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# Bounds on the force between black holes

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## Abstract

We treat the problem of  $N$  interacting, axisymmetric black holes and obtain two relations among physical parameters of the system including the force between the black holes. The first relation involves the total mass, the angular momenta, the distances and the forces between the black holes. The second one relates the angular momentum and the area of each black hole with the forces acting on it.

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## 1. Introduction

The problem of interacting black holes dates back to the beginnings of general relativity. Although a solution representing multiple black holes in equilibrium is known to exist [4] (the Majumdar–Papapetrou solution, which consists of  $N$  extreme Reissner–Nördstrom black holes, i.e. each black hole has electric charge equal to its mass parameter,  $q_i = m_i$ ), it is in general expected that there should be a force between different components of the horizon in order to prevent the spacetime from collapsing. Nevertheless, based on Newtonian ideas, a main concern was the following. Given two black holes, could the spin–spin repulsion compensate for the gravitational attraction and keep the system in equilibrium? Many attempts have since then been made in order to answer this question (see [2, 3, 10, 17, 20, 22, 25, 26, 28, 30–34, 37]). In particular, in order to rule out the existence of two vacuum axially symmetric stationary black hole configurations, in 1991 Li and Tian [22] proved that when the solution admits an involution interchanging the components of the event horizons, then equilibrium is not possible (see also [12]). On the other hand, between 2000 and 2001, Manko *et al* [25, 26] proved that the two Kerr black hole configurations fail to possess a positive Komar mass. Finally, in a series of papers starting in 2009, Neugebauer and Hennig [17, 30, 31] (see also the references therein) proved that any axially symmetric, stationary, vacuum solution of Einstein equations with disconnected horizon (and only two connected components) violates an inequality between area and angular momentum known to hold for each axisymmetric, regular black hole. In a completely different approach, Weinstein [35–37], using harmonic

maps, showed that stationary, asymptotically flat, vacuum, axially symmetric multiple black hole solutions exist, although possibly with a conical singularity in the bounded component of the symmetry axis.

The appearance of this conical singularity on the axis was clearly seen in the superposition of two Schwarzschild black holes in the early work of Bach [2] and was later discussed by Einstein and Rosen in [13]. Although the existence of a singularity may seem discouraging, the solution so built has a well-defined action and the corresponding thermodynamic potential describes a system of black holes at fixed temperature and at a fixed distance (see [18, 19] for a thermodynamic description of solutions with conical singularities, cf [7]). The long-range interactions in the multiple black hole solutions are unable to provide equilibrium between the various black holes and the conical singularity is interpreted as a boundary condition that keeps the holes at fixed separation preventing the collapse of the system.

The non-existence proof of Neugebauer and Hennig [30] for two rotating black holes in vacuum strongly uses the relation  $A \geq 8\pi|J|$  between the area and the angular momentum of each black hole which is known to hold in the axisymmetric, regular case (see [1]). These types of geometrical inequalities relating physical parameters of black holes have been extensively studied in the last few years as an attempt to explore the allowed values of angular momentum, mass and size a black hole can have, and turned out to be relevant in different contexts as well (see [8] for an excellent recent review on geometrical inequalities). As interesting outcomes of these researches, two main relations have been found.

The first one of such relations in the context of (not necessarily stationary) multiple black hole configurations was proved by Chrusciel *et al* [6] (see also [11] for the inequality in the case of  $N = 1$ ) and gives a lower bound to the ADM mass in the axisymmetric, vacuum case in the form

$$0 \geq f(J_1, \dots, J_N) - m, \quad (1)$$

where  $m$  is the total (ADM) mass,  $J_i$  are the Komar angular momenta of the individual constituents and  $f$  is a function of the  $J_i$  (and possibly depending on other parameters as well), not known explicitly, which reduces in the case of a single black hole to  $\sqrt{|J|}$ .

The second inequality we are interested in, and the one used in [30], is the relation

$$1 \geq \frac{8\pi|J|}{A} \quad (2)$$

between the angular momentum  $J$  and the area  $A$  of an axisymmetric black hole which has recently been proven. In [1, 14] and [9] maximal vacuum initial data (possibly with a positive cosmological constant) containing a minimal stable surface are considered. Also, in [21] the inequality is proven for spacetimes having outermost stable marginally trapped surfaces, not necessarily in vacuum. As we will discuss later, (2) has been extended to incorporate electric and magnetic charges. Inequality (2) holds for each black hole in a regular initial data and, loosely speaking, states that a hole cannot rotate too fast.

An important ingredient in the above results, relations (1) and (2), is the assumed regularity at the symmetry axis. As we mentioned, this is related to the notion of equilibrium, although it does not imply it since one must also require stationarity. The regularity at the symmetry axis only means the absence of conical singularities. A way to incorporate the interaction between the black holes into the problem is by means of relaxing the regularity condition at the axis and therefore allowing a conical singularity between the holes. The value of the deficiency angle will be related to the force between the black holes [37] (see equation (11)).

The purpose of this work is to investigate how inequalities (1) and (2) are modified when one takes explicitly into account the interaction between the black holes. In this way we will obtain bounds on the forces between them (i.e. on the deficit angles) set by the physical

quantities  $A$ ,  $m$  and  $J$ . Although the results presented here are not restricted to stationarity, they are the first step toward understanding, from a different and, in some sense, more general point of view, the possible final states in black hole scenarios, that is, whether stationary multiple black hole configurations might exist in equilibrium, or they would collapse into a unique (possibly Kerr) black hole. From a heuristic perspective one would expect that such multiple black hole solutions should not exist in equilibrium, which would correspond to having a non-zero force between the black holes. Moreover, one would expect the force to be positive, reflecting the attractive character of the gravitational interaction. Therefore, finding appropriate lower bounds for the interaction force between the black holes seems to be a promising and convenient way to prove this conjecture.

The paper is organized as follows. Beginning section 2, we briefly introduce the basic features of multiple black hole solutions to Einstein constraint equations and the explicit relation between the force and deficit angle. Then, in section 2.1 we present the more specific hypothesis needed in order to obtain a relation between mass, angular momentum and force, similar in spirit as (1), theorem 2.1, and finally, in section 2.2, we find a new inequality relating area, angular momentum and force, theorem 2.2, analogous to (2).

## 2. Main result

We begin this section with a brief description of multiple black hole solutions of Einstein equations in the form of maximal axisymmetric initial data.

Consider a maximal, simply connected initial data set  $(M, g_{ab}, K_{ab})$ , invariant under a  $U(1)$  action. It is known [5] that there exists a global coordinate system  $(\rho, z, \phi)$  such that the metric is of the form

$$g = e^{\sigma+2q}(d\rho^2 + dz^2) + \rho^2 e^\sigma (d\phi + \rho A_\rho d\rho + A_z dz)^2, \tag{3}$$

where the functions are taken to be  $\phi$ -independent.

We assume vacuum for simplicity, although matter satisfying some positive energy condition would be also allowed. The constraint equations in the maximal case,  $g^{ab}K_{ab} = 0$ , are

$${}^3R = K_{ab}K^{ab}, \quad \nabla^a K_{ab} = 0. \tag{4}$$

Due to these constraints, it is known that there exists a twist potential  $\omega$  related to the second fundamental form through the expression (see [11])

$$2\epsilon_{abc}K_e^b \eta^c \eta^e dx^a = d\omega, \tag{5}$$

where  $\eta^a$  is a  $2\pi$ -periodic Killing vector field and  $\epsilon_{abc}$  is the Lévi-Civita tensor. Moreover, we have the useful bound [11]

$$K_{ab}K^{ab} \geq e^{-\sigma-2q} \frac{|\partial\omega|^2}{2\eta^2}. \tag{6}$$

The quantities that we are interested in this work are the total mass of the initial data and the area and angular momentum of each black hole. As is well known, the mass is a global quantity, given by a boundary integral at spatial infinity

$$m = \frac{1}{16\pi} \lim_{r \rightarrow \infty} \int_{\partial B_r} (\partial_b g_{ab} - \partial_a g_{bb}) n^a dS, \tag{7}$$

where  $r = \sqrt{\rho^2 + z^2}$  is the Euclidean distance,  $\partial$  denotes partial derivatives,  $B_r$  is the Euclidean ball  $r = \text{const}$ ,  $\partial B_r$  is its border,  $n^a$  is its exterior unit normal and  $dS$  is the unit surface element with respect to the Euclidean metric,  $dS = \sin \theta d\theta d\phi$ .

Now, consider an arbitrary two-surface  $\Sigma$  in  $M$  with an induced metric  $\gamma_{ab}$ ; then the area of  $\Sigma$  is given by the expression

$$A(\Sigma) = \int_{\Sigma} dS_{\gamma}, \quad (8)$$

with  $dS_{\gamma}$  being the surface element with respect to  $\gamma_{ab}$ .

Finally, for axially symmetric and maximal data, one can define the angular momentum  $J$  associated with  $\Sigma$  (the Komar integral of the Killing vector) as

$$J(\Sigma) = \int_{\Sigma} K_{ab} \eta^a n^b dS_{\gamma}, \quad (9)$$

where  $n^a$  is the unit normal vector to  $\Sigma$ .

For concreteness, in what follows, we consider  $N \geq 1$  black holes located at the symmetry axis, which will be represented by punctures in section 2.1 and by minimal stable surfaces in section 2.2. The axis  $\rho = 0$  is denoted by  $\Gamma$ . It contains  $N - 1$  bounded components denoted by  $\Gamma_i$ ,  $i = 1, \dots, N - 1$ , and two unbounded components,  $\Gamma_0$  and  $\Gamma_N$ .

As we mentioned in the introduction, the interaction between the black holes can be realized through the non-regular character of the solution at  $\Gamma$ . The regularity condition at  $\Gamma$  can be stated as

$$\lim_{\rho_0 \rightarrow 0^+} \frac{\sqrt{\eta}}{\int_0^{\rho_0} e^{\sigma/2+q} d\rho} = 1, \quad (10)$$

where  $\eta = \eta^a \eta_a$ , and is translated into the requirement that the metric function  $q$  (see expression (3)) vanishes at the axis  $q|_{\Gamma} = 0$ .

In general, this condition will not be satisfied and the metric will not define a regular solution of Einstein equations. If (10) does not hold, then the singularity at the axis is called a conical singularity. The deficit angle at the axis is given by  $e^{-q_i}$  where  $q_i$  is the constant value of the function  $q$  at the  $i$ th connected component of the axis. These singularities are interpreted as forces between black holes, and in the stationary case, would keep the bodies from falling onto each other. We define this force in terms of  $q_i$  as (see [36] for a motivation and details)

$$\mathcal{F}_i = \frac{1}{4}(e^{-q_i} - 1). \quad (11)$$

As we mentioned before, in the stationary case, one would expect this force to be positive, reflecting the fact that a positive force is needed in order to prevent the black holes from falling on each other. A positive force has been found in a few stationary situations, like in the zero [2] and small angular momentum case [23], the case when the problem admits an involutive symmetry [22], when there are two equal, counter-rotating Kerr black holes [33] and in the limit when one of the black holes becomes extreme and the distance to one of the adjacent black holes tends to zero [36]. Nevertheless, so far there is no general proof stating that the force is always positive (or at least, non-negative).

One can prescribe the value of the function  $q$  on one of the components of the axis [36]; the other values will be determined by the solution of the constraints, and are not known in general. In this paper, we give two results, which are the extensions of inequalities (1) and (2) to include the non-zero values of  $q$  at the axis and are presented in the following sections.

### 2.1. Bound in terms of mass and angular momentum

In order to extend the Chrusciel *et al* result [6], inequality (1), to the non-regular case, we state here their main assumptions. We will retain basically all of them, and only relax the regularity of the function  $q$  at the axis.

As we mentioned before, for the mass-angular momentum inequality we represent the black holes as  $N$  punctures on the axis, separated by Euclidean distances  $L_i$ , each puncture being an asymptotically flat end in the sense that for  $k \geq 6$ ,

$$g_{ab} - \delta_{ab} = o_k(r^{-1/2}), \quad (12)$$

$$\partial_k g_{ab} \in L^2(M), \quad (13)$$

where  $f = o_k(r^\lambda)$  means that  $f$  satisfies

$$\partial_{k_1} \dots \partial_{k_n} f = o(r^{\lambda-n}), \quad 0 \leq n \leq k. \quad (14)$$

The main result of this section gives a relation among the total mass of the initial data, the individual angular momenta of the each asymptotic region and the forces between them.

**Theorem 2.1.** *Consider a maximal, asymptotically flat, axisymmetric initial data  $(M, g_{ab}, K_{ab})$  as described above, with  $N$  extra asymptotically flat ends represented by punctures on the axis  $\Gamma$ , with angular momenta  $J_i$ ,  $i = 1, \dots, N$ . Then, there is a lower bound to the ADM mass  $m$  given by*

$$\sum_{i=1}^{N-1} \frac{L_i}{4} \ln(1 + 4\mathcal{F}_i) \geq f(J_1, \dots, J_N) - m, \quad (15)$$

where  $L_i$  is the Newtonian length of  $\Gamma_i$ ,  $\mathcal{F}_i$  is the force between adjacent punctures and  $f$  is a function of the  $J_i$ s.

Before entering into the proof, we want to make a few remarks about the result. It is clearly seen from inequality (15) that it does not imply, in general, the result  $0 \geq f - m$ . That is only the case when the forces are non-positive. But if the forces are positive, which is the expected situation when the data corresponds to a slice in a stationary spacetime, the total mass  $m$  could, in principle, take values lower than  $f$ , as opposed to the regular case (1).

As the data are asymptotically flat, we must have  $q_0 = q_N = 0$ , and (15) reduces to the Chrusciel *et al* result,  $m \geq f$ , for  $q_i = 0$  and also to Dain's result,  $m \geq \sqrt{|J|}$ , for  $N = 1$  [11].

Note that there is no rigidity statement in this theorem. This is due in part to the lack of a known minimizer for a mass functional appearing in the proof of the Chrusciel *et al* result in the case  $N \geq 2$ . But also, because in inequality (15), two quantities appear that depend on the given parameters of the data. Both, the mass and the force are to be computed after solving the constraints and cannot be prescribed *a priori*. Therefore, for given values of angular momenta  $J_i$  and separation distances  $L_i$ , there might exist different solutions saturating the inequality with different values of  $\mathcal{F}_i$  and mass. In this respect, it is worth mentioning, that there exists an explicit solution describing two equal, counter-rotating extreme Kerr black holes given by Manko *et al* [28] which could saturate (15). We will meet this solution again in the following section.

**Proof.** Since the proof is an adaptation of the Chrusciel *et al* argument, [6], we only sketch the main steps and highlight the point where the non-zero force is incorporated. We refer the reader to [6] and its references for more details.

Begin with the expression for the curvature scalar  $R$  of the metric  $g_{ab}$ , given by

$$-\frac{1}{8}R e^{\sigma+2q} = \frac{1}{4}\Delta\sigma + \frac{1}{16}(\partial\sigma)^2 + \frac{1}{4}\Delta_2 q + \frac{1}{16}\rho^2 e^{-2q}(\rho A_{\rho,z} - A_{z,\rho})^2, \quad (16)$$

where  $\Delta$  is the Laplacian in  $\mathbb{R}^3$  and  $\Delta_2$  is the two-dimensional Laplacian

$$\Delta_2 q = q_{,\rho\rho} + q_{,zz}. \quad (17)$$

Integrate (16) over  $\mathbb{R}^3$  and use the asymptotic flatness condition and the definition of the mass  $m$  (see [11]):

$$\int_{\mathbb{R}^3} \Delta \sigma \, d^3x = \lim_{r \rightarrow \infty} \int_{\partial B_r} \partial_r \sigma r^2 \, dS = -8\pi m \quad (18)$$

to obtain

$$2\pi m = \int_{\mathbb{R}^3} \left[ \frac{1}{16} (\partial \sigma)^2 + \frac{1}{4} \Delta_2 q + \frac{1}{8} R e^{\sigma+2q} + \frac{1}{16} \rho^2 e^{-2q} (\rho A_{\rho,z} - A_{z,\rho})^2 \right] d^3x. \quad (19)$$

Use inequality (6) to bound the Ricci scalar in the above integral and disregard the last, non-negative term to obtain the following lower bound for the mass:

$$m \geq \frac{1}{32\pi} \int_{\mathbb{R}^3} (\partial \sigma)^2 + \frac{|\partial \omega|^2}{\eta^2} \, dx^3 + \frac{1}{8\pi} \int_{\mathbb{R}^3} \Delta_2 q \, d^3x. \quad (20)$$

Note that the first integral is exactly the functional  $\mathcal{M}$  used by Dain [11] and Chrusciel *et al* [6] in the proof of the mass-angular momentum inequalities

$$\mathcal{M} := \frac{1}{32\pi} \int_{\mathbb{R}^3} (\partial \sigma)^2 + \frac{(\partial \omega)^2}{\eta^2} \, dx^3. \quad (21)$$

The second integral on the right-hand side of (20) is zero for regular solutions (i.e. when  $q = 0$  at the axis). But when the solution contains conical singularities, it gives instead

$$\int_{\mathbb{R}^3} \Delta_2 q \, d^3x = 2\pi \sum_{i=1}^{N-1} q_i L_i, \quad (22)$$

where  $L_i$  is the Newtonian length of  $\Gamma_i$  and  $q_i$  is the constant value of  $q$  on it. With this result, we arrive at the inequality

$$m \geq \mathcal{M} + \frac{1}{4} \sum_{i=1}^{N-1} q_i L_i. \quad (23)$$

In [6] (see proposition 2.1), the authors prove that for any set of aligned punctures and of axis values  $\omega_i := \omega|_{\Gamma_i}$ , there exists a solution  $(\tilde{\sigma}, \tilde{\omega})$  of the variational equations associated with the functional  $\mathcal{M}$ , with finite values of  $\mathcal{M}$  and appropriate asymptotic behavior near each puncture. So, denote by  $f(J_1, \dots, J_N)$  the numerical value of  $\mathcal{M}$  at the harmonic map  $(\tilde{\sigma}, \tilde{\omega})$  from  $\mathbb{R}^3 \setminus \Gamma$  to the two-dimensional hyperbolic space. Then we obtain

$$m \geq f(J_1, \dots, J_N) + \frac{1}{4} \sum_{i=1}^{N-1} q_i L_i. \quad (24)$$

Finally expressing the constants  $q_i$  in (24) in terms of the forces (see equation (11)), the statement of the theorem is proven.  $\square$

## 2.2. Bound in terms of area and angular momentum

In this section, we obtain a bound on the forces acting on a black hole in terms of its area and angular momentum. This will generalize inequality (2) to the interacting case (i.e. to the case where conical singularities may be present).

For that purpose, we will follow Dain and Reiris's argument [9], using minimal stable surfaces. It is remarkable that since this result is quasi-local, there is no asymptotically flatness requirement concerning the initial data. We only need the existence of stable minimal surfaces located at the axis, with aligned angular momenta (in order to maintain axial symmetry). It is clear that in order for these surfaces to have a non-vanishing Komar angular momentum,

a non-trivial topology is needed. One way to think about this is imagining that each surface encloses a puncture.

A surface  $\Sigma$  is called minimal if its mean curvature  $\chi$  vanishes, and it is called stable if it is a local minimum of the area functional (8). This can be precisely stated in the following form. Consider a flux of surfaces  $F_t : \mathbb{R} \times S^2 \rightarrow \Sigma$  parametrized by  $t \in \mathbb{R}$  such that  $F|_{t=0}(S^2) = \Sigma$  and  $\dot{F}^a|_{t=0} = \alpha n^a$ , where dot denotes derivative with respect to  $t$ ,  $n^a$  is the unit normal to  $\Sigma$  and  $\alpha$  is in principle an arbitrary function on  $\Sigma$ . Then the stability condition can be written as

$$\ddot{A}|_{t=0} = \int \alpha \dot{\chi} \, dS_\gamma \geq 0, \tag{25}$$

where

$$\dot{\chi} = -\Delta_\gamma \alpha - (R - R_\gamma + \chi_{ab} \chi^{ab}) \alpha, \tag{26}$$

$\chi_{ab}$  is the second fundamental form of  $\Sigma$ ,  $R$  is the Ricci scalar associated with the 3-metric  $g_{ab}$  (see equation (3)) and  $\Delta_\gamma$  and  $R_\gamma$  are the Laplace operator and the Ricci scalar associated with the 2-metric  $\gamma_{ab}$ , respectively.

As in the previous section, we allow the function  $q$  in the 3-metric to have non-zero values at the bounded components  $\Gamma_i$ . This leads us to the following result, which is an extension of theorem 1 in [9].

**Theorem 2.2.** *Consider an axisymmetric, vacuum and maximal initial data, with a non-negative cosmological constant and  $N \geq 1$  orientable closed stable minimal axially symmetric surfaces  $\Sigma_i$ . Then*

$$\sqrt{1 + 4\mathcal{F}_{i-1}} \sqrt{1 + 4\mathcal{F}_i} \geq \frac{8\pi |J_i|}{A_i} \quad i = 1, \dots, N, \tag{27}$$

where  $J_i$  and  $A_i$  are the angular momentum and area of  $\Sigma_i$ , respectively.

Before giving the proof of theorem 2.2, we want to remark that the equality in (27) is achieved by a stationary solution describing two counter-rotating identical extreme Kerr black holes separated by a mass-less strut given by Manko *et al* [28]. The explicit values for the separation distance, ratio  $A/8\pi J$  and positive force in this solution are

$$L = 2m \frac{a^2 + m^2}{a^2 - m^2}, \quad \frac{A}{8\pi |J|} = \frac{2|J|}{a^2 + m^2}, \quad \mathcal{F} = \frac{(a^2 - m^2)^2}{16J^2} > 0, \tag{28}$$

where  $J = am$ ,  $|J|$  is the angular momentum of each black hole and  $m$  its mass parameter ( $2m$  is the total ADM mass). We note that  $a^2 > m^2$  holds for the extreme counter-rotating constituents, unlike in the case of a single Kerr black hole, for which  $a^2 = m^2$ . It is also clear from the above expressions that as  $m \rightarrow a^-$ , the separation distance becomes arbitrarily large and the mutual force becomes arbitrarily small.

In case that the force is positive, which is the expected outcome in the stationary case, the quantity  $A/8\pi |J|$  can be less than unity, a situation that cannot happen for regular black holes. Nevertheless, a type of variational characterization of extreme Kerr throat is still valid in the presence of a conical singularity. This is due to the fact that in the variational principle used to derive the above inequality (see [1]), the function  $q$  plays no role, and the relevant functions are the conformal factor  $e^\sigma$  and the twist potential  $\omega$ .

Inequality (2) has been recently extended in [15, 16] to include electric and magnetic charges and to apply also to stable marginally outermost trapped surfaces satisfying the dominant energy condition. To be more precise, consider a surface  $\Sigma$  embedded in a spacetime, with null normal vectors  $\ell^a$  and  $k^a$  such that  $\ell^a k_a = -1$  and  $\ell^a$  is outward pointing. Let  $\theta^{(\ell)}$  be the expansion associated with the null normal  $\ell^a$ . Then,  $\Sigma$  is a stable marginally outermost trapped surface if  $\theta^{(\ell)} = 0$  and if there exists an outgoing vector  $X^a = \gamma \ell^a - \psi k^a$  with



$\gamma \geq 0$  and  $\psi > 0$  such that  $\delta_X \theta^{(\ell)} \geq 0$ . Here,  $\delta_X$  denotes the deformation operator on  $\Sigma$  that controls the infinitesimal variations of geometric objects defined on  $\Sigma$  under an infinitesimal deformation of the surface along the vector  $X^a$ . By following the same lines as in theorem (2.2) based on the proof of [15, 16], one obtains the following corollary.

**Corollary 2.3.** *Consider an axisymmetric, stable marginally outermost trapped surface  $\Sigma_i$ , with a non-negative cosmological constant, satisfying the dominant energy condition. Then*

$$\sqrt{1 + 4\mathcal{F}_{i-1}}\sqrt{1 + 4\mathcal{F}_i} \geq \frac{\sqrt{(8\pi J_i)^2 + (4\pi Q_{iE}^2 + 4\pi Q_{iB}^2)^2}}{A_i} \quad i = 1, \dots, N, \quad (29)$$

where  $J_i$ ,  $Q_{iE}$ ,  $Q_{iB}$  and  $A_i$  are the angular momentum, electric charge, magnetic charge and area of  $\Sigma_i$ , respectively.  $\mathcal{F}_i$  and  $\mathcal{F}_{i-1}$  are the forces acting on  $\Sigma_i$  along the axis components  $\Gamma_i$  and  $\Gamma_{i-1}$ .

In the case of zero angular momenta, a non-trivial solution satisfying inequality (29) is given by the case of two Reissner–Nördstrom black hole configurations discussed in detail by Manko in [27], and by Manko *et al* in [29]. The Majumdar–Papapetrou solution, with  $J_i = Q_{iB} = 0$ ,  $A_i = 4\pi Q_{iE}^2$  saturates inequality (29) with vanishing forces. The same thing does not seem plausible for Kerr black holes, since numerical results [10] suggest that a configuration with two Kerr black holes having  $a = m$  leads to a positive force.

**Proof of theorem 2.2.** We follow the lines of [9] and refer the reader to this paper for more details. Choose a coordinate system such that the determinant of the induced metric  $\gamma_{ab}$  on  $\Sigma_i$  is

$$\sqrt{\det(\gamma)} = e^{c_i} \sin \theta, \quad (30)$$

where  $c_i$  is a constant. Then, the metric can be written as

$$\gamma_{ab} = e^{\sigma_i} (e^{2q_i} d\theta^2 + \sin^2 \theta d\phi^2) \quad (31)$$

with  $\sigma_i + q_i = c_i$  and

$$(\sigma_i + q_i)|_{\theta=0} = (\sigma_i + q_i)|_{\theta=\pi} = c_i. \quad (32)$$

Now multiply  $\dot{\chi}$ , given by (26) by  $\alpha$ , use the constraint to express the Ricci scalar  $R$  in terms of  $K_{ab}$ , integrate over  $\Sigma_i$  and use condition  $\chi = 0$ , the stability condition (25) and the bound (6). Then we find, disregarding the non-negative terms,

$$\int \left( |D\alpha|^2 + \frac{1}{2} R_\gamma \alpha^2 \right) e^{c_i} dS \geq \int \frac{1}{4} \frac{\omega_i^2}{\eta^2} e^{\sigma_i - 2c_i} \alpha^2 e^{c_i} dS, \quad (33)$$

where a prime denotes derivative with respect to  $\theta$ . Now, using that

$$R_\gamma = \frac{e^{\sigma_i - 2c_i}}{\sin \theta} (2q_i' \cos \theta + \sin \theta \sigma_i' q_i' + 2 \sin \theta - (\sin \theta \sigma_i')') \quad (34)$$

and choosing

$$\alpha = e^{c_i - \sigma_i/2}, \quad (35)$$

we obtain

$$4\pi(c_i + 1) - 2\pi(q_i(\pi) + q_i(0)) - \int \left( \sigma_i + \frac{1}{4} \sigma_i'^2 \right) dS \geq \int \frac{1}{4} \frac{\omega_i^2}{\eta^2} dS, \quad (36)$$

which can be written as

$$4\pi(c_i + 1) - 2\pi(q_i(\pi) + q_i(0)) \geq \frac{\pi}{2} \mathcal{M}_i, \quad (37)$$

where  $\mathcal{M}_i$  is the functional introduced by [1] and used also by [9] and [21],

$$\mathcal{M}_i := \frac{1}{2\pi} \int 4\sigma_i + \sigma_i'^2 + \frac{\omega_i'^2}{\eta^2} dS. \quad (38)$$

Note that in going from (33) to (36) is where we have used the non-regularity of the metric at the axis. Now, in [1] it has been proven that for each  $\mathcal{M}_i$ , the following bounds hold:

$$\mathcal{M}_i \geq 8(\ln(2|J_i|) + 1), \quad (39)$$

where  $J_i$  is the angular momentum of  $\Sigma_i$ . Using the fact that  $A_i = 4\pi e^{c_i}$ , we obtain

$$A_i \geq 8\pi |J_i| e^{\frac{q_i(0)+q_i(\pi)}{2}}. \quad (40)$$

Finally, using the expression for  $\mathcal{F}_i$  in terms of  $q_i$  we complete the proof of the theorem.  $\square$

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