HERTZ POTENTIALS AND ASYMPTOTIC PROPERTIES OF MASSLESS FIELDS

LARS ANDERSSON, THOMAS BÄCKDAHL, AND JÉRÉMIE JOUDIOUX

ABSTRACT. In this paper we analyze Hertz potentials for free massless spin-s fields on the Minkowski spacetime, with data in weighted Sobolev spaces. We prove existence and pointwise estimates for the Hertz potentials using a weighted estimate for the wave equation. This is then applied to give weighted estimates for the solutions of the spin-s field equations, for arbitrary half-integer s. In particular, the peeling properties of the free massless spin-s fields are analyzed for initial data in weighted Sobolev spaces with arbitrary, non-integer weights.

1. INTRODUCTION

The analysis by Christodoulou and Klainerman of the decay of massless fields of spins 1 and 2 on Minkowski space [10] served as an important preliminary for their proof of the non-linear stability of Minkowski space [11]. The method used in [10] was based on energy estimates using the vector fields method, see [19]. This approach was extended to fields of arbitrary spin by Shu [32]. The approach of [11] to the problem of nonlinear stability of Minkowski space was later extended by Klainerman and Nicolo [22] to give the full peeling behavior for the Weyl tensor at $\mathcal{I}$.

The vector fields method makes use of the conformal symmetries of Minkowski space to derive conservation laws for higher order energies, which then via the Klainerman Sobolev inequality [21] give pointwise estimates for the solution of the wave equation. An analogous procedure is used for the higher spin fields in the papers cited above. This procedure gives pointwise decay estimates for the solution of Cauchy problem of the wave equation and the spin-s equation, for initial data of one particular fall-off. The conditions on the initial data originate in the growth properties of the conformal Killing vector fields on Minkowski space, which are used in the energy estimates.

Let H^j_δ be the weighted L^2 Sobolev spaces on $\mathbb{R}^3$. We use the conventions[1] of Bartnik [4]. Consider the Cauchy problem for the wave equation

$$\begin{aligned} \Box\phi &= 0, \\ \phi\big|_{t=0} = f, &\quad \partial_t\phi\big|_{t=0} = g. \end{aligned} \tag{1.1}$$

Then, for $j \geq 2$ one has the estimate [19]

$$|\phi(x,t)| \leq C\langle u\rangle^{-1/2}\langle v\rangle^{-1}(\|f\|_{j,-3/2} + \|g\|_{j-1,-5/2}) \tag{1.2}$$

where $\langle u\rangle = (1+u^2)^{1/2}$, $u = \frac{1}{2}(t-r)$ and $v = \frac{1}{2}(t+r)$. On the other hand, if one considers the wave equation (1.1) on the flat 3+1 dimensional Minkowski spacetime as a special case of the conformally covariant form of the wave equation

$$(\nabla^a\nabla_a + R/6)\phi = 0,$$

the condition on the initial data which is compatible with regular conformal compactification is

$$\partial^\ell f = \mathcal{O}(r^{-2-\ell}), \quad \partial^\ell g = \mathcal{O}(r^{-3-\ell}) \tag{1.3}$$

Since we shall use the 2-spinor formalism, we work here and throughout the paper on Minkowski space with signature $+---$. Making use of standard energy estimates in the conformal

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[1] These spaces H^j_δ are in [4] denoted by $W^{j,2}_\delta$.

compactification of Minkowski space, one arrives after undoing the conformal compactification, at

$$|\phi(x,t)| = \mathcal{O}\left(<u>^{-1}<v>^{-1}\right) \tag{1.4}$$

see the discussion in [17, §6.7]. In particular, there is an extra $r^{-1/2}$ falloff in the condition (1.3) on the initial data as well as an additional factor $<u>^{-1/2}$ decay in the retarded time coordinate u in (1.4) compared to (1.2).

Let us now consider the case of higher spin fields. Let $2s$ be a positive integer and let $\phi_{A\ldots F}$ be a totally symmetric spinor field of spin s, i.e with $2s$ indices. The Cauchy problem for a massless spin-s field is

$$\nabla_{A'}{}^{A}\phi_{A\ldots F} = 0,$$
$$\phi_{A\ldots F}\big|_{t=0} = \varphi_{A\ldots F}.$$

For $s \geq 1$, the Cauchy datum $\varphi_{A\ldots F}$ must satisfy the constraint equation

$$D^{AB}\varphi_{AB\ldots F} = 0$$

where D_{AB} is the intrinsic space spinor derivative on Σ, see section 4.1. The spin $\frac{1}{2}$ case does not have constraints.

One of the main differences in asymptotic behavior between a massless scalar field satisfying a wave equation and a massless higher spin field is the existence of a hierarchy of decay rates for the different null components of the field along the outgoing null directions. This was first pointed out by Sachs in 1961 [31].

Let o_A, ι_A be a spin dyad aligned with the outgoing and ingoing null directions ∂_v, ∂_u, and let ϕ_i be the scalars of $\phi_{A\ldots F}$, defined by

$$\phi_{\mathbf{i}} = \phi_{A_1\ldots A_{\mathbf{i}}B_{\mathbf{i}+1}\ldots B_{2s}}\iota^{A_1}\cdots\iota^{A_{\mathbf{i}}}\cdots o^{B_{\mathbf{i}+1}}\cdots o^{B_{2s}}$$

One says that $\phi_{A\ldots F}$ satisfies the peeling property if the components $\phi_{\mathbf{i}}$ satisfy

$$\phi_{\mathbf{i}} = \mathcal{O}(r^{\mathbf{i}-2s-1}),$$

along outgoing null geodesics.

In [29], Penrose gave two arguments for peeling of massless fields on Minkowski space. The first, cf. [29, §4], makes use of a representation of the field in terms of a Hertz potential of order $2s$, i.e. the field is written as a derivative of order $2s$ of a potential satisfying a wave equation. Penrose assumes that the Hertz potential decays at a specific rate along outgoing null rays. He then infers the peeling property from this decay assumption.

The second approach presented by Penrose, cf. [29, §13], is based on the just mentioned fact together with the conformal invariance of the spin-s field equation. Solving the Cauchy problem in the conformally compactified picture, as was discussed for the wave equation in [17, §6.7], and taking into account the effect of the conformal rescaling, one recovers the peeling property for the solution of the massless spin-s equation on Minkowski space. Based on this analysis, Penrose conjectured that peeling for fields at $\mathcal{I}$ should be a generic property of asymptotically simple space-times.

The estimate proved in [10] for the spin-1 or Maxwell field, can be stated in the present notation as

$$|\phi_{\mathbf{i}}(t,x)| \leq C<u>^{1/2-\mathbf{i}}<v>^{\mathbf{i}-3}\|\varphi_{AB}\|_{j,-5/2}, \qquad \text{for } =1,2$$

while for the component ϕ_0 one has

$$|\phi_0(t,x)| \leq Cr^{-5/2}\|\varphi_{AB}\|_{j,-5/2}$$

along outgoing null rays. Thus, this result does not give the peeling property for all components of ϕ_{AB}, which is due to the fact that the norm $\|\varphi_{AB}\|_{j,-5/2}$ is not compatible with the conformal compactification of Minkowski space. Similarly for the spin-2 case, the result in [10] gives peeling for $\phi_{\mathbf{i}}$, $\mathbf{i} = 2,3,4$ for initial data in $H^j_{-7/2}$, while peeling fails to hold for $\phi_{\mathbf{i}}$, $\mathbf{i} = 0,1$. On the other hand, the condition on the initial datum which is compatible with a regular conformal compactification, and which hence also gives peeling, is for a spin-s field

$$\partial^\ell \varphi_{A\ldots F} = \mathcal{O}(r^{-2s-2-\ell}) \tag{1.5}$$

In this paper we shall follow an approach outlined by Penrose in [29, §6] to give a weighted decay estimate for spin-s fields of arbitrary, half-integer spin s. The result proved here admits conditions on the initial data which include the ones considered in [10, 32], as well as conditions which are compatible with peeling, but also general weights. The results of this paper clarify the relation beween the condition on the inital datum and the peeling property of the solution the spin-s field equation. In this paper we shall make use of some estimates for elliptic equations in weighted Sobolev spaces, and for technical reasons these are not compatible with the integer powers or r as in (1.5).

The method we shall use is based on the notion of Hertz potentials. For background, see Stewart [35], Fayos et al. [14] and references therein, see also Benn et al. [6]. Since Minkowski space is topologically trivial, there is no obstruction to representing a Maxwell field on Minkowski space in terms of a Hertz potential. However, this general fact does not provide estimates for the potential. In this paper we prove the necessary estimates not only for the Maxwell field but for fields with general half-integer spins.

To introduce the method we consider the spin-1 case, i.e. the Maxwell field on 3+1 dimensional Minkowski space. With our choice of signature, the metric on the spatial slice is negative definite.

The Maxwell field is a real differential 2-form F_{ab} which is closed and divergence free. For convenience we consider the complex self-dual form

$$\mathcal{F}_{ab} = F_{ab} - i * F_{ab},$$

which corresponds to a symmetric 2-spinor ϕ_{AB} via

$$\mathcal{F}_{ab} = \phi_{AB}\epsilon_{A'B'}. \tag{1.6}$$

In terms of $\mathcal{F}_{ab}$, the Maxwell equation is simply

$$(d\mathcal{F})_{abc} = 0. \tag{1.7}$$

Let $\xi^a = (\partial_t)^a$ be the unit normal to the Cauchy surface $\Sigma = \{t = 0\}$. Given a complex 1-form $\mathcal{E}_a$ on Σ, with divergence zero there is a unique solution of the Maxwell equation such that

$$(\mathcal{F}_{ab}\xi^b)\big|_\Sigma = \mathcal{E}_a.$$

Now let $\mathcal{H}_{ab}$ be an anti-self-dual 2-form which solves the wave equation

$$\Box\, \mathcal{H}_{ab} = 0, \tag{1.8}$$

where $\Box = dd^* + d^*d$ is the Hodge wave operator, and $d^* = *d*$ is the exterior co-derivative. Defining the form $\mathcal{F}_{ab}$ by

$$\mathcal{F}_{ab} = dd^*\mathcal{H}_{ab}, \tag{1.9}$$

we have using (1.8) that $\mathcal{F}_{ab}$ is self-dual, and solves the Maxwell equation. The form $\mathcal{H}_{ab}$ is called a Hertz-potential for $\mathcal{F}_{ab}$. Since we are working on Minkowski space, the wave equation (1.8) is just a collection of scalar wave equations for the components of $\mathcal{H}_{ab}$, and hence the solution to (1.8) for given Cauchy data can be analyzed using results for the scalar wave equation. Thus, if we are able to relate the Cauchy data for the Maxwell field $\mathcal{F}_{ab}$ to the Cauchy data for $\mathcal{H}_{ab}$, we may use the Hertz potential construction to prove estimates for the solution of the Maxwell field equation, starting from estimates for the wave equation.

Let the complex 1-form $\mathcal{K}_a$ be the "electric field" corresponding to $\mathcal{H}_{ab}$,

$$\mathcal{K}_a = \mathcal{H}_{ab}\xi^b.$$

A calculation shows that if $\mathcal{F}_{ab}$ is defined in terms of $\mathcal{H}_{ab}$ by (1.9), the Cauchy data for (1.8) is related to the Cauchy data for $\mathcal{F}_{ab}$ by

$$\mathcal{E}_a = - * d * d\mathcal{K}_a + i * d\partial_t\mathcal{K}_a, \tag{1.10}$$

where in the right hand side we restrict $\mathcal{K}_a$ and $\partial_t \mathcal{K}_a$ to Σ, and $d, *$ act on objects on Σ. The constraint equation $d^* \mathcal{E}_a = 0$ holds automatically for $\mathcal{E}_a$ given by (1.10).

Now, in order to prove estimates for the Maxwell equation with data $\mathcal{E}_a \in H^j_\delta$, satisfying $d^* \mathcal{E} = 0$, it is sufficient to show that for any such $\mathcal{E}_a$, there exists a 1-form $\mathcal{L}_a \in H^{j+1}_{\delta+1}$ such that

$$\mathcal{E}_a = i * d\mathcal{L}_a. \tag{1.11}$$

Then taking $\mathcal{H}_{ab}$ to be a solution of (1.8) with Cauchy data

$$\mathcal{H}_{ab}\big|_{t=0} = 0, \quad \left(\partial_t \mathcal{H}_{ab} \xi^b\right)\big|_{t=0} = \mathcal{L}_a,$$

gives a solution to the Maxwell equation via (1.9). Estimates for the wave equation can thus be applied to give estimates for the solution of the Maxwell field equation.

The operator $*d$ acting on one forms, which appears in (1.10) and (1.11) is simply the curl operator. This is a special case of the operator $\mathscr{C}_{2s}$,

$$(\mathscr{C}_{2s}\phi)_{AB\ldots F} = D_{(A}{}^{G}\phi_{B\ldots F)G},$$

acting on space spinors of half-integer spin s, see Definition 2.11 below. Below in Proposition 4.7, we shall prove that for non-integer weights $\delta > -4$, $*d : H^{j+1}_{\delta+1} \to \ker d^* \cap H^j_\delta$ is a surjection, and the estimate

$$\|\mathcal{L}_a\|_{j+1,\delta+1} \leq C \|\mathcal{E}_a\|_{j,\delta} \tag{1.12}$$

holds, for some constant C. However, for $\delta < -4$ the operator $*d$ is not a surjection to $\ker d^*$, i.e. the space of solutions to the Maxwell constraint equation, but to a subspace of finite codimension, which we characterize, see section 4.3. This result is based on ideas from Hodge theory, in particular we make use of the fact that the operators d, $*d$, d^* form an elliptic complex, closely related to the de Rahm complex, see section 3. In Proposition 4.7 we give the proof for the spin-1 case in terms of space spinors. This gives an outline for the treatment of the general spin-s case which is given in Proposition 4.8. The latter result is based on a generalization of the above mentioned elliptic complex to higher spin fields and on expressions of powers of the Laplacian in terms of the corresponding operators.

It is instructive to consider two special cases. First we consider the case $\mathcal{E}_a \in \ker d^* \cap H^j_{-5/2}$ which corresponds to the case considered in [10]. In this case, the Cauchy data for the Hertz potential is in $H^{j+2}_{-1/2} \times H^{j+1}_{-3/2}$. Since the Laplacian $\Delta = dd^* + d^*d$ is a surjection $H^{j+2}_{-1/2} \to H^j_{-5/2}$ it is straightforward to show using the fact that the ranges of d and $*d$ are L^2 orthogonal, and $d^* * d = 0$, that $*d$ is a surjection $H^{j+1}_{-3/2} \to \ker d^* \cap H^j_{-5/2}$. Secondly, we consider the case $\mathcal{E}_a \in \ker d^* \cap H^j_{-7/2}$ where full peeling holds. In this case, the Cauchy data for the Hertz potential is in $H^{j+2}_{-3/2} \times H^{j+1}_{-5/2}$. The relevant fact about the Laplacian is now that $\Delta : H^{j+2}_{-3/2} \to H^j_{-7/2}$ is Fredholm with cokernel consisting of constant forms. Since a constant form ξ_a is automatically closed, it is also exact, $\xi_a = df$ for some f. Hence, the cokernel of Δ is automatically L^2-orthogonal to $\ker d^* \cap H^j_{-7/2}$. Using this fact, it follows that $*d : H^{j+1}_{-5/2} \to \ker d^* \cap H^j_{-7/2}$ is a surjection.

The argument for the higher spin case follows the same outline. However in that case, for spin s, one must consider a power of the Laplacian $\Delta^{\lfloor s \rfloor}$ and a decomposition of this in terms of fundamental operators introduced in section 2.3.

The existence of a solution to (1.11) with an estimate (1.12) is then used together with a weighted estimate for the solution of the wave equation with initial $(f, g) \in H^j_\delta \times H^{j-1}_{\delta-1}$ for non-integer δ. As we have not found a sufficiently general result in the literature, in particular which covers the range of weights $\delta > -1$ which we need for the applications to the Hertz potential in the range where full peeling fails to hold (including the situation considered in [10]), we prove the required result in section 5. This result consists of a direct estimate for the solution of the wave equation, using the representation formula. For $\delta < -1$ we have in the exterior region

$$|\phi(t,x)| \leq C \langle v \rangle^{-1} \langle u \rangle^{1+\delta} \left(\|f\|_{3,\delta} + \|g\|_{2,\delta-1}\right),$$

see Proposition 5.2.

The core of the paper is the proof of the existence of a Hertz potential for all massless fields; this Hertz potential has the form:

$$\phi_{A\dots F} = \nabla_{AA'} \dots \nabla_{FF'} \chi^{A'\dots F'}, \quad \text{where } \Box \chi^{A'\dots F'} = 0 \tag{1.13}$$

For the spin-1, or Maxwell case, equation (1.13) takes the form

$$\phi_{AB} = \nabla_{AA'} \nabla_{BB'} \chi^{A'B'}, \quad \text{where } \Box \chi^{A'B'} = 0$$

which is equivalent, when written in terms of differential forms, to equations (1.7) and (1.8), via the correspondence (1.6).

The main result of the paper, see Theorem 7.1, combines the analysis of the Hertz potential Cauchy data in weighted Sobolev spaces with the weighted estimate for the solution of the wave equation to provide a weighted estimate for the solution to the massless spin-s field equation. The peeling properties of the spin-s field with initial data in weigted Sobolev spaces are analyzed in detail. Here it is important to note that the detailed decay estimates for the components of the *massless spin-s field* $\phi_{A\dots F}$ relies on the interplay between the Hertz potential $\chi^{A'\dots F'}$ which behaves according as a solution of the *wave equation*. The decay properties of the components of $\phi_{A\dots F}$ comes about due to their relation to the derivatives of $\chi^{A'\dots F'}$ in terms of a null tetrad.

Overview of this paper. In section 2 we state our conventions and recall some basic facts about elliptic operators on weighted Sobolev space. In particular we introduce the Stein-Weiss operators divergence $\mathscr{D}$, curl $\mathscr{C}$ and the twistor operator $\mathscr{T}$ for higher spin fields, as well as a fundamental higher order operator $\mathcal{G}$ originating in the 3+1 splitting of the Hertz potential equation. In section 3 we use these to introduce a generalization of the deRham sequence for spinor fields. The problem of constructing initial data for the Hertz potential is solved in section 4, and this then gives the existence of Hertz potentials. In the analysis of the initial data for the Hertz potential we make use of the space spinor formalism, see section 4.1. The weighted estimate for the wave equation is given in section 5, and the resulting estimates for the spin-s fields is given in section 6. The main result is stated in section 7. Appendix A contains some results on $\mathcal{G}$ used in the analysis of the elliptic complex introduced in section 3 as well as for the construction of the Hertz potential in section 4.

2. Preliminaries

2.1. **Conventions.** In this paper, we will only work on Minkowski space time. The spinor formalism with the conventions of [30] is extensively used. For important parts of the paper, 3+1 splittings of spinor expressions are performed. The space spinor formalism as introduced in [34] is therefore used. In this case, the conventions of [3] are adopted. We will always consider the space spinors on the $t = const.$ slices of Minkowski space with normal $\tau_{AA'} = \sqrt{2}\nabla_{AA'}t$. Observe that a *negative* definite metric on the slices is used.

The Minkowski space-time $(\mathbb{R}^4, \eta_{\alpha\beta})$ is endowed with its standard connection $\nabla_a = \nabla_{AA'}$. The time slice $\{t = 0\}$ is endowed with the connection $D_a = D_{AB}$ defined by:

$$D_{AB} = \tau_{(A}{}^{A'} \nabla_{B)A'}$$

where $\tau_{AA'}$ is the timelike vector field defined above. Its relation to the connection of the ambient space-time is given by

$$\nabla_{AA'} = \frac{1}{\sqrt{2}} \tau_{AA'} \partial_t - \tau^B{}_{A'} D_{AB}.$$

The following set of spinors are then defined:

Definition 2.1. *Let $\mathcal{S}_k$ denote the set of symmetric valence k space spinor fields on $\mathbb{R}^3$.*

Definition 2.2. *Let $\mathcal{P}_k^{<\delta}$ denote the set of spinor fields in $\mathcal{S}_k$ spanned by constant spinors with polynomial coefficients of degree $< \delta$.*

Observe that with $\delta \le 0$, $\mathcal{P}_k^{<\delta}$ is just the trivial space $\{0\}$.

2.2. **Analytic framework.** We introduce in this section the analytic framework which is necessary to understand the propagation of the field as well as the geometric constraints. We will use the conventions of Bartnik [4]. Even though Bartnik's paper only gives statements for functions, we can easily extend this to space spinors on the flat space.

We recall first the standard norms, coming from the Hermitian space spinor product:

Definition 2.3. *The Hermitian space spinor product is defined via*

$$\langle \zeta_{A\dots F}, \phi_{A\dots F} \rangle = \zeta_{A\dots F}\widehat{\phi}^{A\dots F},$$

where $\widehat{\phi}^{A\dots F} = \tau^{AA'}\dots\tau^{FF'}\overline{\phi}_{A'\dots F'}$ *and* $\tau_{AA'} = \sqrt{2}\nabla_{AA'}t$. *The pointwise norm of a smooth* $\phi_{A\dots F}$ *is defined via*

$$|\phi_{A\dots F}|^2 = \phi_{A\dots F}\widehat{\phi}^{A\dots F}.$$

The pointwise norm of the derivatives of the smooth spinor $\phi_{A\dots F}$ *on* $\mathbb{R}^3$ *is given by*

$$|D_a\phi_{A\dots F}|^2 = \delta^{ab}D_a\phi_{A\dots F}\widehat{D_b\phi}^{A\dots F},$$

where δ_{ab} *is the standard Euclidean metric on* $\mathbb{R}^3$. *The norm of higher order derivatives is defined similarly.*

Remark 2.4. *The identity*

$$D_{AB}\widehat{\phi}_{A\dots F} = -\widehat{D_{AB}\phi}_{A\dots F}$$

holds, due to the reality of the operator D_{AB}.

Definition 2.5. *The* L^2*-norm of a smooth spinor field in* $\mathbb{R}^3$ *is defined by:*

$$\|\phi_{A\dots F}\|_2 = \left(\int_{\mathbb{R}^3} |\phi_{A\dots F}|^2 d\mu_{\mathbb{R}^3}\right)^{\frac{1}{2}},$$

where $d\mu_{\mathbb{R}^3}$ *is the standard volume form on* $\mathbb{R}^3$. *The* L^2*-norm is also defined for derivatives in the same way using the pointwise definition above.*

If u is a real scalar, its japanese bracket is defined by:

$$<u> = (1+u^2)^{\frac{1}{2}}.$$

The weighted Sobolev norms, necessary to describe the asymptotic behavior of initial data at space-like infinity, are defined by:

Definition 2.6 (Weighted Sobolev spaces)**.** *Let* δ *be a real number and* j *a nonnegative integer. The completion of the space of smooth spinor fields in* $\mathcal{S}_{2s}$ *with compact support in* $\mathbb{R}^3$ *endowed with the norm*

$$\|\phi_{A\dots F}\|^2_{j,\delta} = \sum_{n=0}^{j} \left\| <|x|>^{-(\delta+\frac{3}{2})+n} D^n\phi_{A\dots F}\right\|_2^2,$$

is denoted by $H^j_\delta(\mathcal{S}_{2s})$.

Remark 2.7.
- *For* $\delta = -3/2$, *the weighted space* $H^0_{-3/2}(\mathcal{S}_{2s})$ *is the standard Sobolev spaces* $L^2(\mathcal{S}_{2s})$.
- *The derivatives decay faster (or grow slower, accordingly to the sign of* δ*) than the spinor field.*

Many well-known properties can be proved about these spaces – see for instance [4, Theorem 1.2] for more details. The only property, crucial to obtain the pointwise estimates, is the following Sobolev embedding ([4, Theorem 1.2, (iv)], in the specific case of the dimension 3):

Proposition 2.8. *Let* δ *be a real number and* $j \geq 2$ *an integer. Then, any spinor field in* $H^j_\delta(\mathcal{S}_{2s})$ *is in fact continuous and there exists a constant* C *such that, for any* $\phi_{A\dots F}$ *in* $H^j_\delta(\mathcal{S}_{2s})$

$$|\phi_{A\dots F}(x)| \leq C<|x|>^\delta \|\phi_{A\dots F}\|_{2,\delta},$$

and, in fact,

$$|\phi_{A\dots F}(x)| = o\left(<|x|>^\delta\right) \text{ as } <|x|> \to \infty.$$

We finally recall the following properties of elliptic operators, restricting ourself to the case of the powers of the Laplacian. The result stated is a combination of the standard results in [4, 27, 8, 18].

Proposition 2.9. *Let j, l be non-negative integers such that $j \geq 2l$, $s \geq 0$ be in $\frac{1}{2}\mathbb{Z}$ and δ be in $\mathbb{R} \setminus \mathbb{Z}$. The formally self-adjoint elliptic operator of order $2l$:*

$$\Delta_{2s}^{l} : H_{\delta}^{j}(\mathcal{S}_{2s}) \longrightarrow H_{\delta-2l}^{j-2l}(\mathcal{S}_{2s})$$

satisfies:

- *its kernel is a subset of $\mathcal{P}_{2s}^{<\delta}$; in particular, Δ_{2s}^{l} is injective when $\delta < 0$;*
- *its co-kernel is a subset of $\mathcal{P}_{2s}^{<-3-\delta+2l}$; in particular, Δ_{2s}^{l} is surjective when $\delta > 2l - 3$.*

Furthermore, the following closed range estimates holds: there exists a constant C such that, for all spinor fields in $H_{\delta}^{j}(\mathcal{S}_{2s})$,

$$\min_{\psi_{A\dots F} \in \ker(\Delta_{2s}^{l}) \cap H_{\delta}^{j}(\mathcal{S}_{2s})} \left(\|\phi_{A\dots F} + \psi_{A\dots F}\|_{j,\delta} \right) \leq C \|\Delta_{2s}^{l} \phi_{A\dots F}\|_{j-2s,\delta-2s}.$$

Remark 2.10.

- *In the range of weights $[-1, 0]$, the operator Δ is both injective and surjective.*
- *The use of the closed range estimate usually comes with an infimum (see* [18, Theorem 5.2]*). Since this infimum corresponds to the distance to the kernel* $\ker(\Delta_{2s}^{l})$*, which is closed, this infimum is in fact attained.*
- *We recall here that, due to its self-adjointness, the co-kernel of Δ_{2s}^{l} in $H_{\delta-2l}^{j-2l}(\mathcal{S}_{2s})$ is L^2-orthogonal to the kernel of Δ_{2s}^{l} in the dual space $H_{-3-\delta+2l}^{-j+2l}(\mathcal{S}_{2s})$.*
- *The dimension of the spaces can be computed explicitly – see for instance* [25]*. However, for our purpose, we do not need to know the dimension.*

2.3. **Fundamental operators.**

Definition 2.11. *Let $\phi_{A_1\dots A_k} \in \mathcal{S}_k$, that is $\phi_{A_1\dots A_k} = \phi_{(A_1\dots A_k)}$. Let D_{AB} be the intrinsic Levi-Civita connection. Define the operators $\mathfrak{d}_k : \mathcal{S}_k \to \mathcal{S}_{k-2}$, $\mathfrak{c}_k : \mathcal{S}_k \to \mathcal{S}_k$ and $\mathfrak{t}_k : \mathcal{S}_k \to \mathcal{S}_{k+2}$ via*

$$(\mathfrak{d}_k\phi)_{A_1\dots A_{k-2}} \equiv D^{A_{k-1}A_k}\phi_{A_1\dots A_k},$$

$$(\mathfrak{c}_k\phi)_{A_1\dots A_k} \equiv D_{(A_1}{}^{B}\phi_{A_2\dots A_k)B},$$

$$(\mathfrak{t}_k\phi)_{A_1\dots A_{k+2}} \equiv D_{(A_1A_2}\phi_{A_3\dots A_{k+2})}.$$

These operators will be called divergence, curl *and* twistor operator *respectively.*

We suppress the indices of ϕ in the left hand sides. The label k indicates how many indices it has. The importance of these operators comes from the following irreducible decomposition which is valid for any $k \geq 1$

$$\begin{aligned} D_{A_1A_2}\phi_{A_3\dots A_{k+2}} = (\mathfrak{t}_k\phi)_{A_1\dots A_{k+2}} &- \tfrac{k}{k+2}\epsilon_{A_1(A_3}(\mathfrak{c}_k\phi)_{A_4\dots A_{k+2})A_2} \\ &- \tfrac{k}{k+2}\epsilon_{A_2(A_3}(\mathfrak{c}_k\phi)_{A_4\dots A_{k+2})A_1} + \tfrac{1-k}{1+k}\epsilon_{A_1(A_3}(\mathfrak{d}_k\phi)_{A_4\dots A_{k+1}}\epsilon_{A_{k+2})A_2}. \end{aligned}$$

This irreducible decomposition follows from [30, Proposition 3.3.54]. Contraction with ϵs and partial expansion of the symmetries give the actual coefficients.

Lemma 2.12. *The symbol $\sigma_x(\mathfrak{c}_k)$ of $\mathfrak{c}_k$ is Hermitian and has only real eigenvalues.*

Proof. By definition we have

$$(\sigma_x(\mathfrak{c}_k)\phi)_{A_1\dots A_k} \equiv X_{(A_1}{}^{B}\phi_{A_2\dots A_k)B},$$

where X_{AB} is real, i.e. $\widehat{X}_{AB} = -X_{AB}$. For arbitrary $\xi_{A_1\dots A_k}$ and $\zeta_{A_1\dots A_k}$, we have

$$\begin{aligned} \langle (\sigma_x(\mathfrak{c}_k)\xi)_{A_1\dots A_k}, \zeta_{A_1\dots A_k} \rangle &= X_{A_1}{}^{B}\xi_{A_2\dots A_kB}\hat{\zeta}^{A_1\dots A_k} = \xi_{A_2\dots A_kB}\widehat{X}^{C_1B}\hat{\zeta}_{C_1}{}^{A_2\dots A_k} \\ &= \langle \xi_{A_1\dots A_k}, (\sigma_x(\mathfrak{c}_k)\zeta)_{A_1\dots A_k} \rangle. \end{aligned}$$

Hence, the symbol $\sigma_x(\mathfrak{c}_k)$ is Hermitian and, by the spectral theorem, it has only real eigenvalues. $\square$

The operators $\mathfrak{d}_k$, $\mathfrak{c}_k$ and $\mathfrak{t}_k$ are special cases of Stein-Weiss operators. We refer to [7] and references therein for general properties of this class of operators.

2.3.1. *Important higher order operators.* The most important higher order operator is clearly the Laplacian $\Delta \equiv D_{AB}D^{AB}$. Observe that we are using a *negative* definite metric on $\mathbb{R}^3$ which affects the definition of Δ. When the Laplacian acts on a spinor field $\phi_{A\dots F}$ in $\mathcal{S}_k$, we will often use the notation $(\Delta_k\phi)_{A\dots F}$, where k indicates the valence of $\phi_{A\dots F}$.

Definition 2.13. *Define the order $k-1$ operators $\mathcal{G}_k : \mathcal{S}_k \to \mathcal{S}_k$ as*

$$(\mathcal{G}_k\phi)_{A_1\dots A_k} \equiv \sum_{n=0}^{\left\lfloor\frac{k-1}{2}\right\rfloor} \binom{k}{2n+1}(-2)^{-n} \underbrace{D_{(A_1}{}^{B_1}\cdots D_{A_{k-2n-1}}{}^{B_{k-2n-1}}}_{k-2n-1}(\Delta_k^n\phi)_{A_{k-2n}\dots A_k)B_1\dots B_{k-2n-1}}.$$

The first operators are

$$\begin{aligned}
(\mathcal{G}_1\phi)_A &\equiv \phi_A,\\
(\mathcal{G}_2\phi)_{AB} &\equiv 2D_{(A}{}^C\phi_{B)C} = 2(\mathfrak{c}_2\phi)_{AB},\\
(\mathcal{G}_3\phi)_{ABC} &\equiv 3D_{(A}{}^D D_B{}^F\phi_{C)DF} - \tfrac{1}{2}(\Delta_3\phi)_{ABC}\\
&= \tfrac{1}{3}(\mathfrak{t}_1\mathfrak{d}_3\phi)_{ABCD} + 4(\mathfrak{c}_3\mathfrak{c}_3\phi)_{ABC},\\
(\mathcal{G}_4\phi)_{ABCD} &\equiv 4D_{(A}{}^F D_B{}^H D_C{}^L\phi_{D)FHL} - 2D_{(A}{}^F(\Delta_4\phi)_{BCD)F}\\
&= 2(\mathfrak{t}_2\mathfrak{d}_4\mathfrak{c}_4\phi)_{ABCD} + 8(\mathfrak{c}_4\mathfrak{c}_4\mathfrak{c}_4\phi)_{ABCD}.
\end{aligned}$$

These operators naturally appear in Proposition 4.1. The most important properties of these operators are $\mathfrak{d}_k\mathcal{G}_k = 0$ and $\mathcal{G}_k\mathfrak{t}_{k-2} = 0$, which is valid for any $k \geq 2$. The main idea to prove this is to use that $\mathfrak{d}_k\mathcal{G}_k$ and $\mathcal{G}_k\mathfrak{t}_{k-2}$ contains derivatives of the kind $D_A{}^C D_{BC} = \frac{1}{2}\epsilon_{AB}\Delta$. For a complete proof see Proposition A.3. The operators $\mathcal{G}_k$ also commute with $\mathfrak{c}_k$; this is proven in Proposition A.1.

To relate these operators with elliptic theory, we express appropriate powers of the Laplacian in terms of the $\mathcal{G}_k$ operators as

$$(\Delta_{2k}^k\phi)_{A_1\dots A_{2k}} = (\mathfrak{t}_{2k-2}\mathcal{F}_{2k-2}\mathfrak{d}_{2k}\phi)_{A_1\dots A_{2k}} - (-2)^{1-k}(\mathcal{G}_{2k}\mathfrak{c}_{2k}\phi)_{A_1\dots A_{2k}}, \tag{2.1a}$$

$$(\Delta_{2k+1}^k\phi)_{A_1\dots A_{2k+1}} = (\mathfrak{t}_{2k-1}\mathcal{F}_{2k-1}\mathfrak{d}_{2k+1}\phi)_{A_1\dots A_{2k+1}} + (-2)^{-k}(\mathcal{G}_{2k+1}\phi)_{A_1\dots A_{2k+1}}, \tag{2.1b}$$

where the operators $\mathcal{F}_{2s}$ are defined via

$$\begin{aligned}
(\mathcal{F}_{2s}\phi)_{A_1\dots A_{2s}} = 2^{-2s}&\sum_{n=0}^{\lfloor s\rfloor}\sum_{m=0}^{\lfloor s\rfloor-n}\binom{2s+2}{2n+2m+2}(-2)^n\\
&\times \underbrace{D_{(A_1}{}^{B_1}\cdots D_{A_{2n}}{}^{B_{2n}}}_{2n}(\Delta_{2s}^{\lfloor s\rfloor-n}\phi)_{A_{2n+1}\dots A_{2s})B_1\dots B_{2n}}.
\end{aligned}$$

The first operators are

$$\begin{aligned}
(\mathcal{F}_0\phi) &= \phi,\\
(\mathcal{F}_1\phi)_A &\equiv \tfrac{3}{2}\phi_A,\\
(\mathcal{F}_2\phi)_{AB} &= \tfrac{7}{4}(\Delta_2\phi)_{AB} - \tfrac{1}{2}D_{(A}{}^C D_{B)}{}^D\phi_{CD}.
\end{aligned}$$

See Lemma A.4 in the appendix for the proof of (2.1a) and (2.1b).

The operator $\mathfrak{c}_2$ is the spinor equivalent to the operator $*\mathrm{d}$ acting on 1-forms. The tensor equivalent of the operator $\mathcal{G}_4$ is the linearized Cotton-York tensor acting on symmetric trace-free 2-tensors. In the following section, a more detailed description of these relations is given.

3. Integrability properties of spinor fields

A crucial part of this work relies on integrability properties for spinors. In the case of spin 1, these integrability properties are well known since it corresponds to standard integrability properties of 1-forms. For spin 2, one has to resort to a generalization of the de Rham theory to trace free 2-tensors, which happened to have been studied in the context of conformal deformation of flat structure by Gasqui and Goldschmidt [16], whose results were extended by Beig [5]. On $\mathbb{R}^3$, we prove a generalization of these elliptic sequences for arbitrary spin.

We present here the general picture of this integrability result for smooth spinors. It is well known that for 1-forms the integrability conditions is given by the following elliptic complex

$$C^\infty(\mathbb{R}^3,\mathbb{R}) \xrightarrow{d} \Lambda^1 \xrightarrow{\star d} \Lambda^1 \xrightarrow{d^*} C^\infty(\mathbb{R}^3,\mathbb{R}),$$

whose spinorial equivalent is

$$\mathcal{S}_0 \xrightarrow{\mathfrak{t}_0} \mathcal{S}_2 \xrightarrow{\mathscr{C}_2} \mathcal{S}_2 \xrightarrow{\mathfrak{d}_2} \mathcal{S}_0. \tag{3.1}$$

Gasqui and Goldschmidt were interested in the conformal deformation of a metric on a 3-manifold M.

Definition 3.1. *The deformation g_t of a metric g_0 is said to be conformally rigid if there exist a family of diffeomorphisms $\phi_t^\star$ and of functions u_t such that:*

$$\phi_t^\star g_0 = e^{u_t} g_t.$$

The infinitesimal equation corresponding to this definition is given by the conformal Killing equation:

$$\mathcal{L}_X g_0 - \frac{1}{3}\mathrm{Tr}_{g_0}(\mathcal{L}_X g_0) g_0 = h \tag{3.2}$$

where X a vector field on M and h is a trace free 2-tensor. The spinor equivalent of this equation is given by

$$2D_{(AB}X_{CD)} = h_{ABCD}.$$

Solving (3.2) requires that the 2-tensor h satisfies the constraint equation. This is stated in [16, Theorem 6.1, (2.24)] and the following proposition in [5]:

Theorem 3.2 (Gasqui-Goldschmidt). *If (M,g) is a conformally flat 3-dimensional manifold, then the following is an elliptic complex*

$$\Lambda^1(M) \xrightarrow{L} S_0^2(M,g) \xrightarrow{\mathcal{R}} S_0^2(M,g) \xrightarrow{div} \Lambda^1(M),$$

where $\Lambda^1(M)$ is the space of 1-form over M, $S_0^2(M,g)$ is the space of symmetric trace free 2-tensors and

$$\begin{aligned} (LW)_{ab} &= D_{(a}W_{b)} - \frac{1}{3}g_{ab}D^cW_c \\ (div t)_a &= 2g^{bc}D_c t_{ab} \end{aligned}$$

and

$$\mathcal{R}(\psi)_{ab} = \epsilon^{cd}{}_a D_{[c}\sigma_{d]b} \text{ where}$$
$$\sigma_{ab} = D_{(a}D^c\psi_{b)c} - \tfrac{1}{2}\Delta\psi_{ab} - \tfrac{1}{4}g_{ab}D^cD^d\psi_{cd}.$$

Remark 3.3.
- *A consequence of this proposition is that equation (3.2) is integrable provided that*

$$\mathcal{R}(h)_{ab} = 0.$$

- *In terms of spinors, the operator $\mathcal{R}_{ab}$ reads*

$$\mathcal{R}_{ab} = \mathcal{R}_{ABCD} = -\frac{i}{2\sqrt{2}}\mathcal{G}_4.$$

The spinorial equivalent of this sequence is the following elliptic complex

$$\mathcal{S}_2 \xrightarrow{\mathfrak{t}_2} \mathcal{S}_4 \xrightarrow{\mathcal{G}_4} \mathcal{S}_4 \xrightarrow{\mathfrak{d}_4} \mathcal{S}_2. \tag{3.3}$$

We now state, using the fundamental operators $\mathfrak{t}_{2s-2}$, $\mathcal{G}_{2s}$ and $\mathfrak{d}_{2s}$, a generalization of the elliptic complexes (3.1) and (3.3) for arbitrary spin:

Lemma 3.4. *The sequence*

$$\mathcal{S}_{2s-2} \xrightarrow{\mathfrak{t}_{2s-2}} \mathcal{S}_{2s} \xrightarrow{\mathcal{G}_{2s}} \mathcal{S}_{2s} \xrightarrow{\mathfrak{d}_{2s}} \mathcal{S}_{2s-2},$$

is an elliptic complex.

Proof. In view of Proposition A.3, the sequence is a differential complex. It is therefore enough to check that the symbol sequence is exact, i.e. for $x \in \mathbb{R}^3$,

$$T_x^\star \mathcal{S}_{2s-2} \xrightarrow{\sigma_x(\mathfrak{t}_{2s-2})} T_x^\star \mathcal{S}_{2s} \xrightarrow{\sigma_x(\mathcal{G}_{2s})} T_x^\star \mathcal{S}_{2s} \xrightarrow{\sigma_x(\mathfrak{d}_{2s})} T_x^\star \mathcal{S}_{2s-2}$$

This follows from the vanishing properties stated in Proposition A.3 and the expression of powers of the Laplacian in these operators, i.e. (2.1a) and (2.1b).

As we are working only with constant coefficient operators, we use $\sigma(\cdot)$ for the symbols in the rest of the proof in order to avoid clutter. We first notice that the relation $\mathfrak{d}_{2s}\mathcal{G}_{2s} = 0$ (Proposition A.3) implies:

$$\operatorname{im}(\sigma(\mathcal{G}_{2s})) \subset \ker(\sigma(\mathfrak{d}_{2s})).$$

Similarly, the relations $\mathcal{G}_{2s}\mathfrak{t}_{2s-2} = 0$ implies:

$$\operatorname{im}(\sigma(\mathfrak{t}_{2s-2})) \subset \ker(\sigma(\mathcal{G}_{2s}))$$

We then notice that the symbol of the Laplacian Δ_{2s}^k is an invertible symbol which is in the center of the algebra of symbols since its expression is

$$\sigma(\Delta_{2s}^k) = r^{2k} I.$$

Furthermore, using the relations stated in Lemma A.4, we have

$$\begin{aligned}
(\Delta_{2s}^s \phi)_{A_1 \dots A_{2s}} &= (\mathfrak{t}_{2s-2}\mathcal{F}_{2s-2}\mathfrak{d}_{2s}\phi)_{A_1\dots A_{2s}} - (-2)^{1-s}(\mathcal{G}_{2s}\mathfrak{c}_{2s}\phi)_{A_1\dots A_{2s}} \text{ for } s \in \mathbb{Z}, \\
(\Delta_{2s+1}^{s-\frac{1}{2}} \phi)_{A_1 \dots A_{2s+1}} &= (\mathfrak{t}_{2s-2}\mathcal{F}_{2s-2}\mathfrak{d}_{2s}\phi)_{A_1\dots A_{2s}} + (-2)^{-\frac{1}{2}-s}(\mathcal{G}_{2s}\phi)_{A_1\dots A_{2s}} \text{ for } s \in \frac{1}{2} + \mathbb{Z}.
\end{aligned} \tag{3.4}$$

Assume now that the spin is an integer. The proof in the case when the spin is a half integer is left to the reader (the proof is almost identical). Let Y be an element of $\ker(\sigma(\mathfrak{d}_{2s}))$. Using formula (3.4), we get:

$$\begin{aligned}
Y &= \sigma(\Delta_{2s}^s)^{-1}\sigma(\Delta_{2s}^s)Y \\
&= \sigma(\Delta_{2s}^s)^{-1}\left(\sigma(\mathfrak{t}_{2s-2})\sigma(\mathcal{F}_{2s-2})\sigma(\mathfrak{d}_{2s}) - (-2)^{1-s}\sigma(\mathcal{G}_{2s})\sigma(\mathfrak{c}_{2s})\right)Y \\
&= -(-2)^{1-s}\sigma(\Delta_{2s}^s)^{-1}\sigma(\mathcal{G}_{2s})\sigma(\mathfrak{c}_{2s})Y
\end{aligned}$$

Since $\mathcal{G}_{2s}$ and $\mathfrak{c}_{2s}$ commute according to Lemma A.1 and since the symbol of the Laplacian commutes with all other symbols, we consequently get

$$Y = -(-2)^{1-s}\sigma(\mathcal{G}_{2s})\sigma(\Delta_{2s}^s)^{-1}\sigma(\mathfrak{c}_{2s})Y,$$

that is to say that Y belongs to the image of $\sigma(\mathcal{G}_{2s})$. If we now assume that Y is in $\ker(\sigma(\mathcal{G}_{2s}))$. Using formula (3.4), we get

$$\begin{aligned}
Y &= \sigma(\Delta_{2s}^s)^{-1}\sigma(\Delta_{2s}^s)Y \\
&= \sigma(\Delta_{2s}^s)^{-1}\left(\sigma(\mathfrak{t}_{2s-2})\sigma(\mathcal{F}_{2s-2})\sigma(\mathfrak{d}_{2s}) - (-2)^{1-s}\sigma(\mathcal{G}_{2s})\sigma(\mathfrak{c}_{2s})\right)Y.
\end{aligned}$$

Since $\mathcal{G}_{2s}$ and $\mathfrak{c}_{2s}$ commute (Lemma A.1) and since the symbol of the Laplacian commutes with all other symbols, we get

$$Y = \sigma(\mathfrak{t}_{2s-2})\sigma(\Delta_{2s}^s)^{-1}\sigma(\mathcal{F}_{2s-2})\sigma(\mathfrak{d}_{2s}),$$

that is to say that Y belongs to the image of $\sigma(\mathfrak{t}_{2s-2})$. $\square$

Using the ellipticity of the sequence, it is finally possible to prove the existence of solutions of equations involving $\mathcal{G}_{2s}$ and $\mathfrak{k}_{2s}$. This theorem is a direct consequence of [33, Theorem 1.4]:

Proposition 3.5. *For x in $\mathbb{R}^3$, there exists an open neighborhood U of x such that the sequence*

$$\mathfrak{A}(U,\mathcal{S}_{2s-2}) \xrightarrow{\mathfrak{k}_{2s-2}} \mathfrak{A}(U,\mathcal{S}_{2s}) \xrightarrow{\mathcal{G}_{2s}} \mathfrak{A}(U,\mathcal{S}_{2s}) \xrightarrow{\mathfrak{d}_{2s}} \mathfrak{A}(U,\mathcal{S}_{2s-2}),$$

is exact, where $\mathfrak{A}(U,E)$ denotes the set of real analytic functions from U into E.

Remark 3.6. *We in fact only need this result in the context of polynomials: the problem will be to solve, for any real number δ:*

$$\mathfrak{k}_{2s-2}\phi = \psi \text{ when } \psi \in \mathcal{P}_{2s}^{<\delta}$$

and

$$\mathcal{G}_{2s}\xi = \zeta \text{ when } \zeta \in \mathcal{P}_{2s}^{<\delta}.$$

Theorem 3.5 ensures the local existence of solutions to these equations provided that:

$$\mathcal{G}_{2s}\psi = 0 \text{ and } \mathfrak{d}_{2s}\zeta = 0.$$

By integration, these solutions are necessarily polynomials.

Proof. The proof of Theorem 3.5 is a direct consequence of the fact that the fundamental operators $\mathfrak{k}_{2s-2}$, $\mathcal{G}_{2s}$ and $\mathfrak{d}_{2s}$ are operators with constant coefficients, which consist only of higher order homogeneous terms. As a consequence, these operators are all *sufficiently regular* in the terminology of [33] (since they have constant coefficients, cf. [33, Remark 1.16]) and *formally integrable* (since they have only homogeneous terms of the highest possible order, cf. [33, Remark 1.21]). Theorem 3.5 is then a direct consequence of [33, Theorem 1.4]. □

4. Construction of potentials

Consider a free massless spin-s field $\phi_{A\dots F}$, *i.e.* a symmetric valence $2s$ spinor field on Minkowski space that solves

$$\nabla^{AA'}\phi_{A\dots F} = 0. \tag{4.1}$$

In this section we investigate which spin-s fields can be represented by a potential of the form

$$\phi_{A\dots F} = \nabla_{AA'}\cdots\nabla_{FF'}\widetilde{\chi}^{A'\dots F'}, \tag{4.2}$$

where

$$\Box\, \widetilde{\chi}^{A'\dots F'} = 0, \tag{4.3}$$

and at the same time estimate appropriate weighted Sobolev norms of $\widetilde{\chi}^{A'\dots F'}$ in terms of Sobolev norms of $\phi_{A\dots F}$. To achieve this, a 3+1 splitting with respect to the surfaces $t = const.$ is performed and the standard elliptic theory is used to obtain the estimates (see for instance [29, 36], where the standard analytic integration procedure is used).

4.1. Space spinor splitting.

Now, we make a 3+1 splitting of the potential equation (4.2). Let $\tau_{AA'} = \sqrt{2}\nabla_{AA'}t$, which is covariantly constant. The operator $D_{AB} = \tau_{(A}{}^{A'}\nabla_{B)A'}$ is valid everywhere and it coincides with the intrinsic derivative on the slices $t = const.$. We therefore can consider it as an operator both on the spacetime and on a slice. Also all other operators defined for fields on $\mathbb{R}^3$ extend in this way to operators on fields on Minkowski space. With this view, we have the decomposition $\tau_B{}^{A'}\nabla_{AA'} = D_{AB} + \frac{1}{\sqrt{2}}\epsilon_{AB}\partial_t$. Define

$$\chi_{A\dots F} \equiv \tau_{AA'}\cdots\tau_{FF'}\widetilde{\chi}^{A'\dots F'}.$$

The equations (4.1) and (4.3) can be re-expressed as

$$\partial_t\phi_{A\cdots F} = \sqrt{2}D^H{}_{(A}\phi_{B\dots F)H} = \sqrt{2}(\mathfrak{c}_{2s}\phi)_{A\cdots F}, \tag{4.4}$$

$$(\mathfrak{d}_{2s}\phi)_{C\cdots F} = 0 \quad \text{when } s \geq 1, \tag{4.5}$$

$$\partial_t\partial_t\chi_{A\cdots F} = -\Delta_{2s}\chi_{A\cdots F}. \tag{4.6}$$

For the spin $\frac{1}{2}$ case, we immediately get

$$\phi_A = (\mathcal{C}_1\chi)_A + \frac{1}{\sqrt{2}}\partial_t\chi_A = (\mathcal{G}_1\mathcal{C}_1\chi)_A + \frac{1}{\sqrt{2}}(\mathcal{G}_1\partial_t\chi)_A.$$

This simple pattern in fact generalizes to arbitrary spin:

Proposition 4.1. *The equation* (4.2) *together with* (4.6) *implies*

$$\phi_{A_1\dots A_{2s}} = (\mathcal{G}_{2s}\mathcal{C}_{2s}\chi)_{A_1\dots A_{2s}} + \frac{1}{\sqrt{2}}(\mathcal{G}_{2s}\partial_t\chi)_{A_1\dots A_{2s}}.$$

Remark 4.2. *The property $\mathcal{d}_k\mathcal{G}_k = 0$ of the operators directly gives that the constraint $(\mathcal{d}_{2s}\phi)_{C\dots F} = 0$ is automatically satisfied for $s \geq 1$.*

Proof. See Proposition A.2 in the appendix for a proof. □

4.2. **Uniqueness of solutions of the Cauchy problem for the massless free fields.** A key point which will be used later to prove the existence of a representation is the uniqueness of the Cauchy problem for first order hyperbolic systems. For such a result, the reader can refer either to [23] or [9, Appendix 4]. That the massless spin-s field equation has a first order symmetric hyperbolic formulation follows immediately from Equation (4.4) and Lemma 2.12.

$$\begin{cases} \nabla^{AA'}\phi_{A\dots F} = 0, \\ \phi_{A\dots F}|_{t=0} = \varphi_{A\dots F}. \end{cases} \tag{4.7}$$

It is important to note that the initial datum $\varphi_{A\dots F}$ for such a Cauchy problem has to satisfy a geometric constraint (which will be in the sequel referred to as the constraint equation for the initial datum) given by:

$$D^{AB}\varphi_{ABC\dots F} = (\mathcal{d}_{2s}\varphi)_{C\dots F} = 0 \tag{4.8}$$

A key ingredient of the work is the uniqueness of the solution of the Cauchy problem (4.7):

Lemma 4.3. *Consider a spinor field $\varphi_{A\dots F}$ in $L^2_{loc}(\mathcal{S}_{2s})$. Then the Cauchy problem* (4.7) *admits at most one solution.*

Proof. This lemma is a direct consequence of the energy estimate. □

Remark 4.4. *This lemma does not state existence of solutions to the Cauchy problem for the massless free fields with initial datum in weighted Sobolev spaces. However, one can use the representation theorem 4.9 to obtain existence of solutions of this Cauchy problem from standard existence theorems for solutions of the wave equation with initial data in weighted Sobolev spaces.*

As explained in the introduction, one of the purposes of the paper, and a key point to obtain a decay result for massless fields, is to construct a Hertz potential $\chi^{A'\dots F'}$ for the solution of the Cauchy problem (4.7) such as

$$\phi_{A\dots F} = \nabla_{AA'}\cdots\nabla_{FF'}\widetilde{\chi}^{A'\dots F'}, \tag{4.9}$$

where the potential $\widetilde{\chi}^{A'\dots F'}$ satisfies the wave equation

$$\square\widetilde{\chi}^{A'\dots F'} = 0.$$

The construction of the potential $\widetilde{\chi}^{A'\dots F'}$ has to be made compatible with the standard decay result for the solution of the wave equation and the easiest way to do so is to consider a Cauchy problem of the potential itself. The problem is then consequently reduced to construct a set of initial data for a Cauchy problem satisfied by the potential. This initial data for the Cauchy problem for the Hertz potential have to be obtained from the initial datum of corresponding massless field. This is done as follows.

It must now be noticed that, if the initial datum $\varphi_{A\dots F}$ is a space spinor, the initial data for a Cauchy problem for $\widetilde{\chi}^{A'\dots F'}$ will have primed indices and will consequently not be space spinors. This is why the spinor field $\chi_{A\dots F}$, with lowered unprimed indices is introduced:

$$\widetilde{\chi}^{A'\dots F'} = \tau^{AA'}\cdots\tau^{FF'}\chi_{A\dots F}$$

so that the initial data for $\chi_{A\dots F}$ are now space spinors and can be directly related, through purely spatial operations, to the initial datum $\varphi_{A\dots F}$. Consider the Cauchy problem for the wave equation:

$$\begin{cases} \Box\chi_{A\dots F} = 0, \\ \chi_{A\dots F}|_{t=0} = \xi_{A\dots F}, \\ \partial_t\chi_{A\dots F}|_{t=0} = \sqrt{2}\zeta_{A\dots F}, \end{cases} \tag{4.10}$$

The construction of a potential can then be reduced to the construction of solutions to the following equation relating the initial datum for the massless field and the initial data for the potential:

Lemma 4.5. *Let $j \geq 2$ be an integer and $\varphi_{A\dots F}$ be a spinor field in $H^j_\delta(\mathcal{S}_{2s})$ satisfying the constraint equation $D^{AB}\varphi_{A\dots F} = 0$.*
Assume that there exist two spinor fields $\xi_{A\dots F} \in H^{j+2s}_{\delta+2s}(\mathcal{S}_{2s})$ and $\zeta_{A\dots F} \in H^{j+2s-1}_{\delta+2s-1}(\mathcal{S}_{2s})$ satisfying

$$\varphi_{A\dots F} = (\mathcal{G}_{2s}\mathcal{C}_{2s}\xi)_{A\dots F} + (\mathcal{G}_{2s}\zeta)_{A\dots F}.$$

Then the only solution to the Cauchy problem (4.7) for massless free fields is given by

$$\phi_{A\dots F} = \nabla_{AA'}\cdots\nabla_{FF'}\widetilde{\chi}^{A'\dots F'},$$

where $\widetilde{\chi}^{A'\dots F'}$ is obtained through the Cauchy problem (4.10) for $\chi_{A\dots F}$ with the initial data $(\xi_{A\dots F}, \zeta_{A\dots F})$.

Proof. Let

$$\widetilde{\phi}_{A\dots F} = \nabla_{AA'}\cdots\nabla_{FF'}\widetilde{\chi}^{A'\dots F'}.$$

It is a simple calculation to check that $\widetilde{\phi}_{A\dots F}$ satisfies the massless field equation of spin s (see [29], for instance). Furthermore, the restriction of $\phi_{A\dots F}$ and $\widetilde{\phi}_{A\dots F}$ agree on $\{t = 0\}$ and are equal to $\varphi_{A\dots F}$ which lies in $H^j_\delta(\mathcal{S}_{2s})$ and consequently in $L^2_\delta(\mathcal{S}_{2s})$. Using the uniqueness stated in Lemma 4.3, we can conclude that both agree. □

4.3. **Solving the constraints.** In this subsection, we will investigate for which initial data we can solve (4.5). We immediately see that $\varphi_{A\dots F}$ has to be in the image of $\mathcal{G}_{2s}$. Therefore, we can without loss of generality choose $\xi_{A\dots F} = 0$.

The main difficulty now is to obtain an estimate

$$\|\zeta_{A\dots F}\|_{j+2s-1,\delta+2s-1} \leq C\|\varphi_{A\dots F}\|_{j,\delta}.$$

With an estimate like this, $\|\varphi_{A\dots F}\|_{j,\delta}$ controls the initial data for $\chi_{A\dots F}$ which we evolve through (4.10). This can be reduced to the standard scalar wave equation, for which we get decay estimates. These decay estimates for the potential are then translated back to decay estimates for the field $\phi_{A\dots F}$.

For $s = \frac{1}{2}$, we immediately get the desired result by setting $\zeta_A = \varphi_A$. For higher spin, we need to make a more careful analysis.

We begin with a small lemma

Lemma 4.6. *Assume that $\varphi_{A\dots F} \in H^j_\delta(\mathcal{S}_{2s})$ satisfies the constraint $D^{AB}\varphi_{A\dots F} = 0$, and that $\eta_{C\dots F} \in H^1_{-2-\delta}(\mathcal{S}_{2s-2})$. Then $\varphi_{A\dots F}$ is L^2 orthogonal to $D_{(AB}\eta_{C\dots F)} = (\mathcal{E}_{2s-2}\eta)_{A\dots F}$.*

Proof. Let us now consider the 2-sphere $\mathbb{S}^2_r$ centered in the origin and of radius r and $\mathbb{B}^2_r$ the corresponding ball and proceed with integration by parts as follows:

$$\int_{\mathbb{B}^2_r} \varphi_{A\dots F} D^{(\widehat{AB}\eta^{C\dots F)}} \mathrm{d}\mu_{\mathbb{R}^3} = -\int_{\mathbb{S}^2_r} \varphi_{A\dots F} n^{AB}\widehat{\eta}^{C\dots F}\mathrm{d}\mu_{\mathbb{S}^2_r} - \int_{\mathbb{B}^2_r} D^{AB}\varphi_{A\dots F}\widehat{\eta}^{C\dots F}\mathrm{d}\mu_{\mathbb{S}^2_r},$$

where n^{AB} is the outward pointing normal to the sphere $\mathbb{S}^2_r$. The volume integral is vanishing since $\varphi_{A\dots F}$ satisfies the constraint equation. Due to Proposition 2.8, the boundary integral behaves like:

$$\left|\int_{\mathbb{S}^2_r} \varphi_{A\dots F} n^{AB}\widehat{\eta}^{C\dots F}\mathrm{d}\mu_{\mathbb{S}^2_r}\right| \leq C\epsilon(r)r^\delta r^{-2-\delta}r^2 \longrightarrow 0 \text{ as } r \to \infty$$

where $\epsilon(r)$ goes to zero as r grows to ∞. Hence, $\varphi_{A\dots F}$ is L^2 orthogonal to $D_{(AB}\eta_{C\dots F)}$. □

4.3.1. *The spin 1 case.*

Proposition 4.7. *Let δ be in $\mathbb{R}\setminus\mathbb{Z}$, j a positive integer, φ_{AB} in $H^j_\delta(\mathcal{S}_2)$ such that $D^{AB}\varphi_{AB} = 0$. If $\delta < -4$, we furthermore assume that φ_{AB} is L_2-orthogonal to the space $\mathbb{E}_{1,\delta}$, defined by;*

$$\mathbb{E}_{1,\delta} \equiv ((\ker \Delta_2) \setminus (\ker \mathfrak{c}_2)) \cap L^2_{-3-\delta}(\mathcal{S}_2).$$

Then there exist a spinor field $\zeta_{AB} \in H^{j+1}_{\delta+1}(\mathcal{S}_2)$ and a constant C depending only on δ and j such that

$$\varphi_{AB} = (\mathcal{G}_2\zeta)_{AB},$$
$$\|\zeta_{AB}\|_{j+1,\delta+1} \le C\|\varphi_{AB}\|_{j,\delta}.$$

Proof. First we establish that φ_{AB} is orthogonal to $\ker \Delta_2 \cap L^2_{-3-\delta}(\mathcal{S}_2)$ by using the constraint equation and the orthogonality to $\mathbb{E}_{1,\delta}$.

For $\delta > -4$, the set $\mathbb{E}_{1,s}$ is empty because $\ker \Delta_2 \cap L^2_{-3-\delta}(\mathcal{S}_2)$ only contains constant spinors times polynomials with degree $< -3 - \delta < 1$, i.e of maximal degree 0. They are therefore in the kernel of the homogeneous first order operator $\mathfrak{c}_2$.

The set $\ker \Delta_2 \cap L^2_{-3-\delta}(\mathcal{S}_2)$ is trivial if $\delta > -3$; we consequently assume that $\delta < -3$. Let θ_{AB} be in $\ker \Delta_2 \cap \ker \mathfrak{c}_2 \cap L^2_{-3-\delta}(\mathcal{S}_2)$. The field θ_{AB} is then in $\mathcal{P}_2^{<-3-\delta}$ and therefore smooth. Furthermore it is curl-free (i.e. in $\ker \mathfrak{c}_2$). Using the exact sequence (3.1):

$$\mathcal{S}_0 \xrightarrow{\mathfrak{t}_0} \mathcal{S}_2 \xrightarrow{\mathfrak{c}_2} \mathcal{S}_2,$$

it can therefore be written as a gradient $D_{AB}\eta = (\mathfrak{t}_0\eta)_{AB} = \theta_{AB}$.

Since θ_{AB} belongs to $\mathcal{P}_0^{<-3-\delta} \subset L^2_{-3-\delta}(\mathcal{S}_2)$, η can be assumed to belong to $\mathcal{P}_2^{<-2-\delta} \subset H^1_{-2-\delta}(\mathcal{S}_2)$. Then, by Lemma 4.6, φ_{AB} is L_2-orthogonal to θ_{AB}.

Now, since φ_{AB} is orthogonal to $\mathbb{E}_{1,\delta}$, we have that φ_{AB} is orthogonal to $\ker \Delta_2 \cap L^2_{-3-\delta}(\mathcal{S}_2)$. The Laplacian $\Delta_2 : H^{j+2}_{\delta+2}(\mathcal{S}_2) \to H^j_\delta(\mathcal{S}_2)$ is formally self-adjoint and has closed range and finite dimensional kernel – see [8, 27, 24] for details. By Fredholm's alternative there exists a $\widetilde{\theta}_{AB} \in H^{j+2}_{\delta+2}(\mathcal{S}_2)$ such that $\varphi_{AB} = (\Delta_2\widetilde{\theta})_{AB}$. The closed range gives an estimate (using Proposition 2.9):

$$\min_{\Upsilon_{AB}\in\ker \Delta_2\cap H^{j+2}_{\delta+2}(\mathcal{S}_2)} \|\widetilde{\theta}_{AB} + \Upsilon_{AB}\|_{j+2,\delta+2} \le C\|\varphi_{AB}\|_{j,\delta}.$$

Now let $\theta_{AB} = \tilde{\theta}_{AB} + \Upsilon_{AB}$ achieving this minimum for a specific Υ_{AB}. Hence, there exists a constant C depending only on j and δ such that:

$$\|\theta_{AB}\|_{j+2,\delta+2} \le C\|\varphi_{AB}\|_{j,\delta}.$$

Now, we can re-express the Laplacian Δ_2 as

$$\varphi_{AB} = (\Delta_2\theta)_{AB} = -2(\mathfrak{c}_2\mathfrak{c}_2\theta)_{AB} + (\mathfrak{t}_0\mathfrak{d}_2\theta)_{AB}.$$

We know want to show that $(\mathfrak{t}_0\mathfrak{d}_2\theta)_{AB}$ vanishes (for $\delta < 0$) or can be shown to be in the image of $\mathfrak{c}_2$ (for $\delta > 0$).

By the constraint equation and commutations of the divergence and the Laplace operator, we have:

$$0 = (\mathfrak{d}_2\varphi) = D^{AB}\varphi_{AB} = D^{AB}\left(\Delta\theta_{AB}\right) = \Delta\left(D^{AB}\varphi_{AB}\right) = (\Delta_0\mathfrak{d}_2\theta).$$

Hence, $(\mathfrak{d}_2\theta) \in \ker \Delta_0 \cap L^2_{\delta+1}(\mathcal{S}_0)$.

If $\delta < 0$, we know that $\ker \Delta_0 \cap L^2_{\delta+1}(\mathcal{S}_0)$ only contains polynomials with degree < 1, i.e. constants, which means that they are in the kernel of the gradient operator $\mathfrak{t}_0$. Hence,

$$\varphi_{AB} = -2(\mathfrak{c}_2\mathfrak{c}_2\theta)_{AB} = -(\mathcal{G}_2\mathfrak{c}_2\theta)_{AB}$$

and we can therefore choose $\zeta_{AB} = -(\mathfrak{c}_2\theta)_{AB}$, and we get

$$\|\zeta_{AB}\|_{j+1,\delta+1} \le \|\theta_{AB}\|_{j+2,\delta+2} \le C\|\varphi_{AB}\|_{j,\delta}.$$

If $\delta > 0$, we need to be more careful. Let $\Omega \equiv \ker \Delta_0 \cap L^2_{\delta+1}(\mathcal{S}_0)$, i.e. the set of harmonic polynomials with degree strictly smaller than $\delta + 1$. $\mathfrak{f}_0(\Omega) \subset L^2_\delta(\mathcal{S}_2)$ is also a finite dimensional space of smooth fields. Since $\mathfrak{d}_2\mathfrak{f}_0 = \Delta_0$, we have the following:

$$\begin{array}{ccccc} \Omega \subset \ker(\Delta_0) \subset \mathcal{S}_0 & \xrightarrow{\quad\mathfrak{f}_0\quad} & \mathfrak{f}_0(\Delta_0) \subset \mathcal{S}_2 & \xrightarrow{\quad\mathfrak{d}_2\quad} & \{0\} \subset \mathcal{S}_2 \\ & & \Big\uparrow{\scriptstyle \mathfrak{c}_2} & & \\ & & (\mathfrak{c}_2)^{-1}(\mathfrak{f}_0(\ker(\Delta_0))) \subset \mathcal{S}_2 & & \end{array}$$

Using the integrability condition stated by the exact sequence (3.1), and more specifically by:

$$\mathcal{S}_2 \xrightarrow{\mathfrak{c}_2} \mathcal{S}_2 \xrightarrow{\mathfrak{d}_2} \mathcal{S}_0,$$

we can define an a priori non unique linear mapping $\mathcal{T} : \mathfrak{f}_0(\Omega) \to \mathcal{S}_2$, such that $\mathcal{T}\mathfrak{c}_2$ acts as the identity on $\mathfrak{f}_0(\Omega)$. As a linear operator from the finite dimensional space $\mathfrak{f}_0(\Omega) \subset H^j_\delta(\mathcal{S}_2)$ into $H^{j+2}_{\delta+2}(\mathcal{S}_2)$ (endowed with their respective induced Sobolev norms), $\mathcal{T}$ is bounded and, therefore, there exists a constant C, depending on the choice of the mapping $\mathcal{T}$, such that

$$\|(\mathcal{T}\mathfrak{f}_0\mathfrak{d}_2\theta)_{AB}\|_{j+1,\delta+1} \leq C\|(\mathfrak{f}_0\mathfrak{d}_2\theta)_{AB}\|_{j,\delta},$$
$$(\mathfrak{c}_2\mathcal{T}\mathfrak{f}_0\mathfrak{d}_2\theta)_{AB} = (\mathfrak{f}_0\mathfrak{d}_2\theta)_{AB}.$$

Now, let

$$\zeta_{AB} = -(\mathfrak{c}_2\theta)_{AB} + (\mathcal{T}\mathfrak{f}_0\mathfrak{d}_2\theta)_{AB}.$$

This gives the desired relations:

$$(\mathcal{G}_2\zeta)_{AB} = 2(\mathfrak{c}_2\zeta)_{AB} = -2(\mathfrak{c}_2\mathfrak{c}_2\theta)_{AB} + 2(\mathfrak{f}_0\mathfrak{d}_4\theta)_{AB} = \varphi_{AB},$$
$$\|\zeta_{AB}\|_{j+1,\delta+1} \leq \|\theta_{AB}\|_{j+2,\delta+2} + C\|(\mathfrak{f}\mathfrak{d}_2\theta)_{AB}\|_{j,\delta} \leq C\|\varphi_{AB}\|_{j,\delta}.$$

□

4.3.2. *The spin s case.*

Proposition 4.8. *Let δ be in $\mathbb{R}\setminus\mathbb{Z}$, j a positive integer, $\varphi_{A\ldots F}$ in $H^j_\delta(\mathcal{S}_{2s})$ such that $D^{AB}\varphi_{A\ldots F} = 0$. Let $m = \lfloor s \rfloor$, i.e. the largest integer such that $m \leq s$. If $\delta < -2s-2$ we furthermore assume that $\varphi_{A\ldots F}$ is L_2-orthogonal to the space $\mathbb{E}_{s,\delta}$, defined by;*

$$\mathbb{E}_{s,\delta} \equiv ((\ker \Delta^m_{2s}) \setminus (\ker \mathcal{G}_{2s})) \cap L^2_{-3-\delta}(\mathcal{S}_{2s}).$$

Then there exist a spinor field $\zeta_{A\ldots F} \in H^{j+2s-1}_{\delta+2s-1}(\mathcal{S}_{2s})$ and a constant C depending only on δ and j such that

$$\varphi_{A\ldots F} = (\mathcal{G}_{2s}\zeta)_{A\ldots F},$$
$$\|\zeta_{A\ldots F}\|_{j+2s-1,\delta+2s-1} \leq C\|\varphi_{A\ldots F}\|_{j,\delta}.$$

Proof. First we establish that $\varphi_{A\ldots F}$ is orthogonal to $\ker \Delta^m_{2s} \cap L^2_{-3-\delta}(\mathcal{S}_{2s})$ by using the constraint equation and the orthogonality to $\mathbb{E}_{s,\delta}$.

For $\delta > -2s-2$, the set $\mathbb{E}_{s,\delta}$ is empty because $\ker \Delta^m_{2s} \cap L^2_{-3-\delta}(\mathcal{S}_{2s})$ only contains fields in $\mathcal{P}^{<-3-\delta}_{2s} \subset \mathcal{P}^{<2s-1}_{2s}$, i.e. constant spinors times polynomials with maximal degree $2s-2$. They are therefore in the kernel of the homogeneous $2s-1$ order operator $\mathcal{G}_{2s}$.

The set $\ker \Delta^m_{2s} \cap L^2_{-3-\delta}(\mathcal{S}_{2s})$ is trivial if $\delta > -3$, so we assume that $\delta < -3$. Let $\theta_{A\ldots F}$ be in $\ker \Delta^m_{2s} \cap \ker \mathcal{G}_{2s} \cap L^2_{-3-\delta}(\mathcal{S}_{2s})$ be arbitrary. We know that $\ker \Delta^m_{2s} \cap L^2_{-3-\delta}(\mathcal{S}_{2s})$ only contains fields in $\mathcal{P}^{<-3-\delta}_{2s}$. Theorem 3.5 (followed by Remark 3.6) states that the sequence

$$\mathcal{P}^{-2-\delta}_{2s-2} \xrightarrow{\mathfrak{f}_{2s-2}} \mathcal{P}^{-3-\delta}_{2s} \xrightarrow{\mathcal{G}_{2s}} \mathcal{P}^{-2s-2-\delta}_{2s}$$

is exact; as a consequence, there exists a spinor field $\eta_{C\ldots F} \in \mathcal{P}^{-2-\delta}_{2s-2}$ such that

$$D_{(AB}\eta_{C\ldots F)} = (\mathfrak{f}_{2s-2}\eta)_{A\ldots F} = \theta_{A\ldots F}.$$

Since $\eta_{C\dots F}$ belongs to $\mathcal{P}^{<-2-\delta}_{2s-2}$, it is therefore in $H^1_{-2-\delta}(\mathcal{S}_{2s-2})$. Then, by Lemma 4.6, $\varphi_{A\dots F}$ is orthogonal to $\theta_{A\dots F}$.

Now, because $\varphi_{A\dots F}$ is also orthogonal to $\mathbb{E}_{s,\delta}$ we have that $\varphi_{A\dots F}$ is orthogonal to $\ker \Delta^m_{2s} \cap L^2_{-3-\delta}(\mathcal{S}_{2s})$. The operator $\Delta^m_{2s} : H^{j+2m}_{\delta+2m}(\mathcal{S}_{2s}) \to H^j_\delta(\mathcal{S}_{2s})$ is formally self-adjoint and has closed range and finite dimensional kernel – see [8, 27, 24] for details. By the Fredholm alternative, there exists a $\widetilde{\theta}_{A\dots F} \in H^{j+2m}_{\delta+2m}(\mathcal{S}_{2s})$ such that $\varphi_{A\dots F} = (\Delta^m_{2s}\widetilde{\theta})_{A\dots F}$. The closed range gives an estimate (Proposition 2.9)

$$\min_{\Upsilon_{A\dots F} \in \ker \Delta^m_{2s} \cap H^{j+2m}_{\delta+2m}(\mathcal{S}_{2s})} \|\widetilde{\theta}_{A\dots F} + \Upsilon_{A\dots F}\|_{j+2m,\delta+2m} \leq C\|\varphi_{A\dots F}\|_{j,\delta}.$$

Now, let $\theta_{A\dots F} = \widetilde{\theta}_{A\dots F} + \Upsilon_{A\dots F}$ achieving this minimum for a specific $\Upsilon_{A\dots F}$. Hence, there exists a constant C, depending only on s, j and δ such that

$$\|\theta_{A\dots F}\|_{j+2m,\delta+2m} \leq C\|\varphi_{A\dots F}\|_{j,\delta}.$$

For integer spin we can express the Δ^m_{2s} operator as

$$(\Delta^m_{2s}\theta)_{A\dots F} = (\mathfrak{t}_{2s-2}\mathcal{F}_{2s-2}\mathfrak{d}_{2s}\theta)_{A\dots F} - (-2)^{1-m}(\mathcal{G}_{2s}\mathfrak{c}_{2s}\theta)_{A\dots F}.$$

For half integer spin we can express the Δ^m_{2s} operator as $2s = 2m+1$

$$(\Delta^m_{2s}\theta)_{A\dots F} = (\mathfrak{t}_{2s-2}\mathcal{F}_{2s-2}\mathfrak{d}_{2s}\theta)_{A\dots F} + (-2)^{-m}(\mathcal{G}_{2s}\theta)_{A\dots F}.$$

We know want to show that $(\mathfrak{t}_{2s-2}\mathcal{F}_{2s-2}\mathfrak{d}_{2s}\theta)_{A\dots F}$ vanishes (for $\delta < 0$) or is in the image of $\mathcal{G}_{2s}$ (for $\delta > 0$).

By the constraint equation and the commutation of the divergence and the Laplace operator, we have

$$0 = (\mathfrak{d}_{2s}\varphi)_{C\dots F} = D^{AB}\varphi_{A\dots F} = D^{AB}\left(\Delta^m_{2s}\theta_{A\dots F}\right) = \Delta^m_{2s-2}\left(D^{AB}\varphi_{A\dots F}\right) = (\Delta^m_{2s-2}\mathfrak{d}_{2s}\theta)_{C\dots F}.$$

Hence, $(\mathfrak{d}_{2s}\theta)_{C\dots F}$ is in $\ker \Delta^m_{2s-2} \cap L^2_{\delta+2m-1}(\mathcal{S}_{2s-2})$.

If $\delta < 0$, we know that fields in $\ker \Delta^m_{2s-2} \cap L^2_{\delta+2m-1}(\mathcal{S}_2)$ are in $\mathcal{P}^{<2m-1}_2$, i. e. they are spanned by constant spinors times polynomials with maximal degree $2m-2$. They therefore belongs to the kernel of the homogeneous order $2m-1$ operator $\mathfrak{t}_{2s-2}\mathcal{F}_{2s-2}$. Hence, $(\mathfrak{t}_{2s-2}\mathcal{F}_{2s-2}\mathfrak{d}_{2s}\theta)_{A\dots F} = 0$ and we get

$$\varphi_{A\dots F} = -(-2)^{1-m}(\mathcal{G}_{2s}\mathfrak{c}_{2s}\theta)_{A\dots F},$$

for integer spin, and

$$\varphi_{A\dots F} = (-2)^{-m}(\mathcal{G}_{2s}\theta)_{A\dots F},$$

for half integer spin. For integer spin we can therefore choose $\zeta_{A\dots F} = -(-2)^{1-m}(\mathfrak{c}_{2s}\theta)_{A\dots F}$, and we get

$$\|\zeta_{A\dots F}\|_{j+2s-1,\delta+2s-1} \leq \|\theta_{A\dots F}\|_{j+2m,\delta+2m} \leq C\|\varphi_{A\dots F}\|_{j,\delta}.$$

For half integer spin we can choose $\zeta_{A\dots F} = (-2)^{-m}\theta_{A\dots F}$, and we get

$$\|\zeta_{A\dots F}\|_{j+2s-1,\delta+2s-1} = (-2)^{-m}\|\theta_{A\dots F}\|_{j+2m,\delta+2m} \leq C\|\varphi_{A\dots F}\|_{j,\delta}.$$

If $\delta > 0$, we need to be more careful. Let $\Omega \equiv \ker(\Delta^m_{2s-2}) \cap \operatorname{im}(\mathfrak{d}_{2s}) \cap L^2_{\delta+2m-1}(\mathcal{S}_{2s-2})$. We know that it is a finite dimensional space of polynomial fields. $\mathfrak{t}_{2s-2}\mathcal{F}_{2s-2}(\Omega) \subset L^2_\delta(\mathcal{S}_{2s})$ is therefore also a finite dimensional space in $\mathcal{P}^{<\delta}_{2s}$.

We notice then the following formula holds: since $\mathfrak{d}_{2s}\mathcal{G}_{2s} = 0$ (cf. Proposition A.3), using the expression of the powers of the Laplacian given in Lemma A.4, we get:

$$\Delta^m_{2s-2}\mathfrak{d}_{2s} = \mathfrak{d}_{2s}\Delta^m_{2s} = \mathfrak{d}_{2s}\mathfrak{t}_{2s-2}\mathcal{F}_{2s-2}\mathfrak{d}_{2s}.$$

As consequence, on $\Omega \subset \operatorname{im}(\mathfrak{d}_{2s})$, the following relation holds:

$$\mathfrak{d}_{2s}\mathfrak{t}_{2s-2}\mathcal{F}_{2s-2}\big|_\Omega = \Delta^m_{2s}\big|_\Omega = 0.$$

The relations between the considered operators can be summarized by:

$$\begin{array}{ccccc}\Omega \subset \ker\left(\Delta_{2s-2}^{m}\right) \subset \mathcal{S}_{2s-2} & \xrightarrow{\mathfrak{k}_{2s-2}\mathcal{F}_{2s-2}} & \mathfrak{k}_{2s-2}\mathcal{F}_{2s-2}\left(\Omega\right) \subset \mathcal{S}_{2s} & \xrightarrow{\mathfrak{d}_{2s}} & \{0\} \subset \mathcal{S}_{2s-2} \\ & & \Big\uparrow \mathcal{G}_{2s} & & \\ & & \left(\mathcal{G}_{2s}\right)^{-1}\left(\mathfrak{k}_{2s-2}\mathcal{F}_{2s-2}\left(\Omega\right)\right) \subset \mathcal{S}_{2s} & & \end{array}$$

Using the integrability condition stated by the exact sequence stated in Theorem 3.5 applied to polynomials (cf. Remark 3.6), and more specifically using the following part of the exact sequence coming from

$$\mathcal{S}_{2s} \xrightarrow{\mathcal{G}_{2s}} \mathcal{S}_{2s} \xrightarrow{\mathfrak{d}_{2s}} \mathcal{S}_{2s-2},$$

we can define an a priori non unique linear mapping $\mathcal{T} : \mathfrak{k}_{2s-2}\mathcal{F}_{2s-2}\left(\Omega\right) \to \mathcal{S}_{2s}$ such that $\mathcal{G}_{2s}\mathcal{T}$ is the identity operator on $\mathfrak{k}_{2s-2}\mathcal{F}_{2s-2}(\Omega)$. As a linear operator on the finite dimensional space $\mathfrak{k}_{2s-2}\mathcal{F}_{2s-2}(\Omega) \subset H_{\delta}^{j}(\mathcal{S}_{2s})$ into $H_{\delta+2s-1}^{j+2s-1}(\mathcal{S}_{2s})$ (endowed with their respective induced Sobolev norms), $\mathcal{T}$ is bounded and, therefore, there exists a constant C, depending on the choice of the operator $\mathcal{T}$, such that

$$\begin{aligned}\|(\mathcal{T}\mathfrak{k}_{2s-2}\mathcal{F}_{2s-2}\mathfrak{d}_{2s}\theta)_{A\ldots F}\|_{j+2s-1,\delta+2s-1} &\leq C\|(\mathfrak{k}_{2s-2}\mathcal{F}_{2s-2}\mathfrak{d}_{2s}\theta)_{A\ldots F}\|_{j,\delta},\\ (\mathcal{G}_{2s}\mathcal{T}\mathfrak{k}_{2s-2}\mathcal{F}_{2s-2}\mathfrak{d}_{2s}\theta)_{A\ldots F} &= (\mathfrak{k}_{2s-2}\mathcal{F}_{2s-2}\mathfrak{d}_{2s}\theta)_{A\ldots F}.\end{aligned}$$

Now, for integer spin we can therefore choose

$$\zeta_{A\ldots F} = (\mathcal{T}\mathfrak{k}_{2s-2}\mathcal{F}_{2s-2}\mathfrak{d}_{2s}\theta)_{A\ldots F} - (-2)^{1-m}(\mathfrak{c}_{2s}\theta)_{A\ldots F}.$$

This gives the desired relations:

$$\begin{aligned}(\mathcal{G}_{2s}\zeta)_{A\ldots F} &= (\mathfrak{k}_{2s-2}\mathcal{F}_{2s-2}\mathfrak{d}_{2s}\theta)_{A\ldots F} - (-2)^{1-m}(\mathcal{G}_{2s}\mathfrak{c}_{2s}\theta)_{A\ldots F}\\ &= (\Delta_{2s}^{m}\theta)_{A\ldots F} = \varphi_{A\ldots F},\\ \|\zeta_{A\ldots F}\|_{j+2s-1,\delta+2s-1} &\leq C(\|\theta_{ABCD}\|_{j+2m,\delta+2m} + \|(\mathfrak{k}_{2s-2}\mathcal{F}_{2s-2}\mathfrak{d}_{2s}\theta)_{ABCD}\|_{j,\delta})\\ &\leq C\|\varphi_{ABCD}\|_{j,\delta}.\end{aligned}$$

Now, for half integer spin we can choose

$$\zeta_{A\ldots F} = (\mathcal{T}\mathfrak{k}_{2s-2}\mathcal{F}_{2s-2}\mathfrak{d}_{2s}\theta)_{A\ldots F} + (-2)^{-m}\theta_{A\ldots F}.$$

This gives the desired relations:

$$\begin{aligned}(\mathcal{G}_{2s}\zeta)_{A\ldots F} &= (\mathfrak{k}_{2s-2}\mathcal{F}_{2s-2}\mathfrak{d}_{2s}\theta)_{A\ldots F}(-2)^{-m}(\mathcal{G}_{2s}\theta)_{A\ldots F}\\ &= (\Delta_{2s}^{m}\theta)_{A\ldots F} = \varphi_{A\ldots F},\\ \|\zeta_{A\ldots F}\|_{j+2s-1,\delta+2s-1} &\leq C(\|\theta_{ABCD}\|_{j+2m,\delta+2m} + \|(\mathfrak{k}_{2s-2}\mathcal{F}_{2s-2}\mathfrak{d}_{2s}\theta)_{ABCD}\|_{j,\delta})\\ &\leq C\|\varphi_{ABCD}\|_{j,\delta}.\end{aligned}$$

□

4.4. **The representation theorem.** This section aims at making a synthetic presentation of the representation theorem of massless fields of arbitrary spin. We would especially like to emphasize the fact that the discussion in the main result arises on the decay properties of the initial datum for the Cauchy problem for the massless fields.

Theorem 4.9. *Let s be in $\frac{1}{2}\mathbb{N}$, δ be in $\mathbb{R}\setminus\mathbb{Z}$ and $j \geq 2$ an integer. We consider $\varphi_{A\ldots F}$ in $H_{\delta}^{j}(\mathcal{S}_{2s})$ satisfying the constraint equation $D^{AB}\varphi_{A\ldots F} = 0$.*
We furthermore assume that, for $\delta < -2s-2$, $\varphi_{A\ldots F}$ is orthogonal to the finite dimensional spaces

$$\left(\ker(\Delta_{2s}^{\lfloor s\rfloor}) \setminus \ker(\mathcal{G}_{2s})\right) \cap L_{-3-\delta}^{2}(\mathcal{S}_{2s}).$$

Then there exists a spinor field $\zeta_{A\ldots F}$, solution of the equation:

$$\varphi_{A\ldots F} = (\mathcal{G}_{2s}\zeta)_{A\ldots F}$$

satisfying the estimates:

$$\|\zeta_{A\dots F}\|_{j+2s-1,\delta+2s-1} \leq C\|\varphi_{A\dots F}\|_{j,\delta}.$$

such that the unique solution of the Cauchy problem for massless fields (4.7) *with the initial datum* $\varphi_{A\dots F}$ *is given by:*

$$\phi_{A\dots F} = \nabla_{AA'}\dots\nabla_{FF'}\widetilde{\chi}^{A'\dots F'}.$$

where the spinor field $\chi_{A\dots F}$*, defined by:*

$$\chi_{A\dots F} = \tau_{AA'}\cdots\tau_{FF'}\widetilde{\chi}^{A'\dots F'}$$

satisfies the Cauchy problem (4.10) *for the wave equation with initial data* $(0,\zeta_{A\dots F})$.

Remark 4.10. *An important remark is that, for the weight considered in* [10, 32] *(*$\delta = -\frac{5}{2}$ *for spin 1 and* $\delta = -\frac{7}{2}$ *for spin 2, respectively), all the fields can be represented by a Hertz potential.*

Proof. The proof of this theorem is a direct consequence of Lemma 4.5, of Propositions 4.7 and 4.8. □

5. Estimates for solutions of the scalar wave equation with initial data with arbitrary weight

This section contains the complementary results for the study of the decay of the solution of the wave equation in the exterior using representation formulae for the Cauchy problem such as the one stated in [19, 20] or in [2, 12] and the standard result of decay of weighted Sobolev spaces given in Proposition 2.8.

In the following, one considers the Cauchy problem:

$$\begin{cases} \Box\phi = 0 \\ \phi|_{t=0} = f \in H^j_\delta(\mathbb{R}^3,\mathbb{C}) \\ \partial_t\phi|_{t=0} = g \in H^{j-1}_{\delta-1}(\mathbb{R}^3,\mathbb{C}). \end{cases} \tag{5.1}$$

The following representation formula then holds ([13] on flat space-time or [15, theorem 5.3.3] for arbitrary curved background):

Lemma 5.1. *The solution of the Cauchy problem 5.1 is given by the representation formula:*

$$\phi(t,x) = \frac{1}{4\pi}\left(\int_{\mathbb{S}^2} t\left(g(x+t\omega)+\partial_\omega f(x+t\omega)\right)+f(x+t\omega)d\mu_{\mathbb{S}^2}\right)$$

where $\mathbb{S}^2$ *is the unit 2-sphere and* ∂_ω *has to be understood as being the derivation in the unit outer normal direction* ω.

The use of the representation formula stated in Lemma 5.1 and of Proposition 2.8 gives the following result for solutions of the wave equation:

Proposition 5.2. *Let* $j\geq 3$ *and* δ *in* $\mathbb{R}\smallsetminus\mathbb{Z}$. *Then the solution* ϕ *of the Cauchy problem* (5.1) *decays as follows, in the exterior region,* i.e. *in the region* $\frac{t}{3}\leq r\leq 3t$:

$$|\phi(t,x)| \leq C\left(\|f\|_{3,\delta}+\|g\|_{2,\delta-1}\right)\begin{cases} <v>^{-1}<u>^{1+\delta} & \text{if } \delta<-1 \\ <v>^{\delta} & \text{if } \delta>-1. \end{cases}$$

If, furthermore, (k,l,m) *is a triplet of non-negative integers,* $j\geq 3+k+l+m$*, the following pointwise inequality holds:*

- *if* $1+\delta-k<0$*, then:*
$$\|\partial_u^k\partial_v^l\nabla\!\!\!/^m\phi\| \leq C<u>^{1+\delta-k}<v>^{-1-l-m}\left(\|f\|_{3+k+l+m,\delta}+\|g\|_{2+k+l+m,\delta-1}\right)$$
- *if* $1+\delta-k>0$*, then:*
$$\|\partial_u^k\partial_v^l\nabla\!\!\!/^m\phi\| \leq C<v>^{\delta-l-m-k}\left(||f||_{3+k+l+m,\delta}+||g||_{2+k+l+m,\delta-1}\right)$$

where $\partial_u = \frac{1}{2}(\partial_t-\partial_r)$ *and* $\partial_v = \frac{1}{2}(\partial_t+\partial_r)$ *are respectively the outgoing and ingoing null directions.*

Remark 5.3. *It must be noticed that these estimates are sharp: it suffices to take as a set of initial data the smooth spherically symmetric functions:*

$$f(r,\omega) = \langle r\rangle^{\delta} \text{ and } g(r,\omega) = \langle r\rangle^{\delta-1}.$$

These functions will give rise to a solution of the wave equation which can be expressed directly in terms of the hypergeometric function used in the proof and which is the sharpest upper bound for the decay of the solution of the wave equation.

For the proof we will need some integral estimates.

Lemma 5.4. *For any δ in $\mathbb{R}\setminus\mathbb{Z}$, we have the following integral estimates*

$$\int_{\mathbb{S}^2} \langle|x+t\omega|^{\delta}\rangle d\mu_{\mathbb{S}^2} \leq \begin{cases} C(\langle u\rangle^2+\langle v\rangle^2)^{\frac{\delta}{2}} & \text{for } \delta > -1 \\ C\langle u\rangle^{\delta+1}\langle v\rangle^{-1}\left(\dfrac{\langle v\rangle^2}{\langle u\rangle^2+\langle v\rangle^2}\right)^{1+\frac{\delta}{2}} & \text{for } \delta < -1, \end{cases}$$

where C only depends on δ.

Remark 5.5. *It has been assumed that δ does not belong to $\mathbb{Z}$, for technical reasons. For the weight $\delta = -1$, we expect logarithmic terms.*

Proof. Let (t,x) be fixed and consider the sphere $S(x,t)$ of center x and of radius t. Let q be a point of the sphere $S(x,t)$. The coordinates of q is then given by (θ,ϕ) defined by:

- in the 2-plane containing the origin o, the point x and q, θ is the oriented angle:
$$\theta = (\vec{xq},\vec{oq}) \in (0,\pi);$$
- in the plane orthogonal to $\vec{ox}$ and passing through x, one chooses a direction of origin. The direction of the orthogonal projection of $\vec{xq}$ on this plane is labeled by an angle ϕ belonging to $(0,2\pi)$.

The integral can now be rewritten as

$$\begin{aligned}\int_{\mathbb{S}^2} \langle|x+t\omega|^{\delta}\rangle \mathrm{d}\mu_{\mathbb{S}^2} &= \int_0^{\pi}\int_0^{2\pi}(1+r^2+t^2-2tr\cos\theta)^{\frac{\delta}{2}}\sin^2\phi\mathrm{d}\theta\mathrm{d}\phi \\ &= \frac{\pi}{2}\int_0^{\pi}(1+r^2+t^2-2tr\cos\theta)^{\frac{\delta}{2}}\mathrm{d}\theta.\end{aligned}$$

This integral can be re-expressed using the hypergeometric function:

$$\int_{\mathbb{S}^2} \langle|x+t\omega|^{\delta}\rangle \mathrm{d}\mu_{\mathbb{S}^2} = \pi^2(1+r^2+t^2)^{\frac{\delta}{2}}{}_2F_1\left(\frac{2-\delta}{4},-\frac{\delta}{4},1,\frac{4r^2t^2}{(1+r^2+t^2)^2}\right)$$

or, using the variables u and v:

$$\int_{\mathbb{S}^2} \langle|x+t\omega|^{\delta}\rangle \mathrm{d}\mu_{\mathbb{S}^2} = 2^{-\frac{\delta}{2}}\pi^2(\langle u\rangle^2+\langle v\rangle^2)^{\frac{\delta}{2}}{}_2F_1\left(\frac{2-\delta}{4},-\frac{\delta}{4},1,1-\frac{4\langle u\rangle^2\langle v\rangle^2}{(\langle u\rangle^2+\langle v\rangle^2)^2}\right)$$

The hypergeometric function ${}_2F_1(a,b,c,z)$ is defined as:

$${}_2F_1(a,b,c,z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\frac{z^n}{n!}.$$

Γ being the Γ Euler function. We refer to [1, Section 15.1.1] for properties of this hypergeometric function. The radius of convergence of the series is equal to 1 and the convergence is absolute on the closed disc if

$$\mathfrak{R}(c-a-b) > 0.$$

For $\mathfrak{R}(c) > \mathfrak{R}(b) > 0$, the following integral representation holds

$${}_2F_1(a,b;c,z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\mathrm{d}t,$$

Two cases are now distinguished: $\delta > -1$ and $\delta < -1$.

For $\delta > -1$, we immediately have, using the convergence properties on the boundary on the disc of convergence.

$$\int_{\mathbb{S}^2} <|x + t\omega|^\delta> \mathrm{d}\mu_{\mathbb{S}^2} \leq C(<u>^2 + <v>^2)^{\frac{\delta}{2}},$$

where the constant C is given by:

$$C = 2^{-\frac{\delta}{2}}\pi^2 \sum_{n=0}^{\infty} \left| \frac{\Gamma\left(\frac{2-\delta}{4} + n\right)\Gamma\left(-\frac{\delta}{4} + n\right)}{\Gamma\left(\frac{2-\delta}{4}\right)\Gamma\left(-\frac{\delta}{4}\right) n!(n+1)!} \right|$$

For $\delta < -1$, the linear transformation [1, Equation 15.3.3] can be used:

$$ {}_2F_1\left(\frac{2-\delta}{4}, -\frac{\delta}{4}, 1, 1 - \frac{4<u>^2<v>^2}{(<u>^2 + <v>^2)^2}\right) =$$

$$\left(\frac{4<u>^2<v>^2}{(<u>^2 + <v>^2)^2}\right)^{\frac{\delta+1}{2}} {}_2F_1\left(\frac{2+\delta}{4}, \frac{4+\delta}{4}, 1, 1 - \frac{4<u>^2<v>^2}{(<u>^2 + <v>^2)^2}\right)$$

The later considered hypergeometric function satisfies the absolute convergence criterium on the unit disc so that:

$$\int_{\mathbb{S}^2} <|x + t\omega|^\delta> \mathrm{d}\mu_{\mathbb{S}^2} \leq C<u>^{\delta+1}<v>^{-1}\left(\frac{<v>^2}{<u>^2 + <v>^2}\right)^{1+\frac{\delta}{2}},$$

where the constant is, this time, given by:

$$C = 2^{1+\frac{\delta}{2}}\pi^2 \sum_{n=0}^{\infty} \left| \frac{\Gamma\left(\frac{2+\delta}{4} + n\right)\Gamma\left(-\frac{4+\delta}{4} + n\right)}{\Gamma\left(\frac{2+\delta}{4}\right)\Gamma\left(-\frac{\delta}{4}\right) n!(n+1)!} \right|.$$

□

Remark 5.6. *It is interesting to remark that the these inequalities actually provides us with* global *inequalities for the decay of the wave equation, that is to say an inequality valid both on the exterior and the interior regions.*

Proof of Proposition 5.2. Using Proposition 2.8, one knows that if $f \in H^j_\delta$, $g \in H^{j-1}_{\delta-1}$, $j \geq 2 + n$ and $j \geq 3 + m$, there is constant C such that:

$$\begin{aligned} |D^n f(y)| &\leq C<y>^{\delta-n}\|f\|_{2+n,\delta} \\ |D^m g(y)| &\leq C<y>^{\delta-1-m}\|g\|_{2+m,\delta-1}. \end{aligned}$$

Using the representation formula 5.1, one gets immediately:

$$|\phi(t,x)| \leq C\left(\|f\|_{3,\delta} + \|g\|_{2,\delta-1}\right)\int_{\mathbb{S}^2}\left(<|x + t\omega|>^\delta + t<|x + t\omega|>^{\delta-1}\right)\mathrm{d}\mu_{\mathbb{S}^2}.$$

We can use the estimate $t \leq (<u>^2 + <v>^2)^{1/2}$ and Lemma 5.4 to obtain *global* estimates for solutions of the wave equation.

In the exterior region, where the value of $|<u>/<v>|$ is bounded, this simplifies to the following asymptotic behavior for the solution ϕ: there exists a constant C depending on δ and the Sobolev embeddings constants such that, in the exterior region:

$$|\phi(t,x)| \leq C\left(\|f\|_{3,\delta} + \|g\|_{2,\delta-1}\right)\begin{cases} \dfrac{<u>^{1+\delta}}{<v>} & \text{for } \delta < -1 \\ <v>^\delta & \text{for } \delta > -1 \end{cases} \tag{5.2}$$

The same process can be applied to the derivatives of ϕ in the direction of u and v. The integral representations of the derivatives are then

$$\begin{aligned}\partial_v\phi(t,x) &= \frac{1}{8\pi}\int_{\mathbb{S}^2}\big(t\left(\partial_r g(x+t\omega)+\partial_\omega g(x+t\omega)\right)+g(x+t\omega)\\ &\quad +t\left(\partial_r\partial_\omega f(x+t\omega)+\partial_\omega^2 f(x+t\omega)\right)+\partial_r f(x+t\omega)+\partial_\omega f(x+t\omega)\big)\mathrm{d}\mu_{\mathbb{S}^2},\\ \partial_u\phi(t,x) &= \frac{1}{8\pi}\int_{\mathbb{S}^2}\big(t\left(-\partial_r g(x+t\omega)+\partial_\omega g(x+t\omega)\right)+g(x+t\omega)\\ &\quad +t\left(-\partial_r\partial_\omega f(x+t\omega)+\partial_\omega^2 f(x+t\omega)\right)-\partial_r f(x+t\omega)+\partial_\omega f(x+t\omega)\big)\mathrm{d}\mu_{\mathbb{S}^2}.\end{aligned}$$

Using Sobolev embeddings, one gets immediately:

$$\begin{aligned}|\partial_v\phi(t,x)| &\le C\left(\|f\|_{4,\delta}+\|g\|_{3,\delta-1}\right)\int_{\mathbb{S}^2}\left(<|x+t\omega|>^{\delta-1}+t<|x+t\omega|>^{\delta-2}\right)\mathrm{d}\mu_{\mathbb{S}^2},\\ |\partial_u\phi(t,x)| &\le C\left(\|f\|_{4,\delta}+\|g\|_{3,\delta-1}\right)\int_{\mathbb{S}^2}\left(<|x+t\omega|>^{\delta-1}+t<|x+t\omega|>^{\delta-2}\right)\mathrm{d}\mu_{\mathbb{S}^2}.\end{aligned}$$

Again using Lemma 5.4, we get in the exterior region, for $\delta<0$:

$$|\partial_v\phi(t,x)| \le C\frac{<u>^{\delta}}{<v>}\left(\|f\|_{4,\delta}+\|g\|_{3,\delta-1}\right) \tag{5.3}$$

$$|\partial_u\phi(t,x)| \le C\frac{<u>^{\delta}}{<v>}\left(\|f\|_{4,\delta}+\|g\|_{3,\delta-1}\right) \tag{5.4}$$

and for $\delta>0$:

$$|\partial_v\phi(t,x)| \le C<v>^{\delta-1}\left(\|f\|_{4,\delta}+\|g\|_{3,\delta-1}\right) \tag{5.5}$$

$$|\partial_u\phi(t,x)| \le C<v>^{\delta-1}\left(\|f\|_{4,\delta}+\|g\|_{3,\delta-1}\right). \tag{5.6}$$

Using these decay results, one can now refine the decay result for the derivatives of the function u, using the commutators properties of the wave equation with the vector fields generating the symmetries of the metric. Introducing:

$$K=u\partial_u+v\partial_v \text{ which satisfies } [K,\Box]=-4\Box,$$

the function $K\cdot\phi$ satisfies Cauchy problem for the linear wave equation:

$$\begin{cases}\Box(K\cdot\phi)=0\\ K\phi|_{t=0}\in H^{j-1}_{\delta}(\mathbb{R}^3)\\ \partial_t K\phi|_{t=0}\in H^{j-2}_{\delta-1}(\mathbb{R}^3).\end{cases}$$

As a consequence, one can apply the decay result (5.2), (5.3) and (5.5) to $K\cdot\phi$. This gives:

$$|K\cdot\phi(t,x)|\le C\left(\|f\|_{4,\delta}+\|g\|_{3,\delta-1}\right)\begin{cases}\left(\dfrac{<u>^{\delta+1}}{<v>}\right) & \text{if } \delta<-1\\ <v>^{\delta} & \text{if } \delta>-1\end{cases}$$

This consequently gives the following decay for the partial derivatives $\partial_u\phi$ and $\partial_v\phi$, using the decay result for these derivatives (5.3):

$$|\partial_v\phi(t,x)|\le C\left(\|f\|_{4,\delta}+\|g\|_{3,\delta-1}\right)\begin{cases}\left(\dfrac{<u>^{\delta+1}}{<v>^2}\right) & \text{if } \delta<-1\\ <v>^{\delta-1} & \text{if } \delta>-1.\end{cases}$$

Finally to obtain the decay result for the derivatives, the commutating properties of $\Box$ with the generators of $SO(3)$:

$$x^i\partial_j-x^j\partial_i$$

can then be used to obtain:

$$|\not\nabla\phi|\le C\left(\|f\|_{4,\delta}+\|g\|_{3,\delta-1}\right)<u>^{\delta+1}<v>^{-2} \text{ if } \delta<-1$$

and

$$|\not\nabla\phi|\le C\left(\|f\|_{4,\delta}+\|g\|_{3,\delta-1}\right)<v>^{\delta-1} \text{ if } \delta>-1$$

Remark 5.7. (1) *It is important to notice at this stage that the different derivatives ∂_u and ∂_v, $\not\nabla$ play different roles when considering the full scale of weights. This difference is at the origin of the failing of the peeling for higher spin fields when the decay at i^0 of the initial data is too low.*

(2) *This difference between the derivatives can be explained by considering the derivatives of the fundamental solution of the wave equation.*

The result is finally completed by a recursion which is not written there, in the exterior region, where the value of $|<u>/<v>|$ is bounded, by the following discussion on δ:

- if $1+\delta-k<0$, then:
$$|\partial_u^k \partial_v^l \not\nabla^m \phi| \le C<u>^{1+\delta-k}<v>^{-1-l-m}\left(\|f\|_{3+k+l+m,\delta} + \|g\|_{2+k+l+m,\delta-1}\right)$$
- if $1+\delta-k>0$, then:
$$|\partial_u^k \partial_v^l \not\nabla^m \phi| \le C<v>^{\delta-k-l-m}\left(\|f\|_{3+k+l+m,\delta} + \|g\|_{2+k+l+m,\delta-1}\right)$$

□

6. Estimates for spinor fields represented by potentials

Penrose, in his original paper on zero-rest mass fields [29], proved to the following two results:

- the existence for analytic massless fields of arbitrary spin of representation of the form:
$$\phi_{A\ldots F} = \xi_1^{A'}\ldots\xi_{2s}^{F'}\nabla_{AA'}\ldots\nabla_{FF'}\chi,$$
where the $\xi^{A'}$ are constant spinors and χ is a complex function satisfying satisfying the wave equation:
$$\Box\chi = 0.$$
- from a decay ansatz for χ along outgoing null light rays, he deduced the full peeling result for the considered field.

The purpose of this section is to give an equivalent result for massless field admitting a potential of the form considered by Penrose. The decay result for the solution of the wave equation which is used in this section is given by Proposition 5.2.

6.1. Geometric background and preliminary lemmata.

The geometric framework and notations are introduced in this section. The geometric background is the Minkowski spacetime. We consider on this space time the normalized null tetrad defined by:
$$\begin{aligned} l^a &= \sqrt{2}\partial_v = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial r}\right) & m^a &= \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial\theta}+\frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}\right)\\ n^a &= \sqrt{2}\partial_u = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial r}\right) & \overline{m}^a &= \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial\theta}-\frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}\right)\end{aligned}$$
so that we have:
$$\begin{array}{ccc} \mathrm{span}\,(l^a,n^a) & \text{orthogonal to} & \mathrm{span}\,(m^a,\overline{m}^a)\\ l_a n^a = 1 & \text{and} & m_a\overline{m}^a = -1.\end{array}$$
The derivatives in the directions $l^a, n^a, m^a, \overline{m}^a$ are denoted by D, D', δ, δ' respectively. Consider finally a spin basis (o^A, ι^A) arising from this tetrad, i.e;
$$\begin{aligned} l^a &= o^A\overline{o}^{A'} & m^a &= o^A\overline{\iota}^{A'}\\ n^a &= \iota^A\overline{\iota}^{A'} & \overline{m}^a &= \iota^A\overline{o}^{A'}\end{aligned}$$

This basis satisfies:
$$\begin{aligned} Do^A &= 0 & D\iota^A &= 0\\ D'o^A &= 0 & D'\iota^A &= 0\\ \delta o^A &= \frac{\cot\theta}{2r\sqrt{2}}o^A & \delta\iota^A &= \frac{\cot\theta}{2r\sqrt{2}}\iota^A - \frac{1}{r\sqrt{2}}o^A\\ \delta' o^A &= -\frac{\cot\theta}{2r\sqrt{2}}o^A + \frac{\iota^A}{r\sqrt{2}} & \delta'\iota^A &= \frac{\cot\theta}{2r\sqrt{2}}\iota^A\end{aligned} \tag{6.1}$$

As a consequence of this the following commutators relations hold:

Lemma 6.1 (Commutators)**.** *The following commutator relations hold:*

- *D and D' commute.*
- *consider the gradient $\nabla\!\!\!/$ on the sphere of radius r:*

$$\nabla\!\!\!/ f = \frac{1}{r}\left(\frac{\partial f}{\partial\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin\theta}\frac{\partial f}{\partial\varphi}\frac{\partial}{\partial\varphi}\right)$$

then, for any positive integer m:

$$\begin{aligned}\nabla\!\!\!/^m D &= D\nabla\!\!\!/^m + \frac{m}{r}\nabla\!\!\!/^m\\ \nabla\!\!\!/^m D' &= D'\nabla\!\!\!/^m + \frac{m}{r}\nabla\!\!\!/^m\end{aligned}$$

Proof. The proof essentially relies on a direct calculation of the first order commutator and an induction. □

Remark 6.2. *Note that the following property holds:*

$$\begin{aligned}\nabla\!\!\!/ f &= \frac{1}{\sqrt{2}}\left(\Re(\delta f)\frac{\partial}{\partial\theta} + \Im(\delta f)\frac{\partial}{\partial\varphi}\right)\\ &= e^{-i\frac{\pi}{4}}\delta f\frac{\partial}{\partial\theta} + e^{i\frac{\pi}{4}}\delta' f\frac{\partial}{\partial\varphi}.\end{aligned} \tag{6.2}$$

A final result deals with the asymptotic behaviour of the coefficients in the decomposition of a constant spinor over the spin basis (o^A, ι^A):

Lemma 6.3 (Asymptotic behaviour of the decomposition of a constant spinor)**.** *Let ξ^A be a constant spinor over $\mathbb{M}$ and consider its decomposition over the basis (o^A, ι^A):*

$$\xi^A = \alpha o^A + \beta\iota^A.$$

Then, for any integer n, $\nabla^n\alpha$ and $\nabla^n\beta$ are smooth bounded functions on $\mathbb{M}\backslash\{\mathbb{R}\times B(0,1)\}$. Furthermore, considering the derivatives in the null directions, the following estimates hold for $\theta\in[c,\pi-c]$ $(c>0)$:

$$\begin{array}{ccc} D\alpha = D'\alpha = 0 & , & D\beta = D'\beta = 0\\ |\delta^n\alpha| \le \frac{C}{r^n} & , & |\delta^n\beta| \le \frac{C}{r^n}\\ |\delta'^n\alpha| \le \frac{C}{r^n} & , & |\delta'^n\beta| \le \frac{C}{r^n}\end{array}$$

Proof. To prove that α and β are bounded functions, it suffices to consider the decomposition of the real vector field $\xi^A\bar{\xi}^{A'}$ in Cartesian coordinates. The time component of the vector field is $|\alpha^2| + |\beta|^2$ and it is constant. As a consequence, α and β are smooth bounded functions.

The second step consists in calculating the derivatives of the components in ξ^A. Since o^A and ι^A are constant along outgoing and ingoing null rays, the following identities hold:

$$D\alpha = D'\beta = D\alpha = D'\beta = 0.$$

For the angular derivatives, we have:

$$\begin{aligned}\left(\delta\alpha + \alpha\frac{\cot\theta}{2r\sqrt{2}} - \frac{\beta}{r\sqrt{2}}\right)o^A + \left(\delta\beta + \beta\frac{\cot\theta}{2r\sqrt{2}}\right)\iota^A = 0\\ \left(\delta'\alpha - \alpha\frac{\cot\theta}{2r\sqrt{2}}\right)o^A + \left(\delta'\beta + \beta\frac{\cot\theta}{2r\sqrt{2}} - \frac{\alpha}{r\sqrt{2}}\right)\iota^A = 0\end{aligned}$$

An immediate induction using these recursive relations gives the desired results. □

6.2. **Proof of the decay result.** We consider in this section a spin-s field represented as:

$$\phi_{A\ldots F} = \xi_1^{M'}\ldots\xi_{2s}^{N'}\nabla_{AM'}\ldots\nabla_{FN'}\chi, \tag{6.3}$$

where χ is a complex scalar Hertz potential satisfying the wave equation:

$$\Box\chi = 0$$

and $\xi_1^{M'},\ldots,\xi_{2s}^{N'}$ are constants spinors.

The purpose of this section is to give a result which is similar to the one obtained for the wave (or 0-spin) equation in order to retrieve similar decay estimates as in the pioneering work of Christodoulou-Klainerman [10].

Proposition 6.4 (Decay estimates for arbitrary spin)**.** *Let (k, l, m) be a triplet of non-negative integers and denote by n their sum. We assume that the Hertz potential is a solution of the Cauchy problem, for $j \geq 2 + 2s + n$ and $\delta \notin \mathbb{Z}$*

$$\begin{cases} \Box\chi = 0 \\ \chi|_{t=0} \in H^j_\delta(\mathbb{R}^3, \mathbb{C}) \\ \partial_t\chi|_{t=0} \in H^{j-1}_{\delta-1}(\mathbb{R}^3, \mathbb{C}) \end{cases}$$

The norm of the initial conditions is denoted by $I_{\delta,j}$:

$$I_{\delta,j} = \|\chi|_{t=0}\|_{\delta,j} + \|\partial_t\chi|_{t=0}\|_{\delta-1,j-1}$$

Then, the following inequalities hold, for all $\boldsymbol{i}$ in $\{0, 2s\}$:

(1) *for any $t \geq 0$, $x \in \mathbb{R}^3$, such that $t > 3r$, that is to say in the interior region,*

$$|\nabla^n \phi_{A\ldots F}| \leq c{<}t{>}^{\delta-2s-n} I_{\delta,n+2s+2}.$$

(2) *for $\boldsymbol{i}$ such that $1 + \delta - k - \boldsymbol{i} < 0$, for any $t \geq 0$, $x \in \mathbb{R}^3$, such that $3r > t > \frac{r}{3}$, that is to say in the exterior region,*

$$|D^k D'^l \not\nabla^m \phi_{\boldsymbol{i}}| \leq \frac{c{<}u{>}^{\delta+1-\boldsymbol{i}-l}}{{<}v{>}^{1+2s-\boldsymbol{i}+k+m}} I_{\delta,n+2s+2};$$

(3) *for $\boldsymbol{i}$ such that $1 + \delta - k - \boldsymbol{i} > 0$, for any $t \geq 0$, $x \in \mathbb{R}^3$, such that $3r > t > \frac{r}{3}$, that is to say in the exterior region,*

$$|D^k D'^l \not\nabla^m \phi_{\boldsymbol{i}}| \leq c{<}v{>}^{\delta-2s-l-k-m} I_{\delta,n+2s+2}.$$

Remark 6.5.

- *The symbol $\not\nabla$ denotes the gradient to the sphere of radius r. As a consequence the rescaled operator $\frac{1}{r}\not\nabla$ which is the gradient on the unit sphere is independent of r.*
- *For the spin 1 and $\delta = -\frac{1}{2}$, that is to say for initial data for the Maxwell fields lying in $H_{-\frac{5}{2}}$, which is the case considered in* [10]*, one recovers the decay result stated in this paper. For the spin 2 and for $\delta = \frac{1}{2}$, that is to say for initial data in $H_{-\frac{7}{2}}$, which is the case considered by Christodoulou-Klainerman, their results are recovered.*
- *It must be noticed that in the case when the potential does not decay enough (for $\delta > -2s - 2$), the decay of some components of the field cannot distinguished: the peeling fails.*
- *The case of integer weights can also be handled similarly, provided that the corresponding decay results for the solution of the wave equation are made available; to avoid a too complicated result, this question has been put aside.*
- *The peeling result obtained by Penrose* [29] *was relying on the assumption that the Hertz potential, solution of the wave equation, was decaying as $\chi \sim \frac{1}{r}$ where r is a parameter along the outgoing null rays. For such a decay result to hold, the initial data for the potential have to lie in H_δ with $\delta < -1$. The peeling result by Mason-Nicolas* [26]*, which holds for the spins 1/2 and 1 on the Schwarzschild space-time, is for initial data lying in a Sobolev space whose weights are not equally distributed on the components.*

Proof. The proof is made by induction on the spin. The result for the spin 0 is exactly the one obtained for the wave equation. The induction does consequently not need to be induced.

We now assume the following induction hypothesis for spin s: assume that any triplet (k, l, m) $(n = k + l + m)$, and for any s-spinor field represented in the following way:

$$\psi_{\underbrace{A\ldots F}_{2s \text{ indices}}} = \xi_1^{A'} \ldots \xi_{2s}^{F'} \nabla_{AA'} \ldots \nabla_{FF'}\chi$$

where χ is a potential whose initial data lie in $H^j_\delta(\mathbb{R}^3,\mathbb{C}) \times H^{j-1}_{\delta-1}(\mathbb{R}^3,\mathbb{C})$ $(j > 1+2s+n)$, the decay results stated in the theorem are true.

Let now (k,l,m) be a triplet of non-negative integers and consider a $s+\frac{1}{2}$-spin field written in the following way

$$\phi_{\underbrace{A\dots FG}_{2s+1 \text{ indices}}} = \xi_1^{A'} \dots \xi_{2s+1}^{G'} \nabla_{AA'} \dots \nabla_{GG'}\chi$$

where χ is a potential whose initial data are in $H^j_\delta(\mathbb{R}^3,\mathbb{C}) \times H^{j-1}_{\delta-1}(\mathbb{R}^3,\mathbb{C})$ $(p > 2+2s+n)$. Consequently, the $2s$-spin field

$$\psi_{\underbrace{B\dots G}_{2s \text{ indices}}} = \xi_2^{B'} \dots \xi_{2s+1}^{G'} \nabla_{BN'} \dots \nabla_{GG'}\chi$$

with χ whose initial data lies in $H^p_\delta(\mathbb{R}^3) \times H^{p-1}_{\delta+1}(\mathbb{R}^3)$ $(p > 1+2s+n)$ satisfies the induction hypothesis. It remains then to prove that

$$\phi_{A\dots G} = \xi^{A'} \nabla_{AA'} \psi_{B\dots G}$$

satisfies the appropriate decay result.

We first start by the interior decay. The result trivially follows from the interior decay result for the wave equation. As a consequence, the following relation holds:

$$|\nabla^n \phi_{A\dots G}| = |\xi^{A'} \nabla_{AA'} \left(\nabla^n \psi_{B\dots G}\right)|$$

which is a derivation of order $M+1$ of a spin field of valence s which satisfies the induction assumptions. As consequence, the following decay result is immediate, in the interior region $3t \leq r$:

$$|\nabla^n \phi_{A\dots F}| \leq C \frac{I_{\delta,n+2s+3}}{<t>^{-\delta+2s+n+1}},$$

where C is a constant depending on M and the spin s. This closes the induction for the part concerning the interior decay.

We consider now the problem of the exterior decay, that is to say the decay in the neighbourhood of an outgoing light cone:

$$\frac{r}{3} \leq t \leq 3r \Leftrightarrow |t-r| \leq \frac{1}{2}|t+r|. \tag{6.4}$$

Recall that the definition of the component of the spinor $\psi_{B\dots F}$ are defined by:

$$\psi_{\mathbf{i}} = \underbrace{\iota^B \dots \iota^C}_{\mathbf{i}} \underbrace{o^D \dots o^F}_{2s-\mathbf{i}} \psi_{B\dots F}.$$

The proof is this region in done by induction as in the first part of the proof. Let (k,l,m) be a given triplet of non negative integers and denote by n their sum. The induction hypothesis is written as follows for spin s:

> For any s-spinor field $\psi_{B\dots G}$ satisfying:
>
> $$\psi_{B\dots G} = \eta^{B'} \dots \zeta^{G'} \nabla_{BB'} \dots \nabla_{GG'}\chi$$
>
> where χ is a complex scalar solution of the massless wave equation whose initial data lies in $H^j_\delta(\mathbb{R}^3,\mathbb{C}) \times H^{j-1}_{\delta+1}(\mathbb{R}^3,\mathbb{C})$ $(j > 1+2s+n)$, the following decay results holds, in the exterior region $\frac{t}{3} \leq r \leq 3t$, for all integer k,l,m:
>
> - for $\mathbf{i}$ such that $1+\delta-k-\mathbf{i} < 0$:
>
> $$|D^k D'^l \eth^m \psi_{\mathbf{i}}| \leq \frac{c<u>^{\delta+1-\mathbf{i}-l}}{<v>^{1+2s-\mathbf{i}+k+m}} I_{\delta,n+2s+2}.$$
>
> - for $\mathbf{i}$ such that $1+\delta-k-\mathbf{i} > 0$:
>
> $$|D^k D'^l \eth^m \psi_{\mathbf{i}}| \leq C<v>^{\delta-2s-l-k-m} I_{\delta,n+2s+2},$$
>
> where the constant C depends on the bounds of the exterior domain and the integers k,l,m.

There is no need to prove the initial step since it is exactly the result for the standard wave equation. Assume that the induction hypothesis holds for spin s in $\frac{1}{2}\mathbb{N}$ and consider the field $\phi_{A\ldots G}$ of spin $s+\frac{1}{2}$ written as:

$$\phi_{A\ldots G} = \xi^{A'}\eta^{B'}\ldots\zeta^{G'}\nabla_{AA'}\ldots\nabla_{GG'}\chi$$

where χ is a complex scalar solution of the massless wave equation whose initial data lies in $H^j_\delta(\mathbb{R}^3,\mathbb{C})\times H^{j-1}_{\delta+1}(\mathbb{R}^3,\mathbb{C})$ ($j>2+2s+n$). As a consequence, the spinor:

$$\psi_{B\ldots G} = \eta^{B'}\ldots\zeta^{G'}\nabla_{BB'}\ldots\nabla_{GG'}\chi$$

is a s-spinor field satisfying the requirements of the induction assumption.

To insure the proof of the induction assumption, a relation between the components of $\phi_{A\ldots F}$ and the components of the field $\psi_{B\ldots G}$ have to established; these relations are given in the following lemma:

Lemma 6.6. *The following relations between the components of ψ (of spin $s+1/2$) and ϕ (of spin s, s being in $\frac{1}{2}\mathbb{N}$) hold:*

$$\begin{aligned}\phi_0 &= \alpha D\psi_0 + \beta\delta\psi_0 - s\beta\frac{\cot\theta}{r\sqrt{2}}\psi_0 &(6.5)\\ \phi_{\boldsymbol{i}} &= \alpha\delta'\psi_{\boldsymbol{i}-1} + \beta D'\psi_{\boldsymbol{i}-1}\\ &\quad +\alpha(s+1-\boldsymbol{i})\frac{\cot\theta}{r\sqrt{2}}\psi_{\boldsymbol{i}-1} - \frac{(2s+1-\boldsymbol{i})\alpha}{r\sqrt{2}}\psi_{\boldsymbol{i}} &(6.6)\end{aligned}$$

for $\boldsymbol{i}>0$.

Proof. The proof is realized using relations (6.1) and is a basic calculation.

$$\begin{aligned}\phi_0 &= o^A\ldots o^F\phi_{A\ldots F}\\ &= o^A\ldots o^F\eta^{A'}\nabla_{AA'}\psi_{B\dot{F}}\\ &= \alpha o^B\ldots o^F D\psi_{B\ldots F} + \beta o^B\ldots o^F\delta\psi_{B\ldots F}\end{aligned}$$

Since $Do^A=0$ and $\delta o^A = \frac{\cot\theta}{2r\sqrt{2}}o^A$, we have:

$$o^B\ldots o^F\delta\psi_{B\ldots F} = \delta\psi_0 - 2s\frac{\cot\theta}{2r\sqrt{2}}\psi_0.$$

and, consequently:

$$\phi_0 = \alpha D\psi_0 + \beta\delta\psi_0 - s\beta\frac{\cot\theta}{r\sqrt{2}}\psi_0.$$

Consider now $\mathbf{i}>0$ fixed; we have:

$$\begin{aligned}\phi_{\mathbf{i}} &= \underbrace{\iota^A\ldots\iota^C}_{\mathbf{i}\text{ times}}\underbrace{o^D\ldots\iota^F}_{2s+1-\mathbf{i}}\phi_{A\ldots F}\\ &= \alpha\,\underbrace{\iota^B\ldots\iota^C}_{\mathbf{i}-1}\underbrace{o^D\ldots\iota^F}_{2s+1-\mathbf{i}}\delta'\psi_{B\ldots F} + \beta\,\underbrace{\iota^B\ldots\iota^C}_{\mathbf{i}-1}\underbrace{o^D\ldots\iota^F}_{2s+1-\mathbf{i}}D'\psi_{B\ldots F}\end{aligned}$$

Since $D'\iota^A = D'o^A = 0$, $\delta' o^A = -\frac{\cot\theta}{2r\sqrt{2}}o^A + \frac{\iota^A}{r\sqrt{2}}$ and $\delta'\iota^A = \frac{\cot\theta}{2r\sqrt{2}}\iota^A$ we have:

$$\begin{aligned}\underbrace{\iota^B\ldots\iota^C}_{\mathbf{i}-1}\underbrace{o^D\ldots\iota^F}_{2s+1-\mathbf{i}}\delta'\psi_{B\ldots F} &= \delta'\psi_{\mathbf{i}-1} - (\mathbf{i}-1)\frac{\cot\theta}{2r\sqrt{2}}\underbrace{\iota^B\ldots\iota^C}_{\mathbf{i}-1}\underbrace{o^D\ldots\iota^F}_{2s+1-\mathbf{i}}\psi_{B\ldots F}\\ -(2s+1-\mathbf{i})&\left(-\frac{\cot\theta}{2r\sqrt{2}}\underbrace{\iota^B\ldots\iota^C}_{\mathbf{i}-1}\underbrace{o^D\ldots\iota^F}_{2s+1-\mathbf{i}}\psi_{B\ldots F} + \frac{1}{r\sqrt{2}}\underbrace{\iota^B\ldots\iota^C}_{\mathbf{i}}\underbrace{o^D\ldots\iota^F}_{2s-\mathbf{i}}\psi_{B\ldots F}\right).\end{aligned}$$

Consequently,

$$\phi_{\mathbf{i}} = \alpha\delta'\psi_{\mathbf{i}-1} + \beta D'\psi_{\mathbf{i}-1} + \alpha(s+1-\mathbf{i})\frac{\cot\theta}{r\sqrt{2}}\psi_{\mathbf{i}-1} - \frac{(2s+1-\mathbf{i})\alpha}{r\sqrt{2}}\psi_{\mathbf{i}}.$$

$\square$

Using Lemma 6.6, the two cases ($\mathbf{i} = 0$ and $\mathbf{i} > 0$) are treated separately although the method is the same. Here we present the case $\mathbf{i} = 0$, the other case follows similarly.

The expression of the derivative $D^k D'^l \eth \phi_0$ is calculated explicitly, one derivative at a time, using the Leibniz rule:

$$\eth^m \phi_0 = \sum_{a=0}^{m} \binom{a}{m} \eth^a \alpha \eth^{m-a} D\psi_0 + \sum_{a=0}^{m} \binom{a}{m} \eth^a \beta \eth^{m-a} \delta\psi_0$$
$$-2s \sum_{a+b+c=m} \frac{m!}{a!b!c!} \left(\frac{\partial_\theta^b (\cot\theta)}{2\sqrt{2} r^{b+1}} \right) \eth^a \beta \partial_\theta^b \eth^c \psi_0$$

since

$$\eth^b \cot\theta = \frac{1}{r^b} \frac{\partial^b \cot\theta}{\partial\theta^b} \partial_\theta^b,$$

the power on the vector field have to be understood as a symmetric tensor exponent.

We then apply simultaneously the derivatives D and D', using the Leibniz rule again. Notice first that $\eth^a \alpha$ and $\eth^a \beta$ depend on r but $r^a \eth^a \alpha$ and $r^a \eth^a \beta$ do not, since both α and β are independent both of time and radius. We have

$$D^k D'^l \eth^m \phi_0 =$$
$$\sum_{d=0}^{k} \sum_{e=0}^{l} \sum_{a=0}^{m} \left[\binom{d}{k} \binom{e}{l} \binom{a}{m} (r^a \eth^a \alpha) \left((-1)^d A_{a+d+e}^{d+e} \right) \right] \left\{ \frac{D^{k-d} D'^{l-e} \eth^{m-a} D\psi_0}{r^{a+d+e}} \right\}$$
$$+ \sum_{d=0}^{k} \sum_{e=0}^{l} \sum_{a=0}^{m} \left[\binom{d}{k} \binom{e}{l} \binom{a}{m} (r^a \eth^a \beta) \left((-1)^d A_{a+d+e}^{d+e} \right) \right] \left\{ \frac{D^{k-d} D'^{l-e} \eth^{m-a} \delta\psi_0}{r^{a+d+e}} \right\}$$
$$+2s \sum_{d=0}^{k} \sum_{e=0}^{l} \sum_{a+b+c=m} \left[\binom{d}{k} \binom{e}{l} \frac{m! \partial_\theta^b (\cot\theta)}{2\sqrt{2} a!b!c!} (r^a \eth^a \beta) \left((-1)^d A_{1+a+d+e}^{d+e} \right) \right]$$
$$\times \left\{ \frac{D^{k-d} D'^{l-e} \eth^c \psi_0}{r^{1+a+b+d+e}} \right\}$$

The factors in the brackets are clearly bounded provided that θ lies in $[c, \pi - c]$ for a given (arbitrarily small) positive constant c. There exists consequently a constant C depending on the spin, the constant c, L^∞ bounds on the coefficients of the spinor field $\xi^{A'}$ and their derivatives, such that:

$$|D^k D'^l \eth^m \phi_0| \le C \left(\sum_{d=0}^{k} \sum_{e=0}^{l} \sum_{a=0}^{m} \left| \frac{D^{k-d} D'^{l-e} \eth^{m-a} D\psi_0}{r^{a+d+e}} \right| \right.$$
$$\left. + \sum_{d=0}^{k} \sum_{e=0}^{l} \sum_{a=0}^{m} \left| \frac{D^{k-d} D'^{l-e} \eth^{m-a} \delta\psi_0}{r^{a+d+e}} \right| + \sum_{d=0}^{k} \sum_{e=0}^{l} \sum_{a+b+c=m} \left| \frac{D^{k-d} D'^{l-e} \eth^c \psi_0}{r^{1+a+b+d+e}} \right| \right) \tag{6.7}$$

Each of these terms is treated separately.

The first term can be transformed to fit the induction hypothesis using lemma 6.1:

$$D^{k-d} D'^{l-e} \eth^{m-a} D\psi_0 = D^{k-d+1} D'^{l-e} \eth^{m-a} \psi_0 + D^{k-d} D'^{l-e} \left(\frac{m-a}{r} \eth^{m-a} \psi_0 \right)$$
$$= D^{k-d+1} D'^{l-e} \eth^{m-a} \psi_0$$
$$+ \sum_{f=0}^{k-d} \sum_{g=0}^{l-e} \binom{f}{k-d} \binom{g}{l-e} \frac{(-1)^f (m-a)(k-d+l-e)!}{r^{1+f+g}} D^{k-d-f} D'^{l-e-g} \eth^{m-a} \psi_0.$$

In order to use the decay result stated in the induction hypothesis, the number of derivatives in the ingoing direction has to be taken into account:

a) if $1 + \delta - k < 0$, then:

$$|D^k D'^l \eth^m \psi_0| \le C \langle u \rangle^{1+\delta-k} \langle v \rangle^{-1-2s-l-m} I_{n+2s+2}$$

b) if $1+\delta-k>0$, then:

$$|D^k D'^l \nabla\!\!\!/^m \psi_0| \le C{<}v{>}^{\delta-2s-k-l-m} I_{n+2s+2}$$

In order to simplify the presentation of the proof, we deal specifically with the sum:

$$A = \sum_{d=0}^{k}\sum_{e=0}^{l}\sum_{a=0}^{m}\left|\frac{D^{k-d+1}D'^{l-e}\nabla\!\!\!/^{m-a}\psi_0}{r^{a+d+e}}\right|$$

Assume first that $1+\delta-k<0$.

For this case the sum a priori contains terms of both type a and type b. We therefore split up the sum over d into two parts corresponding to the different types. Let d' be the largest integer such that $d' \le k+1$ and $\delta - k + d' - 1 < 0$. This means that if $0 \le d \le d'-1$ we have $1+\delta-(k-d+1)<0$ and if $d' \le d \le k$ we have $1+\delta-(k-d+1)>0$. Using the induction hypothesis, the sum A can then be bounded as follows: there exists a constant C such that:

$$\begin{aligned}
A &\le C\sum_{e=0}^{l}\sum_{a=0}^{m}\frac{I_{\delta,n+2s+2}}{r^{a+e}}\cdot\Big(\sum_{d=0}^{d'-1}\frac{{<}u{>}^{1+\delta-(k-d+1)}}{{<}v{>}^{1+2s+l-e+m-a}r^d}+\sum_{d=d'}^{k}\frac{{<}v{>}^{\delta-2s-k+d-1-l+e-m+a}}{r^d}\Big)\\
&\le C\frac{{<}u{>}^{1+\delta-k}I_{\delta,n+2s+2}}{{<}v{>}^{2s+2+l+m}}\left(\sum_{e=0}^{l}\sum_{a=0}^{m}\left(\frac{{<}v{>}}{r}\right)^{a+e}\right)\\
&\quad\times\left(\sum_{d=0}^{d'-1}\left(\frac{{<}u{>}}{{<}v{>}}\right)^{d+1}\left(\frac{{<}v{>}}{r}\right)^{d}+\sum_{d=d'}^{k}\left(\frac{{<}u{>}}{{<}v{>}}\right)^{-(1+\delta-k)}\left(\frac{{<}v{>}}{r}\right)^{d}\right)
\end{aligned}$$

Since the considered region is the exterior region, that is to say the region defined by:

$$\frac{t}{3} \le r \le 3t,$$

the following inequalities hold (assuming also $r>1$, which is not restrictive, when studying the asymptotic behaviour):

$$\frac{{<}v{>}}{r} \le \sqrt{17} \text{ and } \frac{{<}u{>}}{{<}v{>}} \le \sqrt{10}.$$

As a consequence, there exists a constant C depending only on the considered region and of the number of derivatives such that:

$$A \le C{<}u{>}^{1+\delta-k}{<}v{>}^{-1-2s-1-l-m}I_{\delta,n+2s+3}.$$

In the case when $1+\delta-k>0$, all the indices $1+\delta-k+d$ are a fortiori positive and, as a consequence, the induction hypothesis gives immediately: there exists a constant C depending on the number of derivative and on the bounds of the derivatives of α and β such that:

$$A \le C{<}v{>}^{\delta-2s-1-k-l-m}I_{\delta,n+2s+3}\sum_{d=0}^{k}\sum_{e=0}^{l}\sum_{a=0}^{m}\left(\frac{{<}v{>}}{r}\right)^{a+e+d}.$$

There exists consequently, as previously, a constant C depending on the number of derivatives such that:

$$A \le C{<}v{>}^{\delta-2s-1-k-l-m}I_{\delta,n+2s+3}.$$

The other terms in (6.7) can be studied in a similar way and details are left to the reader. Collecting all the inequalities obtained for these derivatives, one gets that there exists a constant C, depending only on the Sobolev embeddings and the number of derivatives such that:

$$|D^k D'^l \nabla\!\!\!/^m \phi_0| \le C I_{\delta,n+2s+3}\begin{cases}{<}u{>}^{1+\delta-k}{<}v{>}^{-1-2s-1-l-m} & \text{if } 1+\delta-k<0\\ {<}v{>}^{-2s-1-k-l-m} & \text{if } 1+\delta-k>0\end{cases}$$

The other components $\psi_{\mathbf{i}}$ of the field can be studied in a similar way. The discussion will this time occur on the sign of $1+\delta-k-\mathbf{i}$. These complementary computations are left to the reader.

Hence, we have proved that the induction hypothesis holds also for $s+\frac{1}{2}$. We can therefore conclude that it holds for all $s \in \frac{1}{2}\mathbb{N}$. □

7. Main result

This section contains the main result of the paper, which consists in, for arbitrary spin, in a decay result for solutions of the Cauchy problem with initial data in weighted Sobolev spaces. This result extends the result contained in [10] for the fixed weight $\delta = -s - \frac{3}{2}$ (for spin-s fields with $s = 1, 2$) and clarifies the fact that the peeling fails for the fastly decaying components of the field. Furthermore, through Theorem 4.9, it establishes a full correspondence between the decay result of the wave equation and the peeling result for the higher spin fields.

The notations adopted in the formulation of the main theorem is consistent with the ones which are adopted in Section 6.1.

Theorem 7.1. *Let s be in $\frac{1}{2}\mathbb{N}$, δ in $\mathbb{R}\backslash\mathbb{Z}$, $j \geq 2$ an integer and consider the Cauchy problem for the massless free fields of spin s*

$$\begin{cases} \nabla^{AA'}\phi_{A\dots F} = 0 \\ \phi_{A\dots F}|_{t=0} = \varphi_{A\dots F} \in H^j_\delta(\mathcal{S}_{2s}) \\ D^{AB}\varphi_{A\dots F} = 0. \end{cases}$$

One assumes that, for $\delta < -2s-2$, the initial datum is orthogonal to the finite dimensional space $\mathbb{E}_{s,\delta}$.

One finally considers three nonnegative integers k, l, m whose sum is denoted by $n \leq j$.

The following inequalities hold, for all $\boldsymbol{i}$ in $\{0, \dots, 2s\}$: there exists a constant c depending only on a choice of a constant dyad and of k, l, m such that:

(1) *for any $t \geq 0$, $x \in \mathbb{R}^3$ such that $t > 3r$, that is to say, in the interior region,*

$$|\nabla^n \phi_{A\dots F}| \leq c{<}t{>}^{\delta-n} \|\varphi_{A\dots F}\|_{\delta,2+n};$$

(2) *for $\boldsymbol{i}$ such that $1 + 2s + \delta - k - \boldsymbol{i} < 0$, for any $t \geq 0$, $x \in \mathbb{R}^3$, such that $3r > t > \frac{r}{3}$, that is to say in the exterior region,*

$$|D^k D'^l \eth^m \phi_{\boldsymbol{i}}| \leq \frac{c{<}u{>}^{\delta+1+2s-\boldsymbol{i}-l}}{{<}v{>}^{1+2s-\boldsymbol{i}+k+m}} \|\varphi_{A\dots F}\|_{\delta,2+n};$$

(3) *for $\boldsymbol{i}$ such that $1 + 2s + \delta - k - \boldsymbol{i} > 0$, for any $t \geq 0$, $x \in \mathbb{R}^3$, such that $3r > t > \frac{r}{3}$, that is to say in the exterior region,*

$$|D^k D'^l \eth^m \phi_{\boldsymbol{i}}| \leq c{<}v{>}^{\delta-n} \|\varphi_{A\dots F}\|_{\delta,2+n}.$$

A specific case, stated as a corollary, due to its importance, is the following:

Corollary 7.2. *If we assume that*

- *for spin 1, $\delta > -4$ and $\delta \notin \mathbb{Z}$;*
- *for spin 2, $\delta > -6$ and $\delta \notin \mathbb{Z}$.*

Then the decay result stated in Theorem 7.1 holds without restrictions on the initial data.

Remark 7.3. *If $\delta < -2s - 2$, we can of course embed $\varphi_{A\dots F}$ in $H^j_{-2s-2+\epsilon}$ which would give a weaker decay result, but without any orthogonality condition.*

Proof. Let s and δ be such as in the theorem and consider $\varphi_{A\dots F}$ a initial datum in $H^j_\delta(\mathcal{S}_{2s})$ satisfying the constraints equation:

$$D^{AB}\varphi_{A\dots F} = 0.$$

The initial datum $\varphi_{A\dots F}$ satisfies the assumptions stated in Theorem 4.9, so that there exists a potential $\tilde{\chi}^{A'\dots F'}$ of order $2s$ such that the solution of the Cauchy problem with initial datum $\varphi_{A\dots F}$ is given by:

$$\phi_{A\dots F} = \nabla_{AA'} \dots \nabla_{FF'} \tilde{\chi}^{A'\dots F'}$$

and $\chi_{A\dots F} = \tau_{AA'} \dots \tau_{FF'} \tilde{\chi}^{A'\dots F'}$ satisfies the Cauchy problem:

$$\begin{cases} \Box \chi_{A\dots F} = 0 \\ \chi_{A\dots F}|_{t=0} \in H^{j+2s}_{\delta+2s}(\mathcal{S}_{2s}) \\ \partial_t \chi_{A\dots F}|_{t=0} \in H^{j+2s-1}_{\delta+2s-1}(\mathcal{S}_{2s}). \end{cases}$$

Furthermore, the norm of the potential is controlled by the norm of the initial data:

$$\|\chi_{A\ldots F}\|_{n+2s,\delta+2s} \le c\|\varphi_{A\ldots F}\|_{n,\delta}.$$

A constant dyad (e_0^A, e_1^A) on the Minkowski space is chosen. The components of the field $\chi_{A\ldots F}$ are then of the form:

$$\chi\xi_A^1\ldots\xi_F^{2s},$$

where the constant spinor ξ_A^i (for $i\in\{1,\ldots,2s\}$) belongs to $\{\overline{e}_A^0, \overline{e}_A^1\}$ and χ is complex function satisfying a Cauchy problem of the form:

$$\begin{cases} \square\chi = 0 \\ \chi|_{t=0} \in H^{j+2s}_{\delta+2s}(\mathbb{R}^3,\mathbb{C}) \\ \partial_t\chi|_{t=0} \in H^{j+2s-1}_{\delta+2s-1}(\mathbb{R}^3,\mathbb{C}). \end{cases}$$

Proposition 6.4 can then be used, on each of the components of the field. All these components decay exactly in the same way and, consequently, the field $\phi_{A\ldots F}$ decays exactly as the field under consideration in Proposition 6.4. □

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Appendix A. Algebraic properties of the fundamental operators

To prove Proposition 4.1, we need the following relation:

Lemma A.1. *The operators $\mathcal{G}_k$ and $\mathcal{C}_k$ commute and we have*

$$\begin{aligned}
(\mathcal{G}_k\mathcal{C}_k\phi)_{A_1\ldots A_k} &= (\mathcal{C}_k\mathcal{G}_k\phi)_{A_1\ldots A_k} \\
&= \sum_{n=0}^{\lfloor\frac{k}{2}\rfloor}\binom{k}{2n}(-2)^{-n}\underbrace{D_{(A_1}{}^{B_1}\cdots D_{A_{k-2n}}{}^{B_{k-2n}}}_{k-2n}(\Delta_k^n\phi)_{A_{k-2n+1}\ldots A_k)B_1\ldots B_{k-2n}}.
\end{aligned}$$

Proof. We begin by proving that $(\mathcal{G}_k\mathcal{C}_k\phi)_{A_1\ldots A_k}$ has the desired form. By partially expanding the symmetry of the $\mathcal{C}_k$ operator we get

$$\begin{aligned}
&\underbrace{D_{(A_1}{}^{B_1}\cdots D_{A_{k-2n-1}}{}^{B_{k-2n-1}}}_{k-2n-1}(\Delta_k^n\mathcal{C}_k\phi)_{A_{k-2n}\ldots A_k)B_1\ldots B_{k-2n-1}} \\
&= \frac{2n+1}{k}\underbrace{D_{(A_1}{}^{B_1}\cdots D_{A_{k-2n}}{}^{B_{k-2n}}}_{k-2n}(\Delta_k^n\phi)_{A_{k-2n+1}\ldots A_k)B_1\ldots B_{k-2n}} \\
&\quad + \frac{k-2n-1}{k}\underbrace{D_{(A_1}{}^{B_1}\cdots D_{A_{k-2n-1}}{}^{B_{k-2n-1}}}_{k-2n-1}D_{|B_{k-2n-1}|}{}^{B_{k-2n}}(\Delta_k^n\phi)_{A_{k-2n}\ldots A_k)B_1\ldots B_{k-2n}} \\
&= \frac{2n+1}{k}\underbrace{D_{(A_1}{}^{B_1}\cdots D_{A_{k-2n}}{}^{B_{k-2n}}}_{k-2n}(\Delta_k^n\phi)_{A_{k-2n+1}\ldots A_k)B_1\ldots B_{k-2n}} \\
&\quad - \frac{k-2n-1}{2k}\underbrace{D_{(A_1}{}^{B_1}\cdots D_{A_{k-2n-2}}{}^{B_{k-2n-2}}}_{k-2n-2}(\Delta_k^{n+1}\phi)_{A_{k-2n-1}\ldots A_k)B_1\ldots B_{k-2n-2}}.
\end{aligned}$$

Where we used $D_A{}^C D_{BC} = -\frac{1}{2}\epsilon_{AB}\Delta$ in the last step. We therefore get:

$$
\begin{aligned}
&(\mathcal{G}_k \mathcal{C}_k \phi)_{A_1\dots A_k} \\
&= \sum_{n=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k-1}{2n} (-2)^{-n} \underbrace{D_{(A_1}{}^{B_1} \cdots D_{A_{k-2n}}{}^{B_{k-2n}}}_{k-2n} (\Delta_k^n \phi)_{A_{k-2n+1}\dots A_k)B_1\dots B_{k-2n}} \\
&\quad + \sum_{n=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k-1}{2n+1} (-2)^{1-n} \underbrace{D_{(A_1}{}^{B_1} \cdots D_{A_{k-2n-2}}{}^{B_{k-2n-2}}}_{k-2n-2} (\Delta_k^{n+1} \phi)_{A_{k-2n-1}\dots A_k)B_1\dots B_{k-2n-2}} \\
&= \sum_{n=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k-1}{2n} (-2)^{-n} \underbrace{D_{(A_1}{}^{B_1} \cdots D_{A_{k-2n}}{}^{B_{k-2n}}}_{k-2n} (\Delta_k^n \phi)_{A_{k-2n+1}\dots A_k)B_1\dots B_{k-2n}} \\
&\quad + \sum_{n=1}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k-1}{2n-1} (-2)^{-n} \underbrace{D_{(A_1}{}^{B_1} \cdots D_{A_{k-2n}}{}^{B_{k-2n}}}_{k-2n} (\Delta_k^n \phi)_{A_{k-2n+1}\dots A_k)B_1\dots B_{k-2n}}.
\end{aligned}
\tag{A.1}
$$

Where we just changed $n \to n-1$ in the last sum. The Pascal triangle gives the algebraic identity:

$$
\sum_{n=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k-1}{2n} A^{k-n} B^n + \sum_{n=1}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k-1}{2n-1} A^{k-n} B^n = \sum_{n=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2n} A^{k-n} B^n,
$$

which in turn gives the desired form for $(\mathcal{G}_k \mathcal{C}_k \phi)_{A_1\dots A_k}$. To handle $(\mathcal{C}_k \mathcal{G}_k \phi)_{A_1\dots A_k}$ we partially expand the symmetry in the following expression:

$$
\begin{aligned}
&D_{A_1}{}^C \underbrace{D_{(A_2}{}^{B_2} \cdots D_{A_{k-2n}}{}^{B_{k-2n}}}_{k-2n-1} (\Delta_k^n \phi)_{A_{k-2n+1}\dots A_k C)B_2\dots B_{k-2n}} \\
&= \frac{k-2n-1}{k} D_{A_1}{}^C D_C{}^{B_2} \underbrace{D_{(A_2}{}^{B_3} \cdots D_{A_{k-2n-1}}{}^{B_{k-2n}}}_{k-2n-2} (\Delta_k^n \phi)_{A_{k-2n}\dots A_k)B_2\dots B_{k-2n}} \\
&\quad + \frac{2n+1}{k} D_{A_1}{}^C \underbrace{D_{(A_2}{}^{B_2} \cdots D_{A_{k-2n}}{}^{B_{k-2n}}}_{k-2n-1} (\Delta_k^n \phi)_{A_{k-2n+1}\dots A_k)CB_2\dots B_{k-2n}} \\
&= -\frac{k-2n-1}{2k} \underbrace{D_{(A_2}{}^{B_3} \cdots D_{A_{k-2n-1}}{}^{B_{k-2n}}}_{k-2n-2} (\Delta_k^{n+1} \phi)_{A_{k-2n}\dots A_k)A_1 B_3\dots B_{k-2n}} \\
&\quad + \frac{2n+1}{k} D_{A_1}{}^{B_1} \underbrace{D_{(A_2}{}^{B_2} \cdots D_{A_{k-2n}}{}^{B_{k-2n}}}_{k-2n-1} (\Delta_k^n \phi)_{A_{k-2n+1}\dots A_k)B_1\dots B_{k-2n}}.
\end{aligned}
$$

Where we in the last step again used $D_A{}^C D_{BC} = -\frac{1}{2}\epsilon_{AB}\Delta$. Using this in the definition of $\mathcal{G}_k$ yields:

$$
\begin{aligned}
&D_{A_1}{}^C (\mathcal{G}_k \phi)_{CA_2\dots A_k} \\
&= \sum_{n=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k-1}{2n+1} (-2)^{1-n} \underbrace{D_{(A_2}{}^{B_3} \cdots D_{A_{k-2n-1}}{}^{B_{k-2n-2}}}_{k-2n-2} (\Delta_k^{n+1} \phi)_{A_{k-2n}\dots A_k)A_1 B_3\dots B_{k-2n}} \\
&\quad + \sum_{n=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k-1}{2n} (-2)^{1-n} D_{A_1}{}^{B_1} \underbrace{D_{(A_2}{}^{B_2} \cdots D_{A_{k-2n}}{}^{B_{k-2n}}}_{k-2n-1} (\Delta_k^n \phi)_{A_{k-2n+1}\dots A_k)B_1\dots B_{k-2n}}.
\end{aligned}
$$

After symmetrization we get that $(\mathscr{C}_k \mathcal{G}_k \phi)_{A_1 \dots A_k}$ has an expansion identical to the one in the first equation in (A.1). This gives the desired result. $\square$

Proposition A.2. *The equation* (4.2) *together with* (4.6) *implies*

$$\phi_{A_1 \dots A_{2s}} = (\mathcal{G}_{2s} \mathscr{C}_{2s} \chi)_{A_1 \dots A_{2s}} + \tfrac{1}{\sqrt{2}} (\mathcal{G}_{2s} \partial_t \chi)_{A_1 \dots A_{2s}}.$$

Proof. Using $\tau_B{}^{A'} \nabla_{AA'} = D_{AB} + \frac{1}{\sqrt{2}} \epsilon_{AB} \partial_t$ we can express the express the potential equation in terms of D_{AB} and ∂_t.

$$\begin{aligned}
\phi_{A_1 \dots A_{2s}} &= \underbrace{\nabla_{A_1 A'_1} \cdots \nabla_{A_{2s} A'_{2s}}}_{2s} \widetilde{\chi}^{A'_1 \dots A'_{2s}} = \underbrace{\tau^{B_1 A'_1} \nabla_{A_1 A'_1} \cdots \tau^{B_{2s} A'_{2s}} \nabla_{A_{2s} A'_{2s}}}_{2s} \chi_{B_1 \dots B_{2s}} \\
&= \underbrace{(D_{A_1}{}^{B_1} + \tfrac{1}{\sqrt{2}} \epsilon_{A_1}{}^{B_1} \partial_t) \cdots (D_{A_{2s}}{}^{B_{2s}} + \tfrac{1}{\sqrt{2}} \epsilon_{A_{2s}}{}^{B_{2s}} \partial_t)}_{2s} \chi_{B_1 \dots B_{2s}} \\
&= \sum_{n=0}^{2s} \binom{2s}{n} 2^{-n/2} \underbrace{D_{(A_1}{}^{B_1} \cdots D_{A_{2s-n}}{}^{B_{2s-n}}}_{2s-n} \partial_t^n \chi_{A_{2s-n+1} \dots A_{2s}) B_1 \dots B_{2s-n}}.
\end{aligned}$$

We can now use (4.6) to elliminate all higher order time derivatives.

$$\begin{aligned}
&\phi_{A_1 \dots A_{2s}} \\
&= \sum_{n=0}^{\lfloor s \rfloor} \binom{2s}{2n} 2^{-n} \underbrace{D_{(A_1}{}^{B_1} \cdots D_{A_{2s-2n}}{}^{B_{2s-2n}}}_{2s-2n} \partial_t^{2n} \chi_{A_{2s-2n+1} \dots A_{2s}) B_1 \dots B_{2s-2n}} \\
&\quad + \sum_{n=0}^{\lfloor s-\frac{1}{2} \rfloor} \binom{2s}{2n+1} 2^{-n-1/2} \underbrace{D_{(A_1}{}^{B_1} \cdots D_{A_{2s-2n-1}}{}^{B_{2s-2n-1}}}_{2s-2n-1} \partial_t^{2n+1} \chi_{A_{2s-2n} \dots A_{2s}) B_1 \dots B_{2s-2n-1}} \\
&= \sum_{n=0}^{\lfloor s \rfloor} \binom{2s}{2n} (-2)^{-n} \underbrace{D_{(A_1}{}^{B_1} \cdots D_{A_{2s-2n}}{}^{B_{2s-2n}}}_{2s-2n} (\Delta_{2s}^n \chi)_{A_{2s-2n+1} \dots A_{2s}) B_1 \dots B_{2s-2n}} \\
&\quad + \frac{1}{\sqrt{2}} \sum_{n=0}^{\lfloor s-\frac{1}{2} \rfloor} \binom{2s}{2n+1} (-2)^{-n} \underbrace{D_{(A_1}{}^{B_1} \cdots D_{A_{2s-2n-1}}{}^{B_{2s-2n-1}}}_{2s-2n-1} (\Delta_{2s}^n \partial_t \chi)_{A_{2s-2n} \dots A_{2s}) B_1 \dots B_{2s-2n-1}} \\
&= (\mathcal{G}_{2s} \mathscr{C}_{2s} \chi)_{A_1 \dots A_{2s}} + \tfrac{1}{\sqrt{2}} (\mathcal{G}_{2s} \partial_t \chi)_{A_1 \dots A_{2s}}.
\end{aligned}$$

In the last step we used the definition of $\mathcal{G}_{2s}$ and lemma A.1. In fact we have defined $\mathcal{G}_k$ to match the ∂_t part of this expression. $\square$

Proposition A.3. *For $k \geq 2$, the operators $\mathcal{G}_k$ have the properties $\mathscr{D}_k \mathcal{G}_k = 0$ and $\mathcal{G}_k \mathscr{T}_{k-2} = 0$.*

Proof. First we prove that $\mathscr{D}_k \mathcal{G}_k = 0$. By partially expanding the symmetrization in the definition of $\mathcal{G}_k$ and restricting the summation to non-vanishing terms we get

$$\begin{aligned}
&(\mathcal{G}_k \phi)_{A_1 \dots A_k} \\
&= \sum_{n=0}^{\left\lfloor \frac{k-3}{2} \right\rfloor} \binom{k-2}{2n+1} (-2)^{-n} D_{A_1}{}^{B_1} D_{A_2}{}^{B_2} \underbrace{D_{(A_3}{}^{B_3} \cdots D_{A_{k-2n-1}}{}^{B_{k-2n-1}}}_{k-2n-3} (\Delta_k^n \phi)_{A_{k-2n} \dots A_k) B_1 \dots B_{k-2n-1}} \\
&\quad + \sum_{n=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \binom{k-2}{2n} (-2)^{-n} D_{A_1}{}^{B_2} \underbrace{D_{(A_3}{}^{B_3} \cdots D_{A_{k-2n}}{}^{B_{k-2n}}}_{k-2n-2} (\Delta_k^n \phi)_{A_{k-2n+1} \dots A_k) A_2 B_2 \dots B_{k-2n}}
\end{aligned}$$

$$+\sum_{n=0}^{\left\lfloor\frac{k-1}{2}\right\rfloor}\binom{k-2}{2n}(-2)^{-n}D_{A_2}{}^{B_2}\underbrace{D_{(A_3}{}^{B_3}\cdots D_{A_{k-2n}}{}^{B_{k-2n}}}_{k-2n-2}(\Delta_k^n\phi)_{A_{k-2n+1}\dots A_k)A_1B_2\dots B_{k-2n}}$$

$$+\sum_{n=1}^{\left\lfloor\frac{k-1}{2}\right\rfloor}\binom{k-2}{2n-1}(-2)^{-n}\underbrace{D_{(A_3}{}^{B_3}\cdots D_{A_{k-2n+1}}{}^{B_{k-2n+1}}}_{k-2n-1}(\Delta_k^n\phi)_{A_{k-2n+2}\dots A_k)A_1A_2B_3\dots B_{k-2n+1}}.$$

Using $D_A{}^C D_{BC}=-\frac{1}{2}\epsilon_{AB}\Delta$ we get $D^{A_1A_2}D_{A_1}{}^{B_1}D_{A_2}{}^{B_2}=\frac{1}{2}D^{B_1B_2}\Delta$, $D^{A_1(A_2}D_{A_1}{}^{B_2)}=0$ and

$$\begin{aligned}&(\mathfrak{d}_k\mathcal{G}_k\phi)_{A_1\dots A_k}\\
&=-\sum_{n=0}^{\left\lfloor\frac{k-3}{2}\right\rfloor}\binom{k-2}{2n+1}(-2)^{-n-1}D^{B_1B_2}\underbrace{D_{(A_3}{}^{B_3}\cdots D_{A_{k-2n-1}}{}^{B_{k-2n-1}}}_{k-2n-3}(\Delta_k^{n+1}\phi)_{A_{k-2n}\dots A_k)B_1\dots B_{k-2n-1}}\\
&\quad+\sum_{n=1}^{\left\lfloor\frac{k-1}{2}\right\rfloor}\binom{k-2}{2n-1}(-2)^{-n}D^{B_1B_2}\underbrace{D_{(A_3}{}^{B_3}\cdots D_{A_{k-2n+1}}{}^{B_{k-2n+1}}}_{k-2n-1}(\Delta_k^n\phi)_{A_{k-2n+2}\dots A_k)B_1\dots B_{k-2n+1}}.\end{aligned}$$

The the first sum is identical to the second sum after a variable change $n\to n-1$, hence $\mathfrak{d}_k\mathcal{G}_k=0$.

Now, we turn to the proof of $\mathcal{G}_k\mathfrak{t}_{k-2}=0$. Partial expansion of the symmetrization in the definition of $\mathfrak{t}_{k-2}$ gives

$$\begin{aligned}&\underbrace{D^{B_1}_{(A_1}\cdots D_{A_{k-2n-1}}{}^{B_{k-2n-1}}}_{k-2n-1}(\Delta_k^n\mathfrak{t}_{k-2}\phi)_{A_{k-2n}\dots A_k)B_1\dots B_{k-2n-1}}\\
&=\frac{(k-2n-1)(k-2n-2)}{k(k-1)}D_{B_1B_2}\underbrace{D_{(A_1}{}^{B_1}\cdots D_{A_{k-2n-1}}{}^{B_{k-2n-1}}}_{k-2n-1}(\Delta_{k-2}^n\phi)_{A_{k-2n}\dots A_k)B_3\dots B_{k-2n-1}}\\
&\quad+\frac{2(k-2n-1)(2n+1)}{k(k-1)}D_{B_1(A_k}\underbrace{D_{A_1}{}^{B_1}\cdots D_{A_{k-2n-1}}{}^{B_{k-2n-1}}}_{k-2n-1}(\Delta_{k-2}^n\phi)_{A_{k-2n}\dots A_{k-1})B_2\dots B_{k-2n-1}}\\
&\quad+\frac{2n(2n+1)}{k(k-1)}D_{(A_{k-1}A_k}\underbrace{D_{A_1}{}^{B_1}\cdots D_{A_{k-2n-1}}{}^{B_{k-2n-1}}}_{k-2n-1}(\Delta_{k-2}^n\phi)_{A_{k-2n}\dots A_{k-2})B_1\dots B_{k-2n-1}}\\
&=\frac{(k-2n-1)(k-2n-2)}{2k(k-1)}D_{(A_1A_2}\underbrace{D_{A_3}{}^{B_3}\cdots D_{A_{k-2n-1}}{}^{B_{k-2n-1}}}_{k-2n-3}(\Delta_{k-2}^{n+1}\phi)_{A_{k-2n}\dots A_k)B_3\dots B_{k-2n-1}}\\
&\quad+\frac{2n(2n+1)}{k(k-1)}D_{(A_{k-1}A_k}\underbrace{D_{A_1}{}^{B_1}\cdots D_{A_{k-2n-1}}{}^{B_{k-2n-1}}}_{k-2n-1}(\Delta_{k-2}^n\phi)_{A_{k-2n}\dots A_{k-2})B_1\dots B_{k-2n-1}}.\end{aligned}$$

Where we again used $D^{A_1A_2}D_{A_1}{}^{B_1}D_{A_2}{}^{B_2}=\frac{1}{2}D^{B_1B_2}\Delta$ and $D^{A_1(A_2}D_{A_1}{}^{B_2)}=0$. We therefore get

$$\begin{aligned}&(\mathcal{G}_k\mathfrak{t}_{k-2}\phi)_{A_1\dots A_k}\\
&=-\sum_{n=0}^{\left\lfloor\frac{k-3}{2}\right\rfloor}\binom{k-2}{2n+1}(-2)^{1-n}D_{(A_1A_2}\underbrace{D_{A_3}{}^{B_3}\cdots D_{A_{k-2n-1}}{}^{B_{k-2n-1}}}_{k-2n-3}(\Delta_{k-2}^{n+1}\phi)_{A_{k-2n}\dots A_k)B_3\dots B_{k-2n-1}}\\
&\quad+\sum_{n=1}^{\left\lfloor\frac{k-1}{2}\right\rfloor}\binom{k-2}{2n-1}(-2)^{-n}D_{(A_{k-1}A_k}\underbrace{D_{A_1}{}^{B_1}\cdots D_{A_{k-2n-1}}{}^{B_{k-2n-1}}}_{k-2n-1}(\Delta_{k-2}^n\phi)_{A_{k-2n}\dots A_{k-2})B_1\dots B_{k-2n-1}}\end{aligned}$$

The the first sum is identical to the second sum after a variable change $n \to n-1$, hence $\mathcal{G}_k \mathfrak{t}_{k-2} = 0$. □

To use elliptic theory, we need well behaved elliptic operators. $\mathcal{G}_k$ is in general not elliptic but, through the following lemma, it can related to some power of the Laplacian – which of course is elliptic.

Lemma A.4. *The formulae* (2.1b) *and* (2.1a) *hold, that is to say:*

$$(\Delta_{2k}^k \phi)_{A_1 \dots A_{2k}} = (\mathfrak{t}_{2k-2}\mathcal{F}_{2k-2}\mathfrak{d}_{2k}\phi)_{A_1\dots A_{2k}} - (-2)^{1-k}(\mathcal{G}_{2k}\mathfrak{c}_{2k}\phi)_{A_1\dots A_{2k}},$$
$$(\Delta_{2k+1}^k \phi)_{A_1 \dots A_{2k+1}} = (\mathfrak{t}_{2k-1}\mathcal{F}_{2k-1}\mathfrak{d}_{2k+1}\phi)_{A_1\dots A_{2k+1}} + (-2)^{-k}(\mathcal{G}_{2k+1}\phi)_{A_1\dots A_{2k+1}}.$$

Proof. For both formulae, we will use the following help quantity for the spin $k + j/2$ case:

$$I_m^{j,k} \equiv \sum_{n=m}^{k-1} \binom{2k+j}{2n+j-2m} (-2)^{-n} \underbrace{D_{(A_1}{}^{B_1} \cdots D_{A_{2k-2n}}{}^{B_{2k-2n}}}_{2k-2n} (\Delta_{2k+j}^n \phi)_{A_{2k-2n+1}\dots A_{2k+j})B_1\dots B_{2k-2n}}$$

Multiplying $D_{A_1}{}^{B_1} D_{C_1}{}^{B_2} \phi_{A_3\dots A_k C_2 B_2}$ with $\epsilon_{A_2}{}^{C_1}\epsilon_{B_1}{}^{C_2} = \epsilon_{A_2B_1}\epsilon^{C_1C_2} + \epsilon_{A_2}{}^{C_2}\epsilon_{B_2}{}^{C_1}$ and using $D_A{}^C D_{BC} = -\frac{1}{2}\epsilon_{AB}\Delta$, we get

$$D_{A_1}{}^{B_1} D_{A_2}{}^{B_2} \phi_{A_3\dots A_k B_1 B_2} = -\tfrac{1}{2}(\Delta_k \phi)_{A_1\dots A_k} + D_{A_1A_2}(\mathfrak{d}_k\phi)_{A_3\dots A_k}.$$

Using this in the definition of $I_m^{j,k}$ gives

$$\begin{aligned}
I_m^{j,k} = & \sum_{n=m}^{k-1} \binom{2k+j}{2n+j-2m} (-2)^{-n-1} \\
& \times \underbrace{D_{(A_1}{}^{B_1} \cdots D_{A_{2k-2n-2}}{}^{B_{2k-2n-2}}}_{2k-2n-2} (\Delta_{2k+j}^{n+1}\phi)_{A_{2k-2n-1}\dots A_{2k+j})B_1\dots B_{2k-2n-2}} \\
& + \sum_{n=m}^{k-1} \binom{2k+j}{2n+j-2m} (-2)^{-n} \\
& \times D_{(A_1A_2} \underbrace{D_{A_3}{}^{B_3} \cdots D_{A_{2k-2n}}{}^{B_{2k-2n}}}_{2k-2n-2} (\mathfrak{d}_{2k+j}\Delta_{2k+j}^n \phi)_{A_{2k-2n+1}\dots A_{2k+j})B_3\dots B_{2k-2n}} \\
= & \, I_{m+1}^{j,k} + \binom{2k+j}{2k+j-2m}(-2)^{-k}(\Delta_{2k+j}^k\phi)_{A_1\dots A_{2k+j}} + \sum_{n=m}^{k-1}\binom{2k+j}{2n+j-2m}(-2)^{-n} \\
& \times D_{(A_1A_2} \underbrace{D_{A_3}{}^{B_3} \cdots D_{A_{2k-2n}}{}^{B_{2k-2n}}}_{2k-2n-2} (\mathfrak{d}_{2k+j}\Delta_{2k+j}^n \phi)_{A_{2k-2n+1}\dots A_{2k+j})B_3\dots B_{2k-2n}}.
\end{aligned} \tag{A.2}$$

Here, we have changed $n \to n-1$ in the first sum, and identified that as $I_{m+1}^{j,k}$ plus the term where $n = k$, which gives us the Δ^k-term. We can easily solve the recursion(A.2) and get

$$\begin{aligned}
I_0^{j,k} = & \sum_{m=0}^{k-1} \binom{2k+j}{2k+j-2m}(-2)^{-k}(\Delta_{2k+j}^k\phi)_{A_1\dots A_{2k+j}} + \sum_{m=0}^{k-1}\sum_{n=m}^{k-1}\binom{2k+j}{2n+j-2m}(-2)^{-n} \\
& \times D_{(A_1A_2} \underbrace{D_{A_3}{}^{B_3} \cdots D_{A_{2k-2n}}{}^{B_{2k-2n}}}_{2k-2n-2} (\mathfrak{d}_{2k+j}\Delta_{2k+j}^n \phi)_{A_{2k-2n+1}\dots A_{2k+j})B_3\dots B_{2k-2n}} \\
= & \sum_{m=0}^{k-1} \binom{2k+j}{2m}(-2)^{-k}(\Delta_{2k+j}^k\phi)_{A_1\dots A_{2k+j}} + \sum_{n=0}^{k-1}\sum_{m=0}^{k-1-n}\binom{2k+j}{2n+2m+2}(-2)^{n+1-k} \\
& \times D_{(A_1A_2} \underbrace{D_{A_3}{}^{B_3} \cdots D_{A_{2n+2}}{}^{B_{2n+2}}}_{2n} (\Delta_{2k+j-2}^{k-1-n}\mathfrak{d}_{2k+j} \phi)_{A_{2n+3}\dots A_{2k+j})B_3\dots B_{2n+2}} \\
= & \sum_{m=0}^{k-1} \binom{2k+j}{2m}(-2)^{-k}(\Delta_{2k+j}^k\phi)_{A_1\dots A_{2k+j}} - 2^{j-1}(-2)^k(\mathfrak{t}_{2k+j-2}\mathcal{F}_{2k+j-2}\mathfrak{d}_{2k+j}\phi)_{A_1\dots A_{2k+j}}.
\end{aligned}$$

In the second sum we have changed the order of summation followed by the change $n \to k-n-1$.

For the operators acting on odd number of indices we have

$$
\begin{aligned}
&(\mathcal{G}_{2k+1}\phi)_{A_1\dots A_{2k+1}} \\
&\quad = \sum_{n=0}^{k} \binom{2k+1}{2n+1} (-2)^{-n} \underbrace{D_{(A_1}{}^{B_1} \cdots D_{A_{2k-2n}}{}^{B_{2k-2n}}}_{2k-2n} (\Delta^n_{2k+1}\phi)_{A_{2k-2n+1}\dots A_{2k+1})B_1\dots B_{2k-2n}} \\
&\quad = I_0^{1,k} + \binom{2k+1}{2k+1} (-2)^{-k} (\Delta^k_{2k+1}\phi)_{A_1\dots A_{2k+1}} \\
&\quad = \sum_{m=0}^{k} \binom{2k+1}{2k+1-2m} (-2)^{-k} (\Delta^k_{2k+1}\phi)_{A_1\dots A_{2k+1}} - (-2)^k (\mathfrak{f}_{2k-1}\mathcal{F}_{2k-1}\mathfrak{d}_{2k+1}\phi)_{A_1\dots A_{2k+1}} \\
&\quad = (-2)^k (\Delta^k_{2k+1}\phi)_{A_1\dots A_{2k+1}} - (-2)^k (\mathfrak{f}_{2k-1}\mathcal{F}_{2k-1}\mathfrak{d}_{2k+1}\phi)_{A_1\dots A_{2k+1}}.
\end{aligned}
$$

Hence,

$$
(\Delta^k_{2k+1}\phi)_{A_1\dots A_{2k+1}} = (-2)^{-k} (\mathcal{G}_{2k+1}\phi)_{A_1\dots A_{2k+1}} + (\mathfrak{f}_{2k-1}\mathcal{F}_{2k-1}\mathfrak{d}_{2k+1}\phi)_{A_1\dots A_{2k+1}}.
$$

For the operators acting on even number of indices we can use (A.1) to obtain

$$
\begin{aligned}
&(\mathcal{G}_{2k}\mathfrak{c}_{2k}\phi)_{A_1\dots A_{2k}} \\
&\quad = \sum_{n=0}^{k} \binom{2k}{2n} (-2)^{-n} \underbrace{D_{(A_1}{}^{B_1} \cdots D_{A_{2k-2n}}{}^{B_{2k-2n}}}_{2k-2n} (\Delta^n_{2k+1}\phi)_{A_{2k-2n+1}\dots A_{2k})B_1\dots B_{2k-2n}} \\
&\quad = I_0^{0,k} + \binom{2k}{2k} (-2)^{-k} (\Delta^k_{2k}\phi)_{A_1\dots A_{2k}} \\
&\quad = \sum_{m=0}^{k} \binom{2k}{2k-2m} (-2)^{-k} (\Delta^k_{2k}\phi)_{A_1\dots A_{2k}} + (-2)^k (\mathfrak{f}_{2k-2}\mathcal{F}_{2k-2}\mathfrak{d}_{2k}\phi)_{A_1\dots A_{2k}} \\
&\quad = -(-2)^{k-1} (\Delta^k_{2k}\phi)_{A_1\dots A_{2k}} + (-2)^{k-1} (\mathfrak{f}_{2k-2}\mathcal{F}_{2k-2}\mathfrak{d}_{2k}\phi)_{A_1\dots A_{2k}}.
\end{aligned}
$$

Hence,

$$
(\Delta^k_{2k}\phi)_{A_1\dots A_{2k}} = (\mathfrak{f}_{2k-2}\mathcal{F}_{2k-2}\mathfrak{d}_{2k}\phi)_{A_1\dots A_{2k}} - (-2)^{1-k} (\mathcal{G}_{2k}\mathfrak{c}_{2k}\phi)_{A_1\dots A_{2k}}.
$$

□

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E-mail address: `laan@aei.mpg.de`

Albert Einstein Institute, Am Mühlenberg 1, D-14476 Potsdam, Germany

E-mail address: `thomas.backdahl@aei.mpg.de`

Albert Einstein Institute, Am Mühlenberg 1, D-14476 Potsdam, Germany

E-mail address: `jeremie.joudioux@aei.mpg.de`

Albert Einstein Institute, Am Mühlenberg 1, D-14476 Potsdam, Germany