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Holonomy observables in Ponzano–Regge-type state sum models

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Abstract

Observables on group elements in the Ponzano–Regge model are studied. These observables are shown to have a natural interpretation in terms of Feynman diagrams on a sphere and are contrasted to the well-studied observables on the spin labels. The physical interpretation is elucidated by showing how they arise from the no-gravity limit of the Turaev–Viro model and Chern–Simons theory.

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1. Introduction

The Ponzano–Regge model [PR68] is a model for three-dimensional quantum gravity without matter. The partition function is formulated as a state sum model, with a sum over labelling of a triangulated 3-manifold, and is independent of the triangulation chosen.

An important generalization of this idea occurred when Witten proposed to construct three-dimensional quantum gravity as a Chern–Simons-type functional integral [Wit88]. He developed both a version with and a version without a cosmological constant [Wit89b].

The model with a positive cosmological constant and a Euclidean signature gives the same partition function as the state sum model constructed by Turaev and Viro [TV92, Rob95].

This suggests that the Chern–Simons model without a cosmological constant should be the Ponzano–Regge model. Aspects of this have been confirmed [BNG09]; the Chern–Simons model gives a formula in terms of Ray–Singer torsion, whereas the Ponzano–Regge model gives a formula in terms of the equivalent Reidemeister torsion. The model is a quantization of the first-order form of three-dimensional gravity, with action $\int \text{Tr}(e \wedge F(\omega))$, where e is an $su(2)$ -valued 1-form and F is the curvature of the $su(2)$ connection ω . The functional integration is not subjected to the condition that $\det(e)$ be positive, and thus e behaves as a Lagrange multiplier. The functional integral thus reduces to an integration over the moduli space of flat $su(2)$ connections.

Observables in these theories have been studied extensively, starting from [AW91, KS93]. For the Turaev–Viro model, a full account was given in [BGIM07] for the

Ponzano–Regge model in [Bar03, FL04, Bar05, FL06, Bar06]. These ‘edge observables’ were always constructed by coupling spin labels on the edges in these state sum model. In this paper, we consider instead observables of the $SU(2)$ connection in the model. This leads to a different set of observables with a different physical interpretation, which is explained here. To explore the interpretation fully, the new observables are considered as the $G \rightarrow 0$ limit of observables for the Turaev–Viro model. This discussion is a development of the discussion in [FL06]; we supply the observables which generate the Feynman diagram measure factors discussed there. The new observables are contrasted with the standard edge observables. The difference is that they are evaluations of spin networks for $SO(4)$ and $DSU(2)$, respectively, quantum groups that are semi-dual in the sense of Majid–Schroers [MS09].

2. Group variables in the Ponzano–Regge model

The Ponzano–Regge model [PR68, BNG09] on a triangulated 3-manifold M can be expressed with spin variables $l_e \in \{0, \frac{1}{2}, 1, \dots\}$ on each edge e as

$$Z(M) = \sum_{l_e} \prod_{\text{interior edges}} (-1)^{2l_e} (2l_e + 1) \prod_{\text{tetrahedra}} \begin{Bmatrix} l_1 & l_2 & l_3 \\ l_4 & l_5 & l_6 \end{Bmatrix}, \quad (1)$$

the tetrahedral weight being the 6j-symbol for the six spins l_1, \dots, l_6 on its edges. An alternative formula is obtained by writing the product of 6j-symbols as an integral over variables $g_f \in SU(2)$ for each dual edge f , using the holonomy h_e around each edge of the triangulation

$$Z(M) = \sum_l \prod_{\text{dual edges}} \int dg_f \prod_{\text{edges}} (2l_e + 1) \text{Tr}_l(h_e). \quad (2)$$

Summing out the spin variables leads to a formulation entirely in terms of the group variables

$$Z(M) = \prod_{\text{dual edges}} \int dg_f \prod_{\text{edges}} \delta(h_e). \quad (3)$$

The delta functions on $SU(2)$ (with support at the identity) force the g variables to describe flat connections on M .

These expressions (1)–(3) all require regularizing; then they are equal for oriented compact manifolds in the circumstances when they are well defined. They correspond to Lagrangian quantum gravity in the metric formulation, first-order formulation and in a connection representation, respectively. One needs to regularize (3), for example, by systematically removing excess delta functions from the formula. For example, in [FL04], the product over edges is restricted to a subset of edges excluding a maximal tree. For a full discussion of the divergences of this formula and their regularization, we point the reader to [BNG09] and [BS10a, BS10b, BS11]. Such a regularization will be assumed in the following.

Observables are often inserted into the state sum model by including a function of the spin labels j in the formula for the partition function Z [Bar03, FL04, FL06, BNG09]. These observables are called the ‘edge observables’ in the following discussion, and are described more precisely in section 3.2.

In this paper, the alternative possibility of using functions of the group variables is considered. This makes sense with either (2) or (3). As the observables do not depend on the spin labels, one may as well sum these out. The formula with observables is thus

$$Z(M, F) = \prod_{\text{dual edges}} \int dg_f F(g_1, g_2, \dots) \prod_{\text{edges}} \delta(h_e). \quad (4)$$

The observable is specified by the function F of the variables g_1, g_2, \dots on the dual edges.

2.1. Character observables on S^3

The main features of the model with these observables are apparent in the special case where $M = S^3$ and F is a product of character functions χ_j on $SU(2)$ for some irreducible representation j :

$$F = \chi_{j_1}(g_1)\chi_{j_2}(g_2)\chi_{j_3}(g_3)\dots$$

The partition function is then a function of the spin labels j_1, j_2, \dots on some subset of the dual edges. These dual edges form a graph Γ in S^3 , and it is assumed that the edges of the graph are numbered with integers starting from 1. The graph together with the spin labels for its edges is denoted $\Gamma(j_1, j_2, \dots)$.

The main mathematical result of this paper is the following identity:

$$Z(S^3, F) = (-1)^{\sum 2j_e} \langle \Gamma(j_1, j_2, \dots) \rangle, \quad (5)$$

where $\langle \Gamma(j_1, j_2, \dots) \rangle$ is the relativistic spin network evaluation of the labelled graph $\Gamma(j_1, j_2, \dots)$ as defined in [Bar98].

The proof of the formula follows from the fact that any flat connection on S^3 is pure gauge. Pick an arbitrary dual vertex v_0 of the triangulation as the origin. For each dual vertex, v defines a new variable $u_v \in SU(2)$ to be the product of group elements along some path that connects the dual vertex to the origin. This is well defined since any two paths are homotopic via dual faces, and the delta function for a dual face ensures that going along different paths around the dual face gives the same group element.

The original group element on a dual edge e with a source and a target vertex $s(e), t(e)$, is recovered as

$$g_e = u_{t(e)}u_{s(e)}^{-1}.$$

Using a maximal tree of dual edges, it is clear that the u_v can take any values in $SU(2)$, except that $u_{v_0} = 1$, the identity element.

The integration measure after regularization is

$$\prod_{\text{dual edges}} dg_f \prod_{\text{edges}} \delta(h_e) = \prod_{\text{dual vertices}} du_v \delta(u_{v_0}).$$

Thus, we have replaced the integration over flat connections with integration over the local gauge group.

Dropping redundant delta functions, the overall partition function with the observables can now be written as

$$Z(S^3, F) = \prod_{\text{dual vertices of } \Gamma} \int du_k \delta(u_{v_0}) \prod_{\text{dual edges of } \Gamma} \chi_{j_{el}}(u_k u_l^{-1}). \quad (6)$$

Up to the minus signs, this is just the definition of relativistic spin network evaluation given in [Bar98]. This completes the proof of (5).

These observables can also be expressed as a modified state sum model. To see this, note that when integrating out the group elements on dual edges implementing the analogue of turning expression (2) into (1), we no longer have three characters at the end of each dual edge which contract to form $6js$, but instead, we get 4-valent intertwiners with one edge joined up:

$$\sum_{a_4} \int dg D_{a_1, b_1}^{l_1} D_{a_2, b_2}^{l_2} D_{a_3, b_3}^{l_3} D_{a_4, a_4}^j = \sum_{k, a_4} t_{a_1, a_2, a_3, a_4}^k t_{b_1, b_2, b_3, a_4}^k (-1)^{2k} (2k + 1), \quad (7)$$

where k labels some basis of 4-valent intertwiners. The state sum thus does not disconnect into a set of tetrahedral spin networks but into a set of tetrahedra joined by intertwiner labels and lines at the vertices. The result is that part of the state sum model becomes a spin network with the same topology as the graph Γ in the observable, but with tetrahedral spin networks for vertices, see figure 1.

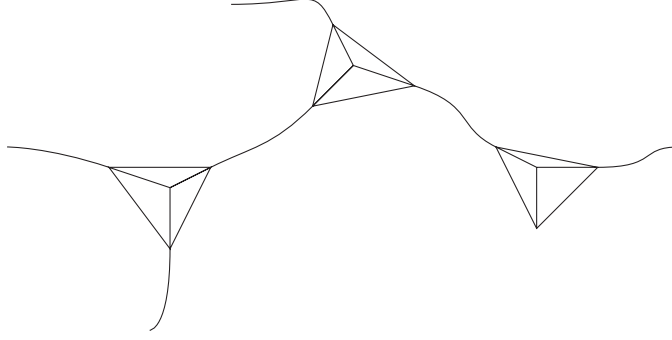


Figure 1. Graph Γ with expanded vertices.

2.2. General observables on S^3

Above we assumed that we were dealing with observables that are conjugation invariant and thus can be expanded in characters. If we chose $F^{gi}(g_1, g_2, \dots)$ to be gauge invariant at the vertices of the graph, it can be expanded instead into spin networks. These are then exactly the observables for which the asymptotic limit was studied in [AW91]. Expressing the group observables at the edges through group elements at the vertices again, we observe immediately that the observable simply is the evaluation at the identity:

$$Z(S^3, F^{gi}) = F^{gi}(1, 1, \dots) Z(S^3). \quad (8)$$

A general function can only be decomposed into matrix elements of representations. The matrix elements as observables on the group elements constitute a generalization of relativistic spin networks. The observable is in general a linear combination of observables of the form

$$\prod_{\text{dual vertices}} \int du_k \prod_{\text{dual edges}} D_{a_{kl}b_{lk}}^{j_{kl}}(u_k u_l^{-1}),$$

with coefficients $O_{kl}^{a_{kl}b_{lk}}$ for each edge kl . Here, we think of the index a_{kl} as living at the vertex k , facing the edge kl . The (almost) general observable on the group variables for fixed spins can then be written in terms of these as

$$\begin{aligned} Z(S^3, F) &= \prod_{\text{dual vertices } i} \int du_i \prod_{\text{dual edges } (kl)} D_{a_{kl}b_{lk}}^{j_{kl}}(u_k u_l^{-1}) O_{a_{kl}b_{lk}}^{kl} \\ &= \prod_{\text{dual vertices } i} \int du_i \prod_{\text{dual edges } (kl)} D_{a_{kl}c_{kl}}^{j_{kl}}(u_k) \epsilon_{c_{kl}d_{lk}} D_{e_{lk}d_{lk}}^{j_{kl}}(u_l) \epsilon_{b_{lk}e_{lk}} O_{a_{kl}b_{lk}}^{kl}, \end{aligned} \quad (9)$$

where ϵ is the bilinear inner product on the j_{kl} representation.

We can now do the integration at each vertex and obtain the usual projectors expressed as a product of 3j symbols. Now note that from this formula it is easy to see that the relativistic spin network evaluation $\langle \cdot \rangle$ is up to signs equal to the square of the ordinary spin network $\{ \cdot \}$ for 3-valent graphs. Setting $O_{kl}^{a_{kl}b_{lk}} = \delta^{a_{kl}b_{lk}}$ and integrating out the vertex variables u_k locally by replacing them with 3-valent intertwiners gives exactly two sets of 3-valent intertwiners contracted according to the combinatorics of Γ . Now restoring the $O_{(kl)}$, we obtain one normal spin network valuation $\{ \cdot \}$, and one network evaluation with operators $O_{(kl)}$ used to contract intertwiners,

$$Z(S^3, \Gamma(j_1, j_2, \dots, O)) = \{ \Gamma(j_1, j_2, \dots) \} \{ \Gamma(j_1, j_2, \dots, O) \}, \quad (10)$$

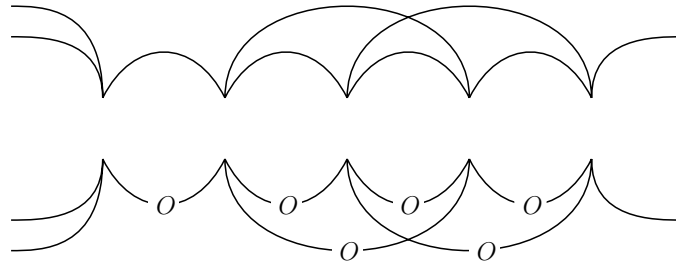


Figure 2. Relativistic spin networks with operators O along the edges.

see figure 2. These observables are therefore straightforward generalizations of relativistic spin networks.

2.3. Generalizing S^3

Although initially formulated for a 3-manifold, the partition function (4) can be generalized by replacing the dual cell complex by an arbitrary cell complex \mathcal{K} . An example would be the dual cell complex of a 4-manifold, giving the Ooguri model [Oog92]. The generalization of (4) is

$$Z(M, F) = \prod_{\text{edges of } \mathcal{K}} \int dg_f F(g_1, g_2, \dots) \prod_{\text{2-cells of } \mathcal{K}} \delta(h_e). \tag{11}$$

The partition function is well defined when a condition on the twisted cohomology is satisfied, as in [BNG09]. Cells of dimension higher than 2 play no role in the formula; however, the partition function for a complex \mathcal{L} containing higher dimensional cells can be formulated by collapsing \mathcal{L} to a 2-skeleton \mathcal{K} , and the regularization of the Ponzano–Regge model can in fact be understood in this way. Thus, formula (11) can be regarded as a generalization of the definition of the relativistic spin network evaluation to an arbitrary manifold. The corresponding formula for $U_qsl(2)$ is studied in [BGIM07].

We have not studied the effect of arbitrary topology on the observables in a systematic way, but limit the discussion to a couple of examples. First, note that if the observable is contractible in \mathcal{K} nothing in the analysis changes. The observable is the product of the S^3 observable with the graph Γ and the evaluation of the partition function for \mathcal{K} with no observable. To see this, simply run the change of integration argument above within this contractible region. Now the integration over vertices that touch the graph factorizes from the rest of the partition function. By introducing spurious integrations, we can run the change of basis argument backwards while keeping the integrations touching the observable separate. This gives the product of the partition function and the integral on the right-hand side of (6).

This is in fact a very general result of surgery theory that also holds in the Turaev–Viro context. Note that the contractible region is connected to the rest of the manifold through a cylinder $S^2 \times [0, 1]$. It was shown in [KMS92] that the partition function for this case does not change if $S^2 \times [0, 1]$ is replaced by $B^3 \cup B^3$, as the range of the projector $\mathbb{Z}(S^2 \times [0, 1])$ is one dimensional. Doing this splits the partition function as $\mathbb{Z}(M, F) = \mathbb{Z}(M)\mathbb{Z}(S^3, F)/\mathbb{Z}(S^3)$. For planar graphs this was already pointed out in theorem 4.2 of [KS93].

For the trivial case of a loop with one vertex and one 1-cell bounding a two-dimensional disc, the character observable (4) is

$$\int dg \delta(g) \chi_j(g) = 2j + 1.$$

If $\mathcal{K} = S^1$, given by a loop with one vertex and one 1-cell with no disc filling it in, then the partition function with the character observable is

$$\int dg \chi_j(g) = \delta_{j0}. \quad (12)$$

In essence, there is no propagation around the non-contractible loop. This result will be interpreted in the next section.

For RP^2 , represented by a generator g and a relation $g^2 = 1$, the corresponding partition function is

$$\int dg \delta(g^2) \chi_j(g) = \frac{1}{8} (\chi_j(1) + \chi_j(-1)) \quad (13)$$

$$= \begin{cases} \frac{1}{4}(2j+1) & 2j \text{ even} \\ 0 & 2j \text{ odd.} \end{cases} \quad (14)$$

3. Particles

3.1. Particle on a sphere

The remaining task is to describe the physics of the new observables.

The first observation is that the new observables have the character of momentum observables. For example, if the graph Γ has a 2-valent vertex, then the labels on either side are forced to be equal (else the partition function is zero). This is the conservation of momentum. For vertices of valence greater than 2, the restrictions on the values of the labels adjacent to a vertex are those compatible with the conservation of momentum where the labels are treated as the length of a momentum vector. Thus, the labels can be considered as the absolute values of momenta, interpreted as (virtual) masses of particles.

One can see the interpretation of the partition function with the new observables directly from (6). The u variables can be considered as points on S^3 . Thus, one has the measure of a Feynman diagram amplitude. The character function on the group $SU(2)$ is the eigenfunction of the Laplacian on the homogeneous S^3 and thus the Feynman propagator (with a fixed virtual mass for the particle [Bar06]).

This interpretation extends to the observables on manifolds other than S^3 . For example, for the case of S^1 considered above in (12) interpreting $\chi_j(g) = \chi_j(u_t u_s^{-1})$ as a propagator we have the interpretation that the observable is given by the amplitude of propagation of a particle on the sphere from u_s to the location u_t . The original observable is then recovered as the average amplitude of propagation over the whole of the spacetime S^3 , which vanishes except for $j = 0$.

3.2. Edge observables

A different set of observables was constructed in [FL06, BNG09]. These are the ‘edge observables’. This is defined by a graph Γ consisting of edges of a triangulation, labelled by an angle $0 \leq \theta_e \leq 2\pi$ associated with each edge e . The amplitudes of these observables are then defined by inserting a factor

$$K_\theta(l) = \frac{\sin\left(\frac{\theta}{2}(2l+1)\right)}{(2l+1) \sin\frac{\theta}{2}}$$

for each edge in the graph. This means that the observable is

$$F(l_1, l_2, \dots) = K_{\theta_1}(l_1)K_{\theta_2}(l_2) \dots,$$

and the Ponzano–Regge state sum (1) with the edge observables is

$$Z(F) = \sum_{l_e} \prod_{\text{interior edges}} (-1)^{2l_e} (2l_e + 1) \prod_{\text{tetrahedra}} \begin{Bmatrix} l_1 & l_2 & l_3 \\ l_4 & l_5 & l_6 \end{Bmatrix} F(l_1, l_2, \dots). \quad (15)$$

These observables are also momentum observables, with the conservation of momentum being respected at each vertex of the graph Γ . This is similar to the conservation of momentum for the new group observables, with the principal difference being that in this case momentum space is curved. The partition function is zero unless there is, at each vertex, a spherical polygon with side lengths given by the angles $\theta_e/2$ incident at the vertex. For a 2-valent vertex, the angles are required to be equal, expressing momentum conservation as before.

3.3. Limits of the Turaev–Viro model

The presence of two different momentum observables in the theory is at first puzzling. We will now elucidate their origin and differences by exhibiting both sets of observables as limits, at least heuristically, of the same observables for the Turaev–Viro model, in which the cosmological constant Λ and the gravitation constant G appear in complementary roles. Our thesis is that the edge observables are obtained from a $\Lambda \rightarrow 0$ limit, and the new observables from a $G \rightarrow 0$ limit of the Turaev–Viro model. Thus, the edge observables pertain to quantum gravity without a cosmological constant, whereas the new observables, at least locally, pertain to quantum field theory on a 3-sphere.

3.3.1. The limits. The Turaev–Viro model is defined by a formula analogous to (1), with the 6j-symbol and dimensions being replaced by their quantum deformations, which depend on an integer parameter r , also written as the deformation parameter $q = e^{i\pi/r}$. The quantum deformation of the dimension factor $(-1)^{2l}(2l + 1)$ is

$$\dim_q l = (-1)^{2l} \frac{\sin \frac{\pi}{r}(2l + 1)}{\sin \frac{\pi}{r}}.$$

The partition function is

$$Z_{\text{TV}} = N^{-v} \sum_{l_e=0}^{(r-2)/2} \prod_{\text{interior edges}} \dim_q l_e \prod_{\text{tetrahedra}} \begin{Bmatrix} l_1 & l_2 & l_3 \\ l_4 & l_5 & l_6 \end{Bmatrix}_q. \quad (16)$$

The main new features are the normalization factor N^{-v} , with v being the number of vertices, and the dependence on the integer r , both explicitly in the limit of the sums, and via the deformation parameter $q = e^{i\pi/r}$. The model is finite and well defined for any compact manifold, and is independent of the choice of triangulation.

Observables for the model are defined by multiplying the summand of the partition function by a function $F(l_1, l_2, \dots)$ of some of the l s in the partition function [Bar03]. The l variables on which F depends are those lying on some subgraph Γ of the edges of some triangulation.

The relevant observables for this paper are the ‘momentum space’ observables, where the function introduced is the product of a Fourier kernel

$$\bar{K}_j(l) = (-1)^{2j} \frac{\sin \frac{\pi}{r}(2j + 1)(2l + 1)}{\sin \frac{\pi}{r}(2l + 1)}$$

for each edge, so

$$F = \bar{K}_{j_1}(l_1)\bar{K}_{j_2}(l_2)\dots$$

As announced in [Bar03], and proved in [BGIM07], the partition function for S^3 with these observables is equal to a constant times the q version of the relativistic spin network invariant [Bar98, CY97, Yok96] of the graph Γ with its edges labelled with j_1, j_2, \dots ,

$$Z_{\text{TV}}(S^3, \Gamma) = Z_{\text{TV}}(S^3) \langle \Gamma(j_1, j_2, \dots) \rangle_q. \quad (17)$$

Heuristically, the Turaev–Viro model, which is a quantization of three-dimensional GR with a cosmological constant, can be seen to reduce to the Ponzano–Regge state sum when one takes the limit $r \rightarrow \infty$. However, this limiting process is subtle and there are two different ways in which one can take a limit of the Turaev–Viro momentum space observables.

(i)

$$r \rightarrow \infty, \quad j_i \text{ constant.}$$

(ii)

$$r \rightarrow \infty, \quad \frac{2j_i + 1}{r} \rightarrow \frac{\theta_i}{2\pi}, \text{ a constant.}$$

Limit (i) is the limit which gives the observables in this paper, as is clear from the limit

$$\langle \Gamma(j_1, j_2, \dots) \rangle_q \rightarrow \langle \Gamma(j_1, j_2, \dots) \rangle,$$

which is obvious from the defining spin network formulae.

The second limit, (ii), is somewhat harder to treat rigorously. It is calculated explicitly for Γ a trefoil knot in [Bar05] and its aspects are generalized to any knot in [BNG09, Dub04]. It seems a reasonable conjecture that when all the θ_e are sufficiently small, then a limit of the Turaev–Viro partition function with the Fourier kernel observables gives the Ponzano–Regge partition function with the edge observables. This is consistent with the fact that in this limit, and with l fixed,

$$\frac{\bar{K}_j(l)}{\dim_q l} \rightarrow K_\theta(l).$$

3.4. Physical parameters in the limits

3.4.1. The minimum mass. To understand the physics of these limits we will introduce the dimensionful parameters into the theory. In the Turaev–Viro partition function without observables there is just one parameter, r . On the other hand, quantum gravitational physics would seem to require three parameters, the gravitation constant G , Planck’s constant \hbar and the cosmological constant Λ . The constant r can be written in terms of these three parameters, but then it would seem that two of the parameters are redundant.

The resolution of this paradox is that the two additional parameters play a role when observables are introduced. In fact, in three dimensions $1/G$ is a unit of mass and both $\hbar G$ and $1/\sqrt{\Lambda}$ are units of length. Therefore, one can multiply the purely numerical measures of mass or length in Turaev–Viro observables by one of these units to get ‘physical’ masses or lengths. The reason this is worthwhile is that in considering the scaling behaviour, it is more useful to consider the physical masses (for example) to be fixed and scale G than it is to consider G fixed and scale the numerical masses. This means that the physical mass converges to a definite value in a scaling limit (rather than, say, 0 or ∞).

In this paper, scales of lengths are not considered. Since the role of \hbar is to relate the scale of masses to the scale of lengths, its scaling is irrelevant. It can be assumed to be a constant throughout. The two units of interest are therefore the two mass scales $1/G$ and $\hbar\sqrt{\Lambda}$.

In the Turaev–Viro model, the numerical values of the masses for the momentum observables range from the minimum $j + 1/2 = 1/2$ to maximum $j + 1/2 = (r - 1)/2$, the ratio between them being r , to leading order. The physical models for these masses [Bar03] are the zonal spherical functions on S^3 , which for a sphere of radius $1/\sqrt{\Lambda}$ have masses $m = \hbar\sqrt{\Lambda}(2j + 1)$, using as definition for mass m the eigenvalue equation

$$\nabla^2\phi = \left(-\frac{m^2}{\hbar^2} + \text{const.}\right)\phi.$$

Thus, the numerical masses are multiplied by the physical unit $2\hbar\sqrt{\Lambda}$. This unit is considered to be of cosmological origin, since the minimum mass corresponds to a particle with a wavelength given by the circumference of S^3 .

3.4.2. The maximum mass. The model of a particle wavefunction on a classical geometry given in the previous section does not account for the maximum mass $\hbar\sqrt{\Lambda}(r - 1)$ in the Turaev–Viro model. A completely different argument based on general relativity can be used to identify this maximum mass in terms of G .

The Einstein equation in three dimensions is written as

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi GT_{\mu\nu},$$

the convention being that the constant $8\pi G$ is the same as in four dimensions.

The model for a particle is a conical defect in a spherical geometry. From the Einstein equation, the defect angle is given by $8\pi Gm$ [DJH84]. Since the defect angle is less than 2π , the maximum mass is therefore just below $1/4G$. Hence,

$$\frac{1}{4G} = \hbar\sqrt{\Lambda}r.$$

This can be rearranged to give

$$r = \frac{1}{4\sqrt{\Lambda G\hbar}}. \quad (18)$$

Now it is possible to rewrite the limits using the physical constants. The physical mass on the i th edge is defined as

$$m_i = (2j_i + 1)\hbar\sqrt{\Lambda} = \theta_i/8\pi G,$$

and according to the above argument, θ_i is the defect angle of the corresponding geometry.

The two limits are as follows.

- m_i, j_i, Λ constant, $G \rightarrow 0, \theta_i \rightarrow 0$.
- m_i, θ_i, G constant, $\Lambda \rightarrow 0, j_i \rightarrow \infty$.

3.5. The functional integral

A more precise relation with the physical constants can be determined using the functional integral picture. The Turaev–Viro model can be written as a functional integral with action given by the difference of two Chern–Simons actions for $SU(2)$ connections A_a^+ and A_a^- [Wit88],

$$S = \frac{k}{4\pi} \int \text{CS}(A_a^+) - \text{CS}(A_a^-),$$

where a is an $su(2)$ Lie-algebra index, that is $[A_a^\pm, A_b^\pm] = \epsilon_{abc} A^{c\pm}$, and

$$\text{CS}(A) = \int \text{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right).$$

Introducing the physical scales of a gravitation constant G , Planck constant \hbar and cosmological constant Λ by

$$k = \frac{1}{4\sqrt{\Lambda G \hbar}},$$

and changing variables to the usual fields of first-order gravity,

$$\omega_a = \frac{1}{2} (A_a^+ + A_a^-),$$

$$e_a = \frac{1}{2\sqrt{\Lambda}} (A_a^+ - A_a^-)$$

gives the familiar form

$$\begin{aligned} S &= \frac{1}{4\pi G \hbar} \int e_a \wedge d\omega^a + \epsilon^{abc} e_a \wedge \omega_b \wedge \omega_c + \frac{\Lambda}{3} \epsilon^{abc} e_a \wedge e_b \wedge e_c \\ &= \frac{\pm 1}{16\pi G \hbar} \int (R - 2\Lambda) dV. \end{aligned} \quad (19)$$

Of course, at this stage, two of the three scales are redundant, since the partition function without observables contains only one parameter, r . But the momentum observables considered here introduce a second parameter and the length observables (if we were included them) would introduce a third parameter.

An observable for the Chern–Simons functional integral framework is given by a generalized Wilson loop that is supported on a graph. It is specified by a representation of the gauge group for each edge of the graph and an intertwining operator for each vertex. According to Witten [Wit89a], the expectation value of this observable, in the case of S^3 , is the corresponding quantum group evaluation for these data labelling a plane projection of the graph, using also the R -matrix for the quantum group at crossings. This evaluation is called a spin network evaluation for the quantum group.

The quantum group relativistic spin network evaluation $\langle \Gamma(j_1, j_2, \dots) \rangle_q$, which appears in (17), is an example of a spin network evaluation [Yet98]. The quantum group is $U_q sl_2 \times U_q sl_2$ (with one factor using q^{-1} in place of q), and the representations are (j, j) . The Wilson loop observable $W(j_1, j_2, \dots)$ for the functional integral that generates this expectation value is therefore the function of the connection given by representing each edge as the parallel transport operator for $SU(2) \times SU(2)$ in the representation (j, j) , and each vertex by the canonical intertwining operator for the $SU(2)$ relativistic spin networks.

The functional integral with the observable is

$$Z(W) = \int [de][d\omega] e^{iS} W(j_1, j_2, \dots).$$

The $G \rightarrow 0$ limit is explained in a conceptual way using this formula. As the j in the observable are fixed in this limit, it simply consists of taking the semiclassical limit in the functional integral. For this, the functional integral can be replaced by an integral over the space of classical fields given by the critical points of the effective action. The effective action for Chern–Simons is the same formula as the classical action, but with k replaced by $r = k + 2$ [AGLR90]. This means that the physical constants are related by (18), as before.

For the case of S^3 , the critical points of the effective action are all gauge equivalent to the trivial connection, and so, since $W(j_1, j_2, \dots)$ is gauge invariant, the functional integral just

amounts to evaluating W on the trivial connection. This is just the alternative definition of the relativistic spin network for $SU(2)$, and so leads to formula (5).

Note that we can make direct contact with the formulation of the observables in the Ponzano–Regge context. To see this consider the observable given by inserting $\chi_j(g_e^+(g_e^-)^{-1})$ into the Chern–Simons path integral for each edge of the graph, where g_e^\pm is the parallel transport with respect to A^\pm along e . As the path integral is gauge invariant, one can replace g_e^+ with $u_{t(e)}g_e^+u_{s(e)}^{-1}$. Therefore, this observable evaluates to the same as $\chi_j(u_{t(e)}g_e^+u_{s(e)}^{-1}(g_e^-)^{-1})$. By gauge averaging over the u for each vertex we obtain the observable $W(j_1, j_2, \dots)$. This is done using the same argument from [Bar98] as that leading to equation (10). On the other hand, the critical points are simply the connections gauge equivalent to the trivial one, so the non-gauge-invariant form of the observable reduces to (6) considered in the Ponzano–Regge model.

The $\Lambda \rightarrow 0$ limit by contrast is not a semiclassical limit. Setting $\Lambda = 0$ in the gravitational action (19) gives the Chern–Simons action for $ISU(2)$ [Wit89b]. This is an Inönü–Wigner contraction of the gauge group, which suggests the corresponding contraction of the representation matrix elements [IW53]. In such a contraction, the representation labels are scaled simultaneously with the group contraction so that the matrix elements of the representation converge. This is exactly the situation in the $\Lambda \rightarrow 0$ limit; however, studying this systematically would stray too far from the main aim of this paper.

4. Discussion

In this paper, we discussed observables coupled to the group elements in the holonomy formulation of the Ponzano–Regge model. Specializing to observables that are products of characters, we showed that we recover the relativistic spin network evaluation. This also shows very simply that these have an interpretation as the evaluation of momentum-labelled Feynman diagrams on the sphere.

The previously considered observables coupled to the spin labels also have the character of momentum observables, we elucidate the presence of two types of momentum observables by showing that they arise as two different limits of the momentum observables in the Turaev–Viro model. To understand the physics of this limit, we reintroduced the dimensionful quantities. This demonstrates that the observables introduced in this paper, and the Ponzano–Regge model on the whole, can be understood as the semiclassical limit of the Turaev–Viro model.

To further see how this happens, we consider the Chern–Simons path integral formulation of the Turaev–Viro model. The analogue of the holonomy observables can be seen to be given by introducing characters depending on the product of holonomies $g^+(g^-)^{-1}$. This has a natural interpretation in terms of the geometry of a sphere as it parametrizes the coset space of $\text{spin}(4) \cong SU(2) \times SU(2)$ with respect to the diagonal subgroup $SU(2)_d$. This coset space is just the homogeneous 3-sphere with $\text{spin}(4)$ as it is the global group of symmetries, our scalar observables do not see the local rotational symmetry $SU(2)_d$ but only the translational part.

This suggests that these observables on general 3-manifolds can be understood as particles propagating in a locally flat Cartan geometry modelled on $SU(2) \times SU(2)/SU(2)_d \cong S^3$. The overall amplitude would then be obtained by averaging over the moduli space of flat Cartan geometries.

One can also consider doing a further limit that takes us from the model based on $SU(2)$ to the Abelian model based on \mathbb{R}^3 . These should be the $G \rightarrow 0$ limit for the edge observables which can be considered as a semiclassical limit. This was discussed as a commutative limit of the effective field theory in [FL06], and the $\Lambda \rightarrow 0$ limit of the group observables, which

should give the same physics again in a dual picture. As conjectured above, this should be achieved by an Inönü–Wigner contraction of the group $SU(2)$.

Note that in this work, we find some of the dualities found by Majid and Schroers in [MS09] at the level of the particles propagating on (non-commutative) spacetimes at the level of the state sums. In particular, the dual limits of the Turaev–Viro observables lead to spin network evaluations that are semi-dual in the sense of [MS09].

Finally, it would be interesting to consider inserting edge-length observables in order to reveal the third physical parameter in the model. An example of the limit of length variables in the Turaev–Viro model is considered in [TW06], where it is shown that the value of the quantum $6j$ -symbol is asymptotic to a formula in spherical geometry. In our language, the limit is taking the Planck length $G\hbar \rightarrow 0$ while keeping Λ fixed. However, this limit cannot be compared directly with the limits of momentum observables considered in this paper, and to combine them is left to future work.

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