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ON THE MASS RADIATED BY COALESCING BLACK-HOLE BINARIES

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ABSTRACT

We derive an analytic phenomenological expression that predicts the final mass of the black-hole remnant resulting from the merger of a generic binary system of black holes on quasi-circular orbits. Besides recovering the correct test-particle limit for extreme mass-ratio binaries, our formula reproduces well the results of all the numerical-relativity simulations published so far, both when applied at separations of a few gravitational radii, and when applied at separations of tens of thousands of gravitational radii. These validations make our formula a useful tool in a variety of contexts ranging from gravitational-wave physics to cosmology. As representative examples, we first illustrate how it can be used to decrease the phase error of the effective-one-body waveforms during the ringdown phase. Second, we show that, when combined with the recently computed self-force correction to the binding energy of nonspinning black-hole binaries, it provides an estimate of the energy emitted during the merger and ringdown. Finally, we use it to calculate the energy radiated in gravitational waves by massive black-hole binaries as a function of redshift, using different models for the seeds of the black-hole population.

Subject headings: black-hole physics — relativity — gravitational waves — galaxies

1. INTRODUCTION

Black-hole (BH) mergers play a central role in gravitational-wave (GW) astrophysics, because they are expected to be among the main sources for existing and future detectors. More specifically, the LIGO/Virgo detectors (Abbott et al. 2009; Acernese et al. 2008) are expected to detect mergers of stellar-mass BHs happening within several hundred Mpc, when operating in their advanced configurations. Similarly, future space-based detectors such as LISA (Amaro-Seoane et al. 2012) or DECIGO (Kawamura et al. 2011) will detect mergers of massive BHs (MBHs) up to redshifts as high as $z \sim 10$ or beyond. Even intermediate-mass BHs (IMBHs), provided they exist, will be within reach of GW detectors, e.g. IMBH-MBH binaries will be detectable by LISA or DECIGO, while IMBH-IMBH binaries will be detectable with DECIGO or with the planned ground-based Einstein Telescope (Punturo et al. 2012; Sathyaprakash et al. 2012).

Given their relevance for GW astrophysics, it is not surprising that BH binaries have received widespread attention over the past few years. Because a detailed understanding of the dynamics of these systems is crucial in order to predict accurately the gravitational waveforms, which, in turn, is necessary to detect the signal and extract information on the physical parameters of the binaries, numerical simulations have been performed by a number of groups for a variety of mass ratios, BH spin magnitudes and orientations [see Pfeiffer (2012) for a recent review].

However, even today, numerical-relativity simulations are computationally very expensive and not able to cover the full seven-dimensional space of parameters of quasi-circular BH binaries. Fortunately, phenomenological models have been very successful at reproducing many aspects of the dynamics of BH binaries as revealed by the numerical simulations. For instance, hybrid "phenomenological waveforms" (Ajith et al. 2008; Santamaría et al. 2010), i.e., templates that represent phenomenological combinations of Post-Newtonian (PN) and numerical-relativity (NR) waveforms, can reproduce with high precision the NR waveforms for a wide range of binary parameters. Similar results are achieved by the effective-one-body (EOB) model, which attempts to reproduce not only the gravitational waveforms, but also the full dynamics of BH binaries during the inspiral, merger and ringdown phases, by resumming the PN dynamics (Buonanno & Damour 1999; Damour et al. 2009), and more recently the self-force dynamics (Barausse et al. 2012).

Other aspects of the dynamics of BH binaries have been phenomenologically understood by using combinations of PN theory, symmetry arguments, as well as hints from the test-particle limit and fits to numerical simulations. For instance, the final spin magnitude of the BH remnant can be predicted by a number of phenomenological formulas (Rezzolla et al. 2008a,b; Tichy & Marronetti 2008; Buonanno et al. 2008; Kesden 2008; Rezzolla et al. 2008c; Barausse & Rezzolla 2009), starting from the configuration of the binary either at small separations $r \lesssim 10M$, or at large separations[2] $r \sim 10^4 M$. These formulas also predict the orientation

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[2] For MBHs, the latter are roughly the separations at which the dynamics starts being dominated by GW emission, and represent therefore the separations at which these phenomenological formulas should work in order to be useful in cosmological contexts.

of the final spin with good accuracy when applied to small-separation binaries, while the formula of Barausse & Rezzolla (2009) is also accurate when the binary has a large separation, e.g. $r \sim 10^4 M$, in a large portion of the parameter space (Barausse & Rezzolla 2009; Kesden et al. 2010). Similar phenomenological formulas have also been proposed for the recoil imparted to the final BH remnant from the anisotropic emission of GWs (Herrmann et al. 2007b; Koppitz et al. 2007; Rezzolla et al. 2008b; Campanelli et al. 2007a; Gonzalez et al. 2007; Campanelli et al. 2007b; Lousto & Zlochower 2009, 2011; Baker et al. 2007, 2008a; van Meter et al. 2010). Because most of the anisotropic GW emission takes place as a result of the strongly nonlinear merger dynamics, these recoil formulas are not predictive, as they depend on quantities that can only be derived with full NR simulations, but they are still useful in the statistical studies usually performed in a cosmological context (Barausse 2012; Lousto et al. 2010b, 2012).

The dependence of the final mass of the BH remnant on the binary's initial parameters has also been investigated systematically in the literature (Tichy & Marronetti 2008; Boyle & Kesden 2008; Reisswig et al. 2009; Kesden 2008; Lousto et al. 2010a)[3], but the knowledge of this dependence is far less detailed. For instance, the formula of Tichy & Marronetti (2008) [who built upon previous work by Boyle & Kesden (2008)] is calibrated to reproduce NR results for comparable-mass binaries, but does not have the correct test-particle limit and is therefore inaccurate for binaries with small mass ratios. The formula of Kesden (2008), on the contrary, has the correct test-particle limit, but does not reproduce accurately the NR results for comparable-mass binaries. Finally, the formula of Lousto et al. (2010a) depends, for generic binary configurations, on quantities that can only be calculated using full NR simulations, and is therefore only useful in statistical studies.

We here introduce a new phenomenological formula for the final mass of the BH remnant (Section 2), which, by construction, reproduces both the test-particle limit and the regime of binaries with comparable masses and aligned or antialigned spins, which has been extensively investigated by NR calculations. In Section 3 we show that this novel formula reproduces accurately all of the available NR data (even for generic spin orientations and mass ratios), both when applied to small- and large-separation binary configurations. Furthermore, in Section 4, we consider three different areas where our formula can be useful: *(i)* we show that it can help reduce the phase error of the EOB waveforms during the ringdown; *(ii)* we combine it with the results of Le Tiec et al. (2012) for the self-force correction to the binding energy of nonspinning BH binaries and derive an estimate for the energy emitted during the merger and ringdown by nonspinning binaries; *(iii)* using a semi-analytical galaxy-formation model to follow the coevolution of MBHs and their host galaxies, we use our formula to predict the energy emitted in GWs by MBH binaries as a function of redshift, and show that these predictions are strongly dependent on the model for the seeds of the MBH population at high redshifts. Our final conclusions are drawn in Section 5.

Throughout this paper, geometrized units $G = c = 1$ are used.

2. THE DEPENDENCE OF THE FINAL MASS ON THE SPINS AND THE MASS RATIO

When deriving a simple algebraic formula that expresses, with a given precision, the mass/energy radiated by a binary system of BHs, two regimes are particularly well-understood. On the analytic side, in fact, the test-particle limit yields predictions that are well-known and simple to derive. On the numerical side, the simulations of binaries with equal-masses and spins aligned or antialigned with the orbital angular momentum are comparatively simpler to study, and have been explored extensively over the last few years. Hence, it is natural that any attempt to derive an improved expression for the radiated energy should try and match both of these regimes. This is indeed what our formula will be built to do.

Let us therefore start by considering the test-particle limit and, in particular, a Kerr spacetime with mass m_1 and spin parameter $a \equiv S_1/m_1^2$, and a particle (or small BH) with mass m_2 on a equatorial circular orbit with radius $r \gg m_1$[4]. To first approximation (i.e., for mass ratios $q \equiv m_2/m_1 \ll 1$), the particle will inspiral towards the BH under the effect of GW emission, moving slowly ("adiabatically") through a sequence of equatorial circular orbits (Kennefick & Ori 1996) until it reaches the innermost stable circular orbit (ISCO), where it starts plunging, eventually crossing the horizon. The energy $E_{\rm rad}$ emitted by the particle during the inspiral from $r \gg m_1$ to the moment it merges with the central BH can be written as

$$\frac{E_{\rm rad}}{M} = [1 - \tilde{E}^{\rm eq}_{\rm ISCO}(a)]\,\nu + o(\nu)\,, \tag{1}$$

$$\tilde{E}^{\rm eq}_{\rm ISCO}(a) = \sqrt{1 - \frac{2}{3\tilde{r}^{\rm eq}_{\rm ISCO}(a)}}\,, \tag{2}$$

$$\tilde{r}^{\rm eq}_{\rm ISCO}(a) = 3 + Z_2 - {\rm sign}(a)\sqrt{(3-Z_1)(3+Z_1+2Z_2)}\,, \tag{3}$$

$$Z_1 = 1 + (1-a^2)^{1/3}\left[(1+a)^{1/3} + (1-a)^{1/3}\right]\,, \tag{4}$$

$$Z_2 = \sqrt{3a^2 + Z_1^2}\,. \tag{5}$$

Here, $M \equiv m_1 + m_2$ is the total mass, $\nu \equiv m_1 m_2/M^2$ is the symmetric mass ratio, $\tilde{E}_{\rm ISCO}$ and $\tilde{r}_{\rm ISCO}$ are respectively the energy per unit mass at the ISCO and the ISCO radius in units of m_1 (Bardeen et al. 1972), while the remainder, $o(\nu)$, contains the higher-order corrections to the radiated energy[5]. These corrections account, for instance, for the conservative self-force effects, which affect the ISCO position and energy (Barack & Sago 2009; Le Tiec et al. 2012), but also for the deviations from adiabaticity, which arise because of the finiteness of the mass m_2 and which blur the sharp transition between inspiral and plunge (Buonanno & Damour 2000; Ori & Thorne 2000; Kesden 2011), and, more in general, for the energy emitted during the plunge and merger (Berti et al. 2007; Buonanno et al. 2007a,b).

If the particle is initially on an inclined (i.e., non-equatorial) circular orbit, GW emission will still cause it to adiabatically inspiral through a sequence of circular orbits (Kennefick & Ori 1996). Also, the inclination of these orbits relative to the

[3] An initial expression for the radiated energy was also suggested by Buonanno et al. (2007a), but was restricted to nonspinning binaries and based on early NR calculations.

[4] Without loss of generality, we can assume that the particle moves on a prograde orbit (i.e. in the positive-ϕ direction), and let the spin of the Kerr BH point up ($a > 0$) or down ($a < 0$).

[5] We here use the Landau symbol o, so that $f = o(g)$ indicates that $f/g \to 0$ when $g \to 0$. Similarly, we will also use the Landau symbol $\mathcal{O}$, where instead $f = \mathcal{O}(g)$ indicates that $f/g \to {\rm const}$ when $g \to 0$.

equatorial plane, which can be defined as (Hughes 2001) [6]

$$\cos(\iota) \equiv \frac{L_z}{\sqrt{Q + L_z^2}}\,, \tag{6}$$

with Q and L_z being respectively the Carter constant and the azimuthal angular momentum, will remain approximately constant during the inspiral (Hughes 2001; Barausse et al. 2007). As in the equatorial case, the particle plunges when it reaches the ISCO corresponding to its inclination ι. Unlike in the equatorial case, though, the radius of the ISCO as a function of a and ι can only be found numerically. An analytical expression, however, can be derived if one considers only the spin-orbit coupling of the particle to the Kerr BH, i.e., if one considers small spins $a \ll 1$. In that case, in fact, one can explicitly check [using, for instance, equations (4)–(5) of Barausse et al. (2007)] that the ISCO location and energy depend only on the combination $a\cos(\iota)$, so that at $\mathcal{O}(a)^2$, the generalization of expressions (1)–(5) to inclined orbits is given by

$$\frac{E_{\rm rad}}{M} = [1 - \tilde{E}_{\rm ISCO}(a,\iota)]\,\nu + o(\nu)\,, \tag{7}$$

$$\tilde{E}_{\rm ISCO}(a,\iota) \approx \sqrt{1 - \frac{2}{3\tilde{r}_{\rm ISCO}(a,\iota)}}\,, \tag{8}$$

$$\tilde{r}_{\rm ISCO}(a,\iota) \approx \tilde{r}^{\rm eq}_{\rm ISCO}(a\cos(\iota))\,, \tag{9}$$

where $\tilde{r}^{\rm eq}_{\rm ISCO}$ is given by (3). Expressions (7)–(9) reduce to equations (1)–(5) in the case of equatorial orbits ($\iota = 0$) and are therefore exact in that limit, with the exception of the higher-order terms in ν.

As mentioned above, another case in which we know accurately the total energy emitted in GWs is given by binaries of BHs with equal masses and spins aligned or antialigned with the orbital angular momentum. Reisswig et al. (2009), for instance, showed that the energy emitted by these binaries during their inspiral (from infinite separation), merger and ringdown can be well described by a polynomial fit

$$\frac{E_{\rm rad}}{M} = p_0 + p_1(a_1 + a_2) + p_2(a_1 + a_2)^2\,, \tag{10}$$

where a_1 and a_2 are the projections of the spin parameters along the direction $\hat{\boldsymbol{L}}$ of the orbital angular momentum (a_i is therefore respectively positive/negative when the spin is aligned/antialigned with $\hat{\boldsymbol{L}}$), and where the fitting coefficients were found to be (Reisswig et al. 2009) $p_0 = 0.04826$, $p_1 = 0.01559$ and $p_2 = 0.00485$, with uncertainties on the order of $\sim 5\%$ (Reisswig et al. 2009). We recall that the coefficient p_0 can be interpreted as the nonspinning orbital contribution to the energy loss (which is the largest one and $\sim 50\%$ of the largest possible mass loss, which happens for $a_1 = a_2 = 1$), p_1 can instead be interpreted as the spin-orbit contribution (which is $\lesssim 30\%$ of the largest possible loss), while p_2 can be associated to the spin-spin contribution (which is $\lesssim 20\%$ of the largest possible loss). Although the fit proposed by Reisswig et al. (2009) predicts a (shallow) minimum for the radiated energy $E_{\rm rad}$ at $(a_1 + a_2)/2 \sim -0.8$, this minimum is (very likely) just an artifact of the fit due to the scarce data available at that time (Reisswig et al. 2009). Having now more data to analyze, we can enforce the monotonicity of $E_{\rm rad}$ as a function of $a_1 + a_2$ by assuming $p_2 = p_1/4$, which constrains the minimum of $E_{\rm rad}$ to be at $(a_1 + a_2)/2 = -1$. Interestingly, a fitting expression of the type

$$\frac{E_{\rm rad}}{M} = p_0 + p_1(a_1 + a_2) + \frac{p_1}{4}(a_1 + a_2)^2\,, \tag{11}$$

provides an estimate of the radiated energy which is as accurate as the one obtained with (10). Indeed, with fitting parameters

$$p_0 = 0.04827 \pm 0.00039\,, \quad p_1 = 0.01707 \pm 0.00032\,, \tag{12}$$

this expression reproduces all of the available NR data[7] for the energy emitted by equal-mass binaries with aligned or antialigned spins, to within $\sim 0.005M$ (except for almost maximal spins, see below). Such an accuracy is comparable to the typical accuracy of the data themselves, so we can conclude that expressions (11)–(12) summarize our complete knowledge of the GW emission from this class of binaries to date.

We note, however, that higher-order terms in the spins may be needed in equation (11) to reproduce the data for nearly extremal spins. In fact, the maximum value for the radiated energy predicted by our fit, i.e., 9.95% of the total mass of the binary at infinite separation when $a_1 = a_2 = 1$, is significantly less than the 10.95% found by Lovelace et al. (2012) for $a_1 = a_2 \approx 0.97$. Such a large value for $E_{\rm rad}$ is somewhat off the general trend shown by the other NR data for large aligned spins. However, it is clear that higher-order spin

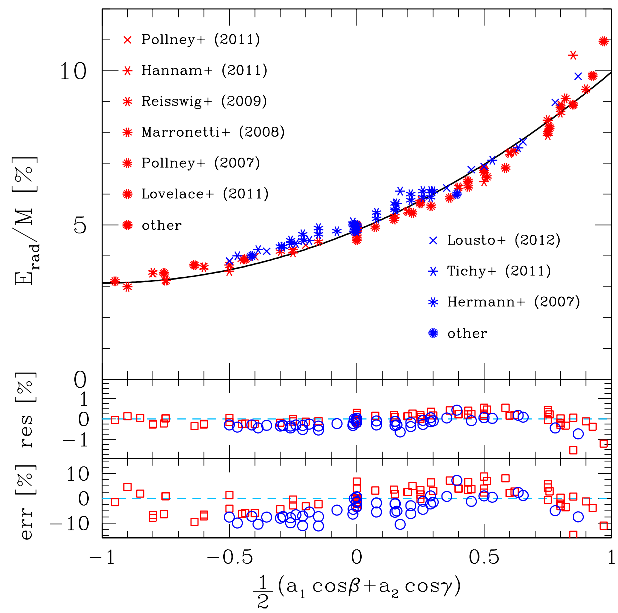

FIG. 1.— *Top panel:* Radiated energy, $E_{\rm rad}/M$, as a function of the total spin of the system along the orbital angular momentum, $|\boldsymbol{a}_1|\cos\beta + |\boldsymbol{a}_2|\cos\gamma$, for all published NR simulations with $q = 1$, both with aligned/antialigned spins (in red) and for misaligned spins (in blue). Shown instead with a black solid line is the prediction of expression (14) with the coefficients fitted from aligned/antialigned binaries. *Bottom panels:* residuals of the NR data from the fitting expression and the corresponding error relative to $E_{\rm rad}/M$.

[6] As in the equatorial case, we can consider only prograde orbits ($0 \le \iota \le \pi/2$) and allow a to be either positive or negative.

[7] The NR data considered is relative to the following references listed in alphabetical order: Baker et al. (2008b); Berti et al. (2007, 2008); Campanelli et al. (2006); Chu et al. (2009); Chu (2012); Hannam et al. (2008, 2010); Kelly et al. (2011); Lovelace et al. (2011, 2012); Marronetti et al. (2008); Pollney et al. (2007); Pollney & Reisswig (2011); Reisswig et al. (2009).

terms may have to be added if more numerical data for high-spin configurations becomes available and confirm this result.

Using therefore the knowledge of the radiated energy from the test-particle limit and from the equal-mass aligned/antialigned configurations, we derive an expression valid for generic binaries. As a first step, let us note that the PN binding energy of an equal-mass binary of spinning BHs depends on the spins, at 1.5 PN order, i.e., at leading order in the spins (Barker & O'Connell 1975), only through the combination

$$\frac{\hat{\boldsymbol{L}} \cdot (\boldsymbol{S}_1 + \boldsymbol{S}_2)}{M^2} = \frac{|\boldsymbol{a}_1| \cos\beta + |\boldsymbol{a}_2| \cos\gamma}{4}\,, \tag{13}$$

where $|\boldsymbol{a}_1|$ and $|\boldsymbol{a}_2|$ are the spin magnitudes, and β, γ are the angles between the orbital angular momentum unit vector $\hat{\boldsymbol{L}}$ and the spins of the first and second BH, respectively. We can therefore attempt to extend expression (11) to generic equal-mass binaries simply by replacing $a_1 + a_2$ with $|\boldsymbol{a}_1| \cos\beta + |\boldsymbol{a}_2| \cos\gamma$, i.e.,

$$\begin{aligned} \frac{E_{\rm rad}}{M} = p_0 &+ p_1 (|\boldsymbol{a}_1| \cos\beta + |\boldsymbol{a}_2| \cos\gamma) \\ &+ \frac{p_1}{4} (|\boldsymbol{a}_1| \cos\beta + |\boldsymbol{a}_2| \cos\gamma)^2\,. \end{aligned} \tag{14}$$

As a check of this ansatz, in the top panel of Fig. 1 we have plotted the radiated energy, $E_{\rm rad}/M$, as a function of the total spin along the orbital angular momentum, $|\boldsymbol{a}_1| \cos\beta + |\boldsymbol{a}_2| \cos\gamma$, for all published NR simulations with $q = 1$, both with aligned/antialigned spins (in red; see footnote 7) and with misaligned spins (in blue[8]). Also, we show with a black solid line the prediction of expression (14) with the coefficients fitted from *aligned/antialigned* binaries [equation (12)]. In the bottom panels we show instead the residuals of the NR data from the same curve and the corresponding errors relative to $E_{\rm rad}/M$. Clearly, while future simulations that are more accurate or describe more involved configurations may present deviations from our simple ansatz, all published simulations for equal-mass binaries are in reasonable agreement with equation (14), with residuals of $\lesssim 1\%$ and errors of $\lesssim 10\%$ relative to the radiated mass. Note that these errors are comparable with the intrinsic scatter of the different NR data.

Because in the test-particle limit the angle β becomes the angle between the spin $\boldsymbol{S}_1$ of the Kerr BH and the orbital angular momentum of the particle, thus coinciding with the angle ι defined in (6), it is natural to rewrite equations (7)–(9) as

$$\frac{E_{\rm rad}}{M} = [1 - \tilde{E}_{\rm ISCO}(\tilde{a})]\,\nu + o(\nu)\,, \tag{15}$$

$$\tilde{E}_{\rm ISCO}(\tilde{a}) = \sqrt{1 - \frac{2}{3\tilde{r}^{\rm eq}_{\rm ISCO}(\tilde{a})}}\,, \tag{16}$$

where we have defined

$$\tilde{a} \equiv \frac{\hat{\boldsymbol{L}} \cdot (\boldsymbol{S}_1 + \boldsymbol{S}_2)}{M^2} = \frac{|\boldsymbol{a}_1| \cos\beta + q^2 |\boldsymbol{a}_2| \cos\gamma}{(1+q)^2}\,. \tag{17}$$

If we now assume that the higher-order term $o(\nu)$ in equation (15) is quadratic in ν, we can determine it by imposing that we recover the equal-mass expression (14) for $q = 1$, thus obtaining the final expression

$$\begin{aligned} \frac{E_{\rm rad}}{M} =& [1 - \tilde{E}_{\rm ISCO}(\tilde{a})]\,\nu \\ &+ 4\,\nu^2 [4p_0 + 16 p_1 \tilde{a}(\tilde{a}+1) + \tilde{E}_{\rm ISCO}(\tilde{a}) - 1]\,, \end{aligned} \tag{18}$$

where $\tilde{E}_{\rm ISCO}(\tilde{a})$ is given by (16). By construction, therefore, expression (18) has the correct behavior both in the test-particle limit and for equal-mass binaries. Also, we stress that the fitting coefficients [given by (12)] are obtained using only a subset of the NR data (i.e., those for equal-mass binaries with *aligned/antialigned* spins).

3. COMPARISON TO DATA: BINARIES AT SMALL AND LARGE SEPARATIONS

In order to test the accuracy of expression (18), we used the data of 186 numerical simulations of inspiralling and merging BH binaries[9], which have reported the ratio $M_f/M \equiv 1 - E_{\rm rad}/M$ between the final mass of the BH remnant, M_f, and the mass $M = m_1 + m_2$ of the binary at infinite separation. In cases where this ratio was not reported explicitly, we have reconstructed it from the energy radiated during the numerical simulation using PN expressions. More specifically, we calculate the radiated energy as

$$E_{\rm rad} = E^{\rm NR}_{\rm rad} + |E^{\rm 3PN}_{\rm bind}(\Omega_0)|\,, \tag{19}$$

where $E^{\rm 3PN}_{\rm bind}(\Omega)$ is the 3PN binding energy as function of the orbital frequency Ω (Buonanno et al. 2003b), and Ω_0 is the initial orbital frequency of the simulation (either reported explicitly or, when unavailable, reconstructed from the initial puncture data). In addition, in those cases where the simulation results are normalized in terms of the Arnowitt-Deser-Misner mass $M_{\rm ADM}$, we approximate it as $M_{\rm ADM} = M - |E^{\rm 3PN}_{\rm bind}(\Omega_0)|$.

For each binary, we apply our expression (18) to the initial configuration of the numerical simulation (where the binary typically has a " small separation" $r \lesssim 10M$). However, in order to check whether our expression predicts the final mass correctly also for widely separated binaries, we have also integrated the initial binary back to a "large separation" $r = 2 \times 10^4\,M$ using the quasi-circular PN evolution equations of Buonanno et al. (2003b) (which are accurate through 3.5PN order for the adiabatic evolution of the orbital frequency, and through 2PN order for the dynamics of the spins). For massive BHs, this is roughly the separation at which the dynamics starts being dominated by GW emission, and is therefore the separation at which our expression (18) ought to work if we want it to be useful in cosmological contexts [cf. Barausse & Rezzolla (2009) and the discussion in Barausse (2012)].

The results of these comparisons are summarized in Fig. 2, which shows the difference between the data and our expression, both for small (left panel) and large separations (right

[8] The NR data for equal-mass misaligned binaries are relative to the following references listed in alphabetical order: Herrmann et al. (2007a); Lousto et al. (2012); Tichy & Marronetti (2007, 2008).

[9] The data is relative to the following references listed in alphabetical order Baker et al. (2008b); Berti et al. (2008, 2007); Buchman et al. (2012); Campanelli et al. (2006, 2009); Chu et al. (2009); Chu (2012); Gonzalez et al. (2009); Hannam et al. (2008, 2010); Herrmann et al. (2007a); Kelly et al. (2011); Lousto & Zlochower (2009); Lousto et al. (2012); Lovelace et al. (2011, 2012); Marronetti et al. (2008); Nakano et al. (2011); Pollney et al. (2007); Pollney & Reisswig (2011); Reisswig et al. (2009); Tichy & Marronetti (2007, 2008).

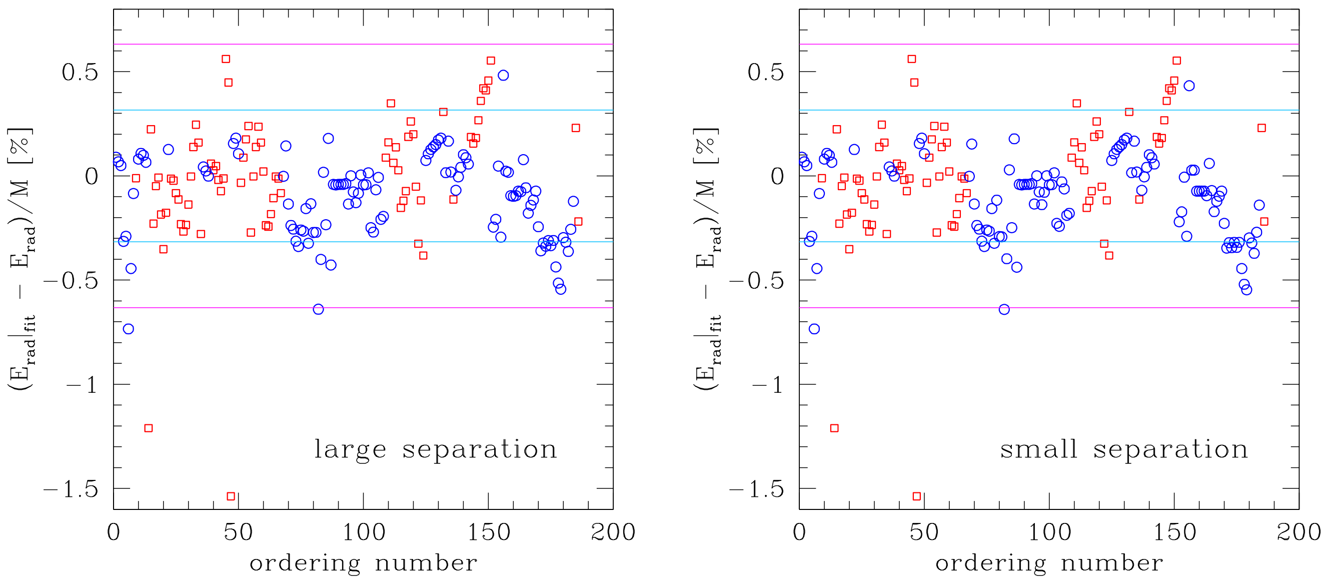

FIG. 2.— Residuals for the fitting formula at small and large separations as a function of a dummy index representing the binaries in our dataset. Binaries with spins aligned/antialigned with the orbital angular momentum are plotted in red, while binaries with misaligned spins are shown in blue. Cyan and violet lines represent the 1σ and 2σ errors (estimated a posteriori) of the data with spins aligned/antialigned with the orbital angular momentum. The first 50 points correspond to the simulations performed after 2010.

panel) as a function of a dummy index representing the ordering of the binaries in our dataset. As can be seen the results at small and large separations are almost indistinguishable. This does not come as a surprise, because the projection of the total spin on the direction of the angular momentum [equation (13)] is approximately conserved during the inspiral, in most of the parameter space of quasi-circular binaries (see discussion in Barausse & Rezzolla (2009) for more details). Binaries with spins aligned/antialigned with the orbital angular momentum are plotted in red, while binaries with misaligned spins are shown in blue. Also shown are cyan and violet lines representing the 1σ and 2σ errors of the data with spins aligned/antialigned with the orbital angular momentum, as obtained a posteriori by comparing them to the fit (11) (cf. Fig. 1). Also, the first 50 points correspond to the simulations performed after 2010, while others correspond to the simulations performed in 2006-2009. Although the quality of numerical simulations improved substantially in last few years, the "old" data give residuals comparable to the "new" ones. Furthermore, the residuals for the binaries with spins aligned/antialigned with the orbital angular momentum appear to be comparable with those for the binaries with misaligned spins.

We stress that while our expression is in reasonable agreement with all the published data, both at large and small separations, there are still large gaps in the parameter space of BH binaries that prevent us from testing our approach more thoroughly. This is best seen in Fig. 3, where we plot the final mass of the remnant for all the published data for binaries with $a_1 \cos\beta = a_2 \cos\gamma$ (blue circles), as well as the predictions of our expression when applied to the "small-separation" initial data of the simulations (meshed surface). Clearly, spinning binaries with unequal mass ratios are essentially absent, and simulations for such binaries will provide a very significant check of our expression (18). Nevertheless, the simple functional dependence shown by the available data, whose behaviour can be well captured with low-order polynomials (with the possible exception, as we already stressed, of almost maximally spinning configurations), is quite remarkable.

The graphical representation of the data in Fig. 3 also allows to reinforce a remark already made by Reisswig et al. (2009), namely, that the largest radiated energy, $E_{\rm rad}(a = 1)/M = 9.95\%$, is lost by binaries with equal-mass and maximally spinning BHs with spins aligned with the orbital angular momentum. Hence, BH binaries on quasi-circular orbits are among the most efficient sources of energy in the Universe. Note, however, that equal-mass binaries are not always the systems that lose the largest amount of energy. Indeed, unequal-mass systems with sufficiently large spins aligned with the angular momentum can lead to emissions larger than those from equal-mass binaries but with large antialigned spins. For instance, a binary with $\nu = 0.15$ and $a_1 = a_2 = 1$ will radiate more than a binary with $\nu = 0.25$ and $a_1 = -a_2$.

Notwithstanding the limited coverage of the parameter space, we can note that our approach substantially improves upon earlier formulas for the final mass. For instance, Tichy & Marronetti (2008) [building on the work of Boyle & Kesden (2008)] suggested a formula linear in the symmetric mass ratio ν, but the coefficients needed to fit NR results are such that the test-particle limit (1)–(9) is not recovered. As mentioned earlier, because we recover the test-particle limit exactly, our expression reproduces the published data more accurately, especially for small mass-ratio configurations (cf. the discussion on the effective-one-body model in the next section). Another example is given by the formula of Lousto et al. (2010a), which has the correct test-particle limit but depends, for generic configurations, on parameters that describe the binary's plunge and merger and which can only be calculated with numerical simulations. Our algebraic formula, instead, allows one to calculate the final mass with reasonable accuracy, using only information on the initial binary config-

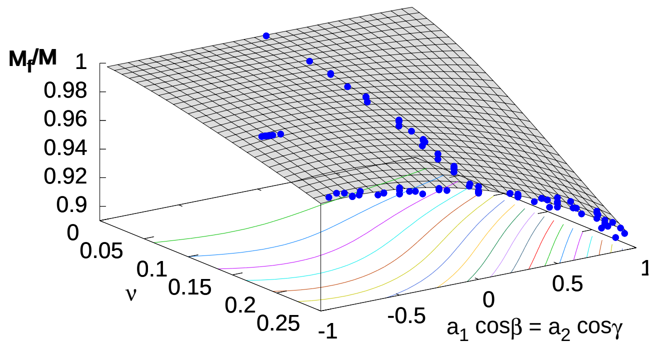

FIG. 3.— Mass of the final BH, $M_f \equiv M - E_{\rm rad}$, and corresponding fit for all the published binaries with $a_1 \cos\beta = a_2 \cos\gamma$. Note the simple functional dependence of the $E_{\rm rad}$, whose behaviour can be well captured with low-order polynomials.

uration, at any separation.

4. APPLICATIONS OF THE NEW FORMULA

In the following Sections we discuss three different examples of how our new expression for the energy radiated during the inspiral, merger and ringdown of two BHs can be used in contexts that range from GW physics to cosmology.

4.1. *Merger-ringdown energy*

We can combine our expression (18) for the total radiated energy with the recent results of Le Tiec et al. (2012) for the binding energy of a binary of nonspinning BHs at next-to-leading order in the mass ratio, and obtain an expression for the energy emitted in the merger and ringdown phases of nonspinning BH binaries. More specifically, for these binaries Le Tiec et al. (2012) found the total energy (i.e., the binary's mass M at infinite separation, plus the binding energy) to be

$$E(x) = M\left[1 + \left(\frac{1-2x}{\sqrt{1-3x}} - 1\right)\nu + \nu^2\, E_{\rm SF}(x)\right] + \mathcal{O}(\nu)^3 \tag{20}$$

where $x \equiv (M\,\Omega)^{2/3}$ and Ω is the orbital frequency. The self-force contribution reads

$$E_{\rm SF}(x) = \frac{z_{\rm SF}(x)}{2} - \frac{x z'_{\rm SF}(x)}{3} - 1 + \sqrt{1-3x} + \frac{x(7-24x)}{6(1-3x)^{3/2}}\,, \tag{21}$$

where $z_{\rm SF}$ is given by the fitting function

$$z_{\rm SF}(x) = \frac{2x\,(1 + a_1 x + a_2 x^2)}{1 + a_3 x + a_4 x^2 + a_5 x^3}\,, \tag{22}$$

which is accurate to within 10^{-5} with $a_1 = -2.18522$, $a_2 = 1.05185$, $a_3 = -2.43395$, $a_4 = 0.400665$, and $a_5 = -5.9991$, and where we use a prime to denote derivatives with respect to x. The minimum of $E(x)$ marks the location of the ISCO [see Le Tiec et al. (2012); Buonanno et al. (2003a)] and lies at

$$x_{\rm ISCO} = \frac{1}{6}\left(1 + \frac{2}{3}\,\nu\, C_\Omega\right) + \mathcal{O}(\nu^2)\,, \tag{23}$$

with $C_\Omega = 1.2510(2)$. Replacing $x_{\rm ISCO}$ in equation (20), one obtains that the energy emitted during the inspiral is

$$\begin{aligned}\frac{E_{\rm rad,insp}}{M} &= 1 - \frac{E(x_{\rm ISCO})}{M}\\ &= \left(1 - \frac{2\sqrt{2}}{3}\right)\nu + 0.037763\,\nu^2 + \mathcal{O}(\nu)^3\\ &\simeq 0.057191\,\nu + 0.037763\,\nu^2 + \mathcal{O}(\nu)^3\,.\end{aligned} \tag{24}$$

Expressing now the total radiated energy as the sum of the one lost during the inspiral and the one lost during the plunge-merger and ringdown, i.e.,

$$E_{\rm rad} = E_{\rm rad,insp} + E_{\rm rad,merger-rd}\,, \tag{25}$$

and using equation (18), we obtain an expression for the energy radiated during the plunge-merger and ringdown of nonspinning BH binaries:

$$\frac{E_{\rm rad,merger-rd}}{M} \approx 0.506\,\nu^2\,. \tag{26}$$

For equal-mass binaries, this energy is almost twice the energy lost during the whole inspiral. In expression (26) we have neglected terms of order $\mathcal{O}(\nu)^3$, so in principle this equation may not hold for comparable-mass binaries. However, the binding energy (20) has been found by Le Tiec et al. (2012) to be in very good agreement with NR results for comparable masses, and we therefore expect the same to hold for expression (26). Indeed, Berti et al. (2007) have estimated that the energy emitted by equal- and unequal-mass nonspinning BH binaries after the 3PN ISCO is given by

$$\frac{E_{\rm rad,merger-rd}}{M} \approx 0.421\,\nu^2\,. \tag{27}$$

Even more strikingly, Berti et al. (2007); Nagar et al. (2007); Bernuzzi & Nagar (2010) showed that the energy emitted by a particle in a Schwarzschild spacetime during its plunge-merger is given by

$$\frac{E_{\rm plunge}}{M} \approx 0.47\,\nu^2\,, \tag{28}$$

in reasonable agreement with (27) and (26). As a result, at least in the nonspinning case, one can reproduce the radiated energy predicted by our final expression (18) for comparable-mass binaries, simply by summing the energy emitted by a particle during the inspiral [equation (24)] and during the plunge-merger [equation (28)].

This confirms previous work showing that perturbative results, when expressed in terms of the symmetric mass ratio ν, are in good agreement with NR results, even for comparable masses [see Detweiler & Szedenits (1979); Anninos et al. (1995); Berti et al. (2007) for the GW fluxes, Le Tiec et al. (2011) for the periastron precession, and Le Tiec et al. (2012) for the binding energy].

4.2. *Tuning of the effective-one-body model*

We recall that the EOB (Buonanno & Damour 1999) is a phenomenological model aiming at describing the dynamics and waveforms of BH binaries, during the inspiral, merger and ringdown phases, combining information from PN theory, perturbative calculations and NR. While developed initially for nonspinning BHs (Buonanno & Damour 1999; Damour et al. 2000; Barausse et al. 2012), the model has been more recently generalized to spinning ones (Damour 2001; Damour et al. 2008; Barausse & Buonanno 2010, 2011; Nagar 2011).

To accurately describe the ringdown phase, the EOB needs expressions predicting the final mass and spin. For non-spinning BHs, these could be estimated self-consistently within the EOB along the lines of Damour & Nagar (2007). However, a generalization of this approach to spinning BHs, especially if precessing, is not straightforward. Moreover, because the final BH's mass and spin are used to calculate the frequencies and decay times of the quasi-normal modes, even small inaccuracies in the prediction of the remnant's mass and spin introduce considerable phase errors in the EOB waveforms. For instance, the EOB model of Taracchini et al. (2012) was compared to NR waveforms for nonspinning BHs with mass ratio $q = 1/6$, using the formula of Tichy & Marronetti (2008) to calculate the final mass. Because that formula does not have the correct test-particle limit [e.g. for non-spinning BHs, it predicts $E_{\rm rad}/M = 0.194\,\nu + \mathcal{O}(\nu)^2$ instead of $E_{\rm rad}/M = 0.057\,\nu + \mathcal{O}(\nu)^2$], the EOB waveforms were accumulating a phase difference of ~ 0.2 rad from the NR ones during the ringdown. With our new formula, this phase difference during the ringdown decreases impressively to less than 0.05 rad (Taracchini 2012, private communication).

4.3. *GW emission by MBHs*

According to our present understanding of galaxy formation, most galactic nuclei should host a massive BH, with mass up to $10^{10}\,M_\odot$. Information on the masses and spins of these BHs can be extracted from present electromagnetic observations (see e.g. Li et al. (2012) for recent constraints), but much more accurate data will be provided by future space-based GW detectors such as LISA or DECIGO, or terrestrial ones such as the Einstein Telescope. These detectors will be able to observe the mergers of MBHs, which take place when their host galaxies coalesce. Semi-analytical galaxy-formation models, such as e.g. that of Barausse (2012), have been employed to understand the MBH merger history and therefore the event rates for these detectors, suggesting hundreds of events per year for DECIGO, from a few to hundreds of events per year for LISA, and up to a few tens of events per year for the Einstein Telescope. A detector-independent diagnostics of the importance of massive BH mergers, however, is given by the energy radiated in GWs by massive BH binaries, per unit comoving volume and as a function of cosmic time. We have calculated this quantity using the galaxy-formation model of Barausse (2012), our new expression (18), and two competing models for the seeds of the massive BHs at high redshift – namely a "light-seed" scenario in which the seeds have mass $M_{\rm seed} \sim 100\,M_\odot$ at $z = 15 - 20$, and a "heavy-seed" model in which the seeds form a $z \sim 15$ with mass $M_{\rm seed} \sim 10^5\,M_\odot$ (see Barausse (2012) and references therein for more details).

The results for this quantity are shown in Fig. 4. Clearly, the heavy-seed scenario predicts much stronger GW emission at redshifts $z \gtrsim 3$, which is not surprising since mergers happen initially between BHs with masses $\sim M_{\rm seed}$, and the radiated energy scales with the total mass of the BH binaries. At $z \sim 0$, instead, the two models yield very similar results, because both reproduce the observed local massive BH mass function (Barausse 2012). Given the significantly different yields in the GW emission expected from these two scenarios of galaxy formation, future space-borne and terrestrial interferometers would provide important and unambiguous information on the BH population at early redshifts [see also Sesana et al. (2009, 2011); Gair et al. (2009, 2011)].

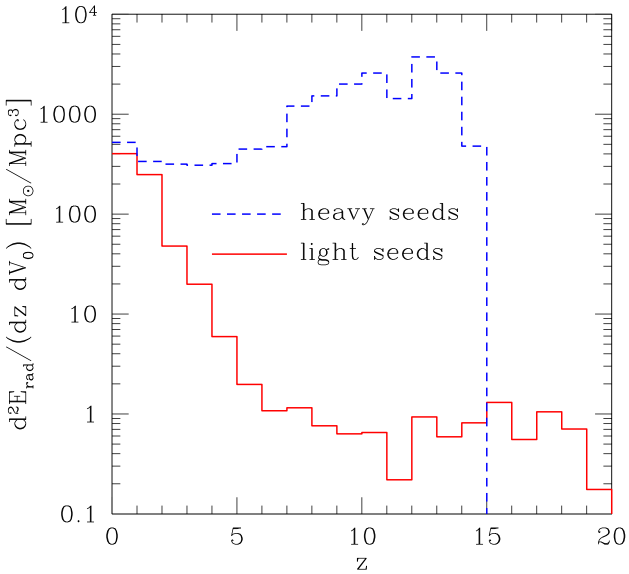

FIG. 4.— The energy emitted by massive BH mergers per unit redshift and unit comoving volume, as a function of redshift. The two lines refer either to the "light-seed" scenario (red solid curve) or to the "heavy-seed" scenario (blue dashed line).

Finally, we note that by integrating the results of Fig. 4, we find that the total energy density in GWs from massive BH binaries at $z = 0$ is $\rho_{\rm GW,mergers} \approx 7.4 \times 10^2\,M_\odot/{\rm Mpc}^3$ in the light-seed scenario and $\rho_{\rm GW,mergers} \approx 1.8 \times 10^4\,M_\odot/{\rm Mpc}^3$ in the heavy-seed scenario, corresponding to a cosmological density parameter $\Omega_{\rm GW,mergers} \equiv \rho_{\rm GW,mergers}/\rho_{\rm crit} \approx 5.4 \times 10^{-9}$ (light-seed scenario) or $\Omega_{\rm GW,mergers} \approx 1.3 \times 10^{-7}$ (heavy-seed scenario).

5. CONCLUSIONS

We have presented a novel algebraic formula to measure the energy radiated by coalescing binary systems of BHs with generic spin magnitudes and orientations and arbitrary mass ratios. Our expression uses information on the binary configuration at an arbitrary separation and reproduces correctly the two regimes in which the radiated energy is known best, namely, the test-particle limit (which is known analytically) and the comparable-mass case (which has been extensively investigated with NR simulations over the last few years). Because it smoothly interpolates these two regimes, we expect our formula to work reasonably well also for intermediate mass ratios. Indeed, we have verified that it reproduces the results of all the NR simulations published so far in the literature, including those with unequal masses, to within an error which is comparable to the typical errors of the simulations. In addition, we have checked that our formula works equally well when applied to binaries starting at small separations (i.e., $r \lesssim 10M$) and at large separations (i.e., $r \sim 10^4 M$), thus opening up the possibility of using our expression also in cosmological contexts.

The algebraic nature of our expression makes it a useful tool in a variety of contexts that range from GW physics to cosmology. As representative examples we have discussed three different applications, namely: *(i)* we have shown that, when combined with the results of Le Tiec et al. (2012) for the self-force correction to the binding energy of nonspinning BH binaries, the new formula provides an estimate for the energy emitted during the merger and ringdown, and that this estimate confirms the conjecture that the results of perturbative calculations may be successfully extrapolated to comparable-mass binaries when expressed in terms of the symmetric mass ratio ν; *(ii)* we have shown that the new formula can help reduce the phase error of the EOB waveforms during the ringdown; *(iii)* using a semi-analytical galaxy-formation model to follow the coevolution of MBHs and their host galaxies, we have used our formula to predict the energy emitted in GWs by MBH binaries as a function of redshift, and found that these predictions strongly depend on the scenario adopted for the MBH seeds at high redshifts, thus making GW emission a powerful cosmological diagnostic.

Additional uses of the new formula can be easily considered and a particularly relevant one is the impact of the mass loss on the accretion disk surrounding the MBH binary. The dynamics of the disk, in fact, can change considerably as a result of the very rapid change in the gravitational mass of the system, with the formation of large shocks, which are potentially detectable via their electromagnetic emission (O'Neill et al. 2009; Rossi et al. 2010; Zanotti et al. 2010).

As a final remark we note that because our approach exploits knowledge derived from NR simulations, the accuracy of the final-mass formula can be improved as additional and more precise NR simulations, especially with highly-spinning BHs, become available.

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