n^{j,k}$. Inside \mathbb{S}_n one can firstly consider the dihedral subgroup, D_n , in which each of the reflections \hat{s} acts as

$$\begin{aligned} \hat{s}(Q^i) &= -Q^i, & \hat{s}(P_{ij}) &= -P_{ij}, \\ \hat{s}(R_{jk}) &= -R_{jk}, \Rightarrow D_n Z_n = 8n \left[\cos^n \theta \cos Q \prod_i^n \frac{\cos P_{i(i+1)}}{x_{i(i+1)}} + \sin^n \theta f_n(Q) \prod_i^n \frac{\sin P_{i(i+1)}}{x_{i(i+1)}} \right], \\ D_n Z_n^{j,k} &= -i \frac{2n}{4^{k-j}} (-\sin \theta)^{n-(k-j+1)} R_{(j-1)(k+1)} \prod_{i=1}^n \frac{1}{x_{i(i+1)}} \prod_{i'}^l \sin P_{i'(i'+1)} f_{n-(k-j+1)}(Q), \end{aligned} \quad (31)$$

where $Q = \sum_{i=1}^n Q^i$, $f_n(x) = \cos x$ for n even and $f_n(x) = i \sin x$ for n odd. As a result,

$$\begin{aligned} \mathbb{S}_n Z_n &= 4 \sum_{\mathbb{S}_n} \left[\cos^n \theta \cos Q \prod_i^n \frac{\cos P_{i(i+1)}}{x_{i(i+1)}} \right. \\ &\quad \left. + \sin^n \theta f_n(Q) \prod_i^n \frac{\sin P_{i(i+1)}}{x_{i(i+1)}} \right], \end{aligned} \quad (32)$$

$$\begin{aligned} \mathbb{S}_n Z_n^{j,k} &= -\frac{i}{4^{n_0-1}} (-\sin \theta)^{n-n_0} \sum_{\mathbb{S}_n} R_{(j-1)(k+1)} f_{n-n_0}(Q) \\ &\quad \times \prod_{i=1}^n \frac{1}{x_{i(i+1)}} \prod_{i'}^l \sin P_{i'(i'+1)}, \end{aligned} \quad (33)$$

where $n_0 (= k - j + 1, \text{ when } k \geq j)$ is the total number of \bar{j}_0 insertions. In the case when all n points are $\Delta = 2$ scalars, we have

$$\langle \tilde{j}_0 \cdots \tilde{j}_0 \rangle = \sum_{\mathbb{S}_n} Z_n^{1,n} = \sum_{\mathbb{S}_n} \text{tr}[\hat{f}_1 \mathcal{R}_{1(n+1)}] 2^{2-2n} \prod_{i=1}^n \frac{1}{x_{i(i+1)}}. \quad (34)$$

Note $\mathbb{S}_n Z_n = V_n = \langle j_s \cdots j_s \rangle$ in cases without a $\Delta = 2$ scalar, but $\mathbb{S}_n Z_n^{j,k}$ is not equivalent to V_n in cases with $\Delta = 2$ scalar operators. The reason is that $Z_n^{j,k}$ only contains contributions from cases where all $\Delta = 2$ scalars are labeled continuously, while other possibilities are not included. In the case with several $\Delta = 2$ scalars, this means there are mixed sequences such as

$$\cdots j_s \tilde{j}_0 \cdots \tilde{j}_0 j_s \cdots j_s \tilde{j}_0 \cdots \tilde{j}_0 j_s \cdots j_s \tilde{j}_0 \cdots \tilde{j}_0 j_s \cdots. \quad (35)$$

In this sequence, the Q , P and R structures can be read off as follows:

$$\begin{aligned} j_s(i) &\rightarrow Q^i, & j_s(i) j_s(i+1) &\rightarrow P_{i(i+1)}, \\ j_s(i) (\text{a sequence of } \tilde{j}_0) j_s(k) &\rightarrow R_{ik}, \end{aligned} \quad (36)$$

where $j_s(i)$ means that j_s is on the i th position in the sequence, and similarly $\tilde{j}_0(i)$ means that \tilde{j}_0 is on the i th position in the sequence. Using $\{\cdots\}$ to denote the above sequence, and using $Z_n^{\{\cdots\}}$ to denote the corresponding generating function, we find

$$\begin{aligned} \mathbb{S}_n Z_n^{\{\cdots\}} &= \frac{(i \sin \theta)^{n-n_0}}{4^{n_0-1}} \sum_{\mathbb{S}_n} f_{n-n_0}(Q) \prod_{i=1}^n \frac{1}{x_{i(i+1)}} \\ &\times Y_{\{\cdots\}}^{-1} \prod R \prod (i \sin P), \\ V_n (\text{with } \tilde{j}_0) &= \sum_{\{\cdots\}} \mathbb{S}_n Z_n^{\{\cdots\}}, \end{aligned} \quad (37)$$

where the last two \prod in $\mathbb{S}_n Z_n^{\{\cdots\}}$ go over all nonvanishing R 's and P 's, which are determined according to (36); and $Y_{\{\cdots\}}$ is the operator that takes the canonical sequence $\{\tilde{j}_0 \cdots \tilde{j}_0 j_s \cdots j_s\}$ to the particular sequence $\{\cdots\}$ that is involved. As an example, let us note

$$\begin{aligned} \langle \tilde{j}_0 \tilde{j}_0 j_s j_s \rangle &= \mathbb{S}_n Z_4^{\{\tilde{j}_0 \tilde{j}_0 j_s j_s\}} + \mathbb{S}_n Z_4^{\{\tilde{j}_0 j_s \tilde{j}_0 j_s\}} \\ &= -\frac{1}{4} \sin^2 \theta \sum_{\mathbb{S}_n} \frac{\cos(Q^3 + Q^4)}{x_{12} x_{23} x_{34} x_{41}} \\ &\times [i R_{03} P_{34} + Y_{\{\tilde{j}_0 j_s \tilde{j}_0 j_s\}}^{-1} R_{02} R_{24}], \end{aligned} \quad (38)$$

where $Y_{\{\tilde{j}_0 j_s \tilde{j}_0 j_s\}}$ is the operator that takes the sequence $\{\tilde{j}_0 \tilde{j}_0 j_s j_s\}$ to $\{\tilde{j}_0 j_s \tilde{j}_0 j_s\}$, cf. the third correlator of (10).

As a side remark, we have explicit i factors floating around in our final results (32) and (37), and also inside R_{ij} as in (A7). The appearance of these i factors is due to the fact that we have neglected some extra phase factors that naturally arise in (21), as is shown in [26]. Consistency requires that the result must be Hermitian and all these i

factors must cancel when all phase factors are taken into account properly. Our main objective here is to obtain the basic structure of correlation functions since an overall factors are undetermined within our approach.

IV. CONCLUSION

In this note we complete the calculation of n -point correlation functions of conserved currents in unbroken $4d$ Vasiliev theory initially carried out for $\Delta = s + 1$ operators in [23]. The missing link that we deal with in our paper is the correlation functions that contain $\Delta = 2$ scalar operator \tilde{j}_0 (5). The obtained results are in agreement with Maldacena-Zhiboedov theorem, with three- and some four-point calculations performed using different methods [7,8,11,21,28] and in agreement with very recent calculation [26] where n -point functions were reproduced from the current operator algebra. Our method is applicable only when HS symmetry is unbroken and it is promising to look whether it can be improved to the case when HS symmetry is broken. The trace formula (1) determines the correlation functions up to overall coefficients and makes the calculations very simple. The price to pay for this simplicity is that we only reproduce the connected part of correlation functions unlike the complete result of [26]. The great advantage of the proposed method is its manifest conformal and HS invariance. HS boundary-to-bulk propagators as well as conformal structures arise in a coordinate independent way. All have beautiful interpretation in HS algebra, the former correspond to projectors in star-product algebra, while the latter appear naturally through the induced \circ -product defined on a projector space. The use of coordinates is thus the matter of presenting the results to make contact with the available ones in the literature.

Our method can be straightforwardly generalized to any dimension although in higher dimensions HS algebra has no longer simple realization similar to lower d spinorial Weyl algebras. That the analogous d -dimensional calculation is not going to be simple at all has been already demonstrated at the level of HS propagators in [12] unless the induced product on the space of projectors is understood.⁷ On the other hand, the spinorial route to HS algebras naturally extends to symplectic HS algebras.

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⁷Three-point functions have been recently found in [29].

APPENDIX: TECHNICALITIES

The $\Delta = 1$ propagator can be constructed from the following:

$$\begin{aligned}\Phi_i &= K_i e^{-iyF_i\bar{y} - iy\xi_i - i\bar{y}\bar{\xi}_i}, & \xi_i &= \Pi_i \eta_i, \\ \bar{\xi}_i &= \bar{\Pi}_i \bar{\eta}_i, & K_i &= K(\rho, x; x_i) = \frac{\rho}{\hat{x}_i^2 + \rho^2}, \\ F_i &= F(\rho, x; x_i) = -\left(2K_i \hat{x}_i + i \frac{\hat{x}_i^2 - \rho^2}{\hat{x}_i^2 + \rho^2}\right), \\ \Pi_i &= \Pi(\rho, x; x_i) = e^{-i\pi/4} K_i \left(\frac{1}{\sqrt{\rho}} \hat{x}_i - i\sqrt{\rho}\right), \\ \bar{\Pi}_i &= \bar{\Pi}(\rho, x; x_i) = e^{-i\pi/4} K_i \left(\frac{1}{\sqrt{\rho}} \hat{x}_i + i\sqrt{\rho}\right),\end{aligned}\quad (\text{A1})$$

where η and $\bar{\eta}$ are *real* constant boundary polarization vectors, and

$$\begin{aligned}\hat{x}_i &\equiv \mathbf{x} - \mathbf{x}_i, \\ \mathbf{x} &= \begin{pmatrix} -x_2 & -x_0 - x_1 \\ x_0 - x_1 & x_2 \end{pmatrix}, \\ \Rightarrow x^2 &= -\det(\mathbf{x}) = x_0^2 - x_1^2 - x_2^2.\end{aligned}\quad (\text{A2})$$

Note the properties

$$\begin{aligned}F_i \check{F}_i &= 1, & \check{\Pi}_i \Pi_i &= \Pi_i \check{\Pi}_i = \check{\Pi}_i \bar{\Pi}_i = \bar{\Pi}_i \check{\Pi}_i = -iK_i, \\ \check{\Pi}_i &= \check{F}_i \Pi_i F_i, & \Pi_i^{-1} F_i \check{\Pi}_i^{-1} &= \Pi_i^{-2} \check{F}_i^{-1} = \frac{1}{K_i}.\end{aligned}\quad (\text{A3})$$

From $\xi = -F\bar{\xi}$, one can derive that $\bar{\xi}^* = \xi$ and $\bar{\eta} = -i\eta$. It will be convenient to write

$$\begin{aligned}\Phi_i &= K_i e^{-\frac{1}{2}Yf_iY - iY\Xi_i}, & Y &= \begin{pmatrix} y \\ \bar{y} \end{pmatrix}, \\ \Xi_i &= \begin{pmatrix} \xi_i \\ \bar{\xi}_i \end{pmatrix} = \Gamma_i \Xi_i^0, & f_i &= \begin{pmatrix} F_i \\ \check{F}_i \end{pmatrix}, \\ \Gamma_i &= \begin{pmatrix} \Pi_i & \\ & \bar{\Pi}_i \end{pmatrix}, & \Xi_i^0 &= \begin{pmatrix} \eta_i \\ \bar{\eta}_i \end{pmatrix}.\end{aligned}\quad (\text{A4})$$

From (A3), one has

$$\begin{aligned}\mathbf{x}_{ij} &\rightarrow \mathbf{x}_i^{-1} - \mathbf{x}_j^{-1} = -\mathbf{x}_i^{-1} \mathbf{x}_{ij} \mathbf{x}_j^{-1} = -\frac{\mathbf{x}_i \mathbf{x}_{ij} \mathbf{x}_j}{x_i^2 x_j^2}, \\ x_{ij}^2 &= -|\mathbf{x}_{ij}| \rightarrow -|\mathbf{x}_i^{-1} \mathbf{x}_{ij} \mathbf{x}_j^{-1}| = -\frac{|\mathbf{x}_{ij}|}{|\mathbf{x}_i| |\mathbf{x}_j|} = \frac{x_{ij}^2}{x_i^2 x_j^2}, \\ \mathbf{x}_{ij}^{-1} &= \frac{\mathbf{x}_{ij}}{x_{ij}^2} \rightarrow -\frac{\mathbf{x}_i \mathbf{x}_{ij} \mathbf{x}_j}{x_i^2 x_j^2} \Big/ \frac{x_{ij}^2}{x_i^2 x_j^2} = -\mathbf{x}_i \mathbf{x}_{ij}^{-1} \mathbf{x}_j, \\ \#_k &\rightarrow (-\Xi_i^0 \mathbf{x}_i^{-1}) (-\mathbf{x}_i \mathbf{x}_{i_1}^{-1} \mathbf{x}_{i_2}^{-1} \mathbf{x}_{i_3}^{-1} \mathbf{x}_{i_4}^{-1} \cdots \mathbf{x}_{i_{k-2}}^{-1} \mathbf{x}_{i_{k-1}}^{-1} \mathbf{x}_{i_k}^{-1} \Xi_i^0) = \#_k,\end{aligned}\quad (\text{A9})$$

$$\check{f}_i = f_i, \quad f_i^2 = 1,$$

$$\Gamma_i^{-1} \hat{f}_i \check{\Gamma}_i^{-1} = i \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \otimes \mathbf{1}_2, \quad (\text{A5})$$

$$\hat{f}_i \equiv -iK_i f_i,$$

where $\mathbf{1}_2$ means two-dimensional unit matrix.

From (25) and (27), we find that $\check{\mathcal{Q}}_{jk}^i = \mathcal{Q}_{jk}^i$, $\text{tr}(f_i \mathcal{Q}_{jk}^i) = -1$ and

$$\begin{aligned}\check{\Gamma}_i \mathcal{Q}_{jk}^i \Gamma_i &= \frac{1}{8} \begin{pmatrix} \mathbf{x}_{ij}^{-1} - \mathbf{x}_{ik}^{-1} & \cdots \\ \cdots & -(\mathbf{x}_{ij}^{-1} - \mathbf{x}_{ik}^{-1}) \end{pmatrix}, \\ \check{\Gamma}_i \mathcal{P}_{ij} \Gamma_j &= -\frac{1}{4} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \otimes \mathbf{x}_{ij}^{-1}, \\ \check{\Gamma}_{j-1} \mathcal{R}_{(j-1)(k+1)} \Gamma_{k+1} &= \frac{c}{2(2i)^{k-j+2}} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \\ &\quad \otimes \left(\prod_{i=j-1}^k \mathbf{x}_{i(i+1)}^{-1} \right),\end{aligned}\quad (\text{A6})$$

where $c = 1$ if \mathbf{x}_{n1}^{-1} is not involved while $c = -1$ if otherwise. As a result,

$$\begin{aligned}Q^i &= -\frac{1}{8} \eta_i [\mathbf{x}_{i(i+1)}^{-1} - \mathbf{x}_{i(i-1)}^{-1}] \eta_i, \\ P_{ij} &= \frac{c}{4} \eta_i \mathbf{x}_{ij}^{-1} \eta_j,\end{aligned}\quad (\text{A7})$$

$$\mathcal{R}_{(j-1)(k+1)} = -\frac{c}{2(2i)^{k-j+2}} \eta_{j-1} \left(\prod_{i=j-1}^k \mathbf{x}_{i(i+1)}^{-1} \right) \eta_{k+1}.$$

Note that the off diagonal elements of $\check{\Gamma}_i \mathcal{Q}_{jk}^i \Gamma_i$ do not contribute to these structures. One can further find that

$$\begin{aligned}Z_n^{1,n} &= \text{tr} \left[-\frac{1}{(2i)^n} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \otimes (\mathbf{x}_{12}^{-1} \mathbf{x}_{23}^{-1} \cdots \mathbf{x}_{n1}^{-1}) \right] 2^{2-2n} \\ &\quad \times \prod_{i=1}^n \frac{1}{x_{i(i+1)}} \\ &= -8 \frac{\text{tr}(\mathbf{x}_{12}^{-1} \mathbf{x}_{23}^{-1} \cdots \mathbf{x}_{n1}^{-1})}{(8i)^n x_{12} x_{23} \cdots x_{n1}}.\end{aligned}\quad (\text{A8})$$

Under the inversion, $\mathbf{x}_i \rightarrow \mathbf{x}_i^{-1}$ and $\Xi_i^0 \rightarrow \mathbf{x}_i^{-1} \Xi_i^0$. One has

where $\#_k = \Xi_{i_1}^0 \mathbf{x}_{i_1 i_2}^{-1} \mathbf{x}_{i_2 i_3} \mathbf{x}_{i_3 i_4}^{-1} \cdots \mathbf{x}_{i_{k-2} i_{k-1}} \mathbf{x}_{i_{k-1} i_k}^{-1} \Xi_{i_k}^0$. This implies, in particular, that P_{ij} and Q^i are conformal invariants. One can then check that under (A9),

$$Z_n \rightarrow \left(\prod_i^n x_i^2 \right) Z_n, \quad Z_n^{j,k} \rightarrow \left(\prod_i^n x_i^2 \prod_{j'=j}^k x_{j'}^2 \right) Z_n^{j,k}, \quad Z_n^{1,n} \rightarrow \left(\prod_i^n x_i^4 \right) Z_n^{1,n}, \quad (\text{A10})$$

which all transform as expected.

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