

DeWitt equation in quantum field theoryParikshit Dutta,¹ Krzysztof A. Meissner,² and Hermann Nicolai¹¹*Max-Planck-Institut für Gravitationsphysik (Albert-Einstein-Institut), Mühlenberg 1, D-14476 Potsdam, Germany*²*Institute of Theoretical Physics, Faculty of Physics, University of Warsaw, Hoża 69, 00-681 Warsaw, Poland*

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We take a new look at the DeWitt equation, a defining equation for the effective action functional in quantum field theory. We present a formal solution to this equation and discuss the equation in various contexts, and in particular for models where it can be made completely well defined, such as the Wess-Zumino model in two dimensions.

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I. INTRODUCTION

In 1965, B. DeWitt wrote down a functional differential equation for the full effective action in quantum field theory [1]. To the best of our knowledge, this result has not received much attention in the existing literature (but see Ref. [2] for a recent exception) nor in quantum field theory textbooks. The equation in question, which in the remainder of this paper we will refer to as the “DeWitt equation”,¹ relates the functional derivative of the full quantum effective action $\Gamma[\varphi]$ to the functional derivative of the classical action and has several remarkable features. First of all, while the usual approach to quantum field theory is based on path integrals and perturbation theory, and thus involves (functional) *integration* (see, e.g., Ref. [3]), the essential information about the quantum field theory is here encoded into a (functional) *differential equation*. If the classical action is polynomial, this equation has only very few terms and therefore assumes a relatively simple form. Second, this equation can serve as the generating equation for an infinite hierarchy of Schwinger-Dyson equations for the theory in question.

The main difficulty, and possibly the reason why this equation has not been much exploited in the past, is that it is even hard to *define* properly. Of course, this is also true of the path integral, but there one has a number of established approximation methods at one’s disposal (such as renormalized perturbation theory), whereas apparently no techniques exist as yet for dealing with a functional differential equation that should contain the complete information about the full renormalized action functional. Among other difficulties, one has to deal with short distance singularities in the equation related to the occurrence of functional derivatives at coincident points that would have to be resolved “in one stroke,” rather than by perturbative methods of the conventional type. Consequently, the proper definition of the equation already requires some knowledge of the properties of the solution. One possible approach here would be to look for formal solutions in a perturbative

expansion of the unrenormalized equation and then renormalize the resulting expression in a second step [2]. A second difficulty is that the equation is not of any known type, even in a discrete approximation with only finitely many degrees of freedom (which we consider in Sec. III).

In this paper we take a new look at the DeWitt equation and will argue that, in spite of the difficulties mentioned above, the equation may provide valuable new insights into quantum field theory, beyond the established results and techniques used so far. Our main motivation here is to be able eventually to develop new methods for future applications, in order to deal with the effective (Coleman-Weinberg) potential [4] in classically conformal versions of the Standard Model of the type considered in Ref. [5], possessing more than 1 scalar degree of freedom. As argued there (see also Refs. [6,7]), classically unbroken conformal symmetry may offer an attractive alternative to low-energy supersymmetry in explaining the stability of the electroweak scale. The main technical problem with this proposal is that, so far, there appear to be no efficient methods to compute the effective potential with more than one physical scalar field beyond one loop. However, such methods are absolutely required in order to reliably assess the existence and stability of nontrivial stationary points of the effective potential because the extremal structure of the potential may delicately depend on higher-order corrections.

In fact, as we will show, there exists a formal solution to the DeWitt equation, which represents the effective action functional $\Gamma[\varphi]$ as an asymptotic series expansion over vacuum diagrams with field-dependent Green’s functions; this result follows from much older results on the effective potential obtained by R. Jackiw [8] (see also Ref. [9]). One interesting new aspect here is that this analysis leads us to consider the question of convergence of such expansions not only in terms of the coupling constants (or running coupling constants), but rather as a question of convergence *in field space*: the value of the classical field φ effectively replaces the renormalization scale of the usual perturbation expansion. In this case, Landau poles and other singularities would manifest themselves as singularities of the effective action in field space, while the

¹This is not to be confused with the more famous Wheeler-DeWitt equation.

couplings are kept fixed and do not run.² As we will show, also in terms of explicit numerical examples (see the appendix), the new expansion may have much better convergence properties even for large coupling constants λ , if the value of the classical field φ is different from zero.

Another new direction opened by this work concerns the formulation of the DeWitt equation in contexts where it can be made completely well defined. Our prime example here is the Wess-Zumino model in two space-time dimensions, where we can exploit the cancellation of UV singularities in a supersymmetric model. As a further application, we derive the DeWitt equation for Liouville theory in two dimensions, as an example of a theory with nonpolynomial action. In this way, we are led to a novel relation between n -point correlators and $(n + 1)$ -point correlators of exponential Liouville operators, a (formal) result that does not rely on conformal symmetry and remains to be exploited in future work. A most interesting future application of the present work would be the formulation and analysis of the DeWitt equation for $N = 4$ super—Yang-Mills theory, the main example of a UV finite interacting quantum field theory in four space-time dimensions [11,12].

II. DERIVATION OF DEWITT EQUATION

For the reader's convenience, we here reproduce the formal derivation of the equation found by B. DeWitt, following Ref. [1] (see also Ref. [2]), restricting attention to the scalar field theory for simplicity, as the extension to more general theories (with fermions and gauge fields) is straightforward, at least in principle. Working with a Euclidean metric for simplicity, we define the generating functional of connected Green's functions $W[J]$ in the standard way via (see, e.g., Refs. [13,14])

$$\begin{aligned} Z[J] &\equiv \exp\left[-\frac{1}{\hbar}W[J]\right] \\ &= \int \mathcal{D}\phi \exp\left[-\frac{1}{\hbar}(S[\phi] + J \cdot \phi)\right], \end{aligned} \quad (1)$$

where $J \cdot \phi \equiv \int dx J(x)\phi(x)$ and the measure $\mathcal{D}\phi$ is formally normalized to unity, that is $W[0] = 0$. The connected Green's functions in the presence of a source J are then given by

²A standard example for this phenomenon is the renormalization-group improved effective potential for ϕ^4 theory, which takes the form

$$V_{\text{eff}}(\varphi) = \frac{\lambda\varphi^4}{1 - a\lambda \log(\varphi^2/v^2)},$$

where a is a positive constant and v a fixed scale [4]. This expression is thus valid only in a compact region in φ space. See also Ref. [10] for further examples and a discussion of this issue in the case with one scalar field but any number of nonscalar fields.

$$W_n(x_1, \dots, x_n; J) \equiv (-\hbar)^{n-1} \frac{\delta^n W[J]}{\delta J(x_1) \dots \delta J(x_n)}, \quad (2)$$

with the full connected n -point functions

$$W_n(x_1, \dots, x_n) \equiv W_n(x_1, \dots, x_n; J)|_{J=0}. \quad (3)$$

Defining the classical field $\varphi(x)$ by

$$\varphi(x) \equiv \varphi(x; J) = \frac{\delta W[J]}{\delta J(x)}, \quad (4)$$

the effective action is the Legendre transform

$$\Gamma[\varphi] = W[J] - \int d^4x J(x)\varphi(x) \quad (5)$$

such that

$$\frac{\delta \Gamma[\varphi]}{\delta \varphi(x)} = -J(x; \varphi). \quad (6)$$

We will assume in the following that the relation between $J = J(x; \varphi)$ and $\varphi(x; J)$ can be freely inverted (although we are aware that this may not be true in many cases of physical interest). As is well known, $\Gamma[\varphi]$ is the generating functional for the one-particle irreducible (\equiv 1PI) Green's functions, with

$$\Gamma_n(x_1, \dots, x_n; \varphi) \equiv \frac{(-1)^n}{\hbar} \frac{\delta^n \Gamma[\varphi]}{\delta \varphi(x_1) \dots \delta \varphi(x_n)}, \quad (7)$$

where the normalization is chosen such that we have the standard relations

$$\int d^4y W_2(x, y; \varphi) \Gamma_2(y, z; \varphi) = \delta^{(4)}(x - z) \quad (8)$$

and

$$\begin{aligned} W_3(x, y, z; \varphi) &= \int d^4u d^4v d^4w W_2(x, u; \varphi) W_2(y, v; \varphi) \\ &\quad \times W_2(z, w; \varphi) \Gamma_3(u, v, w; \varphi) \end{aligned} \quad (9)$$

and so on. Note that here all the Green's functions depend on the classical field $\varphi(x)$. We also recall the expansion of the effective action in powers of \hbar ("loop expansion"),

$$\Gamma[\varphi] = \Gamma^{(0)}[\varphi] + \hbar \Gamma^{(1)}[\varphi] + \dots, \quad (10)$$

where $\Gamma^{(0)}[\varphi] = S[\varphi]$ is the classical action S .

For any functional $Q[\phi]$, we define the expectation value with given source $J(x)$ as

$$\begin{aligned} \langle Q[\phi] \rangle_J &:= \exp\left(\frac{1}{\hbar}W[J]\right) \int \mathcal{D}\phi Q[\phi] \\ &\quad \times \exp\left[-\frac{1}{\hbar}(S[\phi] + J \cdot \phi)\right]. \end{aligned} \quad (11)$$

This can be rewritten as³

³By use of the elementary identity $f(x) = \exp(x\partial/\partial y)f(y)|_{y=0}$.

$$\langle Q[\phi] \rangle_J = \exp\left(\frac{1}{\hbar}W[J]\right) \exp\left(-\frac{1}{\hbar}W\left[J - \hbar \frac{\delta}{\delta\phi}\right]\right) Q[\phi]|_{\phi=0}. \quad (12)$$

Next we expand

$$\begin{aligned} W\left[J - \hbar \frac{\delta}{\delta\phi}\right] &= W[J] - \hbar \int d^4x \frac{\delta W[J]}{\delta J(x)} \frac{\delta}{\delta\phi(x)} \\ &\quad - \hbar \sum_{n=2}^{\infty} \frac{1}{n!} \int d^4x_1 \dots d^4x_n \\ &\quad \times W_n(x_1, \dots, x_n; J) \frac{\delta}{\delta\phi(x_1)} \dots \frac{\delta}{\delta\phi(x_n)}. \end{aligned} \quad (13)$$

Expressing J as a functional of φ , using Eq. (4) and once again the elementary identity from footnote 2 to replace ϕ by φ in Eq. (12), we arrive at

$$\begin{aligned} \langle Q[\phi] \rangle_{J[\varphi]} &= \exp\left[\sum_{n=2}^{\infty} \frac{1}{n!} \int d^4x_1 \dots d^4x_n \right. \\ &\quad \times W_n(x_1, \dots, x_n; J[\varphi]) \frac{\delta}{\delta\varphi(x_1)} \dots \frac{\delta}{\delta\varphi(x_n)} \left. \right]^* \\ &\quad \times Q[\varphi], \end{aligned} \quad (14)$$

where the symbol \ast indicates that the functional differential operators act only on the external factor $Q[\varphi]$ but not on $J[\varphi]$ in G_n . It is important here that the sum in the exponent starts only at $n = 2$. Next recall DeWitt's identity

$$\frac{\delta\Gamma[\varphi]}{\delta\varphi(x)} = \left\langle \frac{\delta S[\phi]}{\delta\phi(x)} \right\rangle_{J=J[\varphi]}, \quad (15)$$

which holds since both sides are equal to $-J(x)$ [a consequence of the formal identity $\int \mathcal{D}\phi \delta/\delta\phi(x)(\dots) = 0$]. DeWitt's equation is now obtained by applying Eq. (14) with $Q[\phi] = \delta S/\delta\phi$. This gives

$$\begin{aligned} \frac{\delta\Gamma[\varphi]}{\delta\varphi(x)} &= \exp\left[\sum_{n=2}^{\infty} \frac{1}{n!} \int d^4x_1 \dots d^4x_n W_n(x_1, \dots, x_n; J[\varphi]) \right. \\ &\quad \times \left. \frac{\delta}{\delta\varphi(x_1)} \dots \frac{\delta}{\delta\varphi(x_n)} \right]^* \frac{\delta S}{\delta\varphi(x)}. \end{aligned} \quad (16)$$

Observe that for polynomial actions $S[\phi]$ the functional differential operator reduces to a finite number of terms upon expansion of the exponential.

To have a concrete example, consider the classically conformal ϕ^4 theory with the action

$$S[\phi] = \int d^4x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{\lambda}{4} \phi^4 \right). \quad (17)$$

This gives

$$\frac{\delta\Gamma[\varphi]}{\delta\varphi(x)} = \langle -\square\phi(x) + \lambda\phi^3(x) \rangle_{J=J[\varphi]} \quad (18)$$

and thus

$$\begin{aligned} \frac{\delta\Gamma[\varphi]}{\delta\varphi(x)} &= -\square\varphi(x) + \lambda\varphi^3(x) + 3\lambda W_2(x, x; \varphi)\varphi(x) \\ &\quad + \lambda W_3(x, x, x; \varphi). \end{aligned} \quad (19)$$

Expressing W_2 and W_3 by means of Eqs. (8) and (9), we see that all quantities in this equation can be expressed in terms of $\Gamma[\varphi]$ and its functional derivatives, so that Eq. (19) indeed becomes a functional differential equation for $\Gamma[\varphi]$.

As they stand these equations, and in particular the basic functional equation (16), are formal. Nevertheless, there is already one useful application: Eq. (19) can be used as a generating equation to derive the Schwinger-Dyson equations. With the standard formula for the one-particle irreducible n -point functions,

$$\Gamma_n(x_1, \dots, x_n) \equiv \Gamma_n(x_1, \dots, x_n; \varphi)|_{\varphi=0}, \quad (20)$$

we obtain, for instance,

$$\begin{aligned} \hbar\Gamma_2(x, y) &= (-\square + 3\lambda W_2(x, x))\delta^{(4)}(x - y) \\ &\quad - \lambda \int d^4u d^4v d^4w W_2(x, u) W_2(x, v) W_2(x, w) \\ &\quad \times \Gamma_4(u, v, w, y), \end{aligned} \quad (21)$$

which can be represented diagrammatically in the usual way. Similar formulas for higher n point functions can be deduced by repeated differentiation.

In principle, Eq. (19) is an *exact* nonlinear functional differential equation for the action functional $\Gamma[\varphi]$. In the full *renormalized* theory, this functional should be well defined on a set of sufficiently well-behaved functions $\varphi(x)$ (say, C^∞ functions, which fall off sufficiently rapidly at infinity). In addition, its functional derivatives should be well defined as distributions. However, this cannot be the case for Eq. (19) as it stands. First of all, the equation is written in terms of bare couplings and correlators and needs to be renormalized. Secondly, even if one assumes that the necessary renormalizations have been performed, and the couplings are replaced by the renormalized (physical) ones, Eq. (19) would still not be well defined as it stands because the rhs of Eq. (19) contains singular contributions in the terms of order \hbar ; recall that $G_2(x, y)$ and higher n -point functions are generally singular at coincident points, even in free field theory. It is for this reason that one conventionally must resort to perturbative methods by considering the n -point functions separately and by rendering them finite order by order in perturbation theory by means of suitable subtractions in momentum space. For instance, this can be easily seen from Eq. (21), where the infinity of $W_2(x, x)$ can be absorbed by an appropriate wave function renormalization $\varphi \rightarrow Z^{1/2}\varphi$ at lowest order.

As already emphasized in the introduction, we here adopt a different strategy by trying to deal with Eq. (19) *directly*. This requires us to look for theories for which the DeWitt equation can be made well defined, that is, free of singularities. Examples of such theories are certain *super-symmetric models* of the type discussed below in Sec. V. We note again that the DeWitt equation (16) is not of any known type. This is so even if one restricts this equation to an “ordinary” partial differential equation for finitely many variables as in the following section. This is one of the reasons for the difficulties in dealing with it and motivates the present effort to gain a better understanding of this equation.

III. “ZERO-DIMENSIONAL FIELD THEORY” EXAMPLE

To bring out the main new features, we now discuss an example from zero-dimensional field theory, that is, a system with finitely many degrees of freedom, in terms of which the results described in the foregoing section can be explicitly illustrated and where we do not have to worry about UV infinities. This example will also allow us to exhibit the vastly improved convergence properties of a new summation scheme over conventional perturbation theory. To this aim, let us consider the “action” of a zero-dimensional ϕ^4 theory,

$$S(x) = \frac{1}{2} \sum_{i,j=1}^n x_i A_{ij} x_j + \frac{\lambda}{4} \sum_{j=1}^n x_j^4, \quad (22)$$

where A_{ij} is a nondegenerate positive definite matrix. The generating function $W(J) \equiv W(J_1, \dots, J_n)$ for the “connected Green’s functions” is then defined in analogy with Eq. (1) as

$$e^{-W(J)} := \int_{\mathbb{R}^n} dx \exp \left[-S(x) - \sum_j x_j J_j \right], \quad (23)$$

where the integration measure dx is normalized in such a way that $W(0) = 0$. The generating function is easily seen to satisfy the differential equation

$$\sum_j A_{ij} \frac{\partial Z}{\partial J_j} + \lambda \frac{\partial^3 Z}{\partial J_i^3} = J_i Z(J) \quad (24)$$

or, in terms of $W(J)$,

$$\sum_j A_{ij} \frac{\partial W}{\partial J_j} + \lambda \left[\frac{\partial^3 W}{\partial J_i^3} - 3 \frac{\partial^2 W}{\partial J_i^2} \frac{\partial W}{\partial J_i} + \left(\frac{\partial W}{\partial J_i} \right)^3 \right] = -J_i. \quad (25)$$

When expressed in terms of the effective action, this is the finite-dimensional analog of the DeWitt equation (16); see below. So in analogy with Eq. (4), let us define the “classical field” by

$$\varphi_i(J) := \frac{\partial W(J)}{\partial J_i} \quad (26)$$

and introduce the “effective action” $\Gamma(\varphi)$ in the usual way by Legendre transformation as in Eq. (5). The DeWitt equation now reduces to a set of partial differential equations:

$$\frac{\partial \Gamma(\varphi)}{\partial \varphi_i} = \exp \left[\sum_{k \geq 2} \frac{1}{k!} \sum_{j_1, \dots, j_k} W_{j_1 \dots j_k}(J) \frac{\partial}{\partial \varphi_{j_1}} \cdots \frac{\partial}{\partial \varphi_{j_k}} \right] \frac{\partial S(\varphi)}{\partial \varphi_i}, \quad (27)$$

where $W_{j_1 \dots j_k} \equiv (-1)^{k-1} \partial_{j_1} \cdots \partial_{j_k} W$, and we have relations analogous to Eqs. (8) and (9), that is, $\sum_j W_{ij}(J) \Gamma_{jk}(\varphi(J)) = \delta_{ik}$, and so on.

We can now produce a formal solution of Eq. (27), rederiving a result that was essentially obtained already long ago [8]. From the general definition, we directly obtain the following differential equation for $\Gamma(\varphi)$:

$$\begin{aligned} & \exp \left[-\Gamma(\varphi) + \sum_j \varphi_j \frac{\partial \Gamma(\varphi)}{\partial \varphi_j} \right] \\ &= \int_{\mathbb{R}^n} dx \exp \left[-S(x) + \sum_j x_j \frac{\partial \Gamma(\varphi)}{\partial \varphi_j} \right]. \end{aligned} \quad (28)$$

To evaluate the integral, we split the effective action into a “classical” part $S(\varphi)$ and a “quantum” part $\tilde{\Gamma}(\varphi)$ according to

$$\Gamma(\varphi) = \frac{1}{2} \sum_{i,j=1}^n \varphi_i A_{ij} \varphi_j + \frac{1}{4} \lambda \sum_{j=1}^n \varphi_j^4 + \tilde{\Gamma}(\varphi). \quad (29)$$

Shifting integration variables as $x_j \rightarrow x_j + \varphi_j$ in Eq. (28), a little algebra gives

$$\begin{aligned} \exp[-\tilde{\Gamma}(\varphi)] &= \int_{\mathbb{R}^n} dx \exp \left[-\frac{1}{2} \sum_{ij} x_i G_{ij}^{-1}(\varphi) x_j - \lambda \sum_j x_j^3 \varphi_j \right. \\ &\quad \left. - \frac{\lambda}{4} \sum_j x_j^4 + \sum_j x_j \frac{\partial \tilde{\Gamma}(\varphi)}{\partial \varphi_j} \right], \end{aligned} \quad (30)$$

with the classical “field-dependent” Green’s function $G_{ij}(\varphi)$

$$\sum_j (A_{ij} + 3\lambda \delta_{ij} \varphi_j^2) G_{jk}(\varphi) = \delta_{ik}. \quad (31)$$

Performing the Gaussian integral and using Wick’s theorem in the form

$$\begin{aligned} & (2\pi)^{-n/2} \int_{\mathbb{R}^n} d^n x f(x) \exp \left(-\frac{1}{2} \sum_{i,j=1}^n C_{ij} x_i x_j \right) \\ &= (\det C)^{-1/2} \exp \left(\frac{1}{2} \sum_{i,j=1}^n C_{ij}^{-1} \frac{\partial}{\partial y_i} \frac{\partial}{\partial y_j} \right) f(y) \Big|_{y=0}, \end{aligned} \quad (32)$$

the expression (30) can be rewritten in the form

$$\begin{aligned} \exp[-\tilde{\Gamma}(\varphi)] &= (\det G_{ij}(\varphi))^{1/2} \exp\left(\frac{1}{2} \sum_{i,j} G_{ij}(\varphi) \frac{\partial}{\partial \eta_i} \frac{\partial}{\partial \eta_j}\right) \\ &\times \exp\left[-\lambda \sum_j \varphi_j \eta_j^3 - \frac{\lambda}{4} \sum_j \eta_j^4\right. \\ &\left. + \sum_j \eta_j \frac{\partial \tilde{\Gamma}(\varphi)}{\partial \varphi_j}\right]_{\eta=0}. \end{aligned} \quad (33)$$

Let us pause to explain this formula. The determinant prefactor just produces the well-known semiclassical (one-loop) correction $\propto \log(\det G_{ij}(\varphi))$ to the classical action. As for the remaining terms, and ignoring the last term $\propto \eta \partial \tilde{\Gamma} / \partial \varphi$, we would get the sum over all connected vacuum diagrams with the field-dependent propagator $G_{ij}(\varphi)$ (as the result of taking the logarithm on both sides). Although this last term would seem to make the equation completely untractable, a little bit of thought shows that this is not so. Because $\tilde{\Gamma}(\varphi)$ contains only one-particle irreducible contributions, the effect of this last term is precisely to remove the one-particle reducible diagrams from the expansion; because this term is linear in η , it can couple to the rest of any diagram only via a single line. In other words, *the quantum effective action is nothing but the sum of the one-loop correction and the sum over one-particle irreducible vacuum diagrams with at least two loops and with the field-dependent Green's function (31)*. This is the result derived in Ref. [8] for the effective potential in quantum field theory.

By construction, this series solution must satisfy the discrete DeWitt equation (27), and this claim can in principle be checked order by order. Equally important is the fact that the expansion, while being asymptotic, can have vastly better convergence properties for nonvanishing φ than the usual perturbation expansion in terms of the coupling constant λ . This is most easily seen by simplifying our zero-dimensional field theory even further to an integral over one variable. In this case the Green's function (31) is simply $G(\varphi) \equiv (1 + 3\lambda\varphi^2)^{-1}$. For a given vacuum diagram with I internal lines, we have

$$I = \frac{3}{2} V_3 + 2V_4, \quad (34)$$

where V_3 and V_4 , respectively, denote the number of three- and four-point vertices in Eq. (33); note that in any vacuum diagram, the number of three-point vertices is *even*. The number of loops is equal to

$$L = \frac{1}{2} V_3 + V_4 + 1. \quad (35)$$

Therefore, an arbitrary vacuum diagram with L loops will be proportional to

$$\frac{\lambda^{V_4} (\lambda\varphi)^{V_3}}{(1 + 3\lambda\varphi^2)^I} \approx (\lambda\varphi^4)^{1-L} \quad (36)$$

[for $L = 1$, the relevant parameter is $\log(1 + 3\lambda\varphi^2)$]. In other words, the loop expansion now coincides with an expansion in $(\lambda\varphi^4)^{-1}$; of course, this expansion should only be used in the appropriate region in field space and the space of couplings, where $\lambda\varphi^4$ is sufficiently large. So we see that the series can converge well even for large λ , provided the value of the classical field φ is not too small (and different from zero). We have checked this claim by numerical integration of a nontrivial example, which we give in the appendix. The important lesson, then, is that it is not simply the coupling constant λ [or its running analog $\lambda(\mu)$, where μ is some renormalization scale] that governs the convergence properties of the effective action functional, but that one should also consider the question of convergence with respect to the value of the field variables φ_j or $\varphi(x)$ as well.

IV. FORMAL SOLUTION

The considerations of the foregoing section can be straightforwardly extended to field theory, enabling us to construct a formal expression for the (unrenormalized) effective action *in terms of a sum over vacuum diagrams with field-dependent classical Green's functions*. For constant field configurations $\varphi(x) = \varphi_0$, this solution reduces to the one found already long ago in Ref. [8], where it was exploited for an efficient determination of higher-order corrections to the Coleman-Weinberg effective potential for various theories. We here present the general solution that allows for arbitrary x dependence of the classical field φ , and that follows directly from the above construction by taking a formal limit $n \rightarrow \infty$, or alternatively by a minor modification of the argument given in Ref. [8]. It is remarkable that in this way an explicit, albeit formal, solution of the (unrenormalized) DeWitt equation that would seem difficult to guess otherwise can be obtained. Of course, even in the full theory, all relevant expressions can be made well defined by regulating the quantum field theory, either by discretization as in the previous section or by suitable continuum regularizations such as smearing.

From Eq. (33), we deduce immediately that the formal solution for the unrenormalized effective action functional can be presented in the form

$$\begin{aligned} \Gamma[\varphi] &= S[\varphi] + \frac{\hbar}{2} \int d^4x \log \left[\frac{\delta^2 S[\varphi]}{\delta \varphi(x) \delta \varphi(x)} \right] \\ &\quad - \hbar \log \left[\exp \left(\frac{\hbar}{2} \int d^4u d^4v G_{\text{cl}}(u, v; \varphi) \frac{\delta^2}{\delta \eta(u) \delta \eta(v)} \right) \right. \\ &\quad \left. \times \exp(-\hbar^{-1} S_{\text{int}}[\varphi, \eta]) \right]_{\eta=0} \Big|_{1\text{PI}}, \end{aligned} \quad (37)$$

where the subscript 1PI means that one-particle reducible diagrams are to be omitted in the expansion, and where the logarithm removes disconnected diagrams from inside the brackets. The interacting part of the action is defined by subtracting the linear and quadratic fluctuations,

$$\begin{aligned}
S_{\text{int}}[\varphi, \eta] &:= S[\varphi + \eta] - S[\varphi] \\
&- \int d^4u \eta(u) \frac{\delta S[\varphi + \eta]}{\delta \varphi(u)} \Big|_{\eta=0} \\
&- \frac{1}{2} \int d^4u d^4v \eta(u) \eta(v) \frac{\delta^2 S[\varphi + \eta]}{\delta \varphi(u) \delta \varphi(v)} \Big|_{\eta=0} \\
&= \frac{1}{3!} \frac{\delta^3 S}{\delta \varphi^3} \eta^3 + \frac{1}{4!} \frac{\delta^4 S}{\delta \varphi^4} \eta^4 + \dots
\end{aligned} \quad (38)$$

Observe that a residual dependence on φ arises from four-point vertices onward, whereas there is no φ dependence if there are only cubic vertices. The expectation values in Eq. (37) are to be computed with the classical *field-dependent* Green's function $G_{\text{cl}}(x, y; \varphi)$, which is defined as

$$\int d^4y G_{\text{cl}}(x, y; \varphi) \frac{\delta^2 S[\varphi]}{\delta \varphi(y) \delta \varphi(z)} = \delta^{(4)}(x - z). \quad (39)$$

Hence, $G_{\text{cl}}(x, y; \varphi)$ is the classical analog of Eq. (8) in the sense that

$$W_2(x, y; \varphi) = \hbar G_{\text{cl}}(x, y; \varphi) + \mathcal{O}(\hbar^2). \quad (40)$$

According to the formula (37), the unrenormalized effective action $\Gamma[\varphi]$ is the *sum over all one-particle-irreducible (1PI) vacuum diagrams with the field-dependent Green's function* (39). The dependence of Γ on the field $\varphi(x)$ thus derives from *two* sources, namely, the field dependence of $G_{\text{cl}}(x, y; \varphi)$ and, second, the residual dependence of S_{int} on φ (which only exists if there are four-point or higher-point vertices). The former can be made more explicit by expanding

$$\begin{aligned}
G_{\text{cl}}(x, y; \varphi) \\
= G_0(x, y) - \int d^4u G_0(x, u) p(\varphi(u)) G_0(u, y) \pm \dots,
\end{aligned} \quad (41)$$

where $p(\varphi)$ is obtained from $\delta^2 S / \delta \varphi^2$ by removing the free part not depending on $\varphi(x)$ and G_0 is the free propagator. The terms in this expansion thus generate the “antennalike” diagrams known from textbook formulas of the effective potential.

By virtue of its definition and the above derivation, the expression (37) must satisfy the DeWitt equation (16) at least formally. This claim is straightforward to check for the semiclassical $\mathcal{O}(\hbar)$ correction by use of the formula

$$\frac{\delta}{\delta \varphi(x)} \text{Tr} \log M = \text{Tr} \left(M^{-1} \frac{\delta M}{\delta \varphi(x)} \right), \quad (42)$$

valid for any functional matrix $M(y, z)$, and by approximating the full two-point function $G(x, y; J(\varphi))$ from Eq. (39) by $G_{\text{cl}}(x, y; \varphi)$. However, a direct verification of Eq. (37) to all orders is cumbersome. We will therefore postpone a discussion of this issue to the following section in terms of an example where the DeWitt equation is well defined. Let us just note that in conjunction with the explicit expression as a sum over $\varphi(x)$ -dependent vacuum diagrams, we can see directly from the DeWitt equation (16) that

$\Gamma[\varphi]$ can only contain one-particle irreducible diagrams; the action of the first functional derivative $\delta \Gamma[\varphi] / \delta \varphi(x)$, in particular, leads to cutting any one of the propagators in a diagram arising in the expansion (37). If we had a diagram which is not 1PI, then there would be at least one propagator which joins two 1PI subdiagrams. The action of the functional derivative on this diagram would thus split the diagram into two parts at this propagator, leaving two disconnected diagrams. But on the rhs of the DeWitt equation, we have only connected Green's functions, $\delta^n W[J] / \delta J(x_1) \dots \delta J(x_n)$. So there can be no disconnected diagrams on the rhs of Ref. [8], and thus we can only have 1PI diagrams contributing to $\Gamma[\varphi]$, as expected.

The effective (Coleman-Weinberg) potential is obtained by specializing all formulas to x -independent fields $\varphi(x) = \varphi_0$ [4] and removing a formally infinite volume factor $\propto \int dx$. The main advantage of writing the effective potential as a sum over vacuum type diagrams is the following: rather than having to do all the combinatorics with “antenna diagrams,” one obtains the answer at each loop order in one stroke. In particular, the renormalization-group improved one-loop potential obtained by summing ladder bubble diagrams is directly obtained. This was, in fact, the first application of this formula in Ref. [8], where the effective potential was also determined to two loops for φ^4 theory. As shown there, the formalism implies considerable simplifications in comparison with the textbook derivations of the Coleman-Weinberg potential.

At the end of this section, we write the solution (37) for the finite-dimensional integral with the action defined by Eq. (22), that is, the solutions to Eq. (28). In accordance with the explanation after Eq. (37), we include only 1PI and connected diagrams in the expansion

$$\Gamma(\varphi_i) = S(\varphi_i) + \Gamma^{(1)}(\varphi_i) + \Gamma^{(2)}(\varphi_i) + \Gamma^{(3)}(\varphi_i) + \dots, \quad (43)$$

where the indices denote the loop order. In this way, we obtain

$$\begin{aligned}
\Gamma^{(1)}(\varphi) &= -\frac{1}{2} \ln \det (G_{ij}) \\
\Gamma^{(2)}(\varphi) &= -\left[-\frac{3\lambda}{4} \sum_i G_{ii}^2 + 3\lambda^2 \sum_{i,j} \varphi_i \varphi_j G_{ij}^3 \right] \\
\Gamma^{(3)}(\varphi) &= -\left[\frac{3\lambda^2}{4} \sum_{i,j} G_{ij}^4 + \frac{9\lambda^2}{4} \sum_{ij} G_{ii} G_{ij}^2 G_{jj} \right. \\
&\quad - 27\lambda^3 \sum_{i,j,k} \varphi_i \varphi_j G_{ij} G_{ik}^2 G_{jk}^2 \\
&\quad - 27\lambda^3 \sum_{i,j,k} \varphi_i \varphi_j G_{ij}^2 G_{ik} G_{jk} G_{kk} \\
&\quad + 54\lambda^4 \sum_{i,j,k,l} \varphi_i \varphi_j \varphi_k \varphi_l G_{ij} G_{jk} G_{kl} G_{li} G_{ik} G_{jl} \\
&\quad \left. + 81\lambda^4 \sum_{i,j,k,l} \varphi_i \varphi_j \varphi_k \varphi_l G_{ij}^2 G_{kl}^2 G_{ik} G_{jl} \right].
\end{aligned}$$

As already pointed out, this is a “nonperturbative expansion” that is restricted to the region of couplings and field space where the “parameter” $G(\varphi) \sim (\lambda\varphi^2)^{-1}$ is small. In the formula above, we included terms up to three loops, i.e., up to sixth order in $G_{ij}(\varphi)$ [one easily checks that all terms are of the appropriate order in $(\lambda\varphi^4)^{-1}$, in agreement with formula (36)]. A numerical comparison of the exact result and this expansion for a one-dimensional integral for several values of λ and φ is given in the appendix. It shows that this expansion can give excellent agreement with the exact result even in regions where λ is very large.

V. WESS-ZUMINO MODEL IN $D = 2$

We next turn to an example where the DeWitt equation (16) can be made completely well defined, that is, free of all short-distance singularities. This is the $N = 1$ Wess-Zumino model in two space-time dimensions, which is UV finite order by order in perturbation theory (the generic nonsupersymmetric theories having only logarithmic divergences in two dimensions, which are removed by imposing supersymmetry).⁴

The Euclidean version of the model can be written in terms of a single superfield $\Phi(z)$ with superspace coordinate $z \equiv (x, \theta)$, where θ is a two-component (anticommuting) Majorana spinor with $\theta = \theta^*$. The superfield contains a real scalar A and a Majorana spinor ψ , as well as the auxiliary field F :

$$\Phi(x, \theta) = A(x) + \bar{\theta}\psi(x) + \frac{1}{2}\bar{\theta}\theta F(x). \quad (44)$$

For simplicity, we restrict attention to the following Lagrangian:

$$\mathcal{L} = -\frac{1}{4}\Phi\bar{D}D\Phi + \frac{1}{2}m\Phi^2 + \frac{1}{3}g\Phi^3. \quad (45)$$

We could replace the last two terms by an arbitrary polynomial $P(\Phi)$ here, but this would only make the formulas more cumbersome and not give any new insights. The supercovariant derivative is defined by

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + (\gamma^\mu\theta)_\alpha\partial_\mu \quad (46)$$

$$\bar{D}^\alpha = -C^{\alpha\beta}D_\beta, \quad (47)$$

where C is the charge conjugation matrix. The Lagrangian in component form is as follows:

$$\begin{aligned} \mathcal{L} = & \frac{1}{2}A\Box A - \frac{1}{2}\bar{\psi}\gamma^\mu\partial_\mu\psi + \frac{1}{2}F^2 + \frac{1}{2}m(2AF - \bar{\psi}\psi) \\ & + g(A^2F - A\bar{\psi}\psi). \end{aligned} \quad (48)$$

⁴See Ref. [15] for a recent treatment of the Wess-Zumino model in $2 + 1$ dimensions.

Writing out the DeWitt equation for the three fields A , ψ , and F , we get

$$\begin{aligned} \frac{\delta\Gamma[A, F, \psi]}{\delta A(x)} &= \Box A(x) + mF(x) + g[2A(x)F(x) - \bar{\psi}(x)\psi(x)] \\ &\quad - g\hbar\left[2\frac{\delta^2 W[J]}{\delta J_A(x)\delta J_F(x)} + \text{Tr}\frac{\delta^2 W[J]}{\delta J_\psi(x)\delta J_{\bar{\psi}}(x)}\right] \\ \frac{\delta\Gamma[A, F, \psi]}{\delta\bar{\psi}(x)} &= -\not{\partial}\psi(x) - m\psi(x) - 2gA(x)\psi(x) \\ &\quad - \hbar g\frac{\delta^2 W[J]}{\delta J_A(x)\delta J_\psi(x)} \\ \frac{\delta\Gamma[A, F, \psi]}{\delta F(x)} &= F(x) + mA(x) + gA^2(x) - \hbar g\frac{\delta^2 W[J]}{\delta J_A(x)\delta J_A(x)}, \end{aligned} \quad (49)$$

with self-explanatory notation. Now we see that the equation for the scalar field A is well defined as it stands because the logarithmic singularities cancel between the two terms in parentheses. More precisely, the latter expression is understood to be

$$\lim_{y \rightarrow x} \left[2\frac{\delta^2 W[J]}{\delta J_A(x)\delta J_F(y)} + \text{Tr}\frac{\delta^2 W[J]}{\delta J_\psi(x)\delta J_{\bar{\psi}}(y)} \right] = \text{finite}. \quad (50)$$

Likewise, the equation for ψ is well defined because $\delta^2 W/\delta A\delta\psi$ is free of short-distance singularities. So the only singularity occurs in the last equation, and this can be removed by replacing the product $A^2(x)$ by the normal ordered product

$$:A(x)A(y): \equiv A(x)A(y) - \underbrace{A(x)A(y)} \quad (51)$$

and taking $x \rightarrow y$ afterward. This singularity simply follows from the fact that if one expresses the auxiliary field F in terms of the physical field A , the nonlinear terms in A must be rendered nonsingular to make F itself well defined as a quantum operator.⁵ Consequently, the last component of the DeWitt equation must be replaced by

$$\frac{\delta\Gamma[A, F, \psi]}{\delta F(x)} = F(x) + mA(x) + g:A^2(x): - \hbar g\frac{\delta^2 W[J]}{\delta J_A(x)\delta J_A(x)}, \quad (52)$$

and then *all* components of the DeWitt equation are free of singularities. In practice, the above replacement simply means that in the formal solution as a sum over vacuum diagrams, there are no tadpole diagrams (these are anyway absent for a theory with only cubic vertices as they would lead to non-1PI diagrams in Γ which cannot be).

⁵But note that, while $:A^2:$ is well defined as an operator, it is singular as a c number, while the converse is true for A^2 .

All these equations can be conveniently recast into superspace equations. A similar normal ordering can be done in the superspace version of the Lagrangian, and, as it is much more convenient to work in it, we would stick to the superspace description. So we have the functional derivative of the action as

$$\frac{\delta S}{\delta \Phi} = -\frac{1}{2}\bar{D}D\Phi + m\Phi + g\Phi^2. \quad (53)$$

The arguments of the foregoing sections generalize directly to superspace. For the cubic Lagrangian above the DeWitt equation, Eq. (16) takes an especially simple form, namely,

$$\frac{\delta \Gamma[\Phi]}{\delta \Phi(z)} = : \frac{\delta S[\Phi]}{\delta \Phi(z)} : - \hbar g \frac{\delta^2 W[J]}{\delta J(z)\delta J(z)} \Big|_{J=J[\Phi]} \quad (54)$$

or, more specifically,

$$\begin{aligned} \frac{\delta \Gamma[\Phi]}{\delta \Phi(z)} = & -\frac{1}{2}\bar{D}D\Phi(z) + m\Phi(z) + g:\Phi^2(z): \\ & - \hbar g \frac{\delta^2 W[J]}{\delta J(z)\delta J(z)} \Big|_{J=J[\Phi]}, \end{aligned} \quad (55)$$

where $z \equiv (x^\mu, \theta)$ and $J(z)$ is the ‘‘supersource field’’ $J(z) \equiv J_F + \bar{\theta}J_\psi + \frac{1}{2}\bar{\theta}\theta J_A$. The normal ordering is understood to be in the sense of the component expressions given above. In the formal solution below, this simply means that all tadpole diagrams are suppressed.

For the free superfield, the superspace propagator is

$$\begin{aligned} G_2^{(0)}(z-z') &= \langle 0|T\left[\left(A(x) + \bar{\theta}\psi(x) + \frac{1}{2}\bar{\theta}\theta F(x)\right)\right. \\ &\quad \times \left.\left(A(x') + \bar{\theta}'\psi(x') + \frac{1}{2}\bar{\theta}'\theta'F(x')\right)\right]|0\rangle \\ &= \exp\left[-\frac{1}{2}(\bar{\theta} - \bar{\theta}')(\gamma_\mu \partial^\mu + m)(\theta - \theta')\right] \\ &\quad \times \Delta_F(x-y). \end{aligned} \quad (56)$$

In analogy with Eq. (39), we define the Green’s function in superspace,

$$\int dz' G_{\text{cl}}(z, z'; \Phi) \frac{\delta^2 S[\Phi]}{\delta \Phi(z')\delta \Phi(z'')} = \delta(z-z'') \quad (57)$$

(where the fermionic part of the δ function is defined in the usual way as $\delta(\theta) = \theta$) so that $G_{\text{cl}}(z, z'; \Phi) = G_2^{(0)}(z-z') + \dots$.

By construction, the supersymmetric DeWitt equation (54) is well defined, and we can therefore take over the formal solution given in the previous section,

$$\begin{aligned} \Gamma[\Phi] = & S[\Phi] + \frac{\hbar}{2} \int d^4z \ln \left[\frac{\delta^2 S}{\delta \Phi(z)\delta \Phi(z)} \right] \\ & - \hbar \ln \left[\exp\left(\frac{\hbar}{2} G_{i,j} \frac{\delta^2}{\delta \tilde{\Phi}_i \delta \tilde{\Phi}_j}\right) \exp\left(-\frac{\tilde{S}_{\text{int}}}{\hbar}\right) \Big|_{\tilde{\Phi}=0} \right], \end{aligned} \quad (58)$$

where G_{ij} is shorthand for $G_{\text{cl}}(z_i, z_j; \Phi)$ and $\tilde{S}_{\text{int}} = \frac{g}{3}\tilde{\Phi}^3$, and all the integrals are understood to be in superspace. Now if we expand the series, we have the following:

$$\left[1 + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{\hbar}{2} G_{i,j} \frac{\delta^2}{\delta \tilde{\Phi}_i \delta \tilde{\Phi}_j} \right)^n \right] \left[1 + \sum_{m=1}^{\infty} \frac{1}{m!} \left(\frac{-\tilde{S}_{\text{int}}}{\hbar} \right)^m \right] \Big|_{\tilde{\Phi}=0}. \quad (59)$$

Because the dummy variable $\tilde{\Phi}$ is put to 0, and the interaction is cubic, only terms with $2n = 3m$ survive. Thus, the first of this will be at two loops for $m = 2, n = 3$. Evaluating the corresponding term, we get

$$\begin{aligned} & \left[\frac{1}{3!} \left(\frac{\hbar}{2} G_{i,j} \frac{\delta^2}{\delta \tilde{\Phi}_i \delta \tilde{\Phi}_j} \right)^3 \right] \left[\frac{1}{2!} \left(\frac{-\tilde{S}_{\text{int}}}{\hbar} \right)^2 \right] \\ &= \frac{\hbar g^2}{3} \int_{z,w} G_{\text{cl}}^3(z, w; \Phi). \end{aligned}$$

At the next order (three loops), we have $n = 6, m = 4$, and

$$\begin{aligned} & \left[\frac{1}{6!} \left(\frac{\hbar}{2} G_{i,j} \frac{\delta^2}{\delta \tilde{\Phi}_i \delta \tilde{\Phi}_j} \right)^6 \right] \left[\frac{1}{4!} \left(\frac{-\tilde{S}_{\text{int}}}{\hbar} \right)^4 \right] \\ &= \hbar^2 \left[\frac{2}{3} g^4 \int_{u,v,w,z} G_{\text{cl}}(u, v; \Phi) G_{\text{cl}}(u, w; \Phi) G_{\text{cl}}(u, z; \Phi) \right. \\ &\quad \times G_{\text{cl}}(v, w; \Phi) G_{\text{cl}}(v, z; \Phi) G_{\text{cl}}(w, z; \Phi) \\ &\quad + g^4 \int_{u,v,w,z} G_{\text{cl}}^2(u, v; \Phi) G_{\text{cl}}^2(u, w; \Phi) G_{\text{cl}}(u, w; \Phi) \\ &\quad \times G_{\text{cl}}(v, z; \Phi) + \left. \frac{1}{2} \left(\frac{g^2}{3} \int_{z,w} G_{\text{cl}}^3(z, w; \Phi) \right)^2 \right]. \end{aligned} \quad (60)$$

We recognize the last term as a square of the term, which we got for $n = 3, m = 2$ (two loops) and which is removed by taking the log of the entire expression as these diagrams are not connected. Hence, summing up, we get the following contribution to the effective action:

$$\begin{aligned} \Gamma = & S + \frac{\hbar}{2} \int d^4z \ln \left[\frac{\delta^2 S}{\delta \Phi(z)\delta \Phi(z)} \right] \\ & - \frac{\hbar^2 g^2}{3} \int_{z,w} G_{\text{cl}}^3(z, w; \Phi) - \frac{2}{3} \hbar^3 g^4 \int_{u,v,w,z} G_{\text{cl}}(u, v; \Phi) \\ &\quad \times G_{\text{cl}}(u, w; \Phi) G_{\text{cl}}(u, z; \Phi) G_{\text{cl}}(v, w; \Phi) G_{\text{cl}}(v, z; \Phi) \\ &\quad \times G_{\text{cl}}(w, z; \Phi) - \hbar^3 g^4 \int_{u,v,w,z} G_{\text{cl}}^2(u, v; \Phi) G_{\text{cl}}^2(w, z; \Phi) \\ &\quad \times G_{\text{cl}}(u, w; \Phi) G_{\text{cl}}(v, z; \Phi) + \mathcal{O}(\hbar^4). \end{aligned} \quad (61)$$

To check this, we first calculate the second functional derivative of Γ , which is, up to order \hbar^2 ,

$$\begin{aligned} \frac{\delta^2 \Gamma}{\delta \Phi(z_1) \delta \Phi(z_2)} &= \frac{\delta^2 S}{\delta \Phi(z_1) \delta \Phi(z_2)} - 2\hbar g^2 G_{\text{cl}}(z_1, z_2; \Phi) G_{\text{cl}}(z_1, z_2; \Phi) \\ &\quad - 8\hbar^2 g^4 \int_{z,w} G_{\text{cl}}(z, z_2; \Phi) G_{\text{cl}}(z_2, z_1; \Phi) G_{\text{cl}}(z_1, w; \Phi) G_{\text{cl}}^2(z, w; \Phi) \\ &\quad - 8\hbar^2 g^4 \int_{z,w} G_{\text{cl}}(z, z_1; \Phi) G_{\text{cl}}(z_1, w; \Phi) G_{\text{cl}}(z, z_2; \Phi) G_{\text{cl}}(z_2, w; \Phi) G_{\text{cl}}(z, w; \Phi). \end{aligned} \quad (62)$$

Inverting the above, we obtain the two-point function up to order \hbar^2 ,

$$\begin{aligned} -\frac{\delta^2 W}{\delta J(z_1) \delta J(z_2)} &= G_{\text{cl}}(z_1, z_2; \Phi) + 2\hbar g^2 \int_{u,v} G_{\text{cl}}(z_1, u; \Phi) G_{\text{cl}}^2(u, v; \Phi) G_{\text{cl}}(v, z_2; \Phi) \\ &\quad + 8\hbar^2 g^4 \int_{u,v,z,w} [G_{\text{cl}}(z_1, u; \Phi) G_{\text{cl}}(z, v; \Phi) G_{\text{cl}}(v, u; \Phi) G_{\text{cl}}(u, w; \Phi) G_{\text{cl}}^2(z, w; \Phi) G_{\text{cl}}(v, z_2; \Phi) \\ &\quad + G_{\text{cl}}(z_1, u; \Phi) G_{\text{cl}}(z, u; \Phi) G_{\text{cl}}(u, w; \Phi) G_{\text{cl}}(z, v; \Phi) G_{\text{cl}}(v, w; \Phi) G_{\text{cl}}(z, w; \Phi) G_{\text{cl}}(v, z_2; \Phi)] \\ &\quad + 4\hbar^2 g^4 \int_{u,v,w,z} G_{\text{cl}}(z_1, u; \Phi) G_{\text{cl}}^2(u, v; \Phi) G_{\text{cl}}(v, w; \Phi) G_{\text{cl}}^2(w, z; \Phi) G_{\text{cl}}(z, z_2; \Phi). \end{aligned} \quad (63)$$

Now putting this in the DeWitt equation from the rhs, we obtain

$$\begin{aligned} \frac{\delta S}{\delta \Phi(z)} - \hbar g \frac{\delta^2 W}{\delta J(z) \delta J(z)} &= \frac{\delta S}{\delta \Phi(z)} + \hbar g G_{\text{cl}}(z, z; \Phi) + 2\hbar^2 g^3 \int_{u,v} G_{\text{cl}}(z, u; \Phi) G_{\text{cl}}^2(u, v; \Phi) G_{\text{cl}}(v, z; \Phi) \\ &\quad + 8\hbar^3 g^5 \int_{u,v,z',w} [G_{\text{cl}}(z, u; \Phi) G_{\text{cl}}(z', v; \Phi) G_{\text{cl}}(v, u; \Phi) G_{\text{cl}}(u, w; \Phi) G_{\text{cl}}^2(z', w; \Phi) G_{\text{cl}}(v, z; \Phi) \\ &\quad + G_{\text{cl}}(z, u; \Phi) G_{\text{cl}}(z', u; \Phi) G_{\text{cl}}(u, w; \Phi) G_{\text{cl}}(z', v; \Phi) G_{\text{cl}}(v, w; \Phi) G_{\text{cl}}(z', w; \Phi) G_{\text{cl}}(v, z; \Phi)] \\ &\quad + 4\hbar^3 g^5 \int_{u,v,w,z'} G_{\text{cl}}(z, u; \Phi) G_{\text{cl}}^2(u, v; \Phi) G_{\text{cl}}(v, w; \Phi) G_{\text{cl}}^2(w, z'; \Phi) G_{\text{cl}}(z', z; \Phi). \end{aligned} \quad (64)$$

This is exactly what we get from the lhs by taking the first functional derivative of Γ .

VI. LIOUVILLE FIELD THEORY

As an example where the DeWitt equation can be worked out explicitly for a theory with a *nonpolynomial* action, we briefly consider Liouville theory in two dimensions. As is well known, the actual construction of this special conformal theory is subtle and has a long history (see, e.g., Refs. [16,17] and references therein), so we here content ourselves with formal arguments and derivations, postponing a more detailed discussion to future work. We note that the derivations given below do not make any use of the conformal symmetry of the theory.

The generating functional $W[J]$ is defined as in Eq. (1) with the action

$$S = \int d^2x \left[\frac{1}{2} (\partial_\mu \phi)^2 + \mu e^{b\phi(x)} \right], \quad (65)$$

where we set $\hbar = 1$ for simplicity. The proper definition of the theory is tricky, not least because the runaway nature of the exponential potential does not allow for a proper classical vacuum. As a consequence, the one-point function, and thus the classical field $\varphi(J)$, may not be well defined in

all circumstances for this reason; in fact, we would expect it to exist only for sources obeying $J(x) < 0$, for which the potential valley is avoided.

For Liouville theory, the main interest is not with expectation values of products of field operators $\phi(x)$, but rather with the expectation values of proper *primary fields*, which are exponential operators of the form

$$V_\alpha(x) \equiv \exp(\alpha \phi(x)). \quad (66)$$

The correlation functions are then given by

$$\langle 0 | V_{\alpha_1}(x_1) \dots V_{\alpha_n}(x_n) | 0 \rangle = \int \mathcal{D}\phi e^{\alpha_1 \phi(x_1)} \dots e^{\alpha_n \phi(x_n)} e^{-S[\phi]}, \quad (67)$$

with the action (65) (where again we assume proper normalization of the path integral). Introducing a source $J(x)$ as before, the correlation functions can be represented by means of a J -dependent partition function, with a distributional source

$$J(x) = - \sum_{j=1}^n \alpha_j \delta(x - x_j). \quad (68)$$

The DeWitt equation can be worked out as before, with the result

$$\begin{aligned} \frac{\delta\Gamma[\varphi]}{\delta\varphi(x)} &= \frac{*}{*} \exp \left[\sum_{n=2}^{\infty} \frac{1}{n!} \int d^4x_1 \dots d^4x_n W_n(x_1, \dots, x_n; J[\varphi]) \right] \\ &\times \frac{\delta}{\delta\varphi(x_1)} \dots \frac{\delta}{\delta\varphi(x_n)} \Big|_{\alpha=0}^* [-\square\varphi(x) + \mu b e^{b\varphi(x)}], \end{aligned} \quad (69)$$

and thus

$$\begin{aligned} \frac{\delta\Gamma[\varphi]}{\delta\varphi(x)} &= -J(x) \\ &= -\square\phi(x) + \exp \left[\sum_{n=2}^{\infty} \frac{b^n}{n!} W_n(x, \dots, x; J[\varphi]) \right] \\ &\quad \times \mu b e^{b\varphi(x)} \\ \Rightarrow \log(-J(x) + \square\phi(x)) &= \left[\sum_{n=2}^{\infty} \frac{b^n}{n!} W_n(x, \dots, x; J[\varphi]) \right] \\ &\quad + \ln(\mu b) + b\varphi(x). \end{aligned} \quad (70)$$

From Eq. (2), we know that

$$W_n(x_1, \dots, x_n) = (-1)^{n-1} \frac{\delta^n W[J]}{\delta J(x_1) \dots \delta J(x_n)}.$$

After some algebra, we obtain

$$\begin{aligned} \log(-J(x) + \square\phi(x)) &= -\exp\left(-b \frac{\delta}{\delta J(x)}\right) W[J] + \ln(\mu b) \\ &\quad + W[J] \\ &= -W[J - b\delta_x] + W[J] + \ln(\mu b), \end{aligned} \quad (71)$$

with $\delta_x(y) \equiv \delta(x - y)$. Equivalently, we can write

$$-J(x) + \square\phi(x) = \mu b e^{-W[J - b\delta_x]} e^{+W[J]} \equiv b\mu \frac{Z[J - b\delta_x]}{Z[J]}.$$

Thus, for J of the form (68), $Z[J - b\delta_x]$ has one more insertion than J , so the ratio appearing on the right-hand side in the previous equation is just

$$\frac{\langle 0 | V_{\alpha_1}(x_1) \dots V_{\alpha_n}(x_n) V_b(x) | 0 \rangle}{\langle 0 | V_{\alpha_1}(x_1) \dots V_{\alpha_n}(x_n) | 0 \rangle}. \quad (72)$$

The usefulness of the equation (71) is still under study, and we intend to return to it in future work. At this point, we only remark that, if we define $\varphi(x)$ as

$$\frac{\partial_{\alpha} \int [D\phi] e^{\alpha\phi(x)} e^{-S[\phi] - J\cdot\phi} |_{\alpha=0}}{\int [D\phi] e^{-S[\phi] - J\cdot\phi}}, \quad (73)$$

the equation can be rewritten as follows:

$$\begin{aligned} (-J(x)) \langle 0 | \prod_{i=1}^n V_{\alpha_i}(x_i) | 0 \rangle + \square \left(\partial_{\alpha} \langle 0 | V_{\alpha}(x) \prod_{i=1}^n V_{\alpha_i}(x_i) | 0 \rangle \right)_{\alpha=0} \\ = b\mu \langle 0 | V_b(x) \prod_{i=1}^n V_{\alpha_i}(x_i) | 0 \rangle. \end{aligned} \quad (74)$$

A similar equation, minus the first term, was used by Ref. [16] to check the proposal for the three-point function

in Liouville theory. If we plug in the Dorn-Otto-Zamolodchikov-Zamolodchikov proposal in this equation, then we find that, neglecting contact terms,

$$4(\Delta_1 - \Delta_2)^2 \partial_{\alpha} C(\alpha, \alpha_1, \alpha_2) |_{\alpha=0} = b\mu C(b, \alpha_1, \alpha_2), \quad (75)$$

where the Δ_i are the conformal dimensions of the primary operators. The other term which arises from the contact term, i.e., the δ functions obtained by the action of the Laplacian, cancels with the term proportional to J , generating the on-shell constraint, $\alpha = \frac{1}{2}Q$.

VII. OUTLOOK

In the introduction, we already mentioned possible further directions. In particular, we would like to apply the DeWitt equation to $N = 4$ Yang-Mills theory, the prime example of a UV finite quantum field theory in four space-time dimensions. However, this is not as straightforward as one might have wished. One main obstacle is the lack of a fully off-shell supersymmetric realization of the theory. If we simply use the on-shell supersymmetric formulation (in the Wess-Zumino gauge), there will appear all kinds of spurious divergences, since only gauge-invariant observables are supposed to be UV finite. The same trouble would arise with formulations where only part of the supersymmetry is realized of shell (for instance, in a formulation of the theory in terms of $N = 1$ superfields) or with harmonic superspace. One could also try the opposite approach, where only the true on-shell degrees of freedom are used, namely, the light-cone superspace formalism proposed in Ref. [18]. There, the Lagrangian is written in terms of a single chiral superfield using only physical degrees of freedom of the theory, using Grassmann parameters θ^m and their complex conjugates $\bar{\theta}_m$. The Lagrangian for $N = 4$ Yang-Mills theory then takes the following form [12,18]:

$$\begin{aligned} L = 72 \left[-\bar{\phi}^a \left(\frac{\square}{\partial^+} \right) \phi^a + \frac{4}{3} g f^{abc} \left[\frac{1}{\partial^+} \bar{\phi}^a \phi^b \bar{\phi}^c \right. \right. \\ \left. \left. + \frac{1}{\partial^+} \phi^a \bar{\phi}^b \partial \bar{\phi}^c \right] - g^2 f^{abc} f^{ade} \left[\frac{1}{\partial^+} (\phi^b \partial^+ \phi^c) \right. \right. \\ \left. \left. \times \frac{1}{\partial^+} (\bar{\phi}^d \partial^+ \bar{\phi}^e) + \frac{1}{2} \phi^b \bar{\phi}^c \phi^d \bar{\phi}^e \right] \right]. \end{aligned} \quad (76)$$

Using this Lagrangian, we can formally write down a well-defined DeWitt equation for this model. However, we have found that the resulting expressions are rather messy, mainly because one has to keep track of all the nonlocal ∂_+^{-1} operator insertions. A more promising avenue seems to be that one should try to link up with very recent advances in the computation of gauge theory and supersymmetric Yang-Mills amplitudes [19,20]. Although this formalism is *on shell*, whereas the effective action functional is by definition off shell, very recent work [21] indicates that it might be possible to arrive at a formulation which is not off shell in the momenta $p_{\alpha\beta} \equiv p_{\mu} \sigma_{\alpha\beta}^{\mu}$ but

would be off shell in the twistorlike variables χ_α and $\tilde{\chi}_\beta$ used to represent on-shell momenta via $p_{\alpha\beta} = \chi_\alpha \tilde{\chi}_\beta$. Clearly, this would lead to an entirely new formulation of quantum field theory and the effective action.

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Note added in proof.—After this article was submitted we learnt of related work by Ludwig Faddeev, to wit [22,23].

APPENDIX: NUMERICAL RESULTS

To illustrate the efficiency of the expansion (37), we present some numerical results for the simple one-dimensional integral

$$\exp(-W(J)) = \int \frac{dx}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}x^2 - \frac{\lambda}{4}x^4 - xJ\right] \quad (\text{A1})$$

in this appendix. To this aim, we go through the same steps as before, with the expansion (43) and $n = 1$ in Eq. (23). The loop expansion (10) here becomes

$$\begin{aligned} \Sigma^{(0)}(\varphi) &\equiv S_{\text{cl}}(\varphi) = \frac{\varphi^2}{2} + \frac{\lambda\varphi^4}{4} \\ \Sigma^{(1)}(\varphi) &= S_{\text{cl}} - \frac{1}{2} \ln(G) \\ \Sigma^{(2)}(\varphi) &= \Sigma^{(1)} - \left(-\frac{3\lambda}{4}G^2 + 3\lambda^2\varphi^2G^3\right) \\ \Sigma^{(3)}(\varphi) &= \Sigma^{(2)} - \left(\frac{3\lambda^2}{4}G^4 + \frac{9\lambda^2}{4}G^4 - 27\lambda^3\varphi^2G^5 \right. \\ &\quad \left. - 27\lambda^3\varphi^2G^5 + 54\lambda^4\varphi^4G^6 + 81\lambda^4\varphi^4G^6\right), \end{aligned} \quad (\text{A2})$$

where $G \equiv 1/(1 + 3\lambda\varphi^2)$ and where we have defined $\Sigma^{(i)} \equiv \sum_{0 \leq j \leq i} \Gamma^{(j)}$. The results for Γ_{exact} and $\Sigma^{(i)}$ for three exemplary values of λ and φ are given in the following table:

λ	1.0	1.0	100.0	200.0
φ	1.0	4.0	1.0	1.0
Γ_{exact}	1.4532	73.9458145683	28.353282939	53.6991599696
S_{cl}	0.75	72.0	25.5	50.5
$\Sigma^{(1)}$	1.4431	73.9459101483	28.353555132	53.6992974659
$\Sigma^{(2)}$	1.4431	73.9458145253	28.353282864	53.6915996015
$\Sigma^{(3)}$	1.4512	73.9458145667	28.353282912	53.6991599663

Evidently, the approximation converges rapidly (in the sense of asymptotic series) even for large values of λ .

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