Gauge Symmetries in Spin-Foam Gravity: The Case for "Cellular Quantization"

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The spin-foam approach to quantum gravity rests on a quantization of *BF* theory using 2-complexes and group representations. We explain why, in dimension three and higher, this spin-foam quantization must be amended to be made consistent with the gauge symmetries of discrete *BF* theory. We discuss a suitable generalization, called "cellular quantization," which (1) is finite, (2) produces a topological invariant, (3) matches with the properties of the continuum *BF* theory, and (4) corresponds to its loop quantization. These results significantly clarify the foundations—and limitations—of the spin-foam formalism and open the path to understanding, in a discrete setting, the symmetry-breaking which reduces *BF* theory to gravity.

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Introduction.—Since it was first advocated by Baez [1], Reisenberger [2], and Rovelli [3], the spin-foam approach to quantum gravity has attracted considerable interest, resulting in more than a hundred papers published on the topic every year. In a nutshell, the idea underlying this activity is that a "spacetime-covariant, Feynman-style" sum-over-histories formulation of background-independent field theories exists in the form of a weighted sum over two-dimensional cell complexes [3]. This approach is believed to provide a successful quantization of the topological *BF* theory [4,5], in the form of the Ponzano-Regge [6] and Ooguri [7] models (in three and four dimensions, respectively), and work is underway to adapt it to general relativity, understood as "*BF* theory with extra constraints" [8]. Standard reviews of the spin-foam formalism are in Refs. [5,9]; the state of the art review is presented in [10].

In spite of strong efforts and promising results [11–18] (and more references in [17]), several outstanding problems with the Ponzano-Regge and Ooguri (PRO) models have remained open so far. We may list them as follows. (1) *Bubble divergences*: the original PRO partition functions are, in general, divergent. How should one regularize them? (2) *Topological invariance*: the PRO partition functions are formally invariant under changes of triangulations, up to divergent factors. How can one turn them into finite topological invariants? (3) *Relationship to the canonical theory*: the connection between the Ponzano-Regge model and loop quantum gravity in three dimensions was established in [13]. Can this connection be extended to four dimensions and higher? (4) *Relationship to the continuum theory*: *BF* theory was quantized in the continuum in [19,20] and was shown to be related to the Ray-Singer torsion. Are the PRO models similarly related to torsion? (See [14] for a positive answer in certain three-dimensional cases.) (5) *Diffeomorphism symmetry*: both the continuum *BF* action and the Einstein-Hilbert action are diffeomorphism-invariant. What is the fate of this symmetry in the PRO models?

Mostly thanks to the work of Freidel *et al.* [11,12], it has become clear that all five problems are related to the issue of identifying the *BF* shift symmetry in a discrete setting and gauge-fixing it. No complete solution to this issue, however, has been proposed in the literature. The purpose of this Letter is to argue that there is a good reason for this: when dealing with two-complexes only, as in the spin-foam formalism, there is no shift symmetry. To identify this symmetry, one must instead resort to an extension of the spin-foam formalism including higher-dimensional cells. This realization paves the way to what we call cellular quantization. This cellular quantization solves problems 1 through 4 and sheds interesting new light on problem 5.

This Letter is organized as follows. We start by reviewing the basic properties of the continuum *BF* theory, emphasizing its gauge symmetries and relationship to analytic torsion. We then describe the spin-foam quantization of *BF* theory, as described, e.g., in Baez's reference paper [5]. We show how to identify the gauge symmetries in a discrete setting and perform a quantization which does preserve the topological features of the continuum theory. Finally, we establish that this cellular quantization corresponds to the loop canonical quantization.

Continuum BF theory.—*BF* theory was introduced by Horowitz [21] and Blau and Thompson [19] as an exactly soluble diffeomorphism-invariant theory, illustrating the connection between quantum gauge systems and manifold topology previously discovered by Schwarz [22]. Defined in terms of a gauge field (or gauge connection), A and a $\mathfrak{g}$-valued $(d-2)$-form B on spacetime M, where $d = \dim M$ and $\mathfrak{g}$ is the Lie algebra of the gauge group G, its classical action reads

$$S_{BF}(B, A) = \int_M \langle B \wedge F(A)\rangle. \qquad (1)$$

Here $F(A)$ is the field strength of A, and the bracket denotes a nondegenerate symmetric bilinear form in $\mathfrak{g}$, typically the Killing form when $\mathfrak{g}$ is semisimple. The corresponding field equations are $F(A) = 0$ and $D_A^{k-2}B = 0$, with D_A^k the covariant exterior derivative associated to A acting on $\mathfrak{g}$-valued k-forms. (Note that D_A^k is not the k-fold of composition of the covariant exterior derivative with itself.)

In addition to the usual gauge symmetry of a gauge field, the action (1) is invariant under the shift symmetry,

$$B \mapsto B + D_A^{d-3}\lambda_{d-3}, \tag{2}$$

where $\lambda_{d-3} \in \Omega^{d-3}(M, \mathfrak{g})$ is any $\mathfrak{g}$-valued $(d-3)$-form. When $d \geq 4$, this gauge symmetry is on-shell reducible: given a flat connection ϕ, i.e., one such that $F(\phi) = 0$, the map $\lambda_{d-3} \mapsto D_\phi^{d-3}\lambda_{d-3}$ is many-to-one. This is to say that the gauge modes λ_{d-3} (the "ghosts") themselves possess a gauge symmetry, namely

$$\lambda_{d-3} \mapsto \lambda_{d-3} + D_\phi^{d-4}\lambda_{d-4}, \tag{3}$$

with $\lambda_{d-4} \in \Omega^{d-4}(M, \mathfrak{g})$ representing "ghosts for ghosts." In turn, these new variables themselves may have a shift symmetry, and so on.

This structure naturally fits in the so-called twisted de Rham complex,

$$0 \to \Omega^0(M, \mathfrak{g}) \xrightarrow{D_\phi^0} \cdots \xrightarrow{D_\phi^{d-1}} \Omega^d(M, \mathfrak{g}) \to 0. \tag{4}$$

In this cochain complex, the coboundary maps are the covariant exterior derivative D_ϕ^k and the k-cochains are elements of $\Omega^k(M, \mathfrak{g})$, viz. $(d-2-k)$-stage ghosts. Also, note that, for a given flat connection ϕ, the space of solutions of the field equation for B is the cohomology space H_ϕ^{d-2} derived from (4).

The path-integral quantization of BF theory requires the gauge-fixing of this shift symmetry. This can be achieved by means of the resolvent method, a generalization of the Faddeev-Popov trick to reducible gauge symmetries devised by Schwarz [22]. Starting from the formal, pre-gauge-fixing expression,

$$Z_{BF} = \int DA \int DB e^{iS_{BF}(B,A)}, \tag{5}$$

the resolvent method consists in extracting the "volume" of the space $\mathrm{Im} D_A^{d-3}$ arising in (2) by means of the complex (4). This method will be outlined below, when we apply it to gauge-fix the discrete counterpart of BF theory. For now, let us simply state the result of this procedure in the continuum: the gauge-fixed partition function Z'_{BF} can be written as a sum over the moduli space $\mathcal{M}$ of flat connections on P, with a summand given by the analytic torsion of the complex (4), viz.

$$\mathrm{Tor}_{[\phi]} = \prod_{j=0}^{d-1} \det((D_\phi^j)^\dagger D_\phi^j)^{(-1)^j/2}. \tag{6}$$

Here $[\phi]$ denotes the gauge equivalence class of a flat connection ϕ, and the dagger denotes the adjoint with respect to arbitrary inner products in the spaces $\Omega^k(M, \mathfrak{g})$; the Ray-Singer torsion is independent of these inner products. In particular, Z'_{BF} is a topological invariant of M [19]. (Strictly speaking, these results hold when the twisted de Rham complex is acyclic, i.e., has vanishing cohomology, for all flat connections ϕ, implying that the moduli space of flat connections is discrete. An extension to the general case is discussed in [19]; see also [23].)

The torsion also provides the measure for transition amplitudes and for the inner product between boundary wave functions. Assume that M has two disconnected boundaries N_1, N_2. Wave functions are square-integrable functions over the moduli space of flat gauge fields on $N_{1,2}$. The transition from an initial state Ψ_1 to a final state Ψ_2 through M reads

$$\langle \Psi_2 | Z'_{BF} | \Psi_1 \rangle = \sum_{[\phi] \in \mathcal{M}} \Psi_2^*([\phi]) \mathrm{Tor}_{[\phi]} \Psi_1([\phi]). \tag{7}$$

From our perspective, the moral of this review is that, if classical BF theory can be thought of as a theory of connections and $(d-2)$-forms, quantum BF theory, on the other hand, involves the entire twisted de Rham complex (4), with forms of all degrees.

Spin-foam quantization.—Let us now describe the spin-foam quantization of BF theory, as presented, e.g., in [5]. Assume that G is compact and that M is equipped with a triangulation Δ and its dual cell complex K. Define a discrete connection on Δ as an assignment of an element g_e of the gauge group G to each edge (1-cell) e of K. Then for each face (2-cell) f of K, define the holonomy H_f along f as the ordered product of g_e attached to the edges on the boundary of f. The set of group elements $H = (H_f)_f$ is the discrete analogue of the field strength $F(A)$ in the continuum.

Now, consider again the formal expression (5) and "integrate over the B-field." This gives

$$Z_{BF} = \int DA \delta(F(A)), \tag{8}$$

with a functional delta function implementing the flatness of the connection. Thanks to the discretization on K, the formal measure DA can be defined by means of the Haar measure dg on G, and we can set

$$Z_{BF} = \int_{G^E} \prod_e dg_e \prod_f \delta(H_f). \tag{9}$$

Now, let us expand the Dirac delta on G in characters,

$$\delta(g) = \sum_j \dim(j) \mathrm{tr} D^j(g), \tag{10}$$

where j ranges over the equivalence classes of unitary irreps $D^j(g)$ of G, and recall the identity

$$\int_G dg \bigotimes_{l=1}^{d} D^{j_l}(g) = \sum_{\iota} |\iota\rangle\langle\iota| \quad (11)$$

for the projector on the G-invariant subspace of the tensor representation $\bigotimes j_l$, of which the "intertwiners" ι spans an orthonormal basis. Plugging (10) and (11) into (9) then gives, after some easy algebra,

$$Z_{BF} = \sum_{(j_f)} \prod_f \dim(j_f) \prod_v \{N_d j\}. \quad (12)$$

Here, $N_d = 3(d+1)(d-2)/2$, and $\{N_d j\}$ is the Wigner $N_d j$ symbol. This expression defines the Ponzano-Regge ($d = 3$) and Ooguri ($d = 4$) models.

Unfortunately, (12) is known to be ill defined in general: when the sum over representations in (12) is truncated to a finite value Λ, the sum diverges as $\Lambda \to \infty$. This phenomenon has been coined "bubble divergences" [24] and was interpreted as an "infrared effect" [5,24]. The connection between these divergences and the BF shift symmetry was understood in three dimensions by Freidel and Louapre [11], and a "gauge-fixing" scheme consisting of removing certain faces of K was proposed [11,12]. For nontrivial topologies, however, this scheme turned out to fail turning (12) into a finite number [25].

It should be clear from the above discussion that the spin-foam scheme, which only relies on the 2-skeleton of K, does not implement any gauge-fixing of the discrete shift symmetry; it simply amounts to a rewriting of the unfixed partition function (5). This is consistent when $d = 2$, in which case BF theory is nothing but the zero-coupling limit of Yang-Mills theory, but it is inconsistent when $d \geq 3$, as the gauge redundancy then makes the expression (12) ill defined. It is these divergences that prevent (12) defining a bona fide topological invariant and cramp any connection with Ray-Singer torsion.

Cellular quantization.—Suppose now that $d \geq 3$. Let $\mathcal{A} = G^E$ denote the space of discrete connections on K, and $\mathcal{F}$ the subspace of flat discrete connections, namely those for which $H_f = 1$ for all faces f. In the neighborhood of $\mathcal{F}$, a discrete connection A can be seen as an element $(\phi, a_\phi) \in \mathcal{F} \times N_\phi \mathcal{F}$ of the normal bundle to $\mathcal{F}$, according to $A = \exp_\phi(a_\phi)$. (We disregard the possibility that $\mathcal{F}$ may contain singularities; see [26] for a discussion of this issue.) Here, $N_\phi \mathcal{F}$ is the set of tangent vectors at ϕ orthogonal to $\mathcal{F}$, and $\exp_\phi : T_\phi \mathcal{A} \to \mathcal{A}$ denotes the Riemannian exponential map. Furthermore, the holonomy can be expanded as

$$H_f = (dH_f)_\phi(a_\phi) + \mathcal{O}(a_\phi^2), \quad (13)$$

and the Haar measure dA on $\mathcal{A}$ splits as

$$dA = d\phi da_\phi, \quad (14)$$

where $d\phi$ is the induced Riemannian measure on $\mathcal{F}$ and da_ϕ is the Lebesgue measure on $N_\phi \mathcal{F}$. Finally, we have

$$\delta(H_f) = \int_{\mathfrak{g}} db_f e^{i\langle b_f, (dH_f)_\phi(a_\phi)\rangle}. \quad (15)$$

Hence (9) can be rewritten as $\int_{\mathcal{F}} d\phi z_\phi$, where z_ϕ has the BF-like form

$$z_\phi = \int_{N_\phi \mathcal{F}} da_\phi \int_{\mathfrak{g}^F} db e^{is(b, a_\phi)}, \quad (16)$$

where $b = (b_f)_f$ and

$$s(b, a_\phi) = \sum_f \langle b_f, (dH_f)_\phi(a_\phi)\rangle. \quad (17)$$

To proceed with the quantization of discrete BF theory, we must now identify the gauge symmetries of (16). To this effect, consider the discrete twisted de Rham complex,

$$0 \to \mathfrak{g}^{c_0} \xrightarrow{\delta_\phi^0} \ldots \xrightarrow{\delta_\phi^{d-1}} \mathfrak{g}^{c_d} \to 0, \quad (18)$$

where c_k is the number of k-cells of K. The cochain space $\mathfrak{g}^{c_k}$ is the discrete analogue of $\Omega^k(M, \mathfrak{g})$, and δ_ϕ^k is the discrete covariant exterior derivative defined in [14,25], satisfying $\delta_\phi^{k+1} \circ \delta_\phi^k = 0$. In particular, if μ is the Maurer-Cartan form on G and $a = \mu(a_\phi)$, then $\delta_\phi^1(a) = dH_\phi(a_\phi)$. Using the bracket in $\mathfrak{g}$, we can also consider the adjoint maps $\partial_k^\phi = (\delta_\phi^{k-1})^\dagger$, defining the dual complex to (18), namely

$$0 \to \mathfrak{g}^{c_d} \xrightarrow{\partial_d^\phi} \ldots \xrightarrow{\partial_1^\phi} \mathfrak{g}^{c_0} \to 0. \quad (19)$$

Thanks to this cohomological structure, it is easy to identify the gauge symmetries of (17): it is simply

$$b \mapsto b + \partial_3^\phi(X_3), \quad (20)$$

with $X_3 \in \mathfrak{g}^{c_3}$. Indeed, we have

$$\langle \partial_3^\phi(X_3), dH_\phi(a_\phi)\rangle = \langle X_3, \delta_\phi^2 \circ \delta_\phi^1(a)\rangle = 0. \quad (21)$$

This is nothing but the discrete counterpart of the shift symmetry (2). When $d \geq 4$, this symmetry is reducible, as $\mathrm{Im}\partial_4^\phi \subset Ker\partial_3^\phi \Omega$, etc. That is, just as in the continuum, the reducible symmetries of the action (17) involves all the chain groups in (19), hence cells of all dimensions.

Let us now use the resolvent method to gauge-fix the discrete shift symmetry. Assume that (18) and (19) are acyclic, so that $\mathrm{Im}\partial_3^\phi$ exhausts the kernel of (17). (In the case where the complex (18) is not acyclic, and in particular when the moduli space of flat connections is not discrete, this method can be amended along the lines of [23]. This yields a similar result, except for a few more determinants.) Then the goal is to restrict the integral over b in (16) to an integral over $\mathrm{Im}(\partial_3^\phi)^\perp$. Write formally

$$\int_{\mathfrak{g}^F} db e^{is(b,a_\phi)} = \mathrm{Vol}(\mathrm{Im}\partial_3^\phi) \int_{(\mathrm{Im}\partial_3^\phi)^\perp} db_\perp e^{is(b,a_\phi)} \quad (22)$$

and observe that, since ∂_3^ϕ provides an isomorphism between $\mathfrak{g}^{c_3}/\mathrm{Ker}\partial_3^\phi = \mathfrak{g}^{c_3}/\mathrm{Im}\partial_4^\phi$ and $\mathrm{Im}\partial_3^\phi$, and moreover $\delta_\phi^2 = (\partial_3^\phi)^\dagger$, we can write

$$\mathrm{Vol}\,(\mathrm{Im}\partial_3^\phi) = \det(\delta_\phi^2 \partial_3^\phi)^{1/2} \frac{\mathrm{Vol}(\mathfrak{g}^{c_3})}{\mathrm{Vol}(\mathrm{Im}\partial_4^\phi)}. \quad (23)$$

Iterating this recursive relation, we get

$$\mathrm{Vol}\,(\mathrm{Im}\partial_3^\phi) = \prod_{j=2}^{d-1} \det(\delta_\phi^j \partial_{j+1}^\phi)^{(-1)^j/2} \mathrm{Vol}(\mathfrak{g}^{c_j})^{(-1)^j}. \quad (24)$$

Now, let us pretend that the chain spaces $\mathfrak{g}^{c_j}$ have unit volume: this is the meaning of the expression "dividing by an infinite volume," underlying the gauge-fixing procedure. (Precisely the same step is taken in the continuum quantization of BF theory.) Then we can replace (22) by the finite quantity

$$\prod_{j=2}^{d-1} \det(\delta_\phi^j \partial_{j+1}^\phi)^{(-1)^j/2} \int_{(\mathrm{Ker}\partial_3^\phi)^\perp} db_\perp e^{is(b_\perp, a_\phi)}. \quad (25)$$

Hence, returning to (16) and performing the integral over $b_\perp$, we get as the definition of gauge-fixed version of z_ϕ

$$z'_\phi = \prod_{j=2}^{d-1} \det(\delta_\phi^j \partial_{j+1}^\phi)^{(-1)^j/2} \int_{N_\phi \mathcal{F}} da_\phi \delta(dH_\phi(a_\phi)). \quad (26)$$

The remaining integral over a_ϕ is now well defined and gives $\det(\delta_\phi^1 \partial_2^\phi)^{-1/2}$. Hence, we obtain for the gauge-fixed partition function $Z'_{BF} = \int_{\mathcal{F}} d\phi z'_\phi$:

$$Z'_{BF} = \int_{\mathcal{F}} d\phi \prod_{j=1}^{d-1} \det(\delta_\phi^j \partial_{j+1}^\phi)^{(-1)^j/2}. \quad (27)$$

The integral over $\mathcal{F}$ can be pulled back to to the moduli space of flat discrete connections $\mathcal{M} = \mathcal{F}/G^{c_0}$ by integrating along the gauge orbits of each flat connection [14,26]. This yields one more determinant $\det(\delta_\phi^0 \partial_1^\phi)^{1/2}$, and thus

$$Z'_{BF} = \sum_{[\phi]\in\mathcal{M}} \mathrm{Tor}_{[\phi]} \quad (28)$$

with

$$\mathrm{Tor}_{[\phi]} = \prod_{j=0}^{d-1} \det(\partial_{j+1}^\phi \delta_\phi^j)^{(-1)^j/2}. \quad (29)$$

The expression (28) is a topological invariant of K. In particular, the quantity $\mathrm{Tor}_{[\phi]}(K, G)$ is the twisted Reidemeister torsion, which is known to coincide with the twisted analytic torsion. Thus, (28) matches with the continuum result, consistently with the general expectation that, for a topological quantum field theory with finitely many degrees of freedom, discretization should play no physical role.

Relation to the loop formalism.—The above method naturally gives rise to the loop quantization of BF theory. In the loop approach, one quantizes before restricting to flat gauge fields. Given an embedded, closed graph γ, cylindrical wave functions are functions of the Wilson lines along the lines of γ. For each graph, there is a Hilbert space whose measure is given by the Haar measure of G on each line, $\prod_e dg_e$. The Hilbert spaces of two different graphs are orthogonal. The standard gauge symmetry requires invariance under G-translation on the source and end nodes of the lines.

Heuristically, the transition amplitudes in the continuum (7) suggest that they can be formulated in the loop approach by taking as boundary states cylindrical functions restricted to the moduli space $\mathcal{M}$, the torsion still providing the measure. Assume M has two disconnected boundaries N_1, N_2, with two closed, embedded graphs γ_1, γ_2 associated with two cylindrical functions Ψ_{γ_1}, Ψ_{γ_2}. The transition is regularized by choosing a cell decomposition K of M such that γ_1, γ_2 are included into the 1-skeleton. The ungauge-fixed transition amplitude reads

$$\langle \Psi_{\gamma_2} | Z_{BF} | \Psi_{\gamma_1} \rangle = \int \prod_e dg_e \Psi^*_{\gamma_2}(g_e) \Psi_{\gamma_1}(g_e) \prod_f \delta(H_f). \quad (30)$$

As the shift symmetry does not act on Wilson lines, the process of the previous section applies. The wavefunctions are evaluated on $\mathcal{M}$ because there are no fluctuations around flat connections, yielding

$$\langle \Psi_{\gamma_2} | Z'_{BF} | \Psi_{\gamma_1} \rangle = \sum_{[\phi]\in\mathcal{M}} \Psi^*_{\gamma_2}([\phi]) \mathrm{Tor}_{[\phi]} \Psi_{\gamma_1}([\phi]). \quad (31)$$

Finally, the regulator K can be removed thanks to the topological invariance of the torsion, which makes the continuum limit result into the above formula. Let us mention an outcome of this result: the loop quantization of the BF model does not distinguish knottings of the graphs $\gamma_{1,2}$.

Conclusion.—We have performed a topological quantization of discrete BF theory, proving its equivalence to the usual quantization in the continuum. This result solves several open problems of the field and extends previous results obtained in dimension 3 to arbitrary dimensions: (1) transition amplitudes are finite, answering the issue of bubble divergences [11,26]; (2) the gauge symmetries in the discrete setting exist, generalizing [11,12]; (3) they can be gauge-fixed to derive the loop quantization, generalizing [13]; and (4) as a result, one gets a topological invariant, which proves that the classical gauge symmetries are correctly promoted to the quantum level.

The crucial steps of our quantization require us to take into account cells of all dimensions in the cell complex and not just its 2-skeleton as in the spin-foam quantization. A challenge for future investigations is to find a representation of (31) as a state-sum, as is done in the latter approach. (This is of direct relevance for nontrivial topologies. But in the spin-foam literature, one is mainly interested in the local degrees of freedom and not topological ones, so it is usually assumed that one can work safely on spheres for which there are no difficulties in gauge-fixing.)

The last issue we mentioned in the introduction is the major difficulty in quantum gravity: understanding the quantum version of diffeomorphism-invariance. It is well known that diffeomorphism-invariance in the *BF* model is contained within its shift symmetry [21]. Hence, the substance of general relativity is to break the topological invariance while preserving diffeomorphism-invariance. Spin-foam models for quantum gravity are very much in line with this idea, as they start by quantizing *BF* theory and then introduce some breaking of the shift symmetry to restore the local degrees of freedom. It is also known that discrete models of gravity generically break diffeomorphism-invariance [17]. Showing that it is restored in the continuum limit (after some coarse-graining or summing over spin-foams appropriately) is one of the main programs in the spin-foam approach. Now that the shift symmetry is correctly controlled in the discrete setting, we feel that the noose is tightening around diffeomorphisms.

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