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Abstract

We construct the spacetime in the vicinity of a general isolated, rotating, charged black hole. The black hole is modeled as a weakly isolated horizon, and we use the characteristic initial value formulation of the Einstein equations with the horizon as an inner boundary. The spacetime metric and other geometric fields are expanded in a power series in a radial coordinate away from the horizon by solving the characteristic field equations in the Newman–Penrose formalism. This is the first in a series of papers which investigate the near-horizon geometry and its physical applications using the isolated horizon framework.

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(Some figures may appear in colour only in the online journal)

1. Introduction

The intrinsic geometry of a classical black hole horizon in equilibrium is well understood. The most general treatment is provided by the framework of isolated horizons which allows for the possibility of a black hole with arbitrary (but time-independent) intrinsic geometry in an otherwise dynamical spacetime [1–9]. Isolated horizons have been applied in various physical circumstances. Some illustrative examples are black hole thermodynamics, black hole entropy calculations in quantum gravity, numerical relativity and hairy black holes. More general situations when the black hole grows due to in-falling matter and/or radiation have also been studied; see e.g. [10, 11]. See [12–15] for reviews and references. Note that in this paper we use the phrase 'isolated black hole' to denote a black hole in equilibrium with its surroundings; this is different from the notion of an isolated system as being far away from other gravitating objects (see e.g. the discussion of asymptotically flatness as in [16], or the work of Kozameh *et al* [17]).

As indicated above, there is an extensive literature on the properties of isolated horizons. Most published results study either: (i) the mechanics of isolated horizons by

considering the infinite-dimensional phase space of solutions to Einstein equations which admit isolated horizon(s) as inner boundaries (see e.g. [1–5]); or (ii) the intrinsic geometry of an isolated horizon in a given individual spacetime (see e.g. [6–9]). However, for a wide variety of astrophysical phenomena, it is the interaction of a black hole with nearby matter fields or other compact objects which leads to important physical phenomena. It is thus necessary to study the behavior of the spacetime metric and other geometrical fields in the neighborhood of an isolated horizon. With some exceptions (as in [4]), this issue has thus far received relatively little attention in the literature on isolated horizons or its non-equilibrium generalizations.

On the other hand, calculations of the spacetime metric in the vicinity of a black hole have been carried out in the astrophysically interesting context of a Schwarzschild or Kerr black hole which is tidally deformed due to its environment (see e.g. [18–22]). Our approach here is closest to the work of Poisson and collaborators [19, 20] who study the tidal deformation of a non-rotating black hole using coordinates based on the past light cones originating from the horizon. These coordinates were originally described in [4] and are similar to the Bondi coordinates near null infinity [23]. This paper goes toward generalizing [19, 20] to include rotation, electric charge and higher multipoles.

We use the characteristic initial value formulation of Einstein's equations where free data are specified on a set of intersecting null hyper-surfaces [24–27]. Consider N-dependent variables $\psi_I (I = 1, \ldots, N)$ on a spacetime manifold with coodinates x^a. We shall be concerned with hyperbolic first-order quasi-linear equations of the form

$$\sum_{J=1}^{N} A_{IJ}^a(x, \psi)\partial_a \psi_J + F_I(x, \psi) = 0. \tag{1}$$

In the standard Cauchy problem, one specifies the ψ_I at some initial time. A solution is then guaranteed to be unique and to exist at least locally in time. The characteristic formulation considers a pair of null surfaces $\mathcal{N}_0$ and $\mathcal{N}_1$ whose intersection is a co-dimension-2 space-like surface S. It turns out to be possible to specify appropriate data on the null surfaces and on S such that the above system of equations is well posed and has a unique solution, at least locally near S.

In our case, the appropriate free data are specified on the horizon and on an outgoing past light cone originating from a cross-section of the horizon. Such a construction in the context of isolated horizons was first studied by Lewandowski [28] who characterized the general solution of Einstein equations admitting an isolated horizon. The general scenario is sketched in figure 1. We consider a portion of the horizon Δ which is isolated, in the sense that no matter and/or radiation is falling into this portion of the horizon. For a cross-section S, the past-outgoing light cone is denoted by $\mathcal{N}$. The null generators of Δ and $\mathcal{N}$ are parameterized by v and r respectively; x^i are coordinates on S. This leads to a coordinate system (v, r, x^i) which is valid till the null geodesics on $\mathcal{N}$ start to cross. The field equations are solved in a power series in r away from the horizon. This construction will be spelled out more precisely in the course of this paper.

Unlike in [19, 20] we use the Newman–Penrose formalism [29] which, as we shall see, is well suited to this problem because of the central role played by null surfaces. Furthermore, including rotation and electric charge does not make the equations much more complicated in the Newman–Penrose framework. The results of this paper cannot yet be directly compared with [19, 20]. This would require us to start with, say a Kerr black hole, perturb it by considering a non-vanishing background curvature, and to expand the perturbation in powers
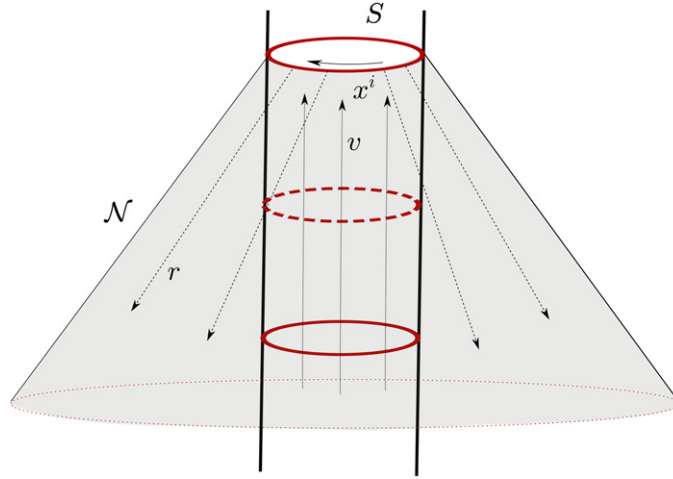
Figure 1. The near-horizon coordinate system. The null generators of the horizon are parameterized by v, surfaces of constant v are spheres. On any such sphere choose coordinates x^i. The past-directed outgoing light cone originating from S is $\mathcal{N}$ and r an affine parameter along the null generators of $\mathcal{N}$.

of r. Perturbation theory turns out to be straightforward in this general set-up, at least in principle and will be studied in a forthcoming paper. Finally, we make no attempt to study the issue of global existence of solutions, and our solutions are valid only in a neighborhood of Δ. There is no guarantee, for example, that our solutions could be extended out to an asymptotically flat region. Nevertheless, we expect that our construction does include most solutions of possible astrophysical relevance. In fact, it can be shown numerically [30] that for a Kerr black hole with the standard Kerr–Schild horizon cross-sections, this coordinate system extends all way out to past null infinity. It is thus reasonable to expect that for perturbations of Kerr, the coordinate system extends sufficiently far away from the horizon.

The plan for this paper is as follows. Section 2 reviews the Newman–Penrose formalism and summarizes the definitions and some basic properties of non-expanding and weakly isolated horizons. This will form the basis of the inner boundary conditions that will be imposed later. Section 3 sets up the near-horizon coordinate system and gauge conditions. We start by ignoring matter fields which will be included later in section 8. Section 4 summarizes the Einstein field equations and the Bianchi identities for vacuum spacetimes in the Newman–Penrose formalism, and section 6 explicitly solves the field equations in powers of the radial coordinate r. Section 8 considers Einstein–Maxwell theory by incorporating the source-free Maxwell equations. This section points out the specific aspects of the previous calculations which need to be modified due to the presence of an electromagnetic field. Finally section 9 concludes with a summary and suggestions for further investigations.

For simplicity, all manifolds and geometric fields shall be assumed to be smooth. We work in units where $G = c = 1$. We use the abstract index notation where lower case Latin letters $a, b, \ldots$ are four-dimensional spacetime indices, and $i, j, \ldots$ denote the two-dimensional angular directions. We take the spacetime metric g_{ab} to have a signature $(- + ++)$. The Riemann tensor R_{abcd} is defined by $(\nabla_a \nabla_b - \nabla_b \nabla_a) X^c = -R_{abd}{}^c X^d$, where ∇_a is the derivative operator compatible with g_{ab} and X^a is an arbitrary smooth vector field.

2. Basic notions

In this section, for completeness and to set up notation, we briefly review the Newman–Penrose formalism and the basic definitions and properties of non-expanding and weakly isolated horizons.

2.1. The Newman–Penrose formalism

The Newman–Penrose formalism [29, 31] is a tetrad formalism where the tetrad elements are null vectors, which makes it especially well suited for studying null surfaces. See [32, 33, 27] for pedagogical treatments (note that these references take the spacetime metric to have a signature of $(+ - - -)$ which is different from ours). Start with a null-tetrad $(\ell, n, m, \bar{m})$ where ℓ and n are real null vectors, and m is a complex null vector and $\bar{m}$ its complex conjugate. The tetrad is such that $\ell \cdot n = -1$, $m \cdot \bar{m} = 1$, with all other inner products vanishing. The spacetime metric is thus given by

$$g_{ab} = -\ell_a n_b - n_a \ell_b + m_a \bar{m}_b + \bar{m}_a m_b. \tag{2}$$

Directional derivatives along the basis vectors are denoted as

$$D := \ell^a \nabla_a, \quad \Delta := n^a \nabla_a, \quad \delta := m^a \nabla_a, \quad \bar{\delta} := \bar{m}^a \nabla_a. \tag{3}$$

(The symbol Δ is used for both the directional derivative along n^a and for the isolated horizon.) The components of the connection are encoded in 12 complex scalars, the spin coefficients, defined via the directional derivatives of the tetrad vectors:

$$D\ell = (\epsilon + \bar{\epsilon})\ell - \bar{\kappa}m - \kappa\bar{m}, \tag{4a}$$

$$Dn = -(\epsilon + \bar{\epsilon})n + \pi m + \bar{\pi}m, \tag{4b}$$

$$Dm = \bar{\pi}\ell - \kappa n + (\epsilon - \bar{\epsilon})m, \tag{4c}$$

$$\Delta\ell = (\gamma + \bar{\gamma})\ell - \bar{\tau}m - \tau\bar{m}, \tag{4d}$$

$$\Delta n = -(\gamma + \bar{\gamma})n + \nu m + \bar{\nu}\bar{m}, \tag{4e}$$

$$\Delta m = \bar{\nu}\ell - \tau n + (\gamma - \bar{\gamma})m, \tag{4f}$$

$$\delta\ell = (\bar{\alpha} + \beta)\ell - \bar{\rho}m - \sigma\bar{m}, \tag{4g}$$

$$\delta n = -(\bar{\alpha} + \beta)n + \mu m + \bar{\lambda}\bar{m}, \tag{4h}$$

$$\delta m = \bar{\lambda}\ell - \sigma n + (\beta - \bar{\alpha})m, \tag{4i}$$

$$\bar{\delta}m = \bar{\mu}\ell - \rho n + (\alpha - \bar{\beta})m. \tag{4j}$$

Many of the spin coefficients have transparent geometric interpretations. Some important ones for us are as follows: the real parts of ρ and μ are the expansion of ℓ and n respectively; σ and λ are the shears of ℓ and n respectively; the vanishing of κ and ν implies that ℓ and n are respectively geodesic; $\epsilon + \bar{\epsilon}$ and $\gamma + \bar{\gamma}$ are respectively the accelerations of ℓ and n, $\alpha - \bar{\beta}$ yields the connection in the m–$\bar{m}$ plane and thus the curvature of the manifold spanned by m–$\bar{m}$.

Since the null-tetrad is typically not a coordinate basis, the above definitions of the spin coefficients lead to non-trivial commutation relations:

$$
\begin{cases}
(\Delta D - D\Delta)f = (\epsilon + \bar{\epsilon})\Delta f + (\gamma + \bar{\gamma})Df - (\bar{\tau} + \pi)\delta f - (\tau + \bar{\pi})\bar{\delta} f, & (5a) \\[2mm]
(\delta D - D\delta)f = (\bar{\alpha} + \beta - \bar{\pi})Df + \kappa\Delta f - (\bar{\rho} + \epsilon - \bar{\epsilon})\delta f - \sigma\bar{\delta} f, & (5b) \\[2mm]
(\delta\Delta - \Delta\delta)f = -\bar{\nu}Df + (\tau - \bar{\alpha} - \beta)\Delta f + (\mu - \gamma + \bar{\gamma})\delta f + \bar{\lambda}\bar{\delta} f, & (5c) \\[2mm]
(\bar{\delta}\delta - \delta\bar{\delta})f = (\bar{\mu} - \mu)Df + (\bar{\rho} - \rho)\Delta f + (\alpha - \bar{\beta})\delta f - (\bar{\alpha} - \beta)\bar{\delta} f. & (5d)
\end{cases}
$$

The Weyl tensor C_{abcd} breaks down into five complex scalars:

$$
\begin{cases}
\Psi_0 = C_{abcd}\ell^a m^b \ell^c m^d, \quad \Psi_1 = C_{abcd}\ell^a m^b \ell^c n^d, \quad \Psi_2 = C_{abcd}\ell^a m^b \bar{m}^c n^d, & (6a) \\[2mm]
\Psi_3 = C_{abcd}\ell^a n^b \bar{m}^c n^d, \quad \Psi_4 = C_{abcd}\bar{m}^a n^b \bar{m}^c n^d. & (6b)
\end{cases}
$$

Similarly, the Ricci tensor is decomposed into four real and three complex scalars Φ_{ij}:

$$
\begin{cases}
\Phi_{00} = \dfrac{1}{2}R_{ab}\ell^a\ell^b, \quad \Phi_{11} = \dfrac{1}{4}R_{ab}(\ell^a n^b + m^a\bar{m}^b), \quad \Phi_{22} = \dfrac{1}{2}R_{ab}n^a n^b, \quad \Lambda = \dfrac{R}{24}, & (7a) \\[3mm]
\Phi_{01} = \dfrac{1}{2}R_{ab}\ell^a m^b, \quad \Phi_{02} = \dfrac{1}{2}R_{ab}m^a m^b, \quad \Phi_{12} = \dfrac{1}{2}R_{ab}m^a n^b, \quad \bar{\Phi}_{ij} = \Phi_{ji}. & (7b)
\end{cases}
$$

The six components of the Maxwell 2-form $\mathbf{F}_{ab}$ are written in terms of three complex scalars:

$$
\phi_0 = -\mathbf{F}_{ab}\ell^a m^b, \quad \phi_1 = \tfrac{1}{2}\mathbf{F}_{ab}(n^a\ell^b + m^a\bar{m}^b), \quad \phi_2 = \mathbf{F}_{ab}n^a\bar{m}^b. \tag{8}
$$

The source-free Maxwell equations $d\mathbf{F} = 0$ and $d\star\mathbf{F} = 0$ are written as four complex scalar equations:

$$
\begin{cases}
D\phi_1 - \bar{\delta}\phi_0 = (\pi - 2\alpha)\phi_0 + 2\rho\phi_1 - \kappa\phi_2, & (9a) \\[2mm]
D\phi_2 - \bar{\delta}\phi_1 = -\lambda\phi_0 + 2\pi\phi_1 + (\rho - 2\epsilon)\phi_2, & (9b) \\[2mm]
\Delta\phi_0 - \delta\phi_1 = (2\gamma - \mu)\phi_0 - 2\tau\phi_1 + \sigma\phi_2, & (9c) \\[2mm]
\Delta\phi_1 - \delta\phi_2 = \nu\phi_0 - 2\mu\phi_1 + (2\beta - \tau)\phi_2. & (9d)
\end{cases}
$$

The stress–energy tensor for the Maxwell field is given by

$$
T_{ab} = \frac{1}{4\pi}\left(\mathbf{F}_{ac}\mathbf{F}_b{}^c - \frac{1}{4}g_{ab}\mathbf{F}_{cd}\mathbf{F}^{cd}\right), \tag{10}
$$

which is seen to be trace free. Using the Einstein equations, we see that for Einstein–Maxwell theory, the Ricci tensor components defined in equation (7) have a simple expression in terms of the ϕ_i:

$$
\Phi_{ij} = 2\phi_i\bar{\phi}_j \ (i,j = 0,1,2), \quad \Lambda = 0. \tag{11}
$$

The relation between the spin coefficients and the curvature components lead to the so-called Newman–Penrose field equations which are a set of 16 complex first-order differential equations. The Bianchi identities, $\nabla_{[a}R_{bc]de} = 0$, are written explicitly as eight complex equations involving both the Weyl and Ricci tensor components, and three real equations involving only Ricci tensor components. See [32, 33, 27] for the full set of field equations and

Bianchi identities. We shall later write these equations after imposing simplifying gauge and coordinate conditions.

It will also be useful to use the notion of spin weights and the $\eth$ operator for derivatives in the m–$\bar{m}$ plane (which will be angular derivatives in our case). A tensor X projected on the m–$\bar{m}$ plane is said to have spin weight s if under a spin rotation $m \to e^{i\psi}m$, it transforms as $X \to e^{is\psi}X$. Thus, m^a itself has spin weight $+1$ while $\bar{m}^a$ has weight -1. For a scalar $X = m^{a_1}\cdots m^{a_p}\bar{m}^{b_1}\cdots\bar{m}^{b_q}X_{a_1\cdots b_q}$, i.e. it has p contractions with m and q with $\bar{m}$, then X has spin weight $s = p - q$. For example, the Weyl tensor component Ψ_k has spin weight $2 - k$. Similarly, the Maxwell field component ϕ_k has weight $1 - k$.

The $\eth$ and $\bar{\eth}$ operators are defined as

$$\eth X = m^{a_1}\cdots m^{a_p}\bar{m}^{b_1}\cdots\bar{m}^{b_q}\delta X_{a_1\cdots b_q}, \tag{12}$$

$$\bar{\eth} X = m^{a_1}\cdots m^{a_p}\bar{m}^{b_1}\cdots\bar{m}^{b_q}\bar{\delta} X_{a_1\cdots b_q}. \tag{13}$$

From equations (4*i*) and (4*j*), after projecting on to the m–$\bar{m}$ plane, we obtain

$$\delta m^a = (\beta - \bar{\alpha})m^a, \qquad \bar{\delta} m^a = (\alpha - \bar{\beta})m^a. \tag{14}$$

A short calculation shows that

$$\eth X = \delta X + s(\bar{\alpha} - \beta)X, \qquad \bar{\eth} X = \bar{\delta} X - s(\alpha - \bar{\beta})X. \tag{15}$$

It is clear that $\eth$ and $\bar{\eth}$ act as spin raising and lowering operators. See [34] for further properties of the $\eth$ operator and its connection to representations of the rotation group.

The transformations of the null-tetrad which preserve the metric are

(i) the boosts:

$$l \to Al, \quad n \to A^{-1}n, \quad m \to m, \tag{16}$$

(ii) the spin transformations in the m–$\bar{m}$ plane:

$$m \to e^{i\psi}m, \quad \ell \to \ell, \quad n \to n, \tag{17}$$

(iii) the null rotations around ℓ:

$$\ell \to \ell, \quad m \to m + a\ell, \quad n \to n + \bar{a}m + a\bar{m} + |a|^2\ell, \tag{18}$$

(iv) the null rotations around n (obtained by interchanging ℓ and n in equation (18)).

Again, we refer to [32, 33, 27] for a more complete discussion.

2.2. Non-expanding and weakly isolated horizons

A black hole in equilibrium with its surroundings is modeled quasi-locally as an isolated horizon. The basic geometrical object is a smooth three-dimensional null surface Δ (which shall be the black hole horizon) in a Lorentzian spacetime $(\mathcal{M}, g_{ab})$. If ℓ is any null-normal of Δ, then it must be geodesic so that $\ell^a\nabla_a\ell^b = \tilde{\kappa}_{(\ell)}\ell^b$; the acceleration $\tilde{\kappa}_{(\ell)}$ is the *surface gravity* associated with ℓ^a. We shall consider only non-extremal horizons here, i.e. we shall always have non-vanishing $\tilde{\kappa}$. The spacetime metric g_{ab} induces a degenerate metric on Δ which we denote q_{ab} which has signature $(0 + +)$. Its (non-unique) inverse will be denoted q^{ab}. There is a volume element $^2\epsilon_{ab}$ on Δ, satisfying $\ell^a\,^2\epsilon_{ab} = 0$, which measures the area of space-like cross-sections of Δ.

A smooth three-dimensional null surface Δ is said to be a *non-expanding horizon* if

- Δ has topology $S^2 \times \mathbb{R}$ and if $\Pi : S^2 \times \mathbb{R} \to S^2$ is the natural projection, then $\Pi^{-1}(x)$ for any $x \in S^2$ are null curves on Δ.
- The expansion $\Theta_{(\ell)} := q^{ab}\nabla_a\ell_b$ of any null-normal ℓ^a of Δ vanishes.

6

• The Einstein field equations hold at Δ, and the matter stress–energy tensor T_{ab} is such that for any future-directed null-normal ℓ^a, $-T_b^a \ell^b$ is future causal.

We shall consider only null-tetrads adapted to Δ such that, at the horizon, ℓ^a coincides with a null-normal to Δ. We shall also consider a foliation of the horizon by space-like spheres S_v with v a coordinate on the horizon which is also an affine parameter along ℓ: $\mathcal{L}_\ell v = 1$; S shall denote a generic spherical cross-section of Δ. Null rotations about ℓ^a correspond to changing the foliation.

This deceptively simple definition of a non-expanding horizon leads to a number of important results which we state here without proof (though some of these will be rederived later).

 (i) The Weyl tensor components Ψ_0 and Ψ_1 vanish on the horizon. This implies that Ψ_2 is an invariant on Δ as long as the null-tetrad is adapted to the horizon; it is automatically invariant under boosts and spin rotations (it has spin weight 0), and it is invariant under null rotations around ℓ because Ψ_0 and Ψ_1 vanish. Similarly, the Maxwell field component ϕ_0 vanishes on the horizon and ϕ_1 is invariant on Δ. Both Ψ_2 and ϕ_1 are also time independent on the horizon.

 (ii) For a general null sub-manifold, there is no unique derivative operator compatible with the metric, and the pull back of the spacetime derivative operator ∇_a does not necessarily induce a connection on the hyper-surface. For non-expanding horizons however, the spacetime connection does induce a unique derivative operator $\mathcal{D}_a$ compatible with q_{ab}. Furthermore, there exists a 1-form ω_a such that, for any vector field X^a tangent to Δ

$$X^a \nabla_a \ell^b = X^a \omega_a \ell^b. \qquad (19)$$

The 1-form ω_a plays a fundamental role in what follows. The pullback of ω_a to the cross-sections S will be denoted $\tilde{\omega}_a$.

 (iii) The surface gravity of ℓ is

$$\tilde{\kappa}_{(\ell)} = \ell^a \omega_a. \qquad (20)$$

The curl and divergence of ω carry important physical information. The curl is related to the imaginary part of the Weyl tensor on the horizon:

$$d\omega = \mathrm{Im}\,[\Psi_2]\,^2\epsilon, \qquad (21)$$

and its divergence specifies the foliation of Δ by spheres [6].

We need to strengthen the conditions of a non-expanding horizon for various physical situations. The minimum extra condition required for black hole thermodynamics and to have a well-defined action principle with Δ as an inner boundary of a portion of spacetime is formulated as a weakly isolated horizon [3]. This is to choose an equivalence class of null-normals $[\ell]$, each related to the other by a constant re-scaling $\ell' = c\ell$, such that

$$\mathcal{L}_\ell \omega_a = 0. \qquad (22)$$

This can be shown to be equivalent to the zeroth law, i.e. $\tilde{\kappa}_{(\ell)} = \ell^a \omega_a$ is constant on the horizon. Note that under a re-scaling $\ell^a \to f\ell^a$, ω_a transforms as $\omega_a \to \omega_a + D_a \ln f$ so that it is invariant under constant re-scalings. This condition is sufficient to ensure a well-defined action principle and to lead to a sensible notion of horizon mass and spin.

• The horizon angular momentum is well defined in the case when there is an axial symmetry φ^a on Δ which preserves q_{ab}, ω_a and the electromagnetic field on the horizon [5]. The angular momentum is given by

$$J = -\frac{1}{4\pi} \oint_S f\,\mathrm{Im}[\Psi_2]\,^2\epsilon + \frac{1}{2\pi} \oint_S g\,\mathrm{Im}[\phi_1]\,^2\epsilon, \qquad (23)$$

where f and g are respectively defined via $\varphi^{a2}\epsilon_{ab} = \partial_b f$ and $\varphi^a \star \mathbf{F}_{ab} = \partial_b g$. Similarly, the electric and magnetic charges of the horizon are defined respectively as

$$Q = \frac{1}{2\pi} \oint_S \mathrm{Re}\,[\phi_1]\,^2\epsilon, \quad P = \frac{1}{2\pi} \oint_S \mathrm{Im}\,[\phi_1]\,^2\epsilon. \tag{24}$$

Hamiltonian methods provide a suitable notion of horizon mass [5]:

$$M = \frac{1}{2R}\sqrt{(R^2 + Q^2)^2 + 4J^2}, \tag{25}$$

where R is the area radius of the horizon so that if A is the area of the horizon cross-sections then $R := \sqrt{A/4\pi}$. In the case when the horizon is not exactly symmetric, one could attempt to find an approximate symmetry to replace φ^a in the above equations [35–37]. It is also possible to define source multipole moments for an axisymmetric-charged isolated horizon [38]. We also note that while these definitions for mass and charge arise for weakly isolated horizons, the expressions themselves are meaningful for non-expanding horizons; this fact is useful in, for example, numerical simulations where it may not be convenient to construct the appropriate null-normal corresponding to a weakly isolated horizon. Also, the values of M, J, Q, P are independent of which cross-section S is used to calculate them.

Any non-expanding horizon can be made into a weakly isolated horizon by suitably scaling the null generators. Thus, the restriction to weakly isolated horizons is not a genuine physical restriction. One could go ahead and impose further physical restrictions on the intrinsic horizon geometry by requiring that not only ω_a, but the full connection $\mathcal{D}_a$ on Δ is preserved by ℓ^a: $[\mathcal{L}_\ell, \mathcal{D}] = 0$ which leads to the definition of an isolated horizon [6]. We shall work only with weakly isolated horizons in this paper.

3. The near-horizon coordinate system and null-tetrad

Let us now assume that the vacuum Einstein equations hold in a neighborhood of the horizon Δ. We will consider electromagnetic fields later in section 8. Following [4], we introduce a coordinate system and null-tetrad in the vicinity of Δ analogous to the Bondi coordinates near null infinity. See figure 1. Choose a particular null-normal ℓ^a on Δ. Let v be the affine parameter along ℓ^a so that $\ell^a \nabla_a v = 1$. Let S_v denote the spheres of constant v. Introduce coordinates x^i ($i = 2, 3$) on any one S_v (call this sphere S_0) and require them to be constant along ℓ^a: $\ell^a \nabla_a x^i = 0$; this leads to a coordinate system (v, x^i) on Δ. Let n^a be a future-directed inward pointing null vector orthogonal to the S_v and normalized such that $\ell \cdot n = -1$. Extend n^a off Δ geodesically, with r being an affine parameter along $-n^a$; set $r = 0$ at Δ. This yields a family of null surfaces $\mathcal{N}_v$ parameterized by v and orthogonal to the spheres S_v. Set (v, x^i) to be constant along the integral curves of n^a to obtain a coordinate system (v, r, x^i) in a neighborhood of Δ. Choose a complex null vector m^a tangent to S_0. Lie drag m^a along ℓ^a:

$$\mathcal{L}_\ell m^a = 0 \quad \text{on } \Delta. \tag{26}$$

We thus obtain a null-tetrad $(\ell, n, m, \bar{m})$ on Δ. Finally, parallel transport ℓ and m along $-n^a$ to obtain a null-tetrad in the neighborhood of Δ. This construction is fixed up to the choice of the x^i and m^a on an initial cross-section S_0. We are allowed to perform an arbitrary spin transformation $m \to e^{i\psi} m$ on S_0.

With the Bondi-like coordinate system in hand, we can now in principle use the coordinate basis vectors in the (v, r, x^i) coordinates to construct an arbitrary null-tetrad near the horizon. The evolution equations for the component functions of the null-tetrad will follow from the above construction. Let us start with n_a and n^a. We have the family of null surfaces $\mathcal{N}_v$

parameterized by v; n_a is normal to the $\mathcal{N}_v$ and r is an affine parameter along $-n^a$. This implies that we can choose

$$n_a = -\partial_a v \qquad \text{and} \qquad n^a \nabla_a := \Delta = -\frac{\partial}{\partial r}. \tag{27}$$

To satisfy the inner-product relations $\ell^a n_a = -1$ and $m^a n_a = 0$, the other basis vectors must be of the form:

$$\ell^a \nabla_a := D = \frac{\partial}{\partial v} + U\frac{\partial}{\partial r} + X^i \frac{\partial}{\partial x^i}, \quad m^a \nabla_a := \delta = \Omega\frac{\partial}{\partial r} + \xi^i\frac{\partial}{\partial x^i}. \tag{28}$$

The frame functions U, X^i are real while Ω, ξ^i are complex. We wish to now specialize to the case when ℓ^a is a null-normal of Δ so that the null-tetrad is adapted to the horizon. Since ∂_v is tangent to the null generators of Δ, this clearly requires that U, X^i must vanish on the horizon. Similarly, we want m^a to be tangent to the spheres S_v at the horizon, so Ω should also vanish on Δ. Thus, U, X^i and Ω are all $\mathcal{O}(r)$ functions.

We expand the spin coefficients, Weyl tensor components and the directional derivatives in a power series in r away from the horizon:

$$X = X^{(0)} + rX^{(1)} + \tfrac{1}{2}r^2 X^{(2)} \cdots. \tag{29}$$

Thus, for example we will have

$$\Psi_k = \Psi_k^{(0)} + r\Psi_k^{(1)} + \tfrac{1}{2}r^2\Psi_k^{(2)} + \cdots. \tag{30}$$

The same notation will be used for the frame fields U and Ω. However, for the frame fields X^i and ξ^i, we will write

$$X^i = X^i_{(0)} + rX^i_{(1)} + \tfrac{1}{2}r^2 X^i_{(2)} + \cdots, \quad \xi^i = \xi^i_{(0)} + r\xi^i_{(1)} + \tfrac{1}{2}r^2\xi^i_{(2)} + \cdots. \tag{31}$$

To avoid clutter we shall not be completely consistent with this notation. Thus, we shall not use any index for the spin-weighted angular derivative $\eth$; it is to be understood that $\eth$ always refers to $\eth^{(0)}$ in this paper. Similarly, where we do not expect any confusion, we shall often drop the index on the directional derivatives such as D and δ; in this case, unless mentioned otherwise, the relevant order of the operator is the same as the order of the operand. For example, $\delta\Psi_2^{(0)}$ refers to $\delta^{(0)}\Psi_2^{(0)}$.

The variables we need to solve for are the frame fields U, X^i, Ω, ξ^i, the 12 spin coefficients and the Weyl tensor components Ψ_k. The equations are the commutation relations, the 16 field equations and 8 of the 11 Bianchi identities. Three of the Bianchi identities involve only the Ricci tensor, thus they will need to be considered when matter fields are present. In section 8, we will consider the Maxwell equations and the three additional Bianchi identities as well. All these are first-order differential equations and each of these sets of equations has possibly three kinds of equations: evolution equations which involve derivatives along v, i.e. D, and do not contain any radial derivatives Δ, equations which contain only purely angular derivatives δ and $\bar{\delta}$ and finally the radial equations involving Δ. In order to integrate these equations, we proceed as follows. We start with suitable data on some initial cross-section S_0 of the horizon and use the non-radial equations to propagate them at all points of the horizon. Starting with this horizon data and the appropriate data on the past light cone $\mathcal{N}_0$ containing S_0, the radial equations then yield the first radial derivatives and, iteratively, all successive higher derivatives as well. At each step, we will need to expand the non-radial equations in powers of r, and ensure that we have consistency order-by-order in r. We now carry out this procedure in detail and spell out the free data.

We start with the conditions on the spin coefficients. Since n^a is an affinely parameterized geodesic, and ℓ and m are parallel propagated along n^a, we have $\Delta n = \Delta \ell = \Delta m = 0$. From equations (4d), (4e) and (4f), this leads to

$$\gamma = \tau = \nu = 0. \tag{32}$$

Imposing equation (32) in the commutation relations (5) leads to

$$
\begin{cases}
(\Delta D - D\Delta)f = (\epsilon + \bar{\epsilon})\Delta f - \pi \delta f - \bar{\pi}\bar{\delta}f, & (33a) \\[2ex]
(\delta D - D\delta)f = (\bar{\alpha} + \beta - \bar{\pi})Df + \kappa \Delta f - (\bar{\rho} + \epsilon - \bar{\epsilon})\delta f - \sigma\bar{\delta}f, & (33b) \\[2ex]
(\delta\Delta - \Delta\delta)f = -(\bar{\alpha} + \beta)\Delta f + \mu\delta f + \bar{\lambda}\bar{\delta}f, & (33c) \\[2ex]
(\bar{\delta}\delta - \delta\bar{\delta})f = (\bar{\mu} - \mu)Df + (\bar{\rho} - \rho)\Delta f + (\alpha - \bar{\beta})\delta f - (\bar{\alpha} - \beta)\bar{\delta}f. & (33d)
\end{cases}
$$

Substituting $f = v$ yields

$$
\pi = \alpha + \bar{\beta}, \quad \mu = \bar{\mu}. \tag{34}
$$

Equations (32) and (34) are the basic conditions on the spin coefficients which hold at all spacetime points where the coordinate system is valid. In addition, we have the boundary condition at the horizon, namely that ℓ is geodesic, normal to a smooth surface, and expansion free. This implies

$$
\rho^{(0)} = \kappa^{(0)} = 0. \tag{35}
$$

The subscript $^{(0)}$ indicates that these are the values at the horizon, i.e. at $r = 0$. From the Raychaudhuri equation applied to the null generators of the horizon, we obtain

$$
D\rho^{(0)} - \bar{\delta}\kappa^{(0)} = (\rho^{(0)})^2 + |\sigma^{(0)}|^2 + (\epsilon^{(0)} + \bar{\epsilon}^{(0)})\rho^{(0)} - 2\alpha^{(0)}\kappa^{(0)} \tag{36}
$$

(this is also one of the Newman–Penrose field equations considered later). This leads to

$$
\sigma^{(0)} = 0. \tag{37}
$$

In addition, we can make a spin transformation on S_0 and set

$$
\epsilon^{(0)} - \bar{\epsilon}^{(0)} = 0. \tag{38}
$$

From equation (33b) and the above conditions on the spin coefficients at the horizon, this also ensures that $\mathcal{L}_\ell m^a = 0$ at the horizon. Finally, we assume that we have scaled the null-normal appropriately (i.e. we have chosen the coordinate v and the foliation S_v) so that the horizon is weakly isolated: $\mathcal{L}_\ell \omega_a = 0$. In terms of the Newman–Penrose spin coefficients, ω_a is written as

$$
\omega_a = -n_b \nabla_a \ell^b = (\epsilon + \bar{\epsilon})n_a + \pi^{(0)}m_a + \bar{\pi}^{(0)}\bar{m}_a. \tag{39}
$$

Thus, Δ will be a weakly isolated horizon if we choose

$$
\tilde{\kappa} := \epsilon^{(0)} + \bar{\epsilon}^{(0)} = \text{constant}, \quad D\pi^{(0)} = 0. \tag{40}
$$

Among the free functions that we need to specify, we can choose a constant for the surface gravity (perhaps determined by the mass and spin [5, 3]), and a function $\pi^{(0)}$ on the initial cross-section S_0.

4. The reduced system of equations

With conditions (32) and (34) on the spin coefficients at hand, we now impose them in the commutator relations, the field equations and the Bianchi identities. The functions U, X^i, Ω, ξ^i are determined by the commutation relations (33) by substituting, in turn, r and x^i for f, and

imposing equations (32) and (34) on the spin coefficients. The radial derivatives are

$$\Delta U = -(\epsilon + \bar{\epsilon}) - \pi\Omega - \bar{\pi}\bar{\Omega}, \tag{41a}$$

$$\Delta X^i = -\pi\xi^i - \bar{\pi}\bar{\xi}^i, \tag{41b}$$

$$\Delta\Omega = -\bar{\pi} - \mu\Omega - \bar{\lambda}\bar{\Omega}, \tag{41c}$$

$$\Delta\xi^i = -\mu\xi^i - \bar{\lambda}\bar{\xi}^i. \tag{41d}$$

The propagation equations along v are

$$D\Omega - \delta U = \kappa + \rho\Omega + \sigma\bar{\Omega}, \tag{42a}$$

$$D\xi^i - \delta X^i = \bar{\rho}\xi^i + \sigma\bar{\xi}^i. \tag{42b}$$

Let us now turn to the field equations. After imposing equations (32) and (34) on the spin coefficients and ignoring matter terms for now, the field equations involving radial derivatives are

$$\Delta\lambda = -2\lambda\mu - \Psi_4, \tag{43a}$$

$$\Delta\mu = -\mu^2 - |\lambda|^2, \tag{43b}$$

$$\Delta\rho = -\mu\rho - \sigma\lambda - \Psi_2, \tag{43c}$$

$$\Delta\sigma = -\mu\sigma - \bar{\lambda}\rho. \tag{43d}$$

$$\Delta\kappa = -\bar{\pi}\rho - \pi\sigma - \Psi_1, \tag{43e}$$

$$\Delta\epsilon = -\bar{\pi}\alpha - \pi\beta - \Psi_2, \tag{43f}$$

$$\Delta\pi = -\pi\mu - \bar{\pi}\lambda - \Psi_3, \tag{43g}$$

$$\Delta\beta = -\mu\beta - \alpha\bar{\lambda}, \tag{43h}$$

$$\Delta\alpha = -\beta\lambda - \mu\alpha - \Psi_3. \tag{43i}$$

The time evolution equations become

$$D\rho - \bar{\delta}\kappa = \rho^2 + |\sigma|^2 + (\epsilon + \bar{\epsilon})\rho - 2\alpha\kappa, \tag{44a}$$

$$D\sigma - \delta\kappa = (\rho + \bar{\rho} + 2\epsilon)\sigma - 2\beta\kappa + \Psi_0, \tag{44b}$$

$$D\alpha - \bar{\delta}\epsilon = (\rho - \epsilon)\alpha + \beta\bar{\sigma} - \bar{\beta}\epsilon - \kappa\lambda + (\epsilon + \rho)\pi, \tag{44c}$$

$$D\beta - \delta\epsilon = (\alpha + \pi)\sigma + (\bar{\rho} - \epsilon)\beta - \mu\kappa - (\bar{\alpha} - \bar{\pi})\epsilon + \Psi_1, \tag{44d}$$

$$D\lambda - \bar{\delta}\pi = (\rho - 2\epsilon)\lambda + \bar{\sigma}\mu + 2\alpha\pi, \tag{44e}$$

$$D\mu - \delta\pi = (\bar{\rho} - 2\epsilon)\mu + \sigma\lambda + 2\beta\pi + \Psi_2. \tag{44f}$$

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The angular field equations are

$$\delta\rho - \bar{\delta}\sigma = \bar{\pi}\rho - (3\alpha - \bar{\beta})\sigma - \Psi_1, \tag{45a}$$

$$\delta\alpha - \bar{\delta}\beta = \mu\rho - \lambda\sigma + |\alpha|^2 + |\beta|^2 - 2\alpha\beta - \Psi_2, \tag{45b}$$

$$\delta\lambda - \bar{\delta}\mu = \pi\mu + (\bar{\alpha} - 3\beta)\lambda - \Psi_3. \tag{45c}$$

Finally, we have the Bianchi identities which, in the NP formalism, are written as a set of nine complex and two real equations; in the absence of matter, only eight complex equations survive. The radial Bianchi identities reduce to

$$\Delta\Psi_0 - \delta\Psi_1 = -\mu\Psi_0 - 2\beta\Psi_1 + 3\sigma\Psi_2, \tag{46a}$$

$$\Delta\Psi_1 - \delta\Psi_2 = -2\mu\Psi_1 + 2\sigma\Psi_3, \tag{46b}$$

$$\Delta\Psi_2 - \delta\Psi_3 = -3\mu\Psi_2 + 2\beta\Psi_3 + \sigma\Psi_4, \tag{46c}$$

$$\Delta\Psi_3 - \delta\Psi_4 = -4\mu\Psi_3 + 4\beta\Psi_4. \tag{46d}$$

Note that there is no equation for the radial derivative of Ψ_4. Among all the fields that we are solving for, this is in fact the only one for which this happens. This means that Ψ_4 (in this case, its radial derivatives) is the free data that must be specified on the null cone $\mathcal{N}_0$ originating from S_0.

Finally, we have the components of the Bianchi equations for evolution of the Weyl tensor components:

$$D\Psi_1 - \bar{\delta}\Psi_0 = (\pi - 4\alpha)\Psi_0 + 2(2\rho + \epsilon)\Psi_1 - 3\kappa\Psi_2, \tag{47a}$$

$$D\Psi_2 - \bar{\delta}\Psi_1 = -\lambda\Psi_0 - 2(\pi - \alpha)\Psi_1 + 3\rho\Psi_2 - 2\kappa\Psi_3, \tag{47b}$$

$$D\Psi_3 - \bar{\delta}\Psi_2 = -2\lambda\Psi_1 + 3\pi\Psi_2 + 2(\rho - \epsilon)\Psi_3 - \kappa\Psi_4, \tag{47c}$$

$$D\Psi_4 - \bar{\delta}\Psi_3 = -3\lambda\Psi_2 + 2(\alpha + 2\pi)\Psi_3 + (\rho - 4\epsilon)\Psi_4. \tag{47d}$$

Having written down the full set of differential equations, we are now ready to solve them locally near the horizon order-by-order in r.

5. The intrinsic horizon geometry

We first consider the intrinsic geometry of the horizon contained in the time evolution equations at the horizon, i.e. the $\mathcal{O}(r^0)$ part of equation (44a). This has been studied elsewhere in great detail [6, 3, 7–9], so we shall be brief. We already have that ℓ^a is geodesic so that $\kappa^{(0)} = 0$, and the expansion and twist of ℓ vanish, which implies $\rho^{(0)} = 0$. The equations (44) in turn lead to

$$\sigma^{(0)} = 0, \qquad \Psi_0^{(0)} = 0, \tag{48a}$$

$$D\alpha^{(0)} - \bar{\delta}\epsilon^{(0)} = -\epsilon^{(0)}\alpha^{(0)} - \epsilon^{(0)}\bar{\beta}^{(0)} + \epsilon^{(0)}\pi^{(0)}, \tag{48b}$$

$$D\beta^{(0)} - \delta\epsilon^{(0)} = -\epsilon^{(0)}\beta^{(0)} - (\bar{\alpha}^{(0)} - \bar{\pi}^{(0)})\epsilon^{(0)} + \Psi_1^{(0)}, \tag{48c}$$

$$D\lambda^{(0)} - \bar{\delta}\pi^{(0)} = -2\epsilon^{(0)}\lambda^{(0)} + 2\alpha^{(0)}\pi^{(0)}, \tag{48d}$$

$$D\mu^{(0)} - \delta\pi^{(0)} = -2\epsilon^{(0)}\mu^{(0)} + 2\beta^{(0)}\pi^{(0)} + \Psi_2^{(0)}. \tag{48e}$$

Adding and subtracting equation (48b) to the complex conjugate of equation (48c), and using the conditions of equations (32), (34), (35), (37), (38) and (40) on the spin coefficients leads to

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$$D\alpha^{(0)} = 0, \quad D\beta^{(0)} = 0, \quad \Psi_1^{(0)} = 0. \tag{49}$$

The remaining two equations tell us about the time dependence of the expansion and shear of n at the horizon:

$$\begin{cases} D\lambda^{(0)} + \tilde{\kappa}\lambda^{(0)} = \bar{\delta}\pi^{(0)} + 2\alpha^{(0)}\pi^{(0)}, & (50) \\ D\mu^{(0)} + \tilde{\kappa}\mu^{(0)} = \delta\pi^{(0)} + 2\beta^{(0)}\pi^{(0)} + \Psi_2^{(0)}. & (51) \end{cases}$$

Using the $\eth$ operator for the angular derivatives, and noting that π has spin weight -1, this is written more cleanly as

$$\begin{cases} D\lambda^{(0)} + \tilde{\kappa}\lambda^{(0)} = \bar{\eth}\pi^{(0)} + (\pi^{(0)})^2, & (52a) \\ D\mu^{(0)} + \tilde{\kappa}\mu^{(0)} = \eth\pi^{(0)} + |\pi^{(0)}|^2 + \Psi_2^{(0)}. & (52b) \end{cases}$$

These two equations can be shown to be identical to equation (3.9) of [6] with the matter terms set to zero.

Now consider the angular field equations (45) at the horizon. The first of these leads to $\Psi_1^{(0)} = 0$ which we already knew. Equation (45c) gives

$$\Psi_3^{(0)} = \bar{\eth}\mu^{(0)} + \pi^{(0)}\mu^{(0)} - \eth\lambda^{(0)} - \bar{\pi}^{(0)}\lambda^{(0)}. \tag{53}$$

The real and imaginary parts of equation (45b) give respectively

$$-2\operatorname{Re}\Psi_2^{(0)} = \delta a + \bar{\delta}\bar{a} - 2|a|^2, \qquad a := \alpha^{(0)} - \bar{\beta}^{(0)}, \tag{54}$$

$$-2\mathrm{i}\operatorname{Im}\Psi_2^{(0)} = \eth\pi^{(0)} - \bar{\eth}\bar{\pi}^{(0)}. \tag{55}$$

Since $a = \alpha^{(0)} - \bar{\beta}^{(0)}$ determines the connection on the horizon cross-section, its derivative must be connected with the scalar 2-curvature of the cross-section. In fact, it is not hard to show that the first equation is equivalent to

$$^2\tilde{\mathcal{R}} = -4\operatorname{Re}\Psi_2^{(0)}. \tag{56}$$

The second equation relates the curl of $\tilde{\omega}$ with the imaginary part of Ψ_2 and thus reproduces equation (21).

Now the Bianchi identities (47a) lead to

$$\begin{cases} D\Psi_2^{(0)} = 0, & (57a) \\ D\Psi_3^{(0)} + \tilde{\kappa}\Psi_3^{(0)} = \bar{\delta}\Psi_2^{(0)} + 3\pi^{(0)}\Psi_2^{(0)}, & (57b) \\ D\Psi_4^{(0)} + 2\tilde{\kappa}\Psi_4^{(0)} = \bar{\eth}\Psi_3^{(0)} + 5\pi^{(0)}\Psi_3^{(0)} - 3\lambda^{(0)}\Psi_2^{(0)}. & (57c) \end{cases}$$

The second of these can be shown to follow from equation (53), so it does not yield any new information, but the last equation gives the time evolution of Ψ_4 at the horizon.

To summarize, here are the free data required at the horizon. We start with a choice of null generator ℓ^a and a constant surface gravity $\tilde{\kappa} := \epsilon^{(0)} + \bar{\epsilon}^{(0)}$, and an affine parameter v. Surfaces of constant v are spheres S_v, and we choose a basis $m, \bar{m}$ tangent to the S_v and Lie dragged along ℓ^a. Then, on any one of the spheres, say S_0, choose the transversal expansion and shear $\mu^{(0)}, \lambda^{(0)}$, the spin coefficient $\pi^{(0)}$, the connection on S_0: $\alpha^{(0)} - \bar{\beta}^{(0)}$, and the transverse gravitational radiation $\Psi_4^{(0)}$. The field equations then show that $\pi^{(0)}, \alpha^{(0)} - \bar{\beta}^{(0)}$ are time independent. The transversal shear and expansion $\lambda^{(0)}(v), \mu^{(0)}(v)$ satisfy equation (52) which means that their time evolution is

$$\begin{cases} \mu^{(0)}(v) = \mu^{(0)}(0)\mathrm{e}^{-\tilde{\kappa}v} + \dfrac{1}{\tilde{\kappa}}[\bar{\eth}\pi^{(0)} + (\pi^{(0)})^2](1 - \mathrm{e}^{-\tilde{\kappa}v}), & (58a) \\ \lambda^{(0)}(v) = \lambda^{(0)}(0)\mathrm{e}^{-\tilde{\kappa}v} + \dfrac{1}{\tilde{\kappa}}[\eth\pi^{(0)} + |\pi^{(0)}|^2 + \Psi_2^{(0)}](1 - \mathrm{e}^{-\tilde{\kappa}v}). & (58b) \end{cases}$$

The Weyl tensor components $\Psi_0^{(0)}$, $\Psi_1^{(0)}$ vanish. $\Psi_2^{(0)}$ is time independent, its real part is determined by $\alpha^{(0)} - \bar{\beta}^{(0)}$ according to equation (54), and its imaginary part by $\pi^{(0)} = \alpha^{(0)} + \bar{\beta}^{(0)}$ according to equation (55); $\Psi_3^{(0)}$ is determined from equation (53) and its time evolution is of the same form as for $\lambda^{(0)}$, $\mu^{(0)}$. From equation (57b) we obtain

$$\Psi_3^{(0)}(v) = \Psi_3^{(0)}(0)e^{-\tilde{\kappa}v} + \frac{1}{\tilde{\kappa}}\big[\bar{\eth}\Psi_2^{(0)} + 3\pi^{(0)}\Psi_2^{(0)}\big](1 - e^{-\tilde{\kappa}v}). \tag{59}$$

Finally, $\Psi_4^{(0)}$ can be freely specified on S_0 and it evolves in time according to equation (57c); this leads to a time dependence of the form

$$\Psi_4^{(0)}(v) = \Psi_4^{(0)}(0)\,e^{-2\tilde{\kappa}v} + \frac{A}{2\tilde{\kappa}}(1 - e^{-2\tilde{\kappa}v}) + \frac{B}{\tilde{\kappa}}\,e^{-\tilde{\kappa}v}(1 - e^{-\tilde{\kappa}v}), \tag{60}$$

where the time-independent angular functions A and B are defined via

$$(\bar{\eth} + 5\pi^{(0)})\Psi_3^{(0)} - 3\lambda^{(0)}\Psi_2^{(0)} = A + B\,e^{-\tilde{\kappa}v}, \tag{61}$$

$$A = \frac{1}{\tilde{\kappa}}[(\bar{\eth} + 5\pi^{(0)})(\bar{\eth} + 3\pi^{(0)}) - 3(\eth\pi^{(0)} + |\pi^{(0)}|^2 + \Psi_2^{(0)})]\Psi_2^{(0)}, \tag{62}$$

$$B = (\bar{\eth} + 5\pi^{(0)})\Psi_3^{(0)}(0) - 3\lambda^{(0)}\Psi_2^{(0)} - A. \tag{63}$$

We note that the time evolution on an extremal horizon ($\tilde{\kappa} = 0$) would be very different; see e.g. [6, 7]. As we shall see in the next section, not only can we specify Ψ_4 freely on S_0, but also all of its radial derivatives. The free data on the transversal null surface $\mathcal{N}_0$ are in fact Ψ_4.

6. The radial equations

With the intrinsic horizon geometry understood, we turn our attention to the radial equations (43) and (46) which determine all radial derivatives for all the non-zero spin coefficients and Weyl tensor components (except Ψ_4). The first radial derivatives are easily obtained by substituting the horizon values on the right-hand sides of equations (43) and (46):

$$\kappa^{(1)} = \sigma^{(1)} = 0, \tag{64a}$$

$$\rho^{(1)} = \Psi_2^{(0)}, \tag{64b}$$

$$\epsilon^{(1)} + \bar{\epsilon}^{(1)} = 2|\pi^{(0)}|^2 + 2\,\mathrm{Re}[\Psi_2^{(0)}], \tag{64c}$$

$$\epsilon^{(1)} - \bar{\epsilon}^{(1)} = \bar{\pi}^{(0)}(\alpha^{(0)} - \bar{\beta}^{(0)}) - \pi^{(0)}(\bar{\alpha}^{(0)} - \beta^{(0)}) + \Psi_2^{(0)} - \bar{\Psi}_2^{(0)}, \tag{64d}$$

$$\lambda^{(1)} = 2\mu^{(0)}\lambda^{(0)} + \Psi_4^{(0)}, \tag{64e}$$

$$\mu^{(1)} = (\mu^{(0)})^2 + |\lambda^{(0)}|^2, \tag{64f}$$

$$\pi^{(1)} = \alpha^{(1)} + \bar{\beta}^{(1)} = \pi^{(0)}\mu^{(0)} + \bar{\pi}^{(0)}\lambda^{(0)} + \Psi_3^{(0)}, \tag{64g}$$

$$\alpha^{(1)} - \bar{\beta}^{(1)} = \mu^{(0)}(\alpha^{(0)} - \bar{\beta}^{(0)}) + \lambda^{(0)}(\beta^{(0)} - \bar{\alpha}^{(0)}) + \Psi_3^{(0)}, \tag{64h}$$

$$\Psi_0^{(1)} = 0, \tag{64i}$$

$$\Psi_1^{(1)} = -\eth\Psi_2^{(0)}, \tag{64j}$$

$$\Psi_2^{(1)} = -(\eth + \bar{\pi}^{(0)})\Psi_3^{(0)} + 3\mu^{(0)}\Psi_2^{(0)}, \tag{64k}$$

$$\Psi_3^{(1)} = -(\eth + 2\bar{\pi}^{(0)})\Psi_4^{(0)} + 4\mu^{(0)}\Psi_3^{(0)}. \tag{64l}$$

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We can continue this process by applying the radial derivative Δ to equation (43) once again, and substituting the first derivatives from above on the right-hand side. This leads to expressions for the second derivatives in terms of the horizon data:

$$\kappa^{(2)} = -(\eth - \bar{\pi}^{(0)})\Psi_2^{(0)}, \tag{65a}$$

$$\sigma^{(2)} = \bar{\lambda}^{(0)}\Psi_2^{(0)}, \tag{65b}$$

$$\rho^{(2)} = 4\mu^{(0)}\Psi_2^{(0)} - (\eth + \bar{\pi}^{(0)})\Psi_3^{(0)}, \tag{65c}$$

$$\lambda^{(2)} = 6(\mu^{(0)})^2\lambda^{(0)} + 2\lambda^{(0)}\left|\lambda^{(0)}\right|^2 + 2\mu^{(0)}\Psi_4^{(0)} + \Psi_4^{(1)}, \tag{65d}$$

$$\mu^{(2)} = 2(\mu^{(0)})^3 + 6\mu^{(0)}\left|\lambda^{(0)}\right|^2 + \lambda^{(0)}\bar{\Psi}_4^{(0)} + \bar{\lambda}^{(0)}\Psi_4^{(0)}, \tag{65e}$$

$$\pi^{(2)} = 2\pi^{(0)}((\mu^{(0)})^2 + |\lambda^{(0)}|^2) + 4\bar{\pi}^{(0)}\lambda^{(0)}\mu^{(0)} + 5\mu^{(0)}\Psi_3^{(0)} + \lambda^{(0)}\bar{\Psi}_3^{(0)} - (\eth + \bar{\pi}^{(0)})\Psi_4^{(0)}, \tag{65f}$$

$$\alpha^{(2)} - \bar{\beta}^{(2)} = 2(\alpha^{(0)} - \bar{\beta}^{(0)})((\mu^{(0)})^2 + \left|\lambda^{(0)}\right|^2) + 4\mu^{(0)}\lambda^{(0)}(\beta^{(0)} - \bar{\alpha}^{(0)})$$
$$+ \Psi_4^{(0)}(\beta^{(0)} - \bar{\alpha}^{(0)}) + 5\mu^{(0)}\Psi_3^{(0)} - \lambda^{(0)}\bar{\Psi}_3^{(0)} - (\eth + 2\bar{\pi}^{(0)})\Psi_4^{(0)}. \tag{65g}$$

Finally we can also investigate the non-radial equations to $\mathcal{O}(r)$. For this, we need the directional derivatives to $\mathcal{O}(r)$ which we have already calculated:

$$\Delta = -\frac{\partial}{\partial r}, \tag{66}$$

$$D = \frac{\partial}{\partial v} + r(\epsilon^{(0)} + \bar{\epsilon}^{(0)})\frac{\partial}{\partial r} + r\pi^{(0)}\xi_{(0)}^i\frac{\partial}{\partial x^i} + r\bar{\pi}^{(0)}\bar{\xi}_{(0)}^i\frac{\partial}{\partial x^i} + \mathcal{O}(r^2), \tag{67}$$

$$\delta = \xi_{(0)}^i\frac{\partial}{\partial x^i} + r\mu^{(0)}\xi_{(0)}^i\frac{\partial}{\partial x^i} + r\bar{\lambda}^{(0)}\bar{\xi}_{(0)}^i\frac{\partial}{\partial x^i} + r\bar{\pi}^{(0)}\frac{\partial}{\partial r} + \mathcal{O}(r^2). \tag{68}$$

We substitute these on the left-hand sides of equations (42), (44), (45) and (47), and also expand the right-hand sides in powers of r. Equating powers of r on both sides yields differential equations at each order. The $\mathcal{O}(r^0)$ terms have already been studied in section 5 since these deal with the intrinsic horizon geometry. The $\mathcal{O}(r)$ equations are straightforward in some cases. For example, equation (44b) is trivial because κ, σ, Ψ_0 are all $\mathcal{O}(r^2)$ quantities as we have already seen in equation (64). Similarly, equation (44a) is also easy. Using $\rho = r\Psi_2^{(0)} + \mathcal{O}(r^2)$ and the vanishing of κ, σ to $\mathcal{O}(r)$, we obtain

$$\frac{\partial}{\partial v}\Psi_2^{(0)} = 0, \tag{69}$$

which we already knew. The remaining equations are similarly straightforward, but a lot more tedious. In the end, it turns out that equations (42), (44), (45) and (47) do not yield any new information at $\mathcal{O}(r)$.

7. The expansion of the metric

To expand the metric in powers of r, we start with the frame fields defined in equations (27) and (28), and the radial frame equations (41) derived from the commutation relations. The strategy is the same as for the spin coefficients. Equation (41) give us the first radial derivatives by substituting the horizon values on the right-hand side, and taking higher derivatives leads to the

higher order terms. The calculations are straightforward and lead to the following expansions:

$$\left\{ \begin{aligned} &U = r\tilde{\kappa} + r^2\big(2|\pi^{(0)}|^2 + \mathrm{Re}\big[\Psi_2^{(0)}\big]\big) + \mathcal{O}(r^3), && (70a) \\[2mm] &\Omega = r\bar{\pi}^{(0)} + r^2\Big(\mu^{(0)}\bar{\pi}^{(0)} + \bar{\lambda}^{(0)}\pi^{(0)} + \tfrac{1}{2}\bar{\Psi}_3^{(0)}\Big) + \mathcal{O}(r^3), && (70b) \\[2mm] &X^i = \bar{\Omega}\xi_{(0)}^i + \Omega\bar{\xi}_{(0)}^i + \mathcal{O}(r^3), && (70c) \\[2mm] &\xi^i = [1 + r\mu^{(0)} + r^2((\mu^{(0)})^2 + |\lambda^{(0)}|^2)]\xi_{(0)}^i + [r\bar{\lambda}^{(0)} + r^2(2\mu^{(0)}\bar{\lambda}^{(0)} + \tfrac{1}{2}\bar{\Psi}_4^{(0)})]\bar{\xi}_{(0)}^i + \mathcal{O}(r^3). && (70d) \end{aligned} \right.$$

The contravariant metric is seen to be given in terms of the frame fields as follows:

$$\left\{ \begin{aligned} &g^{rr} = 2(U + |\Omega|^2), && (71a) \\[2mm] &g^{vr} = 1, && (71b) \\[2mm] &g^{ri} = X^i + \bar{\Omega}\xi^i + \Omega\bar{\xi}^i && (71c) \\[2mm] &g^{ij} = \xi^i\bar{\xi}^j + \bar{\xi}^i\xi^j. && (71d) \end{aligned} \right.$$

An explicit calculation yields

$$g^{rr} = 2\tilde{\kappa}r + 2r^2\big(3|\pi^{(0)}|^2 + \mathrm{Re}\big[\Psi_2^{(0)}\big]\big) + \mathcal{O}(r^3). \tag{72}$$

$$\begin{aligned} g^{ir} &= \big[2r\pi^{(0)} + r^2\big(2\mu^{(0)}\pi^{(0)} + 2\lambda^{(0)}\bar{\pi}^{(0)} + \tfrac{1}{2}\Psi_3^{(0)}\big) + \mathcal{O}(r^3)\big]\xi_{(0)}^i \\ &\quad + \big[2r\bar{\pi}^{(0)} + r^2\big(2\mu^{(0)}\bar{\pi}^{(0)} + 2\bar{\lambda}^{(0)}\pi^{(0)} + \tfrac{1}{2}\bar{\Psi}_3^{(0)}\big) + \mathcal{O}(r^3)\big]\bar{\xi}_{(0)}^i. \end{aligned} \tag{73}$$

$$\begin{aligned} g^{ij} &= \big(1 + 2r\mu^{(0)} + 3r^2((\mu^{(0)})^2 + |\lambda^{(0)}|^2)\big)(\xi_{(0)}^i\bar{\xi}_{(0)}^j + \bar{\xi}_{(0)}^i\xi_{(0)}^j) \\ &\quad + \big[r\lambda^{(0)} + r^2\big(3\mu^{(0)}\lambda^{(0)} + \tfrac{1}{2}\Psi_4^{(0)}\big)\big]2\xi_{(0)}^i\xi_{(0)}^j \\ &\quad + \big[r\bar{\lambda}^{(0)} + r^2\big(3\mu^{(0)}\bar{\lambda}^{(0)} + \tfrac{1}{2}\bar{\Psi}_4^{(0)}\big)\big]2\bar{\xi}_{(0)}^i\bar{\xi}_{(0)}^j + \mathcal{O}(r^3). \end{aligned} \tag{74}$$

The null-co-tetrad, i.e. the dual basis for the null-tetrad found above can also be calculated easily up to $\mathcal{O}(r^2)$:

$$\left\{ \begin{aligned} &n = -\mathrm{d}v, && (75) \\[2mm] &\ell = \mathrm{d}r - \big(\tilde{\kappa}r + \mathrm{Re}[\Psi_2^{(0)}]r^2\big)\mathrm{d}v - \big(\pi^{(0)}r + \tfrac{1}{2}\Psi_3^{(0)}r^2\big)\xi_i^{(0)}\mathrm{d}x^i - \big(\bar{\pi}^{(0)}r + \tfrac{1}{2}\bar{\Psi}_3^{(0)}r^2\big)\bar{\xi}_i^{(0)}\mathrm{d}x^i, && (76) \\[2mm] &m = -\big(\bar{\pi}^{(0)}r + \tfrac{1}{2}\Psi_3^{(0)}r^2\big)\mathrm{d}v + (1 - \mu^{(0)}r)\xi_i^{(0)}\mathrm{d}x^i - \big(\bar{\lambda}^{(0)}r + \tfrac{1}{2}\bar{\Psi}_4^{(0)}r^2\big)\bar{\xi}_i^{(0)}\mathrm{d}x^i. && (77) \end{aligned} \right.$$

Here $\xi_i^{(0)}$ are defined by the relations $\xi_i^{(0)}\xi_{(0)}^i = 0$ and $\xi_i^{(0)}\bar{\xi}_{(0)}^i = 1$; it will be convenient to set $m_a^{(0)} := \xi_i^{(0)}\partial_a x^i$. In powers of r, the metric is

$$g_{ab} = -2\ell_{(a}n_{b)} + 2m_{(a}\bar{m}_{b)} = g_{ab}^{(0)} + g_{ab}^{(1)}r + \tfrac{1}{2}g_{ab}^{(2)}r^2 + \cdots, \tag{78}$$

where

$$g_{ab}^{(0)} = 2\partial_{(a}r\partial_{b)}v + 2m_{(a}^{(0)}\bar{m}_{b)}^{(0)}, \tag{79}$$

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$$g_{ab}^{(1)} = -(2\tilde{\kappa}\partial_{(a}v\partial_{b)}v + 4\pi^{(0)}m_{(a}^{(0)}\partial_{b)}v + 4\bar{\pi}^{(0)}\bar{m}_{(a}^{(0)}\partial_{b)}v$$
$$+ 4\mu^{(0)}m_{(a}^{(0)}\bar{m}_{b)}^{(0)} + 2\lambda^{(0)}m_{(a}^{(0)}m_{b)}^{(0)} + 2\bar{\lambda}^{(0)}\bar{m}_{(a}^{(0)}\bar{m}_{b)}^{(0)}), \tag{80}$$

$$g_{ab}^{(2)} = 4(|\pi^{(0)}|^2 - \mathrm{Re}[\Psi_2^{(0)}])\partial_{(a}v\partial_{b)}v + 4(\mu^{(0)}\pi^{(0)} + \lambda^{(0)}\bar{\pi}^{(0)} - \Psi_3^{(0)})m_{(a}^{(0)}\partial_{b)}v$$
$$+ 4(\mu^{(0)}\bar{\pi}^{(0)} + \bar{\lambda}^{(0)}\pi^{(0)} - \bar{\Psi}_3^{(0)})\bar{m}_{(a}^{(0)}\partial_{b)}v$$
$$+ 4((\mu^{(0)})^2 + |\lambda^{(0)}|^2)m_{(a}^{(0)}\bar{m}_{b)}^{(0)}$$
$$+ (4\mu^{(0)}\lambda^{(0)} - 2\Psi_4^{(0)})m_{(a}^{(0)}m_{b)}^{(0)} + (4\mu^{(0)}\bar{\lambda}^{(0)} - 2\bar{\Psi}_4^{(0)})\bar{m}_{(a}^{(0)}\bar{m}_{b)}^{(0)}. \tag{81}$$

We could easily obtain the third-order metric by using equation (65), but we shall not do so here.

8. Inclusion of electromagnetic fields

Let us now go beyond vacuum spacetimes and include electromagnetic fields. As usual, we expand the Maxwell field components ϕ_k in powers of r:

$$\phi_k = \phi_k^{(0)} + r\phi_k^{(1)} + \tfrac{1}{2}r^2\phi_k^{(2)} + \cdots, \quad \text{where} \quad k = 0, 1, 2. \tag{82}$$

Consider first the intrinsic horizon geometry. We start with the behavior of the Maxwell field components at the horizon. The Raychaudhuri equation for the null generators, after imposing the conditions of equations (32) and (34) is just one of the Newman–Penrose field equation (equation 44a) that we have used earlier. In the presence of a Maxwell field, that equation becomes

$$D\rho - \bar{\delta}\kappa = \rho^2 + |\sigma|^2 + (\epsilon + \bar{\epsilon})\rho - 2\alpha\kappa + 2|\phi_0|^2. \tag{83}$$

On the horizon, by definition $\rho = \kappa = 0$. Thus, from the above equation we deduce that we still have $\sigma = 0$ as before, and in addition we also obtain

$$\phi_0^{(0)} = 0. \tag{84}$$

This implies that

$$\Phi_{00}^{(0)} = \Phi_{01}^{(0)} = \Phi_{02}^{(0)} = \Phi_{10}^{(0)} = \Phi_{20}^{(0)} = 0. \tag{85}$$

Furthermore, we also have $\Lambda = 0$ everywhere since the trace of T_{ab} vanishes. Imposing these conditions to determine the time evolution of the geometrical fields on the horizon, the two non-radial Maxwell equations (9a) and (9b) become

$$\begin{cases} D\phi_1^{(0)} = 0, & (86a) \\[2mm] D\phi_2^{(0)} + \tilde{\kappa}\phi_2^{(0)} = \bar{\delta}\phi_1^{(0)} + 2\pi^{(0)}\phi_1^{(0)}. & (86b) \end{cases}$$

Thus, $\phi_1^{(0)}$ is time independent, the charges defined in equation (24) are constant. Since $\phi_0^{(0)}$ vanishes, these charges are in fact independent of which cross-section of Δ is used in the integral. Just like Ψ_3 and Ψ_4 at the horizon, ϕ_2 has a time dependence.

With these simplifications at hand, it turns out that all of equation (48) remain unchanged. The only change in the intrinsic geometry occurs in two of the angular equations (45). The first change is an extra contribution in equation (54):

$$-2\,\mathrm{Re}[\Psi_2^{(0)}] + 4|\phi_1^{(0)}|^2 = \delta a + \bar{\delta}\bar{a} - 2|a|^2 \Rightarrow -4\,\mathrm{Re}[\Psi_2^{(0)}] + 8|\phi_1^{(0)}|^2 = {}^2\tilde{\mathcal{R}}. \tag{87}$$

The second effect is in equation (53) which determines $\Psi_3^{(0)}$:

$$\Psi_3^{(0)} = \bar{\eth}\mu^{(0)} + \pi^{(0)}\mu^{(0)} - \eth\lambda^{(0)} - \bar{\pi}^{(0)}\lambda^{(0)} + 2\phi_2^{(0)}\bar{\phi}_1^{(0)}. \tag{88}$$

The remaining Maxwell equations (9c) and (9d) determine the radial derivatives of ϕ_0 and ϕ_1. As usual, we impose equations (32) and (34) to obtain the radial equations for ϕ_0 and ϕ_1:

$$\begin{cases} \Delta\phi_0 = \delta\phi_1 - \mu\phi_0 + \sigma\phi_2, & (89a) \\[2mm] \Delta\phi_1 = \delta\phi_2 - 2\mu\phi_1 + 2\beta\phi_2. & (89b) \end{cases}$$

Note that there is no equation for the radial derivative of ϕ_2. Thus, just like Ψ_4, we need to specify ϕ_2 on the past light cone $\mathcal{N}_0$ as part of the free data. As for the other radial equations, we can iterate this to obtain an expansion for ϕ_0 and ϕ_1. To first order we need to simply evaluate the right-hand side of these equations at the horizon:

$$\begin{cases} \phi_0^{(1)} = -\delta\phi_1^{(0)}, & (90a) \\[2mm] \phi_1^{(1)} = -\delta\phi_2^{(0)} + 2\mu^{(0)}\phi_1^{(0)} + 2\beta^{(0)}\phi_2^{(0)} = (-\eth + \pi^{(0)})\phi_2^{(0)} + 2\mu^{(0)}\phi_1^{(0)}. & (90b) \end{cases}$$

Let us turn now to the Bianchi identities. In the vacuum case, we had the equations (46) and (47). These equations give here as well the radial and time derivatives of the Weyl tensor components with additional terms for the matter contributions. For the time derivatives at the horizon, we only need to worry about $\Psi_4^{(0)}$ (since $\Psi_3^{(0)}$ has been determined above and the others are time independent):

$$D\Psi_4^{(0)} + 2\tilde{\kappa}\Psi_4^{(0)} = (\bar{\eth} + 5\pi^{(0)})\Psi_3^{(0)} + 2\bar{\phi}_1^{(0)}(\bar{\eth} + \pi^{(0)})\phi_2^{(0)} - 3\lambda^{(0)}\Psi_2^{(0)} - 4\lambda^{(0)}|\phi_1^{(0)}|^2. \quad (91)$$

The radial equations for the Weyl tensor components from the Bianchi identities are not very enlightening, and we shall just write the first derivatives evaluated at the horizon:

$$\begin{cases} \Psi_0^{(1)} = 0, & (92a) \\[2mm] \Psi_1^{(1)} = -\delta\Psi_2^{(0)} - 2\bar{\phi}_1^{(0)}\delta\phi_1^{(0)}, & (92b) \\[2mm] \Psi_2^{(1)} = -(\eth + \bar{\pi}^{(0)})\Psi_3^{(0)} - 2\bar{\phi}_1^{(0)}(\eth + \bar{\pi}^{(0)})\phi_2^{(0)} + 2\bar{\phi}_2^{(0)}\delta\phi_1^{(0)} + 3\mu\Psi_2^{(0)} + 4\mu|\phi_1^{(0)}|^2, & (92c) \\[2mm] \Psi_3^{(1)} = -(\eth + 2\pi^{(0)})\Psi_4^{(0)} + 4\mu^{(0)}\Psi_3^{(0)} + 2\bar{\phi}_1^{(0)}\phi_2^{(1)} + 2\bar{\phi}_2^{(0)}(\eth - 2\pi^{(0)})\phi_2^{(0)} \\[1mm] \qquad\qquad - 4\lambda^{(0)}\phi_1^{(0)}\bar{\phi}_2^{(0)} + 4\pi^{(0)}|\phi_2^{(0)}|^2. & (92d) \end{cases}$$

In the presence of matter fields, there are three (two real and one complex) additional Bianchi identities containing only Ricci tensor components. Let us consider these equations at the horizon after imposing our conditions on the spin coefficients and the vanishing of ϕ_0. The first of these equations yields simply that $\phi_0^{(0)}$ is time independent (which we already knew):

$$D\Phi_{11}^{(0)} = 0 \Rightarrow D\phi_1^{(0)} = 0. \quad (93)$$

The second determines the time evolution of $\Phi_{12}^{(0)}$

$$D\Phi_{12}^{(0)} - \delta\Phi_{11}^{(0)} + \Delta\Phi_{01}^{(0)} = 2\bar{\pi}^{(0)}\Phi_{11}^{(0)} - 2\bar{\epsilon}^{(0)}\Phi_{12}^{(0)}. \quad (94)$$

The third yields the time dependence of Φ_{22}:

$$\begin{aligned} D\Phi_{22}^{(0)} - \delta\Phi_{21}^{(0)} + \Delta\Phi_{11}^{(0)} - \bar{\delta}\Phi_{12}^{(0)} = &-4\mu^{(0)}\Phi_{11}^{(0)} + (2\pi^{(0)} + 2\bar{\beta}^{(0)})\Phi_{12}^{(0)} \\ &+ (2\bar{\pi}^{(0)} + 2\beta^{(0)})\Phi_{21}^{(0)} - 4\epsilon^{(0)}\Phi_{22}^{(0)}. \end{aligned} \quad (95)$$

Using $\Phi_{ij} = 2\phi_i\bar{\phi}_j$, it can be shown that these two equations do not have any new information and can be obtained by combining equations (86) and (89).

In the rest of this section, let us focus on the expansion of the metric up to $\mathcal{O}(r^2)$. First we need the radial derivatives of the spin coefficients as in equation (64). Let us rewrite

equation (64) keeping this time the contributions of ϕ_1 and ϕ_2 at the horizon, and focusing on the spin coefficients which is all that we need for expanding the metric up to second order:

$$
\begin{cases}
\kappa^{(1)} = \sigma^{(1)} = 0, & (96a) \\[2mm]
\rho^{(1)} = \Psi_2^{(0)}, & (96b) \\[2mm]
\epsilon^{(1)} + \bar{\epsilon}^{(1)} = 2|\pi^{(0)}|^2 + 2\,\mathrm{Re}[\Psi_2^{(0)}] + 4|\phi_1^{(0)}|^2, & (96c) \\[2mm]
\epsilon^{(1)} - \bar{\epsilon}^{(1)} = \bar{\pi}^{(0)}(\alpha^{(0)} - \bar{\beta}^{(0)}) - \pi^{(0)}(\bar{\alpha}^{(0)} - \beta^{(0)}) + \Psi_2^{(0)} - \bar{\Psi}_2^{(0)}, & (96d) \\[2mm]
\lambda^{(1)} = 2\mu^{(0)}\lambda^{(0)} + \Psi_4^{(0)}, & (96e) \\[2mm]
\mu^{(1)} = (\mu^{(0)})^2 + |\lambda^{(0)}|^2 + 2\|\phi_2^{(0)}|^2, & (96f) \\[2mm]
\pi^{(1)} = \alpha^{(1)} + \bar{\beta}^{(1)} = \pi^{(0)}\mu^{(0)} + \bar{\pi}^{(0)}\lambda^{(0)} + \Psi_3^{(0)} + 2\phi_2^{(0)}\bar{\phi}_1^{(0)}, & (96g) \\[2mm]
\alpha^{(1)} - \bar{\beta}^{(1)} = \mu^{(0)}(\alpha^{(0)} - \bar{\beta}^{(0)}) + \lambda^{(0)}(\beta^{(0)} - \bar{\alpha}^{(0)}) + \Psi_3^{(0)} + 2\phi_2^{(0)}\bar{\phi}_1^{(0)}. & (96h)
\end{cases}
$$

We now need the radial derivatives for the frame functions given in equation (41) which leads us to the expansion of the frame functions:

$$
\begin{cases}
U = r\tilde{\kappa} + r^2(2|\pi^{(0)}|^2 + \mathrm{Re}[\Psi_2^{(0)}] + 2|\phi_1^{(0)}|^2) + \mathcal{O}(r^3), & (97a) \\[2mm]
\Omega = r\bar{\pi}^{(0)} + r^2\big(\mu^{(0)}\bar{\pi}^{(0)} + \bar{\lambda}^{(0)}\pi^{(0)} + \tfrac{1}{2}\bar{\Psi}_3^{(0)} + \phi_1^{(0)}\bar{\phi}_2^{(0)}\big) + \mathcal{O}(r^3), & (97b) \\[2mm]
X^i = \bar{\Omega}\xi_{(0)}^i + \Omega\bar{\xi}_{(0)}^i + \mathcal{O}(r^3), & (97c) \\[2mm]
\xi^i = \big[1 + r\mu^{(0)} + r^2\big((\mu^{(0)})^2 + |\lambda^{(0)}|^2 + |\phi_2^{(0)}|^2\big)\big]\xi_{(0)}^i & \\[1mm]
\qquad + \big[r\bar{\lambda}^{(0)} + r^2\big(2\mu^{(0)}\bar{\lambda}^{(0)} + \tfrac{1}{2}\bar{\Psi}_4^{(0)}\big)\big]\bar{\xi}_{(0)}^i + \mathcal{O}(r^3). & (97d)
\end{cases}
$$

It is instructive to compare this with equation (70). We see that the electromagnetic field modifies Ψ_2 and Ψ_3 according to

$$
\mathrm{Re}[\Psi_2^{(0)}] \to \mathrm{Re}[\Psi_2^{(0)}] + 2|\phi_1^{(0)}|^2, \qquad \Psi_3^{(0)} \to \Psi_3^{(0)} + 2\phi_1^{(0)}\bar{\phi}_2^{(0)}. \tag{98}
$$

In addition, the $(\mu^{(0)})^2 + |\lambda^{(0)}|^2$ term gets an additive correction by $\left|\phi_2^{(0)}\right|^2$.

We can finally write the expansion of the metric. It is clear that there is no change up to $\mathcal{O}(r)$. Just like the Weyl tensor, the effects of the electromagnetic field first enter the metric at second order. Thus, equations (79) and (80) are unchanged and equation (81) becomes

$$
\begin{aligned}
g_{ab}^{(2)} =\ & 4\big(|\pi^{(0)}|^2 - \mathrm{Re}[\Psi_2^{(0)}] + 2|\phi_1^{(0)}|^2\big)\partial_{(a}v\partial_{b)}v \\
& + 4\big(\mu^{(0)}\pi^{(0)} + \lambda^{(0)}\bar{\pi}^{(0)} - \Psi_3^{(0)} - 2\phi_2^{(0)}\bar{\phi}_1^{(0)}\big)m_{(a}^{(0)}\partial_{b)}v \\
& + 4\big(\mu^{(0)}\bar{\pi}^{(0)} + \bar{\lambda}^{(0)}\pi^{(0)} - \bar{\Psi}_3^{(0)} - 2\bar{\phi}_2^{(0)}\phi_1^{(0)}\big)\bar{m}_{(a}^{(0)}\partial_{b)}v \\
& + 4\big((\mu^{(0)})^2 + |\lambda^{(0)}|^2 + |\phi_2^{(0)}|^2\big)m_{(a}^{(0)}\bar{m}_{b)}^{(0)} \\
& + \big(4\mu^{(0)}\lambda^{(0)} - 2\Psi_4^{(0)}\big)m_{(a}^{(0)}m_{b)}^{(0)} + \big(4\mu^{(0)}\bar{\lambda}^{(0)} - 2\bar{\Psi}_4^{(0)}\big)\bar{m}_{(a}^{(0)}\bar{m}_{b)}^{(0)}. \tag{99}
\end{aligned}
$$

9. Conclusions

In this paper, we have investigated the geometry in the vicinity of a non-extremal weakly isolated horizon using the Newman–Penrose formalism. We allow the horizon to have generic values of mass, spin and charge and higher multipole moments. We have obtained the connection, metric and the Weyl tensor up to $\mathcal{O}(r^2)$. On the horizon, we choose a parameterization of the null generators so that we get a specific value of the surface gravity. In addition, on an initial spherical cross-section S_0 of the horizon, we specify the spin coefficients π, $\alpha - \bar{\beta}$, μ and λ. The spin coefficients π and $\alpha - \bar{\beta}$ ought to be time independent and they specify respectively the Im[Ψ_2] and the scalar curvature of S_0, and thus the spin and mass multipole moments of the horizon respectively. The transversal expansion and shear μ, λ turn out to be time dependent and they determine Ψ_3. In addition, we need to specify Ψ_4 on the past-outgoing light cone $\mathcal{N}_0$ originating from S_0. If we consider charged horizons, then we must specify in addition the Maxwell field component ϕ_1 on S_0 (this determines the electric and magnetic charges of the horizon) and ϕ_2 on $\mathcal{N}_0$.

We emphasize once again that this is a local calculation and not all solutions produced by this construction are physically meaningful; the interesting issue of global existence of solutions has not been addressed here. For example, the uniqueness theorems would imply that not all values of the horizon multipole moments would lead to asymptotically flat solutions. However, the set of solution we have obtained encompasses all physically interesting situations (in so far as the assumption of the horizon being isolated is valid). If necessary, it is straightforward to compute the metric beyond $\mathcal{O}(r^2)$ as we have done here. For example, to obtain the metric at $\mathcal{O}(r^n)$ in vacuum, equations (41) imply that we need the spin coefficients up to $\mathcal{O}(r^{n-1})$ which in turn, from equations (43), require the Weyl tensor components up to $\mathcal{O}(r^{n-2})$ (including in particular Ψ_4, whose radial derivatives can be freely specified).

There are a number of possible extensions and applications of this work. The first is the study of tidally deformed black holes which has, among other things, implications for the equation of motion and self-force in general relativity [39]. This requires us to consider Kerr black holes perturbed by a background curvature, and to interpret the resulting metric in terms of a moving point particle. This will be studied in a forthcoming paper. The near-horizon geometry computed here could be matched to an appropriate far-zone metric to obtain initial data for numerical simulations of binary black hole systems (see e.g. [40]) and finally, future gravitational wave observatories might be able to measure the tidal deformations through observations of signals from extreme mass ratio systems. Knowledge of the near-horizon geometry including deviations from the Kerr multipoles values (which are allowed here) will be useful in the searches for these signals and in testing the Kerr nature of the black hole. Finally beyond the specific calculation of the near-horizon metric, the characteristic initial value formulation based on the horizon might be useful for wave extraction in numerical relativity. If S_0 is taken to be a cross-section of the horizon at very late times, then the light cone $\mathcal{N}_0$ can be viewed as an approximate null infinity. The usefulness of this construction remains to be explored.

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