Nikhef-2013-003
ITP-UU-13/03
AEI-2013-048

Deformations of gauged SO(8) supergravity and supergravity in eleven dimensions

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Abstract

Motivated by the fact that there exists a continuous one-parameter family of gauged SO(8) supergravities, possible eleven-dimensional origins of this phenomenon are explored. Taking the original proof of the consistency of the truncation of $11D$ supergravity to SO(8) gauged supergravity as a starting point, a number of critical issues is discussed, such as the preferred electric-magnetic duality frame in four dimensions and the existence of dual magnetic gauge fields and related quantities in eleven dimensions. Some of those issues are resolved but others seem to point to obstructions in embedding the continuous degeneracy in $11D$ supergravity. While the final outcome of these efforts remains as yet inconclusive, several new results are obtained. Among those is the full non-linear ansatz for the seven-dimensional flux expressed in terms of the scalars and pseudoscalars of $4D$ supergravity, valid for both the S^7 and the T^7 truncations without resorting to tensor-scalar duality.

1 Introduction

Recently it was discovered that there exists a continuous one-parameter family of inequivalent gauged SO(8) supergravities characterized by one angular parameter ω [1]. The new theories were found by using the embedding tensor approach [2, 3, 4] to couple an ω-dependent linear combination of 28 electric and 28 magnetic gauge fields and elevate their gauge group to SO(8). As is well known one can convert these theories by performing an ω-dependent electric-magnetic duality transformation so that the gauging becomes purely electric. The theories thus obtained correspond to a one-dimensional variety of $N = 8$ supergravity Lagrangians in which the 28 abelian gauge transformations have been extended to a non-abelian SO(8) electric gauge group in the conventional way; the consistency of this gauging can be directly inferred by making use of the T-tensor identities presented in [5], which remain applicable for non-zero ω. The inequivalence of the new gauged SO(8) supergravities for different (generic) values of ω was confirmed in [1] by examining stationary points of the potential in a G_2-invariant sector of the theory which showed that the multiplicities of SO(7)-invariant and G_2-invariant stationary points are different from those found for the original gauging [6, 7, 8]. The discovery of the continuous deformations has meanwhile stimulated further work on more general solutions of gauged SO(8) supergravities [9, 10].

The existence of a continuous family of gauged SO(8) supergravities is a rather surprising fact and its discovery demonstrates the power of the embedding tensor method. In this paper we first rederive and clarify this result in the context of the electric duality frame, following as much as possible the original construction of the SO(8) gauging [5]. The analysis in the electric frame is interesting in its own right. It enables us to compare the $SO(7)^{\pm}$ solutions that were found in the electric frame for $\omega = 0$ [6, 7] to the corresponding solutions in the ω-deformed theory. Besides confirming the consistency of the gaugings, it provides an independent verification of the phenomenon, noted in [1], that the independent deformations cover only part of the full interval $\omega \in (0, 2\pi]$. In the electric duality frame this is caused by the fact that certain changes in ω can be compensated for by performing various field redefinitions in the Lagrangian, so that different values of ω will correspond to the same Lagrangians. Ultimately this reduces the interval of inequivalent deformations to $\omega \in (0, \pi/8]$. In establishing this result the diagonal SU(8) subgroup of $E_{7(7)} \times SU(8)$ plays an important role, where $E_{7(7)}$ is the symmetry group of the ungauged theory [11].

The prime motivation for our work is to explore whether the continuous deformation has a possible interpretation from the perspective of $11D$ supergravity [12], or, more precisely, whether the deformed theories can be consistently embedded into $11D$ supergravity. The original gauged SO(8) supergravity has been proven to correspond to a consistent truncation of $11D$ supergravity associated with S^7 [13, 14]; this proof made use of the SL(8) invariant formulation of the $4D$ theory with the SO(8) gauge group embedded into SL(8). Therefore we first address the question whether or not this proof can be extended to the ω-dependent electric duality frame. The answer turns out to be negative. Therefore the only option seems to remain within the context of the

SL(8) covariant duality frame and to investigate whether one can consistently incorporate the magnetic charges in this frame in the context of the higher-dimensional theory. As we intend to show in this paper, the SU(8) covariant reformulation of $11D$ supergravity given in [15, 16] does indeed allow for the necessary dual structures. On the other hand, the assumption that the ω-deformed theories also have a consistent embedding in $11D$ supergravity, would imply that any solution of $11D$ supergravity that is known to have a $4D$ counterpart for $\omega = 0$ will belong to one-parameter family of similar solutions of $11D$ supergravity. In view of the fact that the ω-deformation commutes with SO(8) the solutions belonging to such a family should share the same invariance subgroup of SO(8). For instance, a continuous family should exist of SO(7) invariant solutions associated with the $11D$ solutions of [17, 18] that have been shown to correspond to similar solutions of $4D$ SO(8)-gauged supergravity with $\omega = 0$ [19, 13, 14]. It seems that this is only possible when $11D$ supergravity is somehow extended such that it will be equipped with the deformation parameter ω as an extraneous parameter, which would require an extension of the version of $11D$ supergravity given in [12]. The nature of such an extension is at present not known. We discuss these issues in the concluding section 6.

While a complete resolution of the important question concerning the possible $11D$ relation of the ω-deformed supergravities remains open for the moment, the consideration of dual vectors in the $11D$ context leads us to two unexpected and important new results which generalize the SU(8) invariant reformulation of $11D$ supergravity given in [15, 16] on which the consistency proof of [13, 14] was based. The first one is the existence of a new 'generalized vielbein' that is related to the 28 dual magnetic vectors in the same way as the original generalized vielbein was related to the 28 electric vectors. More specifically, the latter is a soldering form $e^m{}_{AB}$ associated to the Kaluza-Klein vector fields $B_\mu{}^m$ (contained in the elfbein $E_M{}^A$ of $11D$ supergravity (cf. 4.2)), while the new vielbein $e_{mn\,AB}$ is associated to the components $A_{\mu mn}$ and A_{mnp} of the three-form potential A_{MNP} of $11D$ supergravity. [1] The combination of the two generalized vielbeine then yields the formula (5.12) for the *non-linear flux ansatz*, analogous to the non-linear metric ansatz first presented in [21, 13]. A formula for the flux had already been derived in [13, 14], but that formula was in terms of the four-form field strength rather than the three-form potential and appears to be too unwieldy for practical applications. This is not so with the new and much simpler formula (5.12) which is directly in terms of the three-form potential A_{mnp}. It is remarkable that the detour via the ω-deformed gaugings thus yields the answer to a question that has remained open for almost 30 years!

This paper is organized as follows. Section 2 summarizes a number of characteristic features of $N = 8$ supergravity and of the relevant electric-magnetic duality frames. Subsequently the ω-deformed SO(8) gaugings are discussed in the electric frame and we analyze the inequivalence of supergravities corresponding to different values of ω. In section 3 an analysis is presented of the $\mathrm{SO}(7)^\pm$ solutions for arbitrary values of ω. The results are in agreement with those

[1] A similar extension has already appeared in a previous study [20] in the context of $3D$ supergravity and $E_{8(8)}$, where the vectors are dual to scalar fields, but where it is not possible to compare the relevant formulae to non-trivial compactifications of $11D$ supergravity.

presented in [1]. In the subsequent section 4 the possible embedding of the ω-deformed theories is considered. The first conclusion is that such an embedding can only be given in the SL(8) duality frame, which implies that a possible embedding should involve dual magnetic gauge fields as well as related quantities. The search for such dual quantities is then undertaken in section 5. Although such quantities can indeed be identified, it still does not enable the formulation of a consistent embedding scheme of the ω-deformed $4D$ theories into $11D$ supergravity. On the other hand the newly found dual gauge fields and generalized vielbeine give substantial new insights of the embedding of the original $\omega = 0$ theory into $11D$ supergravity. In particular a non-linear expression is found for the tensor field A_{mnp} of $11D$ supergravity in the S^7 and T^7 truncations. Conclusions and a further outlook are presented in section 6. An appendix A presents a number of definitions and the algebraic details related to the supersymmetry transformation rule of the dual generalized vielbein.

2 SO(8) gaugings of maximal D=4 supergravity

As is well known, four-dimensional Lagrangians with abelian gauge fields are ambiguous, as different Lagrangians can lead to equivalent field equations and Bianchi identities. This phenomenon is known as electric-magnetic duality. Generic electric-magnetic duality transformations do not constitute an invariance but an equivalence. These transformations can be effected by performing a real symplectic rotation of the field strengths $F_{\mu\nu}$ and the dual fields strengths $G_{\mu\nu}$. The latter are defined such that the Bianchi identity on the latter equals precisely the field equations of the vector fields. For $N = 8$ supergravity we have 28 vector fields so that the number of field strengths and dual field-strengths equals 56. The general analysis of [22] therefore implies that the electric-magnetic duality group is equal to $\mathrm{Sp}(56;\mathbb{R})$. After applying the symplectic rotation of the field strengths, the new dual field strengths $G_{\mu\nu}$ take a different form that will in turn follow from a different Lagrangian. In the absence of a gauging, all these Lagrangians are physically equivalent as they describe the same set of field equations and Bianchi identities.

The corresponding theory may in principle be invariant under a subgroup of the electric-magnetic dualities combined with related transformations on the other fields, meaning that the Lagrangian will not change under this subgroup (which does not imply that the Lagrangian is invariant in the naive sense, as the Lagrangian does not transform as a function under duality). This happens for ungauged $N = 8$ supergravity where the invariance group corresponds to the non-compact $\mathrm{E}_{7(7)}$ subgroup of $\mathrm{Sp}(56;\mathbb{R})$ [11]. When working with a formulation that is gauge invariant under local chiral SU(8), which acts on the fermions and on the scalars, the theory is invariant under the group $\mathrm{E}_{7(7)} \times \mathrm{SU}(8)$ which is linearly realized. Once a gauge is adopted with respect to the local SU(8), the group action of $\mathrm{E}_{7(7)}$ will be non-linearly realized on the spinors and the scalars of the theory. The latter then parametrize an $\mathrm{E}_{7(7)}/\mathrm{SU}(8)$ coset space; here it is relevant that SU(8) is the maximal compact subgroup of $\mathrm{E}_{7(7)}$. We prefer to work with the linear version of the theory with manifest local SU(8) invariance.

However, the Lagrangian can only be invariant under a subgroup of $\mathrm{E}_{7(7)}$, such as, for instance, SL(8), under which the vector fields transform in the real **28** representation. While the usefulness of real representations is obvious for the gauge fields, it is not convenient for the remaining fields which transform under SU(8) in complex representations. A crucial quantity in the formulation of the theory is the so-called 56-bein $\mathcal{V}$, which is a 56×56 matrix that belongs to the **56** representation of $\mathrm{E}_{7(7)}$. The usual representation of this matrix is given in a pseudo-real decomposition of $\mathrm{E}_{7(7)}$ based on $\mathbf{56} = \mathbf{28} + \overline{\mathbf{28}}$, where **28** and $\overline{\mathbf{28}}$ denote two conjugate representations of the maximal subgroup SU(8). The 56-bein $\mathcal{V}$ will transform under $\mathrm{E}_{7(7)}$ rigid transformations and under lcoal SU(8) by right- and left-multiplication, respectively.[2]

To set the stage let us briefly discuss some properties of the group $\mathrm{E}_{7(7)} \subset \mathrm{Sp}(56;\mathbb{R})$. We start with the fundamental representation **56** of $\mathrm{Sp}(56;\mathbb{R})$, written as a pseudo-real vector (z_{IJ}, z^{KL}) with $z^{IJ} = (z_{IJ})^*$, where the indices are anti-symmetric index pairs $[IJ]$ and $[KL]$ and $I,J,K,L = 1,\ldots,8$. Hence the (z_{IJ}, z^{KL}) span a real 56-dimensional vector space. Consider infinitesimal transformations of the form,

$$\begin{aligned} \delta z_{IJ} &= \Lambda_{IJ}{}^{KL}\, z_{KL} + \Sigma_{IJKL}\, z^{KL}\,, \\ \delta z^{IJ} &= \Lambda^{IJ}{}_{KL}\, z^{KL} + \Sigma^{IJKL}\, z_{KL}\,. \end{aligned} \tag{2.1}$$

where $\Lambda_{IJ}{}^{KL} = \Lambda_{[IJ]}{}^{[KL]}$ and $\Sigma_{IJKL} = \Sigma_{[IJ]\,[KL]}$ are subject to the conditions,

$$(\Lambda_{IJ}{}^{KL})^* = \Lambda^{IJ}{}_{KL} = -\Lambda_{KL}{}^{IJ}\,, \qquad (\Sigma_{IJKL})^* = \Sigma^{IJKL} = \Sigma^{KLIJ}\,. \tag{2.2}$$

Note that complex conjugation is effected by raising or lowering of indices. The corresponding group elements g constitute the group $\mathrm{Sp}(56;\mathbb{R})$ in a pseudo-real basis provided that they satisfy the conditions,

$$g^* = \omega\, g\, \omega\,, \qquad g^{-1} = \Omega\, g^\dagger\, \Omega\,, \tag{2.3}$$

where ω and Ω are given by

$$\omega = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\,, \qquad \Omega = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}\,. \tag{2.4}$$

The above properties ensure that the sesquilinear form, $(z_1, z_2) = z_1^{IJ}\, z_{2IJ} - z_{1IJ}\, z_2^{IJ}$, is invariant. The generators associated with $\Lambda_{IJ}{}^{KL}$ generate the maximal compact U(28) subgroup of $\mathrm{Sp}(56;\mathbb{R})$, and a GL(28) subgroup is generated by real matrices $\Lambda_{IJ}{}^{KL}$ and purely real or purely imaginary Σ^{IJKL}, whose compact subgroup equals SO(28).

Let us now consider the $\mathrm{E}_{7(7)}$ subgroup, for which Σ^{IJKL} is fully anti-symmetric and the generators are further restricted according to

$$\begin{aligned} &\Lambda_{IJ}{}^{KL} = \delta_{[I}^{[K}\, \Lambda_{J]}{}^{L]}\,, \qquad \Lambda_I{}^J = -\Lambda^J{}_I\,, \\ &\Lambda_I{}^I = 0\,, \qquad \Sigma_{IJKL} = \tfrac{1}{24}\varepsilon_{IJKLMNPQ}\, \Sigma^{MNPQ}\,. \end{aligned} \tag{2.5}$$

Obviously the matrices $\Lambda_I{}^J$ generate the group SU(8), which has dimension 63; since the Σ_{IJKL} comprise 70 real parameters, the dimension of $\mathrm{E}_{7(7)}$ equals 63 + 70 = 133. Because SU(8) is

[2] There are different conventions used in the literature. Here we will follow [5].

the maximal compact subgroup, the number of non-compact generators minus the number of compact ones equals $70 - 63 = 7$. It is straightforward to show that these matrices close under commutation and generate the group $\mathrm{E}_{7(7)}$ [11, 5]. To show this one needs a variety of identities for self-dual tensors. Note that $\mathrm{E}_{7(7)}$ has another maximal 63-dimensional subgroup, which is real but not compact, namely the group SL(8). It is generated by those matrices in (2.5) for which the sub-matrices $\Lambda_I{}^J$ and Σ^{IJKL} are both real.

Let is now define the 56-bein $\mathcal{V}$, which describes the scalar fields,

$$\mathcal{V}(x) = \begin{pmatrix} u_{ij}{}^{IJ}(x) & v_{ij\,KL}(x) \\ \\ v^{kl\,IJ}(x) & u^{kl}{}_{KL}(x) \end{pmatrix} , \tag{2.6}$$

and which is an element of $\mathrm{E}_{7(7)}$. Therefore it can transform by left-multiplication under local SU(8) and by right-multiplication under rigid $\mathrm{E}_{7(7)}$. Hence the indices $[ij]$ and $[kl]$ are local SU(8) indices and $[IJ]$ and $[KL]$ are rigid $\mathrm{E}_{7(7)}$ indices. A standard SU(8) gauge condition leads to the following coset representative ('unitary gauge'),

$$\mathcal{V}(x) = \exp\begin{pmatrix} 0 & -\frac{1}{4}\sqrt{2}\,\phi_{ijkl}(x) \\ \\ -\frac{1}{4}\sqrt{2}\,\phi^{mnpq}(x) & 0 \end{pmatrix} , \tag{2.7}$$

where the ϕ^{ijkl} are complex fields transforming as an anti-symmetric four rank tensor under the linearly realized rigid SU(8). The complex conjugate fields, ϕ_{ijkl}, are related to the original fields by a complex self-duality constraint,

$$\phi_{ijkl} = \tfrac{1}{24}\varepsilon_{ijklmnpq}\,\phi^{mnpq} . \tag{2.8}$$

Observe that in this gauge the indices $I, J, K, \ldots$ are no longer distinguishable from the SU(8) indices $i, j, k, \ldots$. We also note that the reflection $\phi^{ijkl} \to -\phi^{ijkl}$ maps $(u, v) \to (u, -v)$ in (2.6) and therefore corresponds to a trivial reparametrization of the $\mathrm{E}_{7(7)}/\mathrm{SU}(8)$ coset space.

Subsequently we consider the 28 field strengths $F_{\mu\nu}{}^{IJ}$ and their dual field strengths,

$$G^{+\mu\nu}{}_{IJ} = -\frac{4}{e}\,\frac{\partial\mathcal{L}}{\partial F^{+}{}_{\mu\nu IJ}} . \tag{2.9}$$

The Bianchi identies and the field equations of the vector fields are summarized in the following equations,

$$\partial_\mu\big[e\,F^{+\mu\nu IJ} - e\,F^{-\mu\nu IJ}\big] = 0 = \partial_\mu\big[e\,G^{+\mu\nu}_{IJ} + e\,G^{-\mu\nu}{}_{IJ}\big] . \tag{2.10}$$

These equations can be written in terms of a 56-component array of selfdual field strengths, $(F^{+}_{1\mu\nu IJ}, F^{+}_{2\mu\nu}{}^{IJ})$, defined by

$$\begin{aligned} F^{+}_{1\mu\nu IJ} &= \tfrac{1}{2}\big(G^{+\mu\nu}{}_{IJ} + F^{+\mu\nu IJ}\big) , \\ F^{+}_{2\mu\nu}{}^{IJ} &= \tfrac{1}{2}\big(G^{+\mu\nu}{}_{IJ} - F^{+\mu\nu IJ}\big) , \end{aligned} \tag{2.11}$$

and their anti-selfdual ones $(F^{-}_{1\mu\nu}{}^{IJ}, F^{-}_{2\mu\nu IJ})$ that follow by complex conjugation, in a form that is manifestly covariant under $\mathrm{Sp}(56;\mathbb{R})$ [22].

What remains is to specify $G_{\mu\nu IJ}$ in terms of $F_{\mu\nu}{}^{IJ}$ and terms depending on the matter fields. This will then determine all terms involving the vector fields of the Lagrangian. As long as we have not switched on the gauging, the matter field contributions come exclusively from fermionic bilinears, which we denote by $\mathcal{O}_{\mu\nu}$. Since the fermions transform under local SU(8) and not under $\mathrm{E}_{7(7)}$, this relation must necessarily involve the 56-bein $\mathcal{V}$ and can be written as follows [5],

$$\mathcal{V}\begin{pmatrix} F^{+}_{1\mu\nu\,IJ} \\ \\ F^{+}_{2\mu\nu}{}^{KL} \end{pmatrix} = \begin{pmatrix} \bar{F}^{+}_{\mu\nu\,ij} \\ \\ \mathcal{O}^{+kl}_{\mu\nu} \end{pmatrix}, \tag{2.12}$$

where $\mathcal{O}^{+}_{\mu\nu}{}^{ij}$ is an SU(8) covariant tensor quadratic in the fermion fields and independent of the scalar fields, which appears as a moment coupling in the Lagrangian. Without going into the details we mention that chirality and self-duality restricts the form of $\mathcal{O}^{+}_{\mu\nu}{}^{ij}$ up to some normalization constants. The tensor $\bar{F}^{+}_{\mu\nu\,ij}$ is an SU(8) covariant field strength which appears in the supersymmetry transformation rules of the spinors, which is simply defined by the above condition. For future reference we give the definition of $\mathcal{O}^{+}_{\mu\nu}{}^{ij}$,

$$\mathcal{O}^{+}_{\mu\nu}{}^{ij} = -\tfrac{1}{288}\sqrt{2}\,\varepsilon^{ijklmnpq}\,\bar{\chi}_{klm}\gamma_{\mu\nu}\chi_{npq} - \tfrac{1}{4}\bar{\psi}_{\rho k}\gamma_{\mu\nu}\gamma^{\rho}\chi^{ijk} + \tfrac{1}{4}\sqrt{2}\,\bar{\psi}_{\rho}{}^{i}\gamma^{[\rho}\gamma_{\mu\nu}\gamma^{\sigma]}\psi_{\sigma}{}^{j}\,. \tag{2.13}$$

The form of (2.12) emphasizes the covariance under the group SL(8), as both $F^{+}_{1\mu\nu IJ}$ and $F^{+}_{2\mu\nu}{}^{IJ}$ defined in (2.11) transform in the **28** and $\overline{\mathbf{28}}$ representations of that group. As long as we have not switched on the gauging, we have the option of changing the basis of these field strengths by a matrix $E \in \mathrm{Sp}(56;\mathbb{R})$. It thus seems that the possible Lagrangians are encoded in these matrices E. However, this is not the case, because, when E belongs to GL(28) or to $\mathrm{E}_{7(7)}$, it can be absorbed into either the field strengths (2.11) or into the 56-bein, respectively. Hence it follows that (2.12), and thus the Lagrangian has an ambiguity encoded in a matrix [3, 23]

$$E \;\in\; \mathrm{E}_{7(7)}\backslash\mathrm{Sp}(56;\mathbb{R})/\mathrm{GL}(28,\mathbb{R})\,. \tag{2.14}$$

When one is interested in SO(8) invariant Lagrangians, the matrix E must preserve the SO(8) subgroup, so that the relevant matrices E are restricted to

$$E = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\omega}\mathbb{1} & 0 \\ \\ 0 & \mathrm{e}^{-\mathrm{i}\omega}\mathbb{1} \end{pmatrix}, \tag{2.15}$$

where $\mathbb{1} \equiv \mathbb{1}_{28}$ denotes the 28×28 unit matrix. Hence these Lagrangians are encoded in a single angle ω. [3] For special values of ω this matrix will constitute an element of $\mathrm{E}_{7(7)}$, because the compact SU(8) subgroup of $\mathrm{E}_{7(7)}$ has a non-trivial center $Z[\mathrm{SU}(8)] = \mathbb{Z}_8$, which is reduced to $\mathbb{Z}_4$ when acting on bosons (as these come with an even number of SU(8) indices). The center $Z[\mathrm{SU}(8)]$ consists of the matrices $\mathrm{e}^{\mathrm{i}\omega/2}\mathbb{1}_8$ with ω a multiple of $\pi/2$. Consequently, SO(8) invariant

[3] Angles such as ω were first introduced in the context of gauged $N = 4$ supergravity in [24]

Lagrangians corresponding to ω-values that differ by an integer times $\pi/2$ must be equivalent, as they are related by an element of SU(8) (and therefore of $\mathrm{E}_{7(7)}$). Other than these there are no matrices E belonging to $\mathrm{E}_{7(7)}$. We return momentarily to a more detailed analysis of possible equivalences.

The exponential factor in (2.15) can now be incorporated directly into the supergravity Lagrangian by simply including ω-dependent phase factors into the submatrices u and v in the Lagrangian according to

$$u_{ij}{}^{IJ} \to \mathrm{e}^{\mathrm{i}\omega} u_{ij}{}^{IJ}\,, \quad v_{ijIJ} \to \mathrm{e}^{-\mathrm{i}\omega} v_{ijIJ}\,. \tag{2.16}$$

This defines the deformed supergravity Lagrangians in the electric frame. As already mentioned in section 1, the inequivalent theories do not cover the full interval $\omega \in (0, 2\pi]$, but are restricted to the smaller interval $\omega \in (0, \pi/8]$, as was shown by [1] in a mixed electric-magnetic duality frame. We will now verify this result in the electric frame. We distinguish three types of equivalence transformations for ω:

i) The shift $\omega \to \omega + \pi/2$, which can be undone by a special SU(8) transformation belonging to Z[SU(8)].

ii) The shift $\omega \to \omega + \pi/4$, which can be undone by an SU(8) transformation that belongs to a square root of an element of Z[SU(8)] accompanied by a linear redefinition of the gauge fields $A_\mu{}^{IJ}$.

iii) The reflection $\omega \to -\omega$, which can be undone by a parity transformation.

To analyze these three equivalences we consider the ω-deformed Lagrangians. The terms that involve the field strengths are encoded in (2.12) subject to the deformation (2.16). Writing this equation in terms of the separate components, one obtains

$$\begin{aligned} \big(u^{ij}{}_{IJ} + \mathrm{e}^{2\mathrm{i}\omega} v^{ijIJ}\big) G^{+}_{\mu\nu IJ} &= \big(u^{ij}{}_{IJ} - \mathrm{e}^{2\mathrm{i}\omega} v^{ijIJ}\big) F^{+\,IJ}_{\mu\nu} + 2\,\mathrm{e}^{\mathrm{i}\omega} \mathcal{O}^{+\,ij}_{\mu\nu}\,, \\ 2\,\mathrm{e}^{-\mathrm{i}\omega} \bar{F}^{+}_{\mu\nu ij} &= \big(u_{ij}{}^{IJ} + \mathrm{e}^{-2\mathrm{i}\omega} v_{ijIJ}\big) G^{+}_{\mu\nu IJ} + \big(u_{ij}{}^{IJ} - \mathrm{e}^{-2\mathrm{i}\omega} v_{ijIJ}\big) F^{+\,IJ}_{\mu\nu}\,. \end{aligned} \tag{2.17}$$

Let us first consider the effect of the shift $\omega \to \omega + \pi/2$ in (2.17), which we can clearly undo by performing the following redefinitions,

$$v^{ijIJ} \to \mathrm{e}^{-\mathrm{i}\pi} v^{ijIJ} = -v^{ijIJ}\,, \quad \mathcal{O}^{+\,ij}_{\mu\nu} \to \mathrm{e}^{-\mathrm{i}\pi/2} \mathcal{O}^{+\,ij}_{\mu\nu}\,, \quad \bar{F}^{+}_{\mu\nu ij} \to \mathrm{e}^{\mathrm{i}\pi/2} \bar{F}^{+}_{\mu\nu ij}\,. \tag{2.18}$$

We have to ensure that these redefinitions are consistent for the full Lagrangian. This follows rather straightforwardly by noting that the redefinitions (2.18) are precisely generated by applying a uniform SU(8) transformation belonging to the diagonal subgroup of $\mathrm{E}_{7(7)} \times \mathrm{SU}(8)$ and equal to $\mathrm{e}^{\mathrm{i}\pi/2}\,\mathbb{1}_{28}$, which constitutes an element of Z[SU(8)]. Note that on $u^{ij}{}_{IJ}$ the effect of this transformation cancels, as it acts on both index pairs $[ij]$ and $[IJ]$, while it correctly accounts for the phase factor in the redefinition of v^{ijIJ}. [4] The SU(8) transformation is also realized on the

[4] Note that the diagonal SU(8) transformations induce a corresponding change on the field ϕ^{ijkl} in the coset representative (2.7). For this reason the pseudo-reality constraint (2.8) will be preserved throughout.

fermions where it takes the form,

$$\psi_\mu{}^i \to \mathrm{e}^{-\mathrm{i}\pi/4}\psi_\mu{}^i\,, \qquad \chi^{ijk} \to \mathrm{e}^{-3\mathrm{i}\pi/4}\chi^{ijk}\,, \tag{2.19}$$

and this generates the desired redefinition of $\mathcal{O}^+_{\mu\nu}{}^{ij}$ and $\bar{F}^+_{\mu\nu ij}$. As far as the ungauged Lagrangian and the supersymmetry transformations are concerned (we remind the reader that $\bar{F}^+_{\mu\nu ij}$ and its anti-selfdual component appear in the supersymmetry transformations), the shift $\omega \to \omega + \pi/2$ combined with a special SU(8) transformation leaves the Lagrangian and the supersymmetry transformations uneffected. Note that the fact that the Lagrangian and the supersymmetry transformations are consistent with respect to *local* SU(8) plays a crucial role for the remaining terms in the Lagrangian.

To prove that the terms depending on the SO(8) gauging are not affected by the shift and the various field redefinitions, we consider the so-called T-tensor associated with the SO(8) gauging, which takes the following form in the ω-deformed theory,

$$\begin{aligned} T_i^{\,jkl}(\omega\,;\,u,v) &= \left(\mathrm{e}^{-\mathrm{i}\omega}\, u^{kl}{}_{IJ} + \mathrm{e}^{\mathrm{i}\omega}\, v^{klIJ}\right)\left(u_{im}{}^{JK}\, u^{jm}{}_{KI} - v_{imJI}\, v^{jmKL}\right) \\ &= \cos\omega\, T^{(\mathrm{e})}{}_i^{\,jkl}(u,v) \;+\; \sin\omega\, T^{(\mathrm{m})}{}_i^{\,jkl}(u,v)\,. \end{aligned} \tag{2.20}$$

where in the second line, we explicitly display the decomposition of the T-tensor into an 'electric' and a 'magnetic' component. As the reader can check, the consistency of the gauging is not affected by the ω-deformation (2.16), because the analysis given in [5] still applies, in the sense that all the 'T-identities' remain valid. [5] This is consistent with the general outline given in [3, 4] and the specific application described in [1]. When applying the shift $\omega \to \omega + \pi/2$ in (2.20) we follow the same strategy as before and obtain the relation,

$$T_i^{\,jkl}\,(\omega + \pi/2\,;\,u,v) \;=\; \mathrm{e}^{-\mathrm{i}\pi/2}\, T_i^{\,jkl}(\omega\,;\,u,\mathrm{e}^{\mathrm{i}\pi}\,v)\,, \tag{2.21}$$

where u and v denote $u^{ij}{}_{IJ}$ and v^{ijIJ}, respectively. Again the changes take the form of an SU(8) transformation, and are precisely cancelled by the redefinitions found previously in (2.18) and (2.19).

The discussion of the second equivalence transformation $\omega \to \omega + \pi/4$ proceeds along the same lines, but there are new features. First of all, because the transformation $\mathrm{e}^{\mathrm{i}\pi/8}\,\mathbb{1}_8$ is clearly not an element of SU(8), we must replace the identity matrix in this product by some other real matrix P_8. Hence we consider $\mathrm{e}^{\mathrm{i}\pi/8}\,P_8$, which constitutes an element of SU(8) provided that P_8 is real and orthogonal with $\det[P_8] = -1$. As its square should belong to $Z[\mathrm{SU}(8)]$, it follows also that $(P_8)^2 = \mathbb{1}_8$. Obviously such matrices P_8 exists! Examples are diagonal matrices with p eigenvalues equal to -1 and $8-p$ eigenvalues equal to $+1$, with p odd, but there exist more matrices that satisfy these requirements. The SU(8) transformation can also be written in the **28** representation, where it takes the form $\mathrm{e}^{\mathrm{i}\pi/4}\,\Pi$, with $\Pi^{ij}{}_{kl} = P_8{}^{[i}{}_{[k}\,P_8{}^{j]}{}_{l]}$.

[5] These identities encode the same information as the linear and quadratic identities that the embedding tensor has to satisfy.

Now let us return to (2.17), but now multiplied by the matrix Π from the left. Furthermore we multiply the field strength tensors with $\Pi^2 = \mathbb{1}_{28}$. Obviously the shift in ω can now be absorbed by making the following redefinitions,

$$\begin{aligned} u^{ij}{}_{IJ} &\to \Pi^{ij}{}_{kl}\, u^{kl}{}_{KL}\, \Pi^{KL}{}_{IJ}\,, & \mathcal{O}^{+\,ij}_{\mu\nu} &\to \mathrm{e}^{-\mathrm{i}\pi/4}\, \Pi^{ij}{}_{kl}\, \mathcal{O}^{+\,kl}_{\mu\nu}\,, \\ v^{ijIJ} &\to \mathrm{e}^{-\mathrm{i}\pi/2}\, \Pi^{ij}{}_{kl}\, v^{ijIJ}\, \Pi^{ij}{}_{kl}\,, & \bar{F}^{+}_{\mu\nu ij} &\to \mathrm{e}^{\mathrm{i}\pi/4}\, \Pi^{kl}{}_{ij}\, \bar{F}^{+}_{\mu\nu kl}\,. \end{aligned} \tag{2.22}$$

combined with a linear redefinition of the vector gauge fields,

$$A_\mu{}^{IJ} \to \Pi^{IJ}{}_{KL}\, A_\mu{}^{KL}\,. \tag{2.23}$$

The latter induces the same redefinition of the field strengths $G^{+}_{\mu\nu IJ}$ and $F^{+\,IJ}_{\mu\nu}$, even in the presence of the non-abelian completion. Obviously the transformations (2.22) correspond to SU(8) transformations belonging to the diagonal subgroup of $\mathrm{SU}(8) \times \mathrm{E}_{7(7)}$, just as before. On the fermions they act according to

$$\psi_\mu{}^i \to \mathrm{e}^{-\mathrm{i}\pi/8}\, P_8{}^i{}_j\, \psi_\mu{}^j\,, \qquad \chi^{ijk} \to \mathrm{e}^{-3\mathrm{i}\pi/8}\, P_8{}^i{}_l\, P_8{}^j{}_m\, P_8{}^k{}_n\, \chi^{lmn}\,. \tag{2.24}$$

For completeness we consider also the change of the T-tensor under the $\omega \to \omega + \pi/4$ transformation,

$$T_i^{jkl}\,(\omega + \pi/4\,;\, u, v) \;=\; \mathrm{e}^{-\mathrm{i}\pi/4}\, P_8{}^m{}_i\, P_8{}^j{}_n\, P_8{}^k{}_p\, P_8{}^l{}_q\, T_m{}^{npq}(\omega\,;\, \Pi\, u\, \Pi, \mathrm{e}^{\mathrm{i}\pi/2}\, \Pi\, v\, \Pi)\,, \tag{2.25}$$

with u and v as defined below (2.21). As a result the redefinitions noted above cancel precisely the effect of the shift in ω, which establishes the equivalence in the same fashion as before.

Finally we consider the third equivalence relation, $\omega \to -\omega$, whose effect can be absorbed by performing parity reversal on the fields. To explain this we note that original gauged SO(8) supergravity is invariant under parity. Under this discrete symmetry anti-selfdual and selfdual field strengths are interchanged simultaneously with the exchange of positive- and negative-chiral fermion components and of scalar fields with their complex conjugates. The ω-deformation breaks the invariance under parity. More precisely, when applying parity reversal to the Lagrangian for finite ω one obtains the same Lagrangian with ω replaced by $-\omega$. Hence, theories related by $\omega \to -\omega$ are equivalent, as the sign change can be undone by applying a parity transformation directly on the fields. Note that the sign change will also apply to the T-tensor given in (2.20), showing that the magnetic embedding tensor will change sign.

The three equivalence transformations analyzed in this section imply that inequivalent Lagrangians are encoded by values of ω in the restricted interval $\omega \in (0, \pi/8]$. This result, derived in the electric frame, is in full agreement with [1], where a fixed duality frame is used and where ω encodes the mixture of the electric and magnetic components of the embedding tensor. In the next section we will analyze the solutions that are invariant under an $\mathrm{SO}(7)^{\pm}$ subgroup of the SO(8) gauge group. As we shall demonstrate those solutions reflect precisely the equivalences exhibited in this section.

3 The potential and $\mathrm{SO}(7)^{\pm}$ invariant solutions

The potential of the gauged theory is constructed from the T-tensor. We recall that this tensor can generally be decomposed into two irreducible SU(8) tensors,

$$T_i^{jkl} = -\tfrac{3}{2} A_1{}^{j[k}\,\delta^{l]}{}_i - \tfrac{3}{4} A_{2\,i}{}^{jkl}\,, \tag{3.1}$$

where $A_1{}^{ij}$ is symmetric in (ij) and $A_{2\,i}{}^{jkl}$ is anti-symmetric in $[jkl]$ and traceless, $A_{2\,i}{}^{ikl} = 0$; together, these two irreducible components can be assigned to the **912** of $\mathrm{E}_{7(7)}$ [7]. The scalar potential equals

$$\mathcal{P} = g^2\left[-\tfrac{3}{4}|A_1{}^{ij}|^2 + \tfrac{1}{24}|A_{2\,i}{}^{jkl}|^2\right]\,, \tag{3.2}$$

where g is the SO(8) gauge coupling constant. As shown in [7], this potential has a stationary point whenever $4\,A_{1\,m[i}\,A_2{}^m{}_{jkl]} - 3\,A_2{}^m{}_{n[ij]}\,A_2{}^n{}_{kl]m}$ is an *anti*-selfdual tensor.

The simplest examples of special scalar field configurations for which stationary points exist, and where the effect of the ω-deformation can be studied in detail, are the backgrounds preserving $\mathrm{SO}(7)^{\pm}$-invariance [6, 7, 8]. For these the 56-bein takes the form

$$\mathcal{V}(t) = \exp\begin{pmatrix} 0 & \alpha t\, C^{\pm}_{ij\,KL} \\ & \\ \alpha^* t\, C^{\pm kl\,IJ} & 0 \end{pmatrix}\,, \tag{3.3}$$

with $t \in \mathbb{R}$ and $\alpha = 1$ for $\mathrm{SO}(7)^+$, and $\alpha = \mathrm{i}$ for $\mathrm{SO}(7)^-$. Here the $\mathrm{SO}(7)^{\pm}$ invariant tensors are (anti-)selfdual,

$$C^{\pm}_{IJKL} = \pm\frac{1}{24}\varepsilon_{IJKLMNPQ}\,C^{\pm}_{MNPQ}\,, \tag{3.4}$$

and obey the condition,

$$C^{\pm}_{IJMN}\,C^{\pm}_{MNKL} = 12\,\delta_{IJ}{}^{KL} \pm 4\,C^{\pm}_{IJKL}\,. \tag{3.5}$$

Note that (3.3) denotes the coset representative so that we make no distinction between rigid SL(8) indices $I, J, \ldots$ and local SU(8) indices $i, j, \ldots$. Note also that field ϕ^{ijkl} appearing in (2.7) is just equal to $-2\sqrt{2}\,t\,C^{+ijkl}$ or $2\sqrt{2}\,\mathrm{i}t\,C^{-ijkl}$, respectively, so that the pseudo-reality relation (2.8) is satisfied.

Using the relations (3.4) and (3.5), one shows that

$$\begin{aligned} u_{ij}{}^{IJ}(t) &= \cosh^3(2t)\,\delta_{ij}{}^{IJ} \pm \tfrac{1}{2}\cosh(2t)\sinh^2(2t)\,C^{\pm}_{ijIJ}\,, \\ v_{ijIJ}(t) &= \pm\,\alpha\,\sinh^3(2t)\,\delta_{ij}{}^{IJ} + \tfrac{1}{2}\alpha\,\sinh(2t)\cosh^2(2t)\,C^{\pm}_{ijIJ}\,. \end{aligned} \tag{3.6}$$

With these results one can evaluate the corresponding T-tensors (2.20). A straightforward calculation yields the following results for the component functions A_1 and A_2,

$$A_1{}^{ij} = \delta^{ij}\,A(t)\,, \qquad A_{2\,i}{}^{jkl} = A_2(t)\,C^{\pm}{}_i{}^{jkl}\,. \tag{3.7}$$

Note that the parameter t parametrizes the vacuum expectation value of either a selfdual or an anti-selfdual field. We will not consider both vacuum-expectation values simultaneously for reasons of simplicity. When allowing both vacuum-expectation values simultaneously, this would define a G_2 invariant background, as $\mathrm{G}_2 = \mathrm{SO}^+(7) \cap \mathrm{SO}^-(7)$. For the special configurations defined by (3.3) the potential takes the simple form,

$$\mathcal{P}(t) = g^2 \big[-6\,|A_1(t)|^2 + 14\,|A_2(t)|^2 \big] \,. \tag{3.8}$$

Its stationary points are determined by the condition that $\alpha A_2(t)\big(A_1(t) + 3\,A_2(t)\big)$ is imaginary.

Making use of (3.6) and inserting the deformation parameter ω according to (2.16), leads to the following expressions for the two functions $A_1(t)$ and $A_2(t)$ defined in (3.7),

$$\begin{aligned} A_1(\omega,t) &= \mathrm{e}^{-\mathrm{i}\omega}\big[c^7 + 7c^3s^4\big] \pm \alpha^* \mathrm{e}^{\mathrm{i}\omega}\big[s^7 + 7c^4s^3\big] \,, \\ A_2(\omega,t) &= \mp\, \mathrm{e}^{-\mathrm{i}\omega}\big[c\,s^6 + 4c^3s^4 + 3c^5s^2\big] - \alpha^* \mathrm{e}^{\mathrm{i}\omega}\big[c^6s + 4c^4s^3 + 3c^2s^5\big] \,, \end{aligned} \tag{3.9}$$

where $c \equiv \cosh(2t)$ and $s \equiv \sinh(2t)$. It is convenient to present these results as follows. For the $\mathrm{SO}(7)^+$-invariant background, we obtain

$$\begin{aligned} A_1^+(\omega,t) &= \mathrm{e}^{-\mathrm{i}\omega}\big[c^7 + 7c^3s^4\big] + \mathrm{e}^{\mathrm{i}\omega}\big[s^7 + 7c^4s^3\big] \,, \\ A_2^+(\omega,t) &= -\,\mathrm{e}^{-\mathrm{i}\omega}\big[c\,s^6 + 4c^3s^4 + 3c^5s^2\big] - \mathrm{e}^{\mathrm{i}\omega}\big[c^6s + 4c^4s^3 + 3c^2s^5\big] \,, \end{aligned} \tag{3.10}$$

whereas for the $\mathrm{SO}(7)^-$-invariant background we write

$$\begin{aligned} A_1^-(\omega,t) &= \mathrm{e}^{\mathrm{i}\pi/4}\left\{\mathrm{e}^{-\mathrm{i}\tilde\omega}\big[c^7 + 7c^3s^4\big] + \mathrm{e}^{\mathrm{i}\tilde\omega}\big[s^7 + 7c^4s^3\big]\right\} \,, \\ A_2^-(\omega,t) &= -\,\mathrm{e}^{\mathrm{i}\pi/4}\left\{-\mathrm{e}^{-\mathrm{i}\tilde\omega}\big[c\,s^6 + 4c^3s^4 + 3c^5s^2\big] - \mathrm{e}^{\mathrm{i}\tilde\omega}\big[c^6s + 4c^4s^3 + 3c^2s^5\big]\right\} \,, \end{aligned} \tag{3.11}$$

with $\tilde\omega = \omega + \pi/4$.

Interestingly the two $\mathrm{SO}^{\pm}(7)$ backgrounds lead to the same expression for the T-tensor, up to an overall phase factor and a shift in ω, although the overall phase factors for $A_1^{\pm}(\omega,t)$ and $A_2^{\pm}(\omega,(t)$ are clearly not the same. Because of this relation the two expressions (3.10) and (3.11) enable us to write the same formula for both potentials, but in terms of different parameters,

$$\begin{aligned} \mathcal{P}^+(\omega,t_+) &= \frac{g^2}{8}\Big\{ \cos^2\omega\big(x_+^{14} - 14x_+^6 - 35x_+^{-2}\big) + \sin^2\omega\big(x_+^{-14} - 14x_+^{-6} - 35x_+^2\big)\Big\} \,, \\ \mathcal{P}^-(\omega,t_-) &= \frac{g^2}{8}\Big\{ \cos^2\tilde\omega\big(x_-^{14} - 14x_-^6 - 35x_-^{-2}\big) + \sin^2\tilde\omega\big(x_-^{-14} - 14x_-^{-6} - 35x_-^2\big)\Big\} \,, \end{aligned} \tag{3.12}$$

with $x_\pm \equiv e^{2t_\pm}$. For $\omega = 0$ and $\omega = \pi/2$, respectively, these formulas reproduce the results of [7]; in particular, for $\tilde\omega = \pi/4$ we re-obtain the $\mathrm{SO}(7)^-$ potential,

$$\mathcal{P}^-(t_-) = -2g^2\cosh^5(4t_-)\,\big[5 - 2\cosh(8t_-)\big] \,. \tag{3.13}$$

Let us first briefly discuss the stationary points of $\mathcal{P}_\pm$, suppressing the distinction between the parameters, $x_\pm$ and between ω and $\tilde\omega$. Defining $z \equiv x^4 = e^{8t} \geq 0$, the condition for the potentials to be stationary (3.12) is

$$(z^2 - 1)\,\big[\cos^2\omega\, z^3(z^2 - 5) - \sin^2\omega\,\big(5\,z^2 - 1\big)\big] = 0 \,. \tag{3.14}$$

As it turns out this equation has three solutions. One is $z=1 \Leftrightarrow t=0$. A second solution exists with $0 \le z \le 1/\sqrt{5}$ and a third one with $\sqrt{5} \le z$. When $\sin\omega = 0$, there is a regular solution with $z=\sqrt{5}$ as well as a 'run-away solution' $z=0 \Leftrightarrow t=-\infty$; the corresponding solutions for $\cos\omega = 0$ are obtained by interchanging $z \leftrightarrow z^{-1}$ or $t \leftrightarrow -t$. For the $\mathrm{SO}(7)^\pm$ solutions, we see that there is only a single $\mathrm{SO}(7)^+$ solution, $z_+ = \sqrt{5}$ or $z_+ = 1/\sqrt{5}$ (as already explained above) when $\omega = 0$ and $\omega = \pi/2$, respectively. For the $\mathrm{SO}(7)^-$ backgrounds we recover the two solutions at $\tilde\omega = \pi/4$ (corresponding to $\omega = 0$ with $\coth 4t_- = \pm\sqrt{5}$. These two solutions are related by parity reversal. For $\tilde\omega = 0$ or $\tilde\omega = \pi/2$, there is again a run-away solution.

Let us now examine the consequences of the various equivalences between different ω-values noted in section 2. First of all, under a shift $\omega \to \omega + \pi/2$ the potentials (3.12) change according to $\mathcal{P}^\pm(\omega+\pi/2, t) = \mathcal{P}^\pm(\omega, -t)$, which is in agreement with what was derived more generally in section 2. Furthermore the functions $A^\pm_{1,2}$ satisfy $A^\pm_{1,2}(\omega+\pi/2,t) = -\mathrm{e}^{\mathrm{i}\pi/2}\, A^\pm_{1,2}(\omega,-t)$ which is consistent with (2.21). Under the other equivalence associated with the reflection $\omega \to -\omega$ the two potentials change according to $\mathcal{P}^\pm(-\omega,t) = \mathcal{P}^\pm(\omega,\pm t)$, which reflects the fact that for $\omega = 0$, t_+ is a scalar and t_- is a pseudoscalar.

It is rather obvious that the *separate* potentials $\mathcal{P}^\pm$ will exhibit no other equivalence relations, and in particular no relation associated with the shift $\omega \to \omega + \pi/4$. Indeed, this equivalence is qualitatively different because it also involves a change of basis for SO(8), as is shown in (2.23). Therefore the two potentials are interchanged! Inspection shows that the actual relation is given by

$$\mathcal{P}^+(\omega+\pi/4,t) = \mathcal{P}^-(\omega,t)\,, \qquad \mathcal{P}^-(\omega+\pi/4,t) = \mathcal{P}^+(\omega,-t)\,. \tag{3.15}$$

Before explaining this relation in more detail, we note that by applying this change twice, one recovers the result noted above for the shift $\omega \to \omega + \pi/2$.

Let us now clarify the details associated with the equivalence shift $\omega \to \omega + \pi/4$. In the new SO(8) basis the duality assignments of the SO(7) invariant tensors change according to

$$C^\pm_{IJKL} \longrightarrow -C^\mp_{IJKL}\,, \tag{3.16}$$

The change in the duality phase is due to the fact that $\det[P_8] = -1$ so that the 8-dimensional Levi-Civita symbol changes sign. Furthermore, the overall sign in (3.16) is required in order to re-establish the normalization condition (3.5). Using the correspondence noted below (3.5), which explains that $\phi^{ijkl} = 2\sqrt{2}\,\big(-t_+\,C^{+ijkl} + \mathrm{i}t_-\,C^{-ijkl}\big)$, we note the relation,

$$\mathrm{e}^{\mathrm{i}\pi/2}\, {P_8}^i{}_m\, {P_8}^j{}_n\, {P_8}^k{}_p\, {P_8}^l{}_q\, \phi^{mnpq} = 2\sqrt{2}\big(t_-\,C^{+ijkl} + \mathrm{i}t_+\,C^{-ijkl}\big)\,, \tag{3.17}$$

where we made use of (3.16). With this result we can evaluate (2.25), which leads to the following result,

$$\begin{aligned} T_i^{jkl}\big(\omega+\pi/4\,;\, t_+ = t,\, t_- = 0\big) &= \mathrm{e}^{-\mathrm{i}\pi/4}\, T_i^{jkl}\big(\omega\,;\, t_+ = 0,\, t_- = t\big)\,,\\ T_i^{jkl}\big(\omega+\pi/4\,;\, t_+ = 0,\, t_- = t\big) &= \mathrm{e}^{-\mathrm{i}\pi/4}\, T_i^{jkl}\big(\omega\,;\, t_+ = -t,\, t_- = 0\big)\,. \end{aligned} \tag{3.18}$$

This result is in line with (3.15) and can also be verified explicity on the functions $A^{\pm}_{1,2}$ shown in (3.10) and (3.11). One then observes that A_2 acquires an extra minus sign, which is due to the fact that in the T-tensor, A_2 is multiplied by the tensor $C^{\pm}$ (cf. 3.7).

Hence we have explicitly verified all the equivalence relations for the SO(7)$^{\pm}$ solutions. While it is clear that the equivalence based on the shift $\omega \to \omega + \pi/4$ is more subtle, these subtleties have been fully accounted for. Our conclusions are in full agreement with those of [1]. Obviously this pattern will persist for solutions with less symmetry.

4 The embedding in eleven dimensional supergravity

An important question concerns the possible relation of the deformed SO(8) gauged supergravities to $11D$ supergravity as originally formulated in [12]. More specifically, can the deformed $4D$ supergravities be understood as consistent truncations of the $11D$ theory? For the undeformed theory this embedding was studied long ago and it was shown to correspond to a consistent truncation of $11D$ supergravity [13]; a particular subtlety related to the $11D$ field strengths was resolved only recently in [14]. By a 'consistent embedding' we mean that the full field configuration space of gauged $N = 8$ supergravity can be obtained by consistently truncating $11D$ supergravity, so that all the solutions of the $4D$ theory (including x-dependent ones) can be uplifted to solutions of the higher-dimensional theory. The original work made use of the SL(8) invariant formulation of $N = 8$ supergravity, and therefore our first task is to investigate whether or not the original approach can be extended to the electric duality basis of the deformed theories based on (2.15).

We first recall that the consistency proof of [13, 14] is based on the reformulation of the $11D$ theory with local SU(8) invariance that has been presented in [15, 16]. This reformulation relies on a 4+7 split of the $11D$ theory [12] where the original tangent space group SO(1, 10) is replaced by SO(1, 3) × SU(8), so that the $4D$ R-symmetry group is realized on the full $11D$ supergravity. In this construction various features associated with $E_{7(7)}$ emerge, although $E_{7(7)}$ is not a symmetry group of the theory. A key ingredient in that construction was the so-called *generalized vielbein*, which is a soldering form defined by

$$e^m{}_{AB}(x, y) = \mathrm{i}\Delta^{-1/2}\, e_a{}^m \left(\Phi^{\mathrm{T}}\Gamma^a\Phi\right)_{AB} \ , \qquad e^{mAB} \equiv (e^m{}_{AB})^* \tag{4.1}$$

where these quantities depend on all eleven coordinates $z^M \equiv (x^\mu, y^m)$. Here $e_m{}^a$ is the internal siebenbein that is part of the elfbein of $11D$ supergravity in a triangular gauge adapted to the 4+7 split of space-time,

$$E_M{}^A(x, y) = \begin{pmatrix} \Delta^{-1/2} e_\mu{}^\alpha & B_\mu{}^m\, e_m{}^a \\ & \\ 0 & e_m{}^a \end{pmatrix} \ , \tag{4.2}$$

where $\Delta \equiv \det[e_m{}^a]$ is the metric determinant for the compact internal space. Tangent-space indices have been denoted by α and a, respectively. Appendix A contains some of the definitions for the gamma matrices and the spinor fields. The indices $A, B, \ldots = 1, 2, \ldots, 8$ are initially Spin(7) indices associated with the spinor indices of the fermions and the gamma matrices, but they are elevated to chiral SU(8) in the reformulation of the theory. This is achieved by means of the matrix $\Phi(x,y) \in \mathrm{SU}(8)$ which is required to rewrite the theory into SU(8) covariant form. While this matrix is thus undetermined prior to truncation, its precise form will be fixed in a specific truncation modulo the residual (x-dependent) local SU(8) symmetry of the $N=8$ theory. The underlying idea here is that the resulting $4D$ spinors can in principle transform under the SU(8) R-symmetry, although only the Spin(7) subgroup is initially realized as a local symmetry. Introducing the compensating phase Φ generalizes the local symmetry to the full R-symmetry group. To make this approach viable, it is required that the bosonic quantities that appear in the supersymmetry transformations of the fermions, constitute SU(8) representations.

Subsequently, consider the supersymmetry transformations as they emerge for the components of the $11D$ metric, evaluated in the context of the standard Kaluza-Klein decompositions [16],

$$
\begin{aligned}
\delta e_\mu{}^\alpha &= \tfrac{1}{2}\bar\epsilon^A\gamma^\alpha\psi_{\mu A} + \text{h.c.}\,,\\
\delta B_\mu{}^m &= \tfrac{1}{8}\sqrt{2}\, e^m{}_{AB}\left(2\sqrt{2}\,\bar\epsilon^A\psi_\mu{}^B + \bar\epsilon_C\gamma_\mu\chi^{ABC}\right) + \text{h.c.}\,,\\
\delta e^m{}_{AB} &= -\sqrt{2}\,\Sigma_{ABCD}\, e^{mCD}\,,
\end{aligned} \tag{4.3}
$$

where

$$
\Sigma_{ABCD} \equiv \bar\epsilon_{[A}\chi_{BCD]} + \tfrac{1}{24}\varepsilon_{ABCDEFG}\,\bar\epsilon^E\chi^{FGH}\,. \tag{4.4}
$$

We stress that at this point the various quantities all depend on the coordinates x^μ and y^m. The fermions have been rewritten according to the same standard Kaluza-Klein procedure; in particular, the spin-$\frac{1}{2}$ fields χ_{ABC} are the chiral components of the 56 fermions that emerge from the $11D$ gravitino fields Ψ_a, see (A.3) for the precise definitions.

To truncate the $11D$ fields to the $4D$ fields the dependence on the extra seven coordinates y^m is extracted in the form of the Killing vectors and Killing spinors of S^7 such as to make contact with the round sphere of a given radius. Then the deviations of the fields away from the S^7 solution are encoded in terms of the x-dependent fields of $4D$ SO(8)-gauged maximal supergravity. The spinors and vierbein fields can be expressed in the corresponding quantities of the $4D$ maximal supergravity by exploiting S^7 Killing spinors $\eta_A{}^i(y)$ with $i=1,2,\ldots,8$ and their inverses obeying $\eta_i{}^A\,\eta_B{}^i = \delta^A{}_B$ (note that the fermionic quantities on the left-hand side have all been supplied with the appropriate compensating SU(8) rotation Φ),

$$
\begin{aligned}
\psi_{\mu A}(x,y) &= \psi_{\mu i}(x)\,\eta_A{}^i(y)\,,\\
\chi_{ABC}(x,y) &= \chi_{ijk}(x)\,\eta_A{}^i(y)\,\eta_B{}^j(y)\,\eta_C{}^k(y)\,,\\
e_\mu{}^\alpha(x,y) &= e_\mu{}^\alpha(x)\,,\\
\epsilon_A(x,y) &= \epsilon_i(x)\,\eta_A{}^i(y)\,,\\
U(x,y)^A{}_B &= U(x)^i{}_j\,\eta_A{}^i(y)\,\eta^B{}_j(y)\,,
\end{aligned} \tag{4.5}
$$

where $U(x,y)^A{}_B$ is the SU(8) transformation matrix of the full $11D$ theory written in the formulation of [16], whereas $U(x)^i{}_j$ is the corresponding matrix in the $4D$ theory.

In accordance with the standard Kaluza-Klein ansatz, the vector gauge fields ${B_\mu}^m$ are assumed to be proportional to the 28 S^7 Killing vectors $K^{mIJ}(y)$, labeled by the 28 antisymmetric index pairs $[IJ]$ (with $I, J = 1, 2, \ldots, 8$), and related to the Killing spinors by

$$K^{mIJ} = \mathrm{i}\overset{\circ}{e}{}_a^{\,m}\, {\eta^I}_A\, \Gamma^{aAB}\, {\eta^J}_B\,, \tag{4.6}$$

where $\overset{\circ}{e}{}_a^{\,m}(y)$ is the S^7 background siebenbein, so that

$$B_\mu{}^m(x,y) = -\tfrac{1}{4}\sqrt{2}\, K^{mIJ}(y)\, A_\mu{}^{IJ}(x)\,. \tag{4.7}$$

Defining as before,

$$e^m{}_{ij}(x,y) \equiv e^m{}_{AB}(x,y)\, \eta_i{}^A(y)\, \eta_j{}^B(y)\,, \qquad e^{m\,ij} \equiv (e^m{}_{ij})^*\,, \tag{4.8}$$

it follows that $B_\mu{}^m$ and $e^m{}_{ij}$ must have the same y-dependence. Comparing with the $4D$ result from [5] for the variation of the 28 electric vectors,[6]

$$\delta A_\mu{}^{IJ} = -\tfrac{1}{2}\big(u_{ij}{}^{IJ} + v_{ijIJ}\big)\big(\bar\epsilon_k\gamma_\mu\chi^{ijk} + 2\sqrt{2}\bar\epsilon^i\psi_\mu^j\big) \;+\; \text{h.c.}\,, \tag{4.9}$$

one infers the following ansatz for the generalized vielbein,

$$\begin{aligned} e^m{}_{ij}(x,y) &= K^{mIJ}(y)\, \big[u_{ij}{}^{IJ}(x) + v_{ijIJ}(x)\big]\,, \\ e^{m\,ij}(x,y) &= K^{mIJ}(y)\, \big[u^{ij}{}_{IJ}(x) + v^{ijIJ}(x)\big]\,, \end{aligned} \tag{4.10}$$

where $u_{ij}{}^{IJ}$ and v_{ijIJ} are defined by the 56-bein $\mathcal{V}$ of the $4D$ theory given in (2.7). With these definitions the reader can easily verify that the y-dependence assigned to both sides of the supersymmetry transformations (4.3), is consistent.

Let us comment on the above results. First of all, it is remarkable that the transformations (4.3), although still based to the full $11D$ theory, reflect already the structure of the known $4D$ results. Undoubtedly the underlying reason for this is that the results were written in a form in which the invariance under the local R-symmetry group SU(8) of the maximal $4D$ theory is manifest. This was, of course, an important motivation for following the approach initiated in [16]. Another point is that the structure exhibited in (4.3) is also present in the general gaugings of $N = 8$ supergravity by means of the embedding tensor approach [4]. We will return to this aspect in due course.

Another aspect that deserves attention concerns the way in which the sub-matrices $u_{ij}{}^{IJ}$ and v_{ijIJ} appear in the ansatz (4.10) for the generalized vielbein. But, as we already explained in section 2, there are alternative possiblities by changing the electric-magnetic duality frame. For instance, the electric frame for the new SO(8) gaugings requires a different linear combination, namely $\mathrm{e}^{\mathrm{i}\omega}u_{ij}{}^{IJ} + \mathrm{e}^{-\mathrm{i}\omega}v_{ijIJ}$, as is indicated by (2.16). As we will now argue, it is, however, no longer possible to have a consistent ansatz with this linear combination, unless $\exp[2\mathrm{i}\omega]$ is real, so

[6] We rescaled the $4D$ supersymmetry parameter ϵ used in *e.g.* [5, 4] with a factor $\frac{1}{2}$ in order to be consistent with the $11D$ definitions in [16].

that ω must be equal to an integer times $\pi/2$. This implies that the embedding of the $4D$ fields into the fields of $11D$ supergravity, according to the scheme followed in [13], can only be defined provided the $4D$ theory is formulated in an SL(8) invariant duality frame. The underlying reason for this restriction is related to the fact that the generalized vielbein $e^m{}_{ij}$, by virtue of its $11D$ origin (4.1), must obey the 'Clifford property',

$$e^m{}_{ik}\, e^{n\,kj} + e^n{}_{ik}\, e^{m\,kj} = \tfrac{1}{4}\delta_i^j\, e^m{}_{kl}\, e^{n\,lk}\,. \tag{4.11}$$

As shown in [13], (4.11) is indeed satisfied with (4.10) as a consequence of the properties of the $\mathrm{E}_{7(7)}$ matrix $\mathcal{V}$ and its submatrices u and v. For non-vanishing angle ω, the obvious generalization of the formula (4.10) would read

$$e^m{}_{ij}(\omega; x, y) = K^{mIJ}(y)\left[\mathrm{e}^{\mathrm{i}\omega} u_{ij}{}^{IJ}(x) + \mathrm{e}^{-\mathrm{i}\omega} v_{ijIJ}(x)\right] \tag{4.12}$$

together with its complex conjugate. However, substituting this ω-dependent ansatz for the vielbein into (4.11), it turns out that this relation no longer holds for arbitrary values of ω. To see why this is the case, let us for instance reconsider equation (2.21) of [13] and the subsequent equations. There (4.11) is proven by showing that $e^{(m}{}_{kl}\, e^{n)ij}$ vanishes upon contraction with an anti-hermitean traceless SU(8) matrix $\Lambda_{ij}{}^{kl} = \Lambda_{[i}{}^{[k}\delta_{j]}{}^{l]}$ where $\Lambda^i{}_j = -\Lambda_j{}^i$ and $\Lambda_i{}^i = 0$. Inserting the modified ansatz (4.12) into the left-hand side of (4.11) the part of the argument involving the ω-independent combination $u\Lambda\bar{u} + v\Lambda\bar{v}$ goes through as before. By contrast, the second part of the argument involves the replacement

$$\begin{aligned} &\big[(u\Lambda\bar{v})^{IJ,KL} + (v\Lambda\bar{u})_{IJ,KL}\big]\big(K^{m\,IJ}K^{n\,KL} \;+\; K^{m\,IJ}K^{n\,KL}\big) \;\to \\ &\quad\to\; \Big[e^{2\mathrm{i}\omega}(u\Lambda\bar{v})^{IJ,KL} + e^{-2\mathrm{i}\omega}(v\Lambda\bar{u})_{IJ,KL}\Big]\big(K^{m\,IJ}K^{n\,KL} \;+\; K^{m\,IJ}K^{n\,KL}\big)\,. \end{aligned} \tag{4.13}$$

While for the first line, one could exploit the complex selfduality of both terms together with the anti-hermiticity of the matrix Λ to show that these terms cancel, this argument fails, however, in presence of the non-trivial phase factor in the second line, even though the supersymmetry variations based on (4.5), (4.7) and (4.12) do remain mutually consistent (provided that ones uses the $4D$ transformations in the corresponding ω-dependent electric-magnetic duality frame). The breaking of U(8) to its subgroup SU(8) through the presence of the ε-tensor also vitiates other parts of the proof in [13]: in fact, all arguments relying on selfduality or anti-selfduality (e.g. in the later equations (5.11) and (5.25)) fail for $\omega \neq 0, \pi/2$ for precisely this reason.

The conclusion is therefore that the embedding of ω-deformed SO(8) gaugings into $11D$ supergravity has to be effected based on the $4D$ theory written in the SL(8) covariant formulation. This implies that one has to deal with an electric-magnetic duality frame that is not purely electric, while the concept of magnetic charges does not exist in the context of eleven dimensions. The formulation of the $4D$ theory that accomplishes this in four dimensions, is the embedding-tensor formulation of maximal $N = 8$ supergravity given in [4]. In this approach all the couplings of the ungauged theory retain their original form given in [5], but the SO(8) generators will change and will involve magnetic components. In the embedding tensor formalism there are also magnetic gauge fields that couple to these magnetic components, but at the same time there are additional tensor fields with certain gauge invariances and constraints that ensure that 28 linear combinations of the electric and magnetic gauge fields are suppressed. Therefore only 28 gauge fields remain which will correspond to ω-dependent linear combinations of the original 28 electric and 28 magnetic gauge fields. A natural question is therefore whether some of the ingredients of the embedding tensor formalism will also play a role in this context and reveal how the magnetic sector of the $4D$ theory can emerge in a possible embedding in $11D$ supergravity for arbitrary values of ω.

Let us therefore further clarify some details of the $4D$ embedding tensor approach in the SL(8) frame. Casting the results of [4] in this frame shows that the electric and magnetic gauge fields transform under supersymmetry as,

$$\begin{aligned}\delta A_\mu{}^{IJ} &= -\tfrac{1}{2}\big(u_{ij}{}^{IJ} + v_{ijIJ}\big)\big(\bar\epsilon_k\gamma_\mu\chi^{ijk} + 2\sqrt{2}\,\bar\epsilon^i\psi_\mu{}^j\big) \;+\; \text{h.c.}\,,\\ \delta A_{\mu IJ} &= -\tfrac{1}{2}\mathrm{i}\big(u_{ij}{}^{IJ} - v_{ijIJ}\big)\big(\bar\epsilon_k\gamma_\mu\chi^{ijk} + 2\sqrt{2}\,\bar\epsilon^i\psi_\mu{}^j\big) \;+\; \text{h.c.}\,. \end{aligned} \tag{4.14}$$

Obviously the identification of these 'magnetic' gauge fields in $11D$ supergravity should be a crucial element in establishing a possible $11D$ origin of the ω-deformed theories.

Another aspect concerns the relation between the $4D$ T-tensors and the $11D$ theory. In $4D$ the T-tensor is generated by the embedding tensor that defines how the 56 gauge fields couple to the generators of the group $\mathrm{E}_{7(7)}$, and therefore to the electric and the magnetic generators. The latter generate composite electric 'connections' $\mathcal{B}_{IJ}$ and $\mathcal{A}_{IJ}$, belonging to the **63** and **70** representations of SU(8), which together comprise the **133** representation of $\mathrm{E}_{7(7)}$. Likewise there are also magnetic 'connections' $\mathcal{B}^{IJ}$ and $\mathcal{A}^{IJ}$. [7] Obviously these connections do not constitute vectors in some underlying continuous space, but nevertheless they are the straightforward generalization of the space-time connections $\mathcal{B}_\mu$ and $\mathcal{A}_\mu$ that are already present in the ungauged supergravity. For instance, the $\mathcal{B}_\mu$ provide the composite gauge fields for the SU(8) gauge group.

In the locally SU(8) invariant formulation of $11D$ supergravity, there is a similar situation, namely there exist connections $\mathcal{B}_M$ and $\mathcal{A}_M$, but now these are vector fields in the $11D$ space-time, decomposing into the $4D$ vectors $\mathcal{B}_\mu$ and $\mathcal{A}_\mu$, and the $7D$ vectors $\mathcal{B}_m$ and $\mathcal{A}_m$. These connections are present in the supersymmetry transformations of the fermion fields. But they also emerge as composite $\mathrm{E}_{7(7)}$ connections in the so-called *generalized vielbein postulate*, which expresses the fact that the generalized vielbein is covariantly constant [16]. Obviously the connections $\mathcal{B}_m$ and $\mathcal{A}_m$ are expected to be related to the analogous 'connections' $\mathcal{B}_{IJ}$ and $\mathcal{A}_{IJ}$ (and possibly their magnetic duals) that appear in the $4D$ theories. Indeed, this expectation is precisely confirmed for the case of the original $\omega = 0$ supergravity as was exhibited in [13], where the connections $\mathcal{B}_m$ and $\mathcal{A}_m$ yield the electric T-tensor in the SL(8) duality frame.

[7] In [4] these 'connections' were denoted by $\mathcal{Q}_M$ and $\mathcal{P}_M$, where the index M is an index belonging to the **56** representation of $\mathrm{E}_{7(7)}$, which decomposes into the electric and magnetic 28-dimensional representations of SL(8).

5 Dual quantities and the non-linear flux ansatz

In the past the question whether $11D$ supergravity possesses certain structures in the context of a lower-dimensional formulation that more fully exhibits the duality symmetry, has been analyzed in the case of $3D$ where the duality group equals $\mathrm{E}_{8(8)}$, implying a kind of 'generalized geometry' based on $\mathrm{E}_{8(8)}$ [20]. This effort (as well as more recent efforts in connection with 'generalized geometry') was based on a quest for further unification, while in the context of this paper one is confronted with a more concrete motivation, namely of how to reconcile the deformed SL(8) supergravities in $4D$ with the full $11D$ theory. Another main difference is that in $4D$ we have the possibility of testing the various formulas for non-trivial compactifications, whereas in $3D$ most gaugings cannot be obtained from (and thus not compared with) spontaneously compactified solutions of the $11D$ theory. A rather surprising consequence of the present analysis is that we are in this way led to a simple candidate formula for the non-linear flux ansatz!

As was pointed out in the previous section, it is obviously important in this context to have both electric gauge fields and their dual magnetic ones. Taking this as a guideline, we are led to ask whether the $11D$ theory contains such dual gauge fields, and whether those have a relation to components of the three-form tensor fields A_{MNP}. The latter fields were avoided in the analysis of [13], because the equations of motion and the supersymmetry variations of $11D$ supergravity only involve the four-form field strengths, and the truncation to $4D$ usually involves tensor-scalar dualities which require more detailed knowledge of the truncated Lagrangian. Furthermore, for the S^7 compactification of $11D$ supergravity all 28 spin-1 degrees of freedom are known to reside in the Kaluza-Klein vector $B_\mu{}^m$ according to (4.7). By contrast, for the toroidal truncation of [11] only seven (electric) spin-1 degrees of freedom originate from $B_\mu{}^m$, while the remaining 21 (magnetic) spin-1 degrees of freedom reside in $A_{\mu mn}$.

We therefore proceed on the assumption that the dual magnetic gauge fields are contained in the fields

$$B_{\mu mn} \equiv A_{\mu mn} - B_\mu{}^p \, A_{pmn} \,, \tag{5.1}$$

which follow from the standard Kaluza-Klein ansatz and define covariant vector fields in $4D$. A somewhat subtle calculation (see [16] and appendix) shows that these fields transform as follows under supersymmetry,

$$\delta B_{\mu mn} = -\,\mathrm{i}\Delta^{-1/2}\big[\tfrac{1}{48}(\Gamma_{mn})_{AB} + \tfrac{1}{8}\sqrt{2}\, A_{mnp}\,(\Gamma^p)_{AB}\big]\,\big(2\sqrt{2}\,\epsilon^A\psi_\mu{}^B + \bar\epsilon_C\gamma_\mu\chi^{ABC}\big) \quad + \text{h.c.}\,, \tag{5.2}$$

where again all redefinitions required in the passage from $11D$ to $4D$ must be taken into account. As for (4.3), this result still reflects the full $11D$ situation since we have not imposed any restrictions on the dependence on the internal coordinates y^m. Remarkably, the spinor bilinears that appear in (5.2) are exactly as in $\delta B_\mu{}^m$, as well as in the $4D$ supersymmetry variations of the electric and magnetic gauge fields, $\delta A_\mu{}^{IJ}$ and $\delta A_{\mu IJ}$, that follow from the embedding tensor formalism (cf. 4.14) . This indicates that we are dealing with a *dual generalized vielbein*, in terms of which the supersymmetry variations of $B_\mu{}^m$ and $B_{\mu mn}$ acquire the same form,

$$\delta B_\mu{}^m = \tfrac{1}{8}\sqrt{2}\, e^m{}_{AB}\,\big(2\sqrt{2}\,\bar\epsilon^A\psi_\mu{}^B + \bar\epsilon_C\gamma_\mu\chi^{ABC}\big) \;+\; \text{h.c.}\,,$$

$$\delta B_{\mu mn} = \tfrac{1}{8}\sqrt{2}\, e_{mn\,AB}\left(2\sqrt{2}\,\bar{\epsilon}^A \psi_\mu{}^B + \bar{\epsilon}_C \gamma_\mu \chi^{ABC}\right) \;+\; \text{h.c.}\,. \tag{5.3}$$

Here we have simply defined the normalization of $e_{mn\,AB}$ such that the overall factors on the right-hand side of the above two equations are equal, but we should stress that at this point there is no intrinsic normalization criterion for the field $B_{\mu mn}$ and $e_{mn\,AB}$. The generalized vielbein (4.1) is thus complemented by the following new vielbein-like object

$$\begin{aligned} e_{mn\,AB} &= -\,\tfrac{1}{12}\mathrm{i}\sqrt{2}\,\Delta^{-1/2}\left[e_m{}^a e_n{}^b\left(\Phi^{\mathrm{T}}\Gamma_{ab}\Phi\right)_{AB} + 6\sqrt{2}\,A_{mnp}\left(\Phi^{\mathrm{T}}\Gamma^p\Phi\right)_{AB}\right], \\ e_{mn}{}^{AB} &\equiv \left(e_{mn\,AB}\right)^*, \end{aligned} \tag{5.4}$$

characterized by a pair of *lower* world indices m, n. Note that this new vielbein is complex even in the special gauge $\Phi = \mathbb{1}$. It remains to determine its supersymmetry variation. In analogy with the third equation of (4.3), which was originally derived in [16], one finds that both vielbeine transform uniformly,

$$\begin{aligned} \delta e^m{}_{AB} &= -\,\sqrt{2}\,\Sigma_{ABCD}\, e^{m\,CD}\,, \\ \delta e_{mn\,AB} &= -\,\sqrt{2}\,\Sigma_{ABCD}\, e_{mn}{}^{CD}\,. \end{aligned} \tag{5.5}$$

We relegate a derivation of this result to appendix A, where we also summarize a number of other relevant definitions. The new vielbein (5.4) and the SU(8) covariant supersymmetry variations (5.5) are in precise analogy with results found for the $3+8$ split appropriate to $D=3$ dimensions [20].

Defining

$$e_{mn\,ij}(x,y) \equiv e_{mn\,AB}(x,y)\,\eta_i{}^A(y)\,\eta_j{}^B(y)\;, \qquad e_{mn}{}^{ij} \equiv (e_{mn\,ij})^*\,, \tag{5.6}$$

one can now derive certain relations for products of the generalized vielbein, in analogy to the Clifford relation (4.11). The most obvious one is,

$$e_{mn\,ij}\, e^{p\,ij} = -8\,\Delta^{-1}\, g^{pq}\, A_{mnq}\,, \tag{5.7}$$

which defines A_{mnp} in terms of the generalized vielbeine. This formula is the analog of the corresponding formula for the inverse densitized metric $\Delta^{-1}g^{mn}$, obtained by tracing the Clifford relation (4.11). An important consequence of that formula was the *non-linear metric ansatz* [21, 13],

$$\Delta^{-1}g^{mn}(x,y) = \tfrac{1}{8}K^{mIJ}(y)\,K^{nKL}(y)\left[(u^{ij}{}_{IJ} + v^{ijIJ})(u_{ij}{}^{KL} + v_{ijKL})\right](x)\,, \tag{5.8}$$

where we note that explicit symmetrization in the indices m and n is not necessary owing to the properties of the matrices u and v. With the previously derived formulas (4.14) and (5.3) we can now deduce, in complete analogy with (4.10), a similar ansatz for the dual guage field and the dual vielbein in the truncation of the $11D$ to the $4D$ fields, *viz.*

$$B_{\mu mn}(x,y) = -\,\tfrac{1}{4}\sqrt{2}\,K_{mn}{}^{IJ}(y)\,A_{\mu IJ}(x)\,,$$

$$e_{mn\,ij}(x,y) = \mathrm{i}K_{mn}{}^{IJ}(y)\,\big[u_{ij}{}^{IJ} - v_{ijIJ}\big](x)\,, \tag{5.9}$$

where

$$K_{mn}{}^{IJ}(y) \equiv \overset{\circ}{e}_{ma}(y)\,\overset{\circ}{e}_{nb}(y)\,\eta^I_A(y)\,\Gamma^{ab\,AB}\,\eta^J_B(y)\,. \tag{5.10}$$

Using (5.7), (5.8) and (5.9) we get [8]

$$\begin{aligned}&\mathrm{i}K_{mn}{}^{IJ}(y)\,K^{pKL}(y)\,\big[(u^{ij}{}_{IJ} - v^{ijIJ})(u_{ij}{}^{KL} + v_{ijKL})\big](x) = \\ &\quad = \; -K^{pIJ}(y)\,K^{qKL}(y)\,\big[(u^{ij}{}_{IJ} + v^{ijIJ})(u_{ij}{}^{KL} + v_{ijKL})\big](x)\,A_{mnq}(x,y)\,,\end{aligned} \tag{5.11}$$

where we remember that the curved indices on the Killing vector K and its derivative are always to be raised and lowered with the *round* S^7 metric. Using properties of the matrices u and v given in [5] this can be rewritten as

$$\begin{aligned}&\mathrm{i}K_{mn}{}^{IJ}\,K^{pKL}\big[v^{ijIJ}v_{ijKL} - v_{ijIJ}v^{ijKL} + u_{ij}{}^{IJ}v^{ijKL} - u^{ij}{}_{IJ}v_{ijKL}\big] = \\ &\quad = \Big[8\,\overset{\circ}{g}{}^{pq} - K^{pIJ}K^{qKL}\big(v^{ijIJ}v_{ijKL} + v_{ijIJ}v^{ijKL} + u_{ij}{}^{IJ}v^{ijKL} + u^{ij}{}_{IJ}v_{ijKL}\big)\Big]\,A_{mnq}\,.\end{aligned} \tag{5.12}$$

Observe that both sides of this equation are purely imaginary provided that A_{mnp} is real, which is precisely as expected. Alternatively the reality can be proven from the fact that $e_{mn\,ij}\,e^{p\,ij} = e_{mn}{}^{ij}\,e^p{}_{ij}$, which follows by making use of the properties of the matrices u and v. The expressions (5.11) and (5.12) are the analog of the non-linear metric ansatz (5.8), but now for the three-form field $A_{mnp}(x,y)$ (*alias* the 'flux field'). The formulae (5.11) and (5.12) are rather similar to the conjectured formula (6.2) in [21]. Both results reproduce the same *linear* ansatz for A_{mnp}. This illustrates the difficulty in obtaining consistent non-linear ansätze: there is no way of guessing the correct answer from the linearized expression!

To verify that (5.9), and hence (5.11) are really correct we perform a number of consistency checks. One such check concerns the constraint,

$$e_{mn\,ij}\,e^{n\,ij} = 0\,, \tag{5.13}$$

which follows from (5.7) and the antisymmetry of A_{mnp}. To prove it we make use of the identity

$$K_{mn}{}^{IJ}K^{nKL} = -4\,\delta^{[J[K}K_m{}^{L]I]} + K_{mn}{}^{[IJ}K^{nKL]} \tag{5.14}$$

Now we observe that the first two terms in brackets on the left-hand side of (5.12) are anti-symmetric under interchange of the index pairs $[IJ]$ and $[KL]$, whence for them, only the first term on the right-hand side of (5.14) contributes, so the result of the index contraction is proportional to

$$\delta^{[J[K}K_m{}^{L]I]}\big[u_{ij}{}^{IJ}u^{ij}{}_{KL} - v_{ijIJ}v^{ijKL}\big] = 0\,. \tag{5.15}$$

[8] Although A_{mnp} is only determined up to an (x,y)-dependent tensor gauge transformation, the truncation fixes the y-dependence so that A_{mnp} is obtained in a particular gauge.

The vanishing of this expression follows from the fact that, with *uncontracted* SU(8) index pairs $[ij]$ and $[kl]$,

$$\delta^{[J[K}K_m^{L]I]}\big[u_{ij}{}^{IJ}u^{kl}{}_{KL} - v_{ijIJ}v^{klKL}\big]$$

must belong to the Lie algebra of $\mathrm{E}_{7(7)}$ and must therefore vanish when traced with δ^{ij}_{kl} over the SU(8) index pairs $[ij]$ and $[kl]$. The same argument applies to the remaining two terms in the bracket on the left-hand side of (5.12) which are each symmetric under the interchange $[IJ] \leftrightarrow [KL]$, leaving us with

$$K_{mn}{}^{[IJ}K^{nKL]}\big[u_{ij}{}^{IJ}v^{ijKL} - u^{ij}{}_{IJ}v_{ijKL}\big] = 0\,, \tag{5.16}$$

because $K_{mn}{}^{[IJ}K^{nKL]}$ is (complex) selfdual.

A stronger test, which implies the previous one, is to verify the complete anti-symmetry of A_{mnp} in the indices $[mnp]$ from the definition (5.11). Since the anti-symmetry in $[mn]$ is manifest we need only ascertain the anti-symmetry with respect to the other index pair $[mp]$, or equivalently $[np]$. This is equivalent to checking the anti-symmetry of $(\Delta^{-1}g^{nr})(\Delta^{-1}g^{ps})A_{mrs}$ in the indices $[np]$. Using (5.14) this requires

$$\begin{aligned} &K^{nKL}K^{pPQ}\big(-4\,\delta^{K'M}K_m^{L'N} + K_m^{K'L'MN}\big) \\ &\qquad\times\big(u_{kl}{}^{KL} + v_{klKL}\big)\big(u^{kl}{}_{K'L'} + v^{klK'L'}\big)\big(u_{ij}{}^{MN} - v_{ijMN}\big)\big(u^{ij}{}_{PQ} + v^{ijPQ}\big) \end{aligned} \tag{5.17}$$

to be anti-symmetric in $[np]$. We now invoke the previous argument to show that the expression involving the $(u+v)(u-v)$ factor in the middle is $\mathrm{E}_{7(7)}$ Lie-algebra valued in the index pairs $[ij]$ and $[kl]$ and hence can be written as $\delta^{[k}{}_{[i}\,\Lambda^{l]}{}_{j]}$, with $\Lambda^i{}_j$ anti-hermitean and traceless. Hence we are left with the task to show that

$$\Lambda^k{}_i\,\times\big(K^{nKL}K^{pPQ} + K^{pKL}K^{nPQ}\big)\big(u_{kl}{}^{KL} + v_{klKL}\big)\big(u^{il}{}_{PQ} + v^{ilPQ}\big) = 0\,. \tag{5.18}$$

Now we invoke the $\mathrm{E}_{7(7)}$ Lie algebra once again: upon symmetrization under $[KL] \leftrightarrow [PQ]$ it follows that

$$\begin{aligned} u^{KL}{}_{ik}\Lambda^i{}_j u^{jk}{}_{PQ} + v_{KLik}\Lambda^i{}_j v^{jkPQ} &\cong u^{KL}{}_{ik}\Lambda^i{}_j u^{jk}{}_{PQ} + v_{PQik}\Lambda^i{}_j v^{jkKL} \\ &= u^{KL}{}_{ik}\Lambda^i{}_j u^{jk}{}_{PQ} - v^{KLik}\Lambda_i{}^j v_{jkPQ} \\ &= \delta^{[K}{}_{[P}X^{L]}{}_{Q]} \end{aligned} \tag{5.19}$$

is Lie-algebra valued in the index pairs $[KL]$ and $[PQ]$ with an anti-hermitean and traceless matrix $X^I{}_J$. Hence, this contribution is proportional to

$$\big(K^{nKL}K^{pKQ} + K^{nKL}K^{pKQ}\big)X^K{}_Q \;\propto\; \overset{\circ}{g}{}^{np}\,X^K{}_K = 0\,. \tag{5.20}$$

For the remaining two terms we use for the first term that

$$u_{ik}{}^{KL}\Lambda^i{}_j v^{jkPQ} + u_{ik}{}^{PQ}\Lambda^i{}_j v^{jkKL} = u_{ik}{}^{KL}\Lambda^i{}_j v^{jkPQ} - v^{ikKL}\Lambda_l{}^i u_{ik}{}^{PQ}$$

$$= \text{complex selfdual in } [KLPQ]\,. \tag{5.21}$$

For the second term we note,

$$v_{ikKL}\Lambda^i{}_j u^{jk}{}_{PQ} + v_{ikPQ}\Lambda^i{}_j u^{jk}{}_{KL} = -\big(u^{ik}{}_{KL}\Lambda_i{}^j v_{jkPQ} - v_{ikKL}\Lambda^i{}_j u^{jk}{}_{PQ}\big)\,, \tag{5.22}$$

which equals minus the hermitean conjugate of the first term (5.21). Hence, after contraction with $K^{n[KL}K^{p\,PQ]}$, the sum of the two terms gives zero. Therefore $A_{mnp}(x,y)$ as determined from (5.11) is indeed fully anti-symmetric.

6 Outlook

The present work opens unexpected new perspectives on $11D$ supergravity, and the link between this theory and the duality symmetries of $4D$ maximal supergravity. Although the duality between electric and magnetic vector fields is normally viewed as a phenomenon strictly tied to four space-time dimensions, our analysis has revealed $11D$ structures directly associated to electric-magnetic vector duality, yielding as a by-product the long sought formula for the non-linear flux ansatz. These new structures appear in the form of a dual generalized vielbein $e_{mn\,AB}$, whose properties need to be explored further. For instance there is the question whether this object obeys a generalized vielbein postulate analogous to the one satisfied by $e^m{}_{AB}$ [16]. The fact that the solution of the vielbein postulate is not unique, but only determined up to an homogeneous contribution [14] is likewise expected to play a role here.

The subtleties regarding the emergence of electric *vs.* magnetic gauge fields have not been explored much in the present Kaluza-Klein context. Therefore we briefly return to the issue of the origin of the dual vector fields from 11 dimensions, and to the question whether and how the ω-rotation might be implemented in eleven dimensions. One important feature here is that the distribution of the 28 *physical* spin-one degrees between electric and magnetic vectors depends on the compactification. This is very similar to what happens in four dimensions in the context of the embedding tensor formalism, where the embedding tensor determines which combination of the electric and magnetic gauge fields will eventually carry the physical spin-one degrees of freedom. For the S^7 compactification, all 28 vector fields reside in the Kaluza-Klein vector field $B_\mu{}^m(x,y)$ and are electric. By contrast, for the torus reduction of [11] there are only seven electric vectors associated to the seven Killing vectors on T^7, while the remaining 21 vectors come from $A_{\mu mn}$ and are magnetic. For the S^7 compactification, this raises the question how the theory manages to prevent the massless excitations contained in $A_{\mu mn}$ from appearing as *independent* spin-one degrees of freedom on the mass shell.

One may wonder why, now that a number of the appropriate dual quantities in the $11D$ theory has been identified, it is not possible to give a more precise scenario of how the ω-deformations might be embedded. Let us recall that in [13], the T-tensor of the $4D$ supergravity followed from the composite connections $\mathcal{B}_m$ and $\mathcal{A}_m$, which belong to the **133** representation of $\mathrm{E}_{7(7)}$. The actual expressions for these connections in the truncation were determined by solving the generalized vielbein postulate. When going through the actual derivation in [13], it is difficult to envisage a modification of the solution that would enable one to include the magnetic duals. On the other hand, as mentioned above, the solution of the generalized vielbein postulates is not unique [14], a fact that could possibly be explored to somehow include the magnetic duals. However, it was also noted in that work that the ambiguities in $\mathcal{B}_m$ and $\mathcal{A}_m$ are such that they will cancel in the final expression for the T-tensor. Clearly, it is still premature to draw any definite conclusions from this, given the fact that the dual structures have not been explored extensively so far, but we expect that the further analysis of these structure, and in particular, of the generalized vielbein postulate for the new vielbein may provide valuable hints as to the 'hiding place' of the embedding tensor in eleven dimensions.

To better understand the possible origin of the full set of 28 vectors and their 28 magnetic duals *from eleven dimensions* it may be helpful to recall that the $11D$ theory also allows for dual fields, although these do not appear in the Lagrangian and transformation rules of [12]. These are the 6-form field A_{MNPQRS} (dual to the three-form field A_{MNP}) and the 'dual graviton' $h_{M|N_1\cdots N_8}$ (which is dual to the linear graviton field h_{MN}; see *e.g.* [25] and references therein). The latter belongs to a non-trivial Young tableau representation, which is fully antisymmetric in the last eight indices $N_1 \cdots N_8$ and obeys the irreducibility constraint $h_{[M|N_1\cdots N_8]} = 0$. We note here that the incorporation of the dual graviton has so far been achieved *only at the linearized level*, and one may therefore anticipate difficulties in re-formulating the $11D$ theory in a way that would consistently incorporate these dual fields *at the interacting level*, and in a way maintaining full $11D$ covariance. [9] Upon dimensional reduction on a 7-torus these fields give rise to the full set of 28 + 28 vector fields (cf. eqs. (4.2) and (5.1) for the first two lines)

$$\begin{array}{rcll} E_M{}^A & \to & B_\mu{}^m & \in \mathbf{7} \\ A_{MNP} & \to & B_{\mu\, mn} & \in \overline{\mathbf{21}} \\ A_{MNPQRS} & \to & \widetilde{B}_{\mu\, npqrs} & \in \mathbf{21} \\ h_{M|N_1\cdots N_8} & \to & B_{m|\mu\, n_1\cdots n_7} \equiv \widetilde{B}_{\mu\, m}\, \varepsilon_{n_1\cdots n_7} & \in \overline{\mathbf{7}} \end{array} \tag{6.1}$$

at least in the linearized analysis (note that $B_{m|\mu n_1\cdots n_7}$ does satisfy the irreducibility constraint appropriate to the dual graviton field, because the Latin indices only run over $1, ..., 7$). Here we have indicated the SL(7) (or GL(7)) representation on the right-hand side. These representations can be re-combined into the proper SL(8) representations of the electric and magnetic vectors of $N = 8$ supergravity in accordance with the decomposition

$$\mathbf{28} \oplus \overline{\mathbf{28}} \quad \to \quad \mathbf{7} \oplus \mathbf{21} \oplus \overline{\mathbf{7}} \oplus \overline{\mathbf{21}} \tag{6.2}$$

This is consistent with the fact that the electric and magnetic fields must transform in conjugate ('dual') representations. However, as we said, the distribution of the physical spin-one degrees

[9] In fact, it has been known for a long time that even the consistent incorporation of the dual 6-form field in the Lagrangian encounters problems, although this field can be incorporated in the equations of motion [26, 27].

of freedom between these fields depends on the compactification. Of course, for the torus reduction the (ungauged) $4D$ theory cannot tell the difference between 'electric' and 'magnetic', but the distinction does become relevant for the gauged theory, as is evident from the existence of inequivalent ω-deformed SO(8) gaugings [1], and from our discussion in section 2.

The decomposition (6.2) suggests that our set of vielbeine ($e^m{}_{AB}$, e_{mnAB}) is still incomplete, and that there should exist a complementary set ($e_{m\,AB}$, $e^{mn}{}_{AB}$) of yet another set of 28 vielbein components that would complete the generalized vielbein to a full 56-bein in $D = 11$ dimensions — this was, in fact, the conclusion reached in [20] for $\mathrm{E}_{8(8)}$ and the 3+8 decomposition of $11D$ supergravity. Accordingly, the supersymmetry transformations (4.3) would have to generalize to this hypothetical 56-bein, and the vector transformations (4.9) and (4.14) would likewise have to follow from a single variation in $11D$. However, in order to derive these relations we would have to know the full *non-linear* $11D$ transformations of the dual fields in (6.1)! The ω-dependent vielbein ansatz (4.12) would then simply follow from

$$e^m{}_{ij}(\omega; x, y) = \cos\omega\, e^m{}_{ij}(x, y) \,+\, \sin\omega\, e_{m\,ij}(x, y) \tag{6.3}$$

thus involving a U(1) rotation between the Kaluza-Klein vector $B_\mu{}^m$ and the dual graviton vector $\widetilde{B}_{\mu\, m}$ from (6.1). This indicates why the ω-rotation may not be implementable in terms of the vielbein components $e^m{}_{AB}$ and $e_{mn\,AB}$ only. We note that the above combination breaks GL(7) invariance (and hence diffeomorphism invariance in the internal dimensions); in fact, it just corresponds to the U(1) rotation coming from the Ehlers $SL(2, \mathbb{R})$ symmetry which enlarges the $\mathrm{E}_{7(7)}$ of the 4D theory to the $\mathrm{E}_{8(8)}$ symmetry of 3D maximal supergravity.

Acknowledgements

We thank Gianluca Inverso, Mario Trigiante and Oscar Varela, as well as Hadi and Mahdi Godazgar for valuable discussions. B.d.W. is supported by the ERC Advanced Grant no. 246974, *"Supersymmetry: a window to non-perturbative physics"*.

A The supersymmetry variation of the dual generalized vielbein

Here we present the evaluation of the supersymmetry transformation (5.5) of the dual generalized vielbein, defined in (5.4). The derivation proceeds in close analogy with the derivation of the third equation in (4.3) given originally in [16] (for further details the reader is invited to consult eqs. (3.10) - (3.15) of that reference).

First let us summarize some of the definitions introduced in [16]. The $11D$ fermion fields Ψ_M and gamma matrices $\tilde{\Gamma}_A$ are decomposed as,

$$\Psi_M(x, y) = \begin{cases} \Psi_\mu(x, y)\,, \\ \Psi_m(x, y)\,, \end{cases} \qquad \tilde{\Gamma}_A = \begin{cases} \tilde{\Gamma}_\alpha = \gamma_\alpha \otimes \mathbb{1}\,, \\ \tilde{\Gamma}_a = \gamma_5 \otimes \Gamma_a\,, \end{cases} \tag{A.1}$$

where the vielbeine $e_\mu{}^\alpha$ and $e_m{}^a$ have been defined in (4.2) and the gamma matrices Γ_a satisfy,

$$\{\Gamma_a, \Gamma_b\} = 2\,\delta_{ab}\,\mathbb{1}\,, \qquad \Gamma^{[a}\Gamma^b \cdots \Gamma^{g]} = -\mathrm{i}\varepsilon^{abcdefg}\,\mathbb{1}. \tag{A.2}$$

Furthermore the resulting chiral spinors, which carry upper and lower SU(8) indices $A, B, \ldots$, are defined by

$$
\begin{aligned}
\psi_\mu{}^A(x,y) &= \tfrac{1}{2}(1+\gamma_5)\, \mathrm{e}^{-\mathrm{i}\pi/4}\, \Delta^{1/4}\big(\Psi_\mu - B_\mu{}^m\Psi_m - \tfrac{1}{2}\gamma_\mu\Delta^{-1/2}\Gamma^m\Psi_m\big)_A\,, \\
\psi_{\mu A}(x,y) &= \tfrac{1}{2}(1-\gamma_5)\, \mathrm{e}^{\mathrm{i}\pi/4}\, \Delta^{1/4}\big(\Psi_\mu - B_\mu{}^m\Psi_m + \tfrac{1}{2}\gamma_\mu\Delta^{-1/2}\Gamma^m\Psi_m\big)_A\,, \\
\epsilon^A(x,y) &= \tfrac{1}{2}(1+\gamma_5)\, \mathrm{e}^{-\mathrm{i}\pi/4}\, \Delta^{1/4}\, \epsilon_A\,, \\
\epsilon_A(x,y) &= \tfrac{1}{2}(1-\gamma_5)\, \mathrm{e}^{\mathrm{i}\pi/4}\, \Delta^{1/4}\, \epsilon_A\,, \\
\chi^{ABC}(x,y) &= \tfrac{3}{4}\sqrt{2}\,(1+\gamma_5)\, \mathrm{e}^{-\mathrm{i}\pi/4}\, \Delta^{-1/4}\mathrm{i}\Gamma_{a[AB}\Psi^a{}_{C]}\,, \\
\chi_{ABC}(x,y) &= \tfrac{3}{4}\sqrt{2}\,(1-\gamma_5)\, \mathrm{e}^{\mathrm{i}\pi/4}\, \Delta^{-1/4}\mathrm{i}\Gamma_{a[AB}\Psi^a{}_{C]}\,,
\end{aligned} \tag{A.3}
$$

where for the $11D$ spinors on the right-hand side we made no distinction between upper and lower spinor indices and suppressed the dependence on x^μ and y^m.

To derive the second equation in (5.5), we first evaluate the right-hand side of the equation, going 'backwards' from the SU(8) covariant expressions as in [11, 16], but suppressing the SU(8) compensating phase Φ. Using SO(8) Fierz identities given in [16], we obtain in this way

$$
\begin{aligned}
&-\sqrt{2}\Big(\bar\epsilon_{[A}\chi_{BCD}+\tfrac{1}{24}\varepsilon_{ABCDEFGH}\,\bar\epsilon^E\chi^{FGH}\Big)\, e_{mn}{}^{CD} \\
&= -\tfrac{1}{12}\mathrm{i}\sqrt{2}\,\Delta^{-1/2}\Big\{ -\tfrac{1}{32}\bar\epsilon(1-\gamma_5)(\Gamma_{bc}\Gamma_a\Gamma_{mn}+\Gamma_{mn}\Gamma_a\Gamma_{bc})\Psi^a\;\Gamma^{bc}{}_{AB} \\
&\qquad + \tfrac{1}{16}\bar\epsilon(1-\gamma_5)(\Gamma_b\Gamma_a\Gamma_{mn}+\Gamma_{mn}\Gamma_a\Gamma_b)\Psi^a\;\Gamma^b{}_{AB} \\
&\qquad - \tfrac{1}{4}\bar\epsilon(1-\gamma_5)\Gamma_{mn}\Psi^a\;\Gamma_{a\,AB} \\
&\qquad - \tfrac{1}{12}\bar\epsilon(1+\gamma_5)\Gamma_{b[m}\Psi_{n]}\;\Gamma^b{}_{AB} \\
&\qquad - \tfrac{1}{12}\bar\epsilon(1+\gamma_5)\Gamma^b\Psi_{[m}\;\Gamma_{n]b\,AB} \\
&\qquad - \tfrac{1}{4}\bar\epsilon(1+\gamma_5)\Gamma_{[mn}\Psi^a\;\Gamma_{a]\,AB} \\
&\qquad - \tfrac{1}{4}\bar\epsilon(1+\gamma_5)\Gamma_{[a}\Psi^a\;\Gamma_{mn]\,AB} \\
&\qquad - \tfrac{1}{96}\bar\epsilon(1+\gamma_5)(\Gamma_{d[m}\Gamma^{bc}\Gamma^d+\Gamma^d\Gamma^{bc}\Gamma_{d[m})\Psi_{n]}\;\Gamma_{bc\,AB} \\
&\qquad + \tfrac{1}{48}\bar\epsilon(1+\gamma_5)(\Gamma_{d[m}\Gamma^b\Gamma^d+\Gamma^d\Gamma^b\Gamma_{d[m})\Psi_{n]}\;\Gamma_{b\,AB} \\
&\qquad - \tfrac{1}{32}\bar\epsilon(1+\gamma_5)(\Gamma_{[a}\Gamma^{bc}\Gamma_{mn]}+\Gamma_{[mn}\Gamma^{bc}\Gamma_{a]})\Psi^a\;\Gamma_{bc\,AB} \\
&\qquad + \tfrac{1}{16}\bar\epsilon(1+\gamma_5)(\Gamma_{[a}\Gamma^b\Gamma_{mn]}+\Gamma_{[mn}\Gamma^b\Gamma_{a]})\Psi^a\;\Gamma_{b\,AB}\Big\}\,,
\end{aligned} \tag{A.4}
$$

This result should be compared to the left-hand side of the second equation in (5.5), which arises from the variation of (5.4) as obtained from the $11D$ variations of the siebenbein $e_m{}^a$ and the three-form field A_{mnp},

$$
\begin{aligned}
&-\tfrac{1}{12}\mathrm{i}\sqrt{2}\big\{\delta\big(\Delta^{-1/2}e_m{}^a e_n{}^b\big)\,\Gamma_{ab\,AB} + 6\sqrt{2}\big(\delta A_{mnp}\big)\,\Delta^{-1/2}\,\Gamma^p{}_{AB}\big\} \\
&= -\tfrac{1}{12}\mathrm{i}\sqrt{2}\,\Delta^{-1/2}\big\{ -\bar\epsilon\gamma_5\Gamma^p\Psi_{[m}\;\Gamma_{n]p\,AB} - \tfrac{1}{4}\bar\epsilon\gamma_5\Gamma_a\Psi^a\;\Gamma_{mn\,AB} - \tfrac{3}{2}\bar\epsilon\Gamma_{[mn}\Psi_{p]}\;\Gamma^p{}_{AB}\big\}\,.
\end{aligned} \tag{A.5}
$$

where we suppressed the contribution proportional to $A_{mnp}\,\delta\big(\Delta^{-1/2}e_a{}^p\big)\Gamma^a{}_{AB}$, as this part of the variation is already taken care of by the calculation in [16], which corresponds to the result in the first line of (5.5). As it turns out, the two contributions (A.4) and (A.5) are equal provided we add an infinitesimal SU(8) transformation acting on e_{mnAB} to (A.5),

$$\begin{aligned}\delta_{\mathrm{SU}(8)} e_{mn\,AB} &= 2\,\Lambda^{C}{}_{[A}\, e_{mn\,B]C} \\ &= -\tfrac{1}{12}\mathrm{i}\sqrt{2}\,\Delta^{-1/2}\Big[-\tfrac{1}{8}\bar{\epsilon}\Gamma^{ab}\Psi^{c}\,\Gamma_{abcmn\,AB} + \tfrac{3}{4}\bar{\epsilon}\Gamma_{[mn}\Psi_{a]}\,\Gamma^{a}_{AB} \\ &\qquad -\tfrac{1}{2}\bar{\epsilon}\gamma_5\Gamma_{a[m}\Psi^{a}\,\Gamma_{n]\,AB} + \tfrac{1}{2}\bar{\epsilon}\gamma_5\Gamma^{a}\Psi_{[m}\,\Gamma_{n]a\,AB} - \tfrac{1}{2}\bar{\epsilon}\gamma_5\Gamma_{[m}\Psi^{a}\,\Gamma_{n]a\,AB}\Big]\,,\end{aligned} \tag{A.6}$$

where the parameter of this transformation takes the form

$$\Lambda_{A}{}^{B} = -\Lambda^{B}{}_{A} \equiv \tfrac{1}{8}\bar{\epsilon}\gamma_5\Gamma_{ab}\Psi^{b}\Gamma^{a}_{AB} - \tfrac{1}{8}\bar{\epsilon}\gamma_5\Gamma_{a}\Psi_{b}\Gamma^{ab}_{AB} - \tfrac{1}{16}\bar{\epsilon}\Gamma_{ab}\Psi_{c}\Gamma^{abc}_{AB}\,. \tag{A.7}$$

The expression for the SU(8) parameter (A.7) is identical to the one given in eq. (3.13) of [16], where it was found by determining the SU(8) covariant form of the supersymmetry transformation of $e^{m}{}_{AB}$. This remarkable coincidence is not only crucial for the correctness of the second equation (5.5), but it is also another non-trivial consistency check of the SU(8) invariant reformulation of $11D$ supergravity presented in [16].

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