Black hole Area-Angular momentum-Charge inequality in dynamical non-vacuum spacetimes

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We show that the area-angular momentum-charge inequality $(A/(4\pi))^2 \geq (2J)^2 + (Q_{\rm E}^2 + Q_{\rm M}^2)^2$ holds for apparent horizons of electrically and magnetically charged rotating black holes in generic dynamical and non-vacuum spacetimes. More specifically, this quasi-local inequality applies to axially symmetric closed outermost stably marginally (outer) trapped surfaces, embedded in non-necessarily axisymmetric black hole spacetimes with non-negative cosmological constant and matter content satisfying the dominant energy condition.

PACS numbers: 04.70.-s, 04.20.Dw, 04.20.Cv

Introduction. Isolated stationary black holes in Einstein-Maxwell theory are completely characterized by their mass M, angular momentum J and electric and magnetic charges, $Q_{\rm E}$ and $Q_{\rm M}$. This *no hair* property is endorsed by the black hole uniqueness theorems leading to Kerr-Newman spacetimes. In these black hole solutions a mass-angular momentum-charge inequality enforces a lower bound for M. Such a constraint among M, J, $Q_{\rm E}$ and $Q_{\rm M}$ is however lost in the extended Kerr-Newman family, including singular solutions without a horizon. In this sense, the mass-angular momentum-charge inequality follows when the physical principle of (weak) cosmic censorship, namely the absence of *naked singularities*, is advocated. Weak cosmic censorship conjecture provides a *dynamical principle* aiming at preserving predictability and playing a crucial role in our understanding of classical gravitational collapse. This picture motivates the study of extensions of the total mass-angular momentum-charge inequality to dynamical contexts, something accomplished in [1] for vacuum axially symmetric spacetimes. In more generic scenarios, in particular incorporating matter, it is natural to consider a quasi-local version of the inequality not involving global spacetime quantities (see [2] for a review). An appropriate starting point is the area-angular momentum-charge inequality $(A/(4\pi))^2 \geq (2J)^2 + (Q_{\rm E}^2 + Q_{\rm M}^2)^2$ also holding in the stationary vacuum case. This inequality (for $Q_{\rm M} = 0$) has been proved to hold for stationary axisymmetric spacetimes with matter in [3–6], although requiring electrovacuum in a neighborhood of the horizon. Regarding the dynamical case [7–10], a proof has been presented for the non-vacuum uncharged case [11] and the area-charge inequality [12] (in absence of any symmetries). Here we extend the full area-angular momentum-charge inequality, in particular incorporating the magnetic charge, to generic non-axisymmetric dynamical non-vacuum black hole spacetimes (axial symmetry only required on the horizon). This completes the discussion of this inequality in the Einstein-Maxwell context.

The result. The area-angular momentum-charge inequality applies to horizon sections satisfying a stability condition. Following the approach in [11], we consider a closed marginally outer trapped surface $\mathcal{S}$ satisfying a (spacetime) *stably outermost* condition in the sense of [13, 14] (see Definition 1 below for details). Then the following result holds:

Theorem 1. *Given an axisymmetric closed marginally trapped surface $\mathcal{S}$ satisfying the (axisymmetry-compatible) spacetime stably outermost condition, in a spacetime with non-negative cosmological constant and matter content fulfilling the dominant energy condition, it holds the inequality*

$$(A/(4\pi))^2 \geq (2J)^2 + (Q_{\rm E}^2 + Q_{\rm M}^2)^2 \tag{1}$$

where A is the area of $\mathcal{S}$ and J, $Q_{\rm E}$ and $Q_{\rm M}$ are, respectively, the total (gravitational and electromagnetic) angular momentum, the electric and the magnetic charges associated with $\mathcal{S}$.

This quasi-local result holds in fully dynamical spacetimes without bulk symmetries and with arbitrary (non-exotic) matter possibly crossing the horizon. In particular, it extends to generic scenarios the inequality proved in [5, 6] for Killing horizons in stationary axisymmetric spacetimes, with electrovacuum around the black hole (matter can surround but not cross the horizon). Axisymmetry is required only on $\mathcal{S}$, so that a canonical notion of angular momentum J can be employed. The stably outermost and dominant energy conditions imply, for some non-vanishing J, $Q_{\rm E}$ or $Q_{\rm M}$ and in our four-dimensional spacetime context, the spherical topology of the surface $\mathcal{S}$. For Killing horizons [3, 5, 6] a rigidity result holds, namely equality in (1) implies the degeneracy of the Killing horizon (vanishing of the *surface gravity*), providing a characterization of extremality. In the present dynamical setting, with no spacetime stationary Killing field, rigidity statements involve rather the characterization of the induced metric on $\mathcal{S}$ as an *extremal throat* (i.e. with the geometry of a horizon section in the extremal Kerr-Newman family) and as a section of an instantaneous (non-expanding) isolated horizon [15]. We postpone the discussion of the rigidity part of the result to [16], where full details of the proof of inequality (1) [required to make the rigidity statement precise] are presented.

Main geometric elements. The proof of (1) proceeds by, first, casting the stably outermost condition for marginally outer trapped surfaces as a geometric inequality leading to an action functional $\mathcal{M}$ on $\mathcal{S}$ and, second, by solving the associated variational problem. Following [11], we start by introducing the general geometric elements and by formulating the geometric inequality following from the stability of $\mathcal{S}$.

Let (M, g_{ab}) be a 4-dimensional spacetime with Levi-Civita connection ∇_a, satisfying the dominant energy condition and with non-negative cosmological constant $\Lambda \geq 0$. Let us consider an electromagnetic field with strength field (Faraday) tensor F_{ab}, so that $F_{ab} = \nabla_a A_b - \nabla_b A_a$ on a local chart (corresponding to a given section of the $U(1)$-fibre-bundle, possibly non-trivial to account for magnetic monopoles).

Let us consider a closed orientable 2-surface $\mathcal{S}$ embedded in (M, g_{ab}). Regarding its intrinsic geometry, let us denote the induced metric as q_{ab} with connection D_a, Ricci scalar as ${}^2R$, volume element ϵ_{ab} and area measure dS. Regarding its extrinsic geometry, we first consider normal (respectively, outgoing and ingoing) null vectors ℓ^a and k^a normalized as $\ell^a k_a = -1$. This fixes ℓ^a and k^a up to (boost) rescaling factor. The extrinsic curvature elements needed in our analysis are the expansion $\theta^{(\ell)}$, the shear $\sigma^{(\ell)}_{ab}$ and the normal fundamental form $\Omega^{(\ell)}_a$ associated with the outgoing null normal ℓ^a

$$\begin{aligned} \theta^{(\ell)} &= q^{ab}\nabla_a \ell_b \ , \ \ \sigma^{(\ell)}_{ab} = q^c{}_a q^d{}_b \nabla_c \ell_d - \frac{1}{2}\theta^{(\ell)} q_{ab} \\ \Omega^{(\ell)}_a &= -k^c q^d{}_a \nabla_d \ell_c \ . \end{aligned} \tag{2}$$

We require the geometry of $\mathcal{S}$ to be axisymmetric with axial Killing vector η^a on $\mathcal{S}$. That is, $\mathcal{L}_\eta q_{ab} = 0$ and η^a has closed integral curves, vanishes exactly at two points on $\mathcal{S}$ and is normalized so that its integral curves have an affine length of 2π. Besides, we demand $\mathcal{L}_\eta \Omega^{(\ell)}_a = \mathcal{L}_\eta A_a = 0$ and adopt a tetrad $(\xi^a, \eta^a, \ell^a, k^a)$ on $\mathcal{S}$ adapted to axisymmetry, namely $\mathcal{L}_\eta \ell^a = \mathcal{L}_\eta k^a = 0$ with ξ^a a unit vector tangent to $\mathcal{S}$ satisfying $\xi^a\eta_a = \xi^a \ell_a = \xi^a k_a = 0$, $\xi^a \xi_a = 1$. We can then write $q_{ab} = \frac{1}{\eta}\eta_a\eta_b + \xi_a\xi_b$ (with $\eta = \eta^a\eta_a$) and $\Omega^{(\ell)}_a = \Omega^{(\eta)}_a + \Omega^{(\xi)}_a$ (with $\Omega^{(\eta)}_a = \eta^b\Omega^{(\ell)}_b \eta_a/\eta$ and $\Omega^{(\xi)}_a = \xi^b \Omega^{(\ell)}_b \xi_a$).

We introduce now the expressions for J, $Q_{\rm E}$ and $Q_{\rm M}$. First, the electric and magnetic field components normal to $\mathcal{S}$ are

$$E_\perp = F_{ab}\ell^a k^b \quad , \quad B_\perp = {}^*F_{ab}\ell^a k^b \ , \tag{3}$$

where ${}^*F_{ab}$ is the Hodge dual of F_{ab}. The above-required axisymmetry allows the introduction of the following canonical notion of angular momentum on $\mathcal{S}$ [2, 17–19]

$$J = J_{\rm K} + J_{\rm EM} = \frac{1}{8\pi}\int_{\mathcal{S}} \Omega^{(\ell)}_a \eta^a dS + \frac{1}{4\pi}\int_S (A_a\eta^a) E_\perp dS \ , \tag{4}$$

where $J_{\rm K}$ and $J_{\rm EM}$ correspond, respectively, to (Komar) gravitational and electromagnetic contributions to the total J. Electric and magnetic charges can be expressed as (e.g. [20, 21])

$$Q_{\rm E} = \frac{1}{4\pi}\int_S E_\perp dS \ , \ Q_{\rm M} = \frac{1}{4\pi}\int_S B_\perp dS \ . \tag{5}$$

We characterize now $\mathcal{S}$ as a stable section of a (quasi-local) black hole horizon. First, we require $\mathcal{S}$ to be a marginally outer trapped surface, that is $\theta^{(\ell)} = 0$. Second, we demand $\mathcal{S}$ to be stably outermost as introduced in [13, 14] (see also [22, 23]). More specifically we require $\mathcal{S}$ to be (axisymmetry-compatible) *spacetime stably outermost* [11, 12]:

Definition 1. *A closed marginally trapped surface $\mathcal{S}$ is referred to as* spacetime stably outermost *if there exists an outgoing ($-k^a$-oriented) vector $X^a = \gamma\ell^a - \psi k^a$, with $\gamma \geq 0$ and $\psi > 0$, with respect to which $\mathcal{S}$ is stably outermost: $\delta_X\theta^{(\ell)} \geq 0$. If, in addition, X^a (i.e. γ, ψ) and $\Omega^{(\ell)}_a$ are axisymmetric, we will refer to $\delta_X\theta^{(\ell)} \geq 0$ as an (axisymmetry-compatible) spacetime stably outermost condition.*

Here, the operator δ_X is the variation operator on the surface $\mathcal{S}$ along the vector X^a discussed in [13, 14] (see also [24, 25]). We formulate now the following lemma:

Lemma 1. *Let $\mathcal{S}$ be a closed marginally trapped surface $\mathcal{S}$ satisfying the (axisymmetry-compatible) spacetime stably outermost condition. Then, for all axisymmetric α on $\mathcal{S}$*

$$\int_{\mathcal{S}} \left[|D\alpha|^2 + \frac{1}{2}\alpha^2\, {}^2R\right] dS \geq \int_{\mathcal{S}} \alpha^2 \left[|\Omega^{(\eta)}|^2 + (E_\perp^2 + B_\perp^2)\right] dS, \tag{6}$$

with $|D\alpha|^2 = D_a\alpha D^a\alpha$ and $|\Omega^{(\eta)}|^2 = \Omega^{(\eta)}_a \Omega^{(\eta)a}$.

The proof is a direct application of Lemma 1 in [11]. Given the vector $X^a = \gamma\ell^a - \psi k^a$, for all α on $\mathcal{S}$ it holds [11]

$$\begin{aligned} &\int_{\mathcal{S}} \left[D_a\alpha D^a\alpha + \frac{1}{2}\alpha^2\, {}^2R\right] dS \geq \\ &\int_{\mathcal{S}} \left[\alpha^2 \Omega^{(\eta)}_a \Omega^{(\eta)a} + \alpha\beta\sigma^{(\ell)}_{ab}\sigma^{(\ell)ab} + G_{ab}\alpha\ell^a(\alpha k^b + \beta\ell^b)\right] dS \ , \end{aligned} \tag{7}$$

with $\beta = \alpha\gamma/\psi$. First, since $\alpha\beta \geq 0$, the positive-definite quadratic term in the shear can be neglected. Second, we insert Einstein equation $G_{ab} + \Lambda g_{ab} = 8\pi(T^{\rm EM}_{ab} + T^{\rm M}_{ab})$, with $T^{\rm EM}_{ab}$ and $T^{\rm M}_{ab}$ the electromagnetic and matter stress-energy tensors. In particular, $T^{\rm EM}_{ab} = \frac{1}{4\pi}\left(F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{cd}F^{cd}\right)$. From the dominant energy condition on $T^{\rm M}_{ab}$, $\Lambda \geq 0$ and the null energy condition applying for $T^{\rm EM}_{ab}$, the Einstein tensor term in inequality (7) is bounded by below by $\alpha^2 8\pi T^{\rm EM}_{ab}\ell^a k^b$. Making use of (see e.g. [12, 21])

$$T^{\rm EM}_{ab}\ell^a k^b = \frac{1}{8\pi}\left[\left(\ell^a k^b F_{ab}\right)^2 + \left(\ell^a k^b\, {}^*F_{ab}\right)^2\right] \ , \tag{8}$$

inequality (6) follows by identifying $E_\perp$ and $B_\perp$ in (3). As a final remark, note that taking $\alpha = \text{const}$ in (6), a non-vanishing angular momentum or charge suffices to conclude the sphericity of $\mathcal{S}$ by applying the Gauss-Bonnet theorem.

The action functional and sketch of the proof. The proof of inequality (1) proceeds by solving a constrained variational problem on $\mathcal{S}$, in which J, $Q_{\rm E}$ and $Q_{\rm M}$ must be kept constant under otherwise arbitrary variations. We construct the corresponding action functional $\mathcal{M}$, by evaluating the geometric expression (6) in a specific coordinate system on $\mathcal{S}$.

First, on an axisymmetric sphere $\mathcal{S}$, a coordinate system can always be chosen such that

$$ds^2 = q_{ab}dx^a dx^b = e^\sigma\left(e^{2q}d\theta^2 + \sin^2\theta d\varphi^2\right) \ , \tag{9}$$

with axisymmetric σ and q satisfying $\sigma + q = c = \text{constant}$. Then $\eta^a = (\partial_\varphi)^a$, $\eta = e^\sigma \sin^2\theta$ and $dS = e^c dS_0$, with $dS_0 = \sin\theta d\theta d\varphi$. In particular, $A = 4\pi e^c$. Second, $\Omega^{(\ell)}_a$ expresses uniquely on a 2-sphere as $\Omega^{(\ell)}_a = \epsilon_{ab}D^b\tilde{\omega} + D_a\lambda$. Since $\Omega^{(\ell)}_a$ is axisymmetric, $\Omega^{(\eta)}_a = \epsilon_{ab}D^b\tilde{\omega}$ [11], and we can write

$$\Omega^{(\eta)}_a = \frac{1}{2\eta}\epsilon_{ab}D^b\bar{\omega} \ , \tag{10}$$

by introducing the potential $\bar{\omega}$, as $d\bar{\omega}/d\theta = (2\eta)d\tilde{\omega}/d\theta$, that satisfies $J_{\mathrm{K}} = [\bar{\omega}(\pi) - \bar{\omega}(0)]/8$ (cf. [11]). Third, from $\ell^a k^{b*}F_{ab} = \frac{1}{2}F_{ab}\epsilon^{ab}$ [12] and the axisymmetry of A_a

$$B_\perp = \frac{1}{e^c \sin\theta}\frac{dA_\varphi}{d\theta} \ . \tag{11}$$

Finally, following [10, 11] we choose $\alpha = e^{c-\sigma/2}$. Inserting it together with (9), (10), (11) into inequality (6), we get

$$8(c+1) \geq \mathcal{M}[\sigma, \bar{\omega}, E_\perp, A_\varphi] \ , \tag{12}$$

where $\mathcal{M}[\sigma, \bar{\omega}, E_\perp, A_\varphi]$ is the action functional

$$\mathcal{M}[\sigma, \bar{\omega}, E_\perp, A_\varphi] = \frac{1}{2\pi}\int_{\mathcal{S}}\left[4\sigma + \left(\frac{d\sigma}{d\theta}\right)^2 + \frac{1}{\eta^2}\left(\frac{d\bar{\omega}}{d\theta}\right)^2 + 4e^{2c-\sigma}E_\perp^2 + 4e^{-\sigma}\left(\frac{1}{\sin\theta}\frac{dA_\varphi}{d\theta}\right)^2\right]dS_0. \tag{13}$$

Inequality (1) follows by solving the variational problem defined by $\mathcal{M}[\sigma, \bar{\omega}, E_\perp, A_\varphi]$. In its form (13), enforcing the constraints on J, Q_{E} and Q_{M} is not straightforward. This is addressed by introducing new potentials ω, χ and ψ on $\mathcal{S}$

$$\begin{aligned} \frac{d\psi}{d\theta} &= E_\perp e^c \sin\theta \quad , \quad \chi = A_\varphi \ , \\ \frac{d\omega}{d\theta} &= 2\eta\frac{d\tilde{\omega}}{d\theta} + 2\chi\frac{d\psi}{d\theta} - 2\psi\frac{d\chi}{d\theta} \ , \end{aligned} \tag{14}$$

with the crucial property that J, Q_{E} and Q_{M} are written as

$$J = \frac{\omega(\pi) - \omega(0)}{8}, Q_{\mathrm{E}} = \frac{\psi(\pi) - \psi(0)}{2}, Q_{\mathrm{B}} = \frac{\chi(\pi) - \chi(0)}{2} \tag{15}$$

Physical parameters in inequality (1) can then be kept constant by fixing ω, χ and ψ on the axis as a boundary condition in the variational problem (note that $\bar{\omega}$ in (10) is an appropriate potential to control the Komar J_{K}, but not for the total J). In terms of σ, ω, χ and ψ the action functional reads

$$\mathcal{M}[\sigma, \omega, \psi, \chi] = \frac{1}{2\pi}\int_{\mathcal{S}}\Big[4\sigma + |D\sigma|^2 + \frac{|D\omega - 2\chi D\psi + 2\psi D\chi|^2}{\eta^2} + \frac{4}{\eta}(|D\psi|^2 + |D\chi|^2)\Big]dS_0 \ , \tag{16}$$

where $\mathcal{M}$ is formally promoted beyond axisymmetry. The proof of (1) proceeds in two steps (see details in [16]). First

$$A \geq 4\pi e^{\frac{\mathcal{M}-8}{8}} \ , \tag{17}$$

follows directly from (12) and $A = 4\pi e^c = 4\pi e^{\sigma(0)}$. Second, by solving the variational problem defined by the action functional (16) with values of ω, ψ, χ fixed on the axis and determined from relations (15), it is shown

$$\mathcal{M} \geq \mathcal{M}_0 = 8\ln\sqrt{(2J)^2 + (Q_{\mathrm{E}}^2 + Q_{\mathrm{M}}^2)^2} + 8 \ , \tag{18}$$

where $\mathcal{M}_0$ corresponds to the evaluation of $\mathcal{M}$ on an extremal solution in the (magnetic) Kerr-Newman family with given J, Q_{E} and Q_{M}. Inequality (1) follows from the combination of inequalities (17) and (18). Full intermediate details of the proof, in particular addressing the resolution of the variational problem along the lines in [8] will be presented in [16].

Explicit proof of the vanishing magnetic charge case. Complementary to the discussion above of the elements in the proof of the full inequality (1), we present a straightforward explicit proof of the case $Q_{\mathrm{M}} = 0$ by matching the reasoning in [5]. The result in [5] states that a *subextremal* stationary black hole, in the sense that trapped surfaces exist in the interior vicinity of the event horizon [28], satisfies the strict inequality (1). Namely,

$$\textit{horizon subextremal condition} \Rightarrow p_J^2 + p_Q^2 < 1 \ . \tag{19}$$

where $p_J = \frac{8\pi J}{A}$ and $p_Q = \frac{4\pi Q_{\mathrm{E}}^2}{A}$. This implication (actually, its logical counter-reciprocal) is cast in [5] as a variational problem on a Killing horizon section. The action functional in [5] is constructed by combining the *horizon subextremal condition* in (19) with the condition $p_J^2 + p_Q^2 < 1$. The key remark here is to show that such variational problem, defined solely on a sphere $\mathcal{S}$, has full applicability in the generic dynamical case beyond the original stationary and spacetime axisymmetric setting of [5]. More specifically, we show that our expressions for p_J, p_Q and the stably outermost condition (12), valid in the generic dynamical non-vacuum case, match exactly the expressions in [5] for the elements in (19). Therefore, the proof in [5] extends *exactly* to the generic case.

From the comparison between the 4-dimensional stationary axisymmetric line element in [5] with our line element (9) on $\mathcal{S}$ and between the respective integrands of the Komar angular momentum, we introduce new fields U and V from σ and $\bar{\omega}$

$$\begin{aligned} \hat{u} &= e^\sigma \ , \ \hat{u}_N = e^c \ , \ U = \frac{1}{2}\ln\left(\frac{\hat{u}}{\hat{u}_N}\right) \ , \\ V &= \frac{e^\sigma \sin\theta}{2\eta^2}\frac{d\bar{\omega}}{d\theta} \ . \end{aligned} \tag{20}$$

Regarding the electromagnetic potentials, we define S and T

$$S = -E_\perp e^{c/2} \quad , \quad T = A_\varphi e^{-c/2} \ . \tag{21}$$

Inserting these fields in (4) and (5) above, using $A = 4\pi e^c$ and changing to variable $x = \cos\theta$ we get

$$\begin{aligned} p_J &= -\frac{1}{2}\int_{-1}^1 Ve^{2U}(1-x^2)dx + \int_{-1}^1 ST dx \\ p_Q &= \frac{1}{4}\left(\int_{-1}^1 S dx\right)^2 \ , \end{aligned} \tag{22}$$

that coincide exactly with expressions in Eqs. (23) and (24) in [5]. Regarding the stability (subextremal) condition, we insert (20) and (21) in condition (12) [with strict inequality]. Using $\int_{-1}^1 U dx = -\int_{-1}^1 U' x dx$ (following from $U(1) = U(-1) = 0$, as a regularity condition for q on the axis) and denoting with a prime the derivative with respect to x, we find

$$1 > \frac{1}{2}\int_{-1}^1 (U'^2 + V^2)(1-x^2) - 2U'x + e^{-2U}(S^2 + T'^2)dx \tag{23}$$

This matches exactly the *horizon subextremal condition* inequality (28) in [5]. Considering expressions (22) and (23), the same variational problem used in the proof of (19) can be defined in the generic case. This is a complete proof of inequality (1) with vanishing $Q_{\rm M}$ in the strictly stably case.

Discussion. We have shown that $(A/(4\pi))^2 \geq (2\pi J)^2 + (Q_{\rm E}^2 + Q_{\rm M}^2)^2$ holds for axisymmetric stable marginally trapped surfaces in generically dynamical, non-necessarily axisymmetric spacetimes with ordinary matter that can be crossing the horizon. More specifically, we have presented a complete proof of the strictly stable case with $Q_{\rm M} = 0$ and provided the key elements for the proof of the general inequality. From the perspective of the *no hair* property of vacuum stationary black holes, the extension of inequality (1) to fully dynamical non-vacuum situations represents a remarkable result. Indeed, although parameters $A, J, Q_{\rm E}$ and $Q_{\rm M}$ do not longer fully characterize the black hole state and new degrees of freedom are required to describe the spacetime geometry, the generic incorporation of the latter is still constrained by inequality (1). Such a constraint represents a valuable probe into non-linear black hole dynamics. As a first remark, it gives support to the physical interpretation of the Christodoulou mass in dynamical settings (cf. discussion in [2]), in particular endorsing dynamical horizon [15] thermodynamics [29]. More generally, whereas inequality (1) follows originally in the Kerr-Newman family under the assumption of (weak) cosmic censorship, the present result is purely quasi-local involving no global condition on the spacetime, namely no asymptotic predictability. This suggests a link between cosmic censorship and marginally trapped surface stability to be further explored. In this context, assuming Penrose inequality (with no surface enclosing $\mathcal{S}$ with area smaller than A), inequality (1) refines the positive of mass theorem in terms of physical quantities: $16\pi M^2 \geq A \geq \sqrt{(8\pi J)^2 + (4\pi[Q_{\rm E}^2 + Q_{\rm M}^2])^2}$. Although for non-axisymmetric horizons we lack a canonical notion of angular momentum, appropriate quasi-local prescriptions for J should provide good estimates for a lower bound of M. Giving closed general expressions seems however difficult since, in contrast with the area-charge inequality [12], incorporating J involves a subtle variational problem (cf. [2]). In this sense, Ref. [16] discusses the close relation between the variational problem (on a 3-slice) employed in [1] for the proof of the spacetime mass-angular momentum-charge inequality and the present action functional $\mathcal{M}$ in (13) and (16), also closely related to (but different from) the functional used in [5]. Regarding the latter, we stress that electromagnetic potentials S and T in (21) follow straightforwardly (with no gauge choices involved) from the geometric formulation of the general stability condition in Lemma 1. This underlines the intrinsic interest of the *flux inequality* in Lemma 1 (and, more generally, its complete expression in [11]) for exploring further geometric aspects of stable black hole horizons.

Acknowledgments. We thank S. Dain, M. Reiris and W. Simon for the in-depth discussion of crucial aspects of this work and for their encouraging support. We would like also to thank A. Aceña, P. Aguirre and M. Ansorg for enlightening discussions. J.L.J. acknowledges the Spanish MICINN (FIS2008-06078-C03-01) and the Junta de Andalucía (FQM2288/219).

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[1] Chrusciel, P.T., Lopes Costa, J., Class. Quant. Grav. **26**, 235013 (2009); Costa, J.L., arXiv:0912.0838 (2009)
[2] Dain, S., arXiv:1111.3615 (2011)
[3] Ansorg, M., Pfister, H., Class. Quant. Grav. **25**, 035009 (2008)
[4] Hennig, J., Ansorg, M., Cederbaum, C., Class. Quantum Grav. **25**, 162002 (2008)
[5] Hennig, J., Cederbaum, C., Ansorg, M., Commun. Math. Phys. **293**, 449–467 (2010)
[6] Ansorg, M., Hennig, J., Cederbaum, C., Gen. Rel. Grav. **43**, 1205–1210 (2011)
[7] Dain, S., Phys. Rev. **D82**, 104010 (2010)
[8] Acena, A., Dain, S., Gabach Clement, M.E., Class. Quant. Grav. **28**, 105014 (2011)
[9] Gabach Clement, M.E., arXiv:1102.3834 (2011)
[10] Dain, S., Reiris, M., arXiv:1102.5215 (2011)
[11] Jaramillo, J.L., Reiris, M., Dain, S., arXiv:1106.3743 (2011)
[12] Dain, S., Jaramillo, J.L., Reiris, M., arXiv:1109.5602 (2011)
[13] Andersson, L., Mars, M., Simon, W., Phys. Rev. Lett. **95**, 111102 (2005)
[14] Andersson, L., Mars, M., Simon, W., Adv. Theor. Math. Phys. **12**, 853-888 (2008)
[15] Ashtekar, A., Krishnan, B., Liv. Rev. Relat. **7**, 10 (2004).
[16] Gabach Clement, M.E., Jaramillo, J.L., Reiris, M., in preparation (2011)
[17] Carter, B., Gen. Rel. Grav. **42**, 653–744 (2010)
[18] Simon, W., Gen. Rel. Grav. **17**, 439 (1985)
[19] Ashtekar, A., Beetle, C., Lewandowski, J., Phys.Rev. **D64**, 044016 (2001)
[20] Ashtekar, A., Fairhurst, S., Krishnan, B., Phys. Rev. **D62**, 104025 (2000)
[21] Booth, I., Fairhurst, S., Phys. Rev. **D77**, 084005 (2008)
[22] Hayward, S., Phys. Rev. D **49**, 6467 (1994)
[23] Racz, I., Class. Quant. Grav. **25**, 162001 (2008)
[24] Booth, I., Fairhurst, S., Phys. Rev. **D75**, 084019 (2007)
[25] Cao, L.M., JHEP **1103**, 112 (2011)
[26] Ashtekar, A., Krishnan, B., Phys. Rev. Lett. **89**, 261101 (2002)
[27] Ashtekar, A., Krishnan, B., Phys. Rev. D **68**, 104030 (2003)
[28] Horizon sections are *strictly stably outermost* with respect to outgoing directions (namely, *outer trapping horizons* [21, 22]).
[29] In this context, we note that inequalities (6) and (7) can be interpreted as upper bounds on certain *energy fluxes* defined by their right-hand-sides (and closely related to dynamical horizon fluxes [26, 27]). In particular, terms proportional to β in (7) correspond to gravitational and electromagnetic radiative degrees of freedom ($T^{\rm EM}_{ab}\ell^a\ell^b$ being the flux of the Poynting vector).