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Black hole area-angular momentum-charge inequality in dynamical nonvacuum spacetimes

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We show that the area-angular momentum-charge inequality $(A/(4\pi))^2 \geq (2J)^2 + (Q_E^2 + Q_M^2)^2$ holds for axisymmetric apparent horizons of electrically and magnetically charged rotating black holes in dynamical and nonvacuum spacetimes. More specifically, this quasilocal inequality applies to axially symmetric closed stably outermost marginally (outer) trapped surfaces, embedded in non-necessarily axisymmetric black hole spacetimes with non-negative cosmological constant and matter content satisfying the dominant energy condition.

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I. INTRODUCTION

Isolated stationary black holes in Einstein-Maxwell theory are completely characterized by their mass M, angular momentum J, and electric and magnetic charges, Q_E and Q_M. This "no hair" property is endorsed by the black hole uniqueness theorems leading to Kerr-Newman spacetimes. In these black hole solutions, a mass-angular momentum-charge inequality enforces a lower bound for M. Such a constraint among M, J, Q_E, and Q_M is, however, lost in the extended Kerr-Newman family, including singular solutions without a horizon. In this sense, the mass-angular momentum-charge inequality follows when the physical principle of (weak) cosmic censorship, namely, the absence of naked singularities, is advocated. Weak cosmic censorship conjecture provides a "dynamical principle" aiming at preserving predictability and playing a crucial role in our understanding of classical gravitational collapse. This picture motivates the study of extensions of the total mass-angular momentum-charge inequality to dynamical contexts, something accomplished in Ref. [1] for vacuum axially symmetric spacetimes. In more general scenarios, in particular incorporating matter, it is natural to consider a quasilocal version of the inequality not involving global spacetime quantities (see Ref. [2] for a review; cf. also Ref. [3]). An appropriate starting point is the area-angular momentum-charge inequality $(A/(4\pi))^2 \geq (2J)^2 + (Q_E^2 + Q_M^2)^2$ also holding in the stationary vacuum case. This inequality (for $Q_M = 0$) has been proven to hold for stationary axisymmetric spacetimes with matter in Refs. [4–7], although requiring electrovacuum in a neighborhood of the horizon. Regarding the dynamical case [8–11], a proof has been presented for the nonvacuum uncharged case [12] and for the area-charge inequality [13], the latter in absence of any symmetries (cf. also Ref. [14] for the inclusion of the cosmological constant and Ref. [15] for a related higher-dimensional result). Here, we extend the full area-angular momentum-charge inequality, in particular, incorporating the magnetic charge, to nonaxisymmetric dynamical nonvacuum black hole spacetimes where axial symmetry is only required on the horizon.

II. RESULT

The area-angular momentum-charge inequality applies to axisymmetric apparent horizons satisfying a stability condition. Following the approach in Ref. [12], we model sections of quasilocal black hole horizons [16–18] in terms of closed marginally outer trapped surfaces $\mathcal{S}$ satisfying a (spacetime) "stably outermost" condition in the sense of Ref. [19,20] (see Definition 1 below for details). Then the following result holds:

Theorem 1. *Given an axisymmetric closed marginally outer trapped surface $\mathcal{S}$ satisfying the (axisymmetry-compatible) spacetime stably outermost condition, in a spacetime with non-negative cosmological constant and matter content fulfilling the dominant energy condition, it holds the inequality*

$$(A/(4\pi))^2 \geq (2J)^2 + (Q_E^2 + Q_M^2)^2, \tag{1}$$

where A is the area of $\mathcal{S}$ and J, Q_E and Q_M are, respectively, the total (gravitational and electromagnetic) angular momentum, the electric and the magnetic charges associated with $\mathcal{S}$.

This quasilocal result for axisymmetric apparent horizons holds in dynamical spacetimes without bulk symmetries and with arbitrary (nonexotic) matter possibly crossing the horizon. In particular, it extends the strict inequality proved in Refs. [6,7] for Killing horizons in stationary axisymmetric spacetimes, with electrovacuum around the black hole (matter can surround but not cross the horizon). Axisymmetry is required only on $\mathcal{S}$, in order to make use of a canonical notion of angular momentum J [21] which, in particular, permits us to make a mathematically rigorous and nonambiguous statement. The stably outermost and dominant energy conditions imply, for some nonvanishing J, Q_E, or Q_M and, in our four-dimensional spacetime context, the spherical topology of the surface $\mathcal{S}$. For axisymmetric Killing horizons, Ref. [4] shows that degeneracy (i.e., vanishing of the surface gravity) implies the equality in inequality (1) [for a reciprocal result, see Ref. [24]]. More generally, in the present dynamical setting with no spacetime stationary Killing field, rigidity statements involve rather the characterization of the induced metric on $\mathcal{S}$ as an extreme Kerr-Newman sphere (i.e., with the geometry of a horizon section in the extremal Kerr-Newman family) and as a section of an instantaneous (nonexpanding) isolated horizon [17]. The discussion of this rigidity issue will be presented elsewhere.

III. MAIN GEOMETRIC ELEMENTS

The proof of inequality (1) proceeds by, first, casting the stably outermost condition for marginally outer trapped surfaces as a geometric inequality leading to an action functional $\mathcal{M}$ on $\mathcal{S}$ and, second, by solving the associated variational problem. Following Ref. [12], we start by introducing the general geometric elements and by formulating the geometric inequality following from the stability of $\mathcal{S}$.

Let (M, g_{ab}) be a four-dimensional spacetime with Levi-Civita connection ∇_a, satisfying the dominant energy condition and with non-negative cosmological constant $\Lambda \geq 0$. Let us consider an electromagnetic field with strength field (Faraday) tensor F_{ab}, so that $F_{ab} = \nabla_a A_b - \nabla_b A_a$ on a local chart (corresponding to a local section of the $U(1)$ fiber bundle, possibly nontrivial to account for magnetic monopoles).

Let us consider a closed orientable two-surface $\mathcal{S}$ embedded in (M, g_{ab}). Regarding its intrinsic geometry, let us denote the induced metric as q_{ab} with connection D_a, Ricci scalar as ${}^2R$, volume element ϵ_{ab}, and area measure dS. Regarding its extrinsic geometry, we first consider normal (respectively, outgoing and ingoing) future-oriented null vectors ℓ^a and k^a normalized as $\ell^a k_a = -1$. This fixes ℓ^a and k^a up to a (boost) rescaling positive factor. The extrinsic curvature elements needed in our analysis are the expansion $\theta^{(\ell)}$, the shear $\sigma^{(\ell)}_{ab}$, and the normal fundamental form $\Omega^{(\ell)}_a$ associated with the outgoing null normal ℓ^a

$$\theta^{(\ell)} = q^{ab}\nabla_a \ell_b, \qquad \sigma^{(\ell)}_{ab} = q^c{}_a q^d{}_b \nabla_c \ell_d - \frac{1}{2}\theta^{(\ell)} q_{ab}$$
$$\Omega^{(\ell)}_a = -k^c q^d{}_a \nabla_d \ell_c. \tag{2}$$

We require the geometry of $\mathcal{S}$ to be axisymmetric with axial Killing vector η^a on $\mathcal{S}$, i.e., $\mathcal{L}_\eta q_{ab} = 0$ with η^a having closed integral curves. Besides, we demand $\mathcal{L}_\eta \Omega^{(\ell)}_a = \mathcal{L}_\eta A_a = 0$ and adopt a tetrad $(\xi^a, \eta^a, \ell^a, k^a)$ on $\mathcal{S}$ adapted to axisymmetry, namely, $\mathcal{L}_\eta \ell^a = \mathcal{L}_\eta k^a = 0$ with ξ^a a unit vector tangent to $\mathcal{S}$ satisfying $\xi^a \eta_a = \xi^a \ell_a = \xi^a k_a = 0$, $\xi^a \xi_a = 1$. We can then write $q_{ab} = \frac{1}{\eta}\eta_a \eta_b + \xi_a \xi_b$ (with $\eta = \eta^a \eta_a$) and $\Omega^{(\ell)}_a = \Omega^{(\eta)}_a + \Omega^{(\xi)}_a$ (with $\Omega^{(\eta)}_a = \eta^b \Omega^{(\ell)}_b \eta_a/\eta$ and $\Omega^{(\xi)}_a = \xi^b \Omega^{(\ell)}_b \xi_a$).

We introduce now the expressions for J, $Q_{\rm E}$, and $Q_{\rm M}$. First, the electric and magnetic field components normal to $\mathcal{S}$ are

$$E_\perp = F_{ab}\ell^a k^b, \qquad B_\perp = {}^*F_{ab}\ell^a k^b, \tag{3}$$

where ${}^*F_{ab}$ is the Hodge dual of F_{ab}. The above-required axisymmetry allows the introduction of the following canonical notion of angular momentum on $\mathcal{S}$ [2,25–27]:

$$J = J_{\rm K} + J_{\rm EM}$$
$$= \frac{1}{8\pi}\int_S \Omega^{(\ell)}_a \eta^a dS + \frac{1}{4\pi}\int_S (A_a\eta^a) E_\perp dS, \tag{4}$$

where $J_{\rm K}$ and $J_{\rm EM}$ correspond, respectively, to (Komar) gravitational and electromagnetic contributions to the total J. Electric and magnetic charges can be expressed as (e.g., Refs. [18,28])

$$Q_{\rm E} = \frac{1}{4\pi}\int_S E_\perp dS, \qquad Q_{\rm M} = \frac{1}{4\pi}\int_S B_\perp dS. \tag{5}$$

We characterize now $\mathcal{S}$ as a stable section of a (quasi-local) black hole horizon. First, we require $\mathcal{S}$ to be a marginally outer trapped surface, that is $\theta^{(\ell)} = 0$. Second, we demand $\mathcal{S}$ to be stably outermost as introduced in Refs. [19,20] (see also Refs. [16,29]). More specifically, we require $\mathcal{S}$ to be an (axisymmetry-compatible) spacetime stably outermost marginally outer trapped surface [12,13]:

Definition 1. *A closed marginally outer trapped surface $\mathcal{S}$ is referred to as spacetime stably outermost if there exists an outgoing ($-k^a$-oriented) vector $X^a = \gamma\ell^a - \psi k^a$, with $\gamma \geq 0$ and $\psi > 0$, with respect to which $\mathcal{S}$ is stably outermost: $\delta_X \theta^{(\ell)} \geq 0$. If, in addition, X^a (i.e., γ, ψ) and $\Omega^{(\ell)}_a$ are axisymmetric, we will refer to $\delta_X\theta^{(\ell)} \geq 0$ as an (axisymmetry-compatible) spacetime stably outermost condition.*

Here, the operator δ_X is the variation operator on the surface $\mathcal{S}$ along the vector X^a discussed in Refs. [19,20] (see also Refs. [30,31]). We formulate now the following lemma:

Lemma 1. *Let $\mathcal{S}$ be a closed marginally trapped surface $\mathcal{S}$ satisfying the (axisymmetry-compatible) spacetime stably outermost condition. Then, for all axisymmetric α on $\mathcal{S}$,*

$$\int_S \left[|D\alpha|^2 + \frac{1}{2}\alpha^2\,{}^2R\right] dS$$
$$\geq \int_S \alpha^2[|\Omega^{(\eta)}|^2 + (E_\perp^2 + B_\perp^2)]dS, \tag{6}$$

with $|D\alpha|^2 = D_a\alpha D^a\alpha$ and $|\Omega^{(\eta)}|^2 = \Omega^{(\eta)}_a \Omega^{(\eta)a}$.

The proof is a direct application of Lemma 1 in Ref. [12]. Given the vector $X^a = \gamma\ell^a - \psi k^a$, for all axisymmetric α on $\mathcal{S}$, it holds [12]

$$\int_{\mathcal{S}}\left[D_a\alpha D^a\alpha+\frac{1}{2}\alpha^2{}^2R\right]dS \geq \int_{\mathcal{S}}[\alpha^2\Omega_a^{(\eta)}\Omega^{(\eta)a}+\alpha\beta\sigma_{ab}^{(\ell)}\sigma^{(\ell)ab} + G_{ab}\alpha\ell^a(\alpha k^b+\beta\ell^b)]dS, \quad (7)$$

with $\beta=\alpha\gamma/\psi$. First, since $\alpha\beta\geq 0$, the positive-definite quadratic term in the shear can be neglected. Second, we insert Einstein equation $G_{ab}+\Lambda g_{ab}=8\pi(T_{ab}^{\rm EM}+T_{ab}^{\rm M})$, with $T_{ab}^{\rm EM}$ and $T_{ab}^{\rm M}$ the electromagnetic and matter stress-energy tensors. In particular, $T_{ab}^{\rm EM}=\frac{1}{4\pi}(F_{ac}F_b{}^c-\frac{1}{4}g_{ab}F_{cd}F^{cd})$. From the dominant energy condition on $T_{ab}^{\rm M}$, $\Lambda\geq 0$ and the null energy condition applying for $T_{ab}^{\rm EM}$, the Einstein tensor term in inequality (7) is bounded below by $\alpha^2 8\pi T_{ab}^{\rm EM}\ell^a k^b$. Making use of (see e.g., Refs. [13,18])

$$T_{ab}^{\rm EM}\ell^a k^b=\frac{1}{8\pi}[(\ell^a k^b F_{ab})^2+(\ell^a k^{b*}F_{ab})^2], \quad (8)$$

inequality (6) follows by identifying $E_\perp$ and $B_\perp$ in Eq. (3).

An important remark concerns the topology of $\mathcal{S}$. Note that taking $\alpha=\text{const}$ in inequality (6), a nonvanishing angular momentum or charge suffices to conclude the sphericity of $\mathcal{S}$ by applying the Gauss-Bonnet theorem. In the toroidal case, which corresponds to the equality in inequality (6) together with the vanishing of its right-hand side so that the Euler characteristic of $\mathcal{S}$ is zero, the angular momentum in Eq. (4) and the charges in Eq. (5) vanish identically, so that inequality (1) follows automatically. Therefore, in the following, we can assume a spherical topology for $\mathcal{S}$. In this spherical context, the axisymmetry structure on $\mathcal{S}$ entails (see e.g., Ref. [32]) that η^a vanishes exactly at two points on $\mathcal{S}$ (north and south poles [33]). Moreover, η^a is normalized so that its integral curves have an affine length of 2π.

IV. ACTION FUNCTIONAL

The proof of inequality (1) proceeds by solving a constrained variational problem on $\mathcal{S}$, in which J, $Q_{\rm E}$, and $Q_{\rm M}$ must be kept constant under otherwise arbitrary variations. We construct the corresponding action functional $\mathcal{M}$, by evaluating the geometric expression inequality (6) in a specific coordinate system on the sphere $\mathcal{S}$.

First, on an axisymmetric sphere $\mathcal{S}$, a coordinate system can always be chosen [11,32] such that

$$ds^2=q_{ab}dx^a dx^b=e^\sigma(e^{2q}d\theta^2+\sin^2\theta d\varphi^2), \quad (9)$$

with axisymmetric σ and q satisfying $\sigma+q=c=$ constant. Then, $\eta^a=(\partial_\varphi)^a$, $\eta=e^\sigma\sin^2\theta$ and $dS=e^c dS_0$, with $dS_0=\sin\theta d\theta d\varphi$. In particular, $A=4\pi e^c$. Second, $\Omega_a^{(\ell)}$ is expressed uniquely on a two-sphere as $\Omega_a^{(\ell)}=\epsilon_{ab}D^b\tilde{\omega}+D_a\lambda$. Since $\Omega_a^{(\ell)}$ is axisymmetric, $\Omega_a^{(\eta)}=\epsilon_{ab}D^b\tilde{\omega}$ [12], and we can write

$$\Omega_a^{(\eta)}=\frac{1}{2\eta}\epsilon_{ab}D^b\bar{\omega}, \quad (10)$$

by introducing the potential $\bar{\omega}$, as $d\bar{\omega}/d\theta=(2\eta)d\tilde{\omega}/d\theta$, which satisfies $J_{\rm K}=[\bar{\omega}(\pi)-\bar{\omega}(0)]/8$ (cf. Ref. [12]). Third, from $\ell^a k^{b*}F_{ab}=\frac{1}{2}F_{ab}\epsilon^{ab}$ [13] and the axisymmetry of A_a,

$$B_\perp=\frac{1}{e^c\sin\theta}\frac{dA_\varphi}{d\theta}. \quad (11)$$

Finally, following Refs. [11,12], we choose $\alpha=e^{c-\sigma/2}$. Inserting it together with Eqs. (9)–(11) into inequality (6), we get

$$8(c+1)\geq\mathcal{M}[\sigma,\bar{\omega},E_\perp,A_\varphi], \quad (12)$$

where $\mathcal{M}[\sigma,\bar{\omega},E_\perp,A_\varphi]$ is the action functional

$$\mathcal{M}[\sigma,\bar{\omega},E_\perp,A_\varphi]=\frac{1}{2\pi}\int_{\mathcal{S}}\left[4\sigma+\left(\frac{d\sigma}{d\theta}\right)^2+\frac{1}{\eta^2}\left(\frac{d\bar{\omega}}{d\theta}\right)^2 +4e^{2c-\sigma}E_\perp^2+4e^{-\sigma}\left(\frac{1}{\sin\theta}\frac{dA_\varphi}{d\theta}\right)^2\right]dS_0. \quad (13)$$

Inequality (1) follows by solving the variational problem defined by $\mathcal{M}[\sigma,\bar{\omega},E_\perp,A_\varphi]$.

V. PROOF OF THE STRICT INEQUALITY, WITH $Q_{\rm M}=0$

A first approach to solve such a variational problem consists in casting inequality (12) and the action functional (13) in a form appropriate for the automatic application of the variational treatment in Ref. [6]. The result in Ref. [6] states that an axisymmetric stationary black hole, subextremal in the sense that trapped surfaces exist in the interior vicinity of the event horizon [34], satisfies the strict inequality (1). The proof in Ref. [6] deals with the $Q_{\rm M}=0$ case. Namely,

$$\text{horizon subextremal condition}\Rightarrow p_J^2+p_Q^2<1, \quad (14)$$

where $p_J=\frac{8\pi J}{A}$ and $p_Q=\frac{4\pi Q_{\rm E}^2}{A}$. This implication (actually, its logical counterreciprocal) is cast in Ref. [6] as a variational problem on a Killing horizon section. The action functional in Ref. [6] is constructed by combining the horizon subextremal condition in implication (14) with the expression $p_J^2+p_Q^2<1$. The key remark here is to show that such a variational problem, defined solely on a sphere $\mathcal{S}$, has actually full applicability in the dynamical case beyond the original stationary and spacetime axisymmetric setting of Ref. [6] (cf. also in this sense Refs. [35,36], where the relation between the variational problem in the stationary setting of Ref. [6] and the dynamical one of Refs. [11,12] is clarified). More specifically, we show that our expressions for p_J, p_Q and the stably outermost condition (12), valid in the dynamical nonvacuum case with axisymmetric horizons, match exactly the expressions in Ref. [6] for the elements in implication (14). Therefore, the proof in Ref. [6] extends exactly for nonvacuum dynamical spacetimes with axially symmetric horizons.

From the comparison between the four-dimensional stationary axisymmetric line element in Ref. [6] with our line element (9) on $\mathcal{S}$ and between the respective integrands of the Komar angular momentum, we introduce new fields U and V from σ and $\bar{\omega}$:

$$\hat{u} = e^{\sigma}, \qquad \hat{u}_N = e^{c}, \qquad U = \frac{1}{2}\ln\left(\frac{\hat{u}}{\hat{u}_N}\right), \qquad V = \frac{e^{\sigma}\sin\theta}{2\eta^2}\frac{d\bar{\omega}}{d\theta}. \tag{15}$$

Regarding the electromagnetic potentials, we define S and T

$$S = -E_{\perp}e^{c/2}, \qquad T = A_{\varphi}e^{-c/2}. \tag{16}$$

Inserting these fields in Eqs. (4) and (5) above, using $A = 4\pi e^{c}$ and changing to variable $x = \cos\theta$, we get

$$p_J = -\frac{1}{2}\int_{-1}^{1} Ve^{2U}(1-x^2)dx + \int_{-1}^{1} STdx \qquad p_Q = \frac{1}{4}\left(\int_{-1}^{1} Sdx\right)^2, \tag{17}$$

which coincide exactly with expressions in Eqs. (23) and (24) in Ref. [6]. Regarding the stability (subextremal) condition, we insert Eqs. (15) and (16) in condition (12) [with strict inequality]. Using $\int_{-1}^{1} Udx = -\int_{-1}^{1} U'xdx$ (following from $U(1) = U(-1) = 0$, as a regularity condition for q on the axis) and denoting with a prime the derivative with respect to x, we find

$$1 > \frac{1}{2}\int_{-1}^{1}(U'^2 + V^2)(1-x^2) - 2U'x + e^{-2U}(S^2 + T'^2)dx. \tag{18}$$

This matches exactly the horizon subextremal condition inequality (28) in Ref. [6]. Considering expressions (17) and (18) as the starting point, the same variational problem used in the proof of implication (14) can be defined in the nonvacuum dynamical case with axisymmetric apparent horizons. This proves inequality (1) with vanishing $Q_{\rm M}$ in the strictly stable case.

VI. DISCUSSION OF THE GENERAL PROOF

A complementary more general approach to inequality (1) follows the strategy in Refs. [11,12] to the variational problem defined by expressions (12) and (13). The enforcement of the constraints on J, $Q_{\rm E}$, and $Q_{\rm M}$ in the variational problem is not straightforward in terms of the fields $\bar{\omega}$, $E_{\perp}$, and A_{φ} in Eq. (13). In order to address this issue, we introduce new potentials ω, χ, and ψ on $\mathcal{S}$ (cf. also Refs. [1,25])

$$\frac{d\psi}{d\theta} = E_{\perp}e^{c}\sin\theta, \qquad \chi = A_{\varphi}, \qquad \frac{d\omega}{d\theta} = 2\eta\frac{d\tilde{\omega}}{d\theta} + 2\chi\frac{d\psi}{d\theta} - 2\psi\frac{d\chi}{d\theta} \tag{19}$$

with the crucial property that J, $Q_{\rm E}$, and $Q_{\rm M}$ are written as

$$J = \frac{\omega(\pi) - \omega(0)}{8}, \qquad Q_{\rm E} = \frac{\psi(\pi) - \psi(0)}{2}, \qquad Q_{\rm B} = \frac{\chi(\pi) - \chi(0)}{2}. \tag{20}$$

Physical parameters in inequality (1) can then be kept constant by fixing ω, χ, and ψ on the axis as a boundary condition in the variational problem [note that $\bar{\omega}$ in Eq. (10) is an appropriate potential to control the Komar angular momentum $J_{\rm K}$, but not for the total J]. In terms of σ, ω, χ, and ψ the action functional reads

$$\mathcal{M}[\sigma,\omega,\psi,\chi] = \frac{1}{2\pi}\int_{\mathcal{S}}\left[4\sigma + |D\sigma|^2 + \frac{|D\omega - 2\chi D\psi + 2\psi D\chi|^2}{\eta^2} + \frac{4}{\eta}(|D\psi|^2 + |D\chi|^2)\right]dS_0, \tag{21}$$

where $\mathcal{M}$ is formally promoted beyond axisymmetry. As in Ref. [12], the proof of inequality (1) then proceeds in two steps. First,

$$A \geq 4\pi e^{(\mathcal{M}-8)/8} \tag{22}$$

follows directly from the above-derived inequality (12), and $A = 4\pi e^{c} = 4\pi e^{\sigma(0)}$. Second, the variational problem defined by the action functional (21) with values of ω, ψ, χ fixed on the axis and determined from relations (20), is solved by following exactly the same steps as in Ref. [9], leading to [37]

$$\mathcal{M} \geq \mathcal{M}_0 = 8\ln\sqrt{(2J)^2 + (Q_{\rm E}^2 + Q_{\rm M}^2)^2} + 8, \tag{23}$$

where $\mathcal{M}_0$ corresponds to the evaluation of $\mathcal{M}$ on an extremal solution in the (magnetic) Kerr-Newman family with given J, $Q_{\rm E}$, and $Q_{\rm M}$. The sharp inequality (1), including, in particular, the magnetic charge, follows from the combination of inequalities (22) and (23).

VII. FINAL DISCUSSION

We have shown that $(A/(4\pi))^2 \geq (2J)^2 + (Q_{\rm E}^2 + Q_{\rm M}^2)^2$ holds for axisymmetric stable marginally outer trapped surfaces in dynamical, non-necessarily axisymmetric spacetimes with ordinary matter which can be crossing the horizon. More specifically, we have presented a complete proof of the strictly stable case with $Q_{\rm M} = 0$ and provided the key elements for the proof of the complete inequality. Furthermore, we have explicitly shown the close relationship between the variational problems in Refs. [4–7] and in Refs. [11,12] (see also Refs. [35,36]), in particular showing that the proof in Ref. [6] extends automatically to the dynamical nonvacuum case with axisymmetric horizons. From the perspective of the no hair property of vacuum stationary black holes, the extension of inequality (1) to fully dynamical nonvacuum situations represents a remarkable result. Indeed, although parameters A, J, $Q_{\rm E}$, and $Q_{\rm M}$ no longer characterize completely the black hole state and new degrees of freedom are required to describe the spacetime geometry, the generic incorporation of the latter is still constrained by inequality (1). Such a constraint represents a valuable probe into nonlinear black hole dynamics. As a first remark, it gives support to the physical interpretation of the Christodoulou mass in dynamical settings (cf. discussion in Ref. [2]), in particular, endorsing dynamical horizon [17] thermodynamics [38]. More generally, whereas inequality (1) follows originally in the Kerr-Newman family under the assumption of (weak) cosmic censorship, the present result is purely quasilocal involving no global condition on the spacetime, namely, no asymptotic predictability. This suggests a link between cosmic censorship and marginally trapped surface stability to be further explored. In this context, assuming Penrose inequality (with no surface enclosing $\mathcal{S}$ with area smaller than A), inequality (1) refines the positive of mass theorem in terms of physical quantities: $16\pi M^2 \geq A \geq \sqrt{(8\pi J)^2 + (4\pi[Q_{\rm E}^2 + Q_{\rm M}^2])^2}$.

A very interesting problem, especially for its practical applications in the astrophysically realistic black hole scenarios considered in numerical relativity, is the study of inequality (1) when relaxing the requirement of axisymmetry on the apparent horizon. In this case, we must resort to some quasilocal prescription [22,23] for J. On the one hand, the lack of a canonical choice for J introduces some degree of arbitrariness in the problem. However, a more serious obstacle arises from the fact that, in contrast with the area-charge inequality [13], the incorporation of the angular momentum involves the resolution of a subtle variational problem (cf. Ref. [2]). The construction of the corresponding action functional $\mathcal{M}$ given by expressions (13) and (21) makes explicit use of axisymmetry (although, remarkably, in the resolution itself of the variational problem, axisymmetry can be relaxed; see details in Ref. [9,37]). In the absence of axisymmetry, the main challenge would consist of identifying the appropriate potentials in terms of which the variational problem can be defined and solved. In this context, the assessment of the general situation in the absence of local axisymmetry is certainly worthy of further research. Finally, Ref. [37] discusses the close relation between the variational problem (on a three-slice) employed in Ref. [1] for the proof of the spacetime mass-angular momentum-charge inequality and the present action functional $\mathcal{M}$ in Eqs. (13) and (21), also closely related to (but different from) the functional used in Ref. [6]. Regarding the latter, we stress that electromagnetic potentials S and T in Eq. (16) follow straightforwardly (with no gauge choices involved) from the geometric formulation of the general stability condition in Lemma 1. This underlines the intrinsic interest of the flux inequality in Lemma 1 (and, more generally, its complete expression in Ref. [12]; cf. also the discussion in Ref. [3]) for exploring further geometric aspects of stable black hole horizons.

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[1] P. T. Chrusciel and J. Lopes Costa, Classical Quant. Grav. 26, 235013 (2009); J. L. Costa, arXiv:0912.0838.
[2] S. Dain, Classical Quant. Grav. 29, 073001 (2012).
[3] J. L. Jaramillo, Proceedings of the VI International Meeting on Lorentzian Geometry, GELOGRA'11, Granada, Spain, 2011, Proceedings in Mathematics and Statistics (Springer, New York), arXiv:1201.2054
[4] M. Ansorg and H. Pfister, Classical Quant. Grav. 25, 035009 (2008).
[5] J. Hennig, M. Ansorg, and C. Cederbaum, Classical Quant. Grav. 25, 162002 (2008).
[6] J. Hennig, C. Cederbaum, and M. Ansorg, Commun. Math. Phys. 293, 449 (2010).
[7] M. Ansorg, J. Hennig, and C. Cederbaum, Gen. Relativ. Gravit. 43, 1205 (2011).
[8] S. Dain, Phys. Rev. D 82, 104010 (2010).
[9] A. Acena, S. Dain, and M. E. Gabach Clément, Classical Quant. Grav. 28, 105014 (2011).
[10] M. E. Gabach Clément, arXiv:1102.3834.
[11] S. Dain and M. Reiris, Phys. Rev. Lett. 107, 051101 (2011).
[12] J. L. Jaramillo, M. Reiris, and S. Dain, Phys. Rev. D 84, 121503(R) (2011).
[13] S. Dain, J. L. Jaramillo, and M. Reiris, Classical Quant. Grav. 29, 035013 (2012).
[14] W. Simon, Classical Quant. Grav. 29, 062001 (2012).
[15] S. Hollands, Classical Quant. Grav. 29, 065006 (2012).

[16] S. Hayward, Phys. Rev. D **49**, 6467 (1994).
[17] A. Ashtekar and B. Krishnan, Living Rev. Relativity **7**, 10 (2004).
[18] I. Booth and S. Fairhurst, Phys. Rev. D **77**, 084005 (2008).
[19] L. Andersson, M. Mars, and W. Simon, Phys. Rev. Lett. **95**, 111102 (2005).
[20] L. Andersson, M. Mars, and W. Simon, Adv. Theor. Math. Phys. **12**, 853 (2008).
[21] Different prescriptions for the notion of quasilocal gravitational (mass and) angular momentum exist in the literature [22,23]. However, for a closed surface admitting an axial Killing a canonical choice can be made [see Eq. (4) below].
[22] L. B. Szabados, Living Rev. Relativity **12**, 4 (2009).
[23] J. L. Jaramillo and E. Gourgoulhon, Fundam. Theor. Phys. **162**, 87 (2011).
[24] J. L. Jaramillo, Classical Quant. Grav. **29**, 177001 (2012).
[25] B. Carter, Gen. Relativ. Gravit. **42**, 653 (2010).
[26] W. Simon, Gen. Relativ. Gravit. **17**, 439 (1985).
[27] A. Ashtekar, C. Beetle, and J. Lewandowski, Phys. Rev. D **64**, 044016 (2001).
[28] A. Ashtekar, S. Fairhurst, and B. Krishnan, Phys. Rev. D **62**, 104025 (2000).
[29] I. Racz, Classical Quant. Grav. **25**, 162001 (2008).
[30] I. Booth and S. Fairhurst, Phys. Rev. D **75**, 084019 (2007).
[31] L. M. Cao, J. High Energy Phys. 03 (2011) 112.
[32] A. Ashtekar, J. Engle, T. Pawlowski, and C. Van Den Broeck, Classical Quant. Grav. **21**, 2549 (2004).
[33] In this case the vector ξ^a is not defined at the poles. However, this does not affect the rest of the discussion.
[34] Horizon sections are strictly stably outermost with respect to outgoing directions (namely, outer trapping horizons [16,18] with $\delta_k\theta^{(\ell)} < 0$ for some ℓ^a and k^a).
[35] P. T. Chrusciel, M. Eckstein, L. Nguyen, and S. J. Szybka, Classical Quant. Grav. **28**, 245017 (2011).
[36] M. Mars, Classical Quant. Grav. **29**, 145019 (2012).
[37] M. E. Gabach Clément, J. L. Jaramillo, and M. Reiris, arXiv:1207.6761.
[38] In this context, we note [3] that inequalities (6) and (7) can be interpreted as upper bounds on certain *energy fluxes* defined by their right-hand-sides (and closely related to dynamical horizon fluxes [39,40]). In particular, terms proportional to β in (7) correspond to gravitational and electromagnetic radiative degrees of freedom ($T^{\rm EM}_{ab}\ell^a\ell^b$ being the flux of the Poynting vector).
[39] A. Ashtekar and B. Krishnan, Phys. Rev. D **68**, 104030 (2003).
[40] A. Ashtekar and B. Krishnan, Phys. Rev. Lett. **89**, 261101 (2002).