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# All order $\alpha'$-expansion of superstring trees from the Drinfeld associator

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We derive a recursive formula for the $\alpha'$-expansion of superstring tree amplitudes involving any number $N$ of massless open string states. String corrections to Yang-Mills field theory are shown to enter through the Drinfeld associator, a generating series for multiple zeta values. Our results apply for any number of spacetime dimensions or supersymmetries and chosen helicity configurations.



## I. INTRODUCTION

Scattering amplitudes are the most fundamental observables to compute in both quantum field theory and string theory. In both disciplines, numerous hidden structures underlying the S-matrix have been revealed in recent years. Several of these discoveries can be attributed to and have benefited from the close interplay between amplitudes of string theory in the low-energy limit and supersymmetric Yang-Mills (YM) field theory.

A main challenge in the study of field theory amplitudes originates from the transcendental functions in their quantum corrections. Novel mathematical techniques such as the symbol [1] helped to streamline the polylogarithms and multiple zeta values (MZVs) in loop amplitudes of (super-)YM theory. In string theory, MZVs appear in the $\alpha'$-corrections already at tree level due to the exchange of infinitely many heavy vibrational modes. These effects are encoded in integrals over world-sheets of genus zero.

The study of $\alpha'$-expansions in the superstring tree-level amplitude is interesting from both a mathematical and a physical point of view. On the one hand, the pattern of MZVs appearing therein can be understood from an underlying Hopf algebra structure [2]. On the other hand, explicit knowledge of the associated string corrections is crucial for the classification of candidate counterterms in field theories with unsettled questions about their UV properties [3].

In spite of technical advances to evaluate $\alpha'$-expansions for any multiplicity [4], a closed formula for string corrections is still lacking. This letter closes this gap by describing a method to recursively determine the $\alpha'$-dependence of $N$-point trees through the generating function of MZVs – the Drinfeld associator. Our techniques are based on the Knizhnik-Zamolodchikov (KZ) differential equation [5] obeyed by world-sheet integrals and thereby resemble ideas in field theory to determine loop integrals. Along the lines of [6], the associator is shown to connect boundary values, given by $N$-point and $(N-1)$-point disk amplitudes, respectively.

### A. The structure of disk amplitudes

The color-ordered $N$-point disk amplitude $A_{\rm open}(\alpha') \equiv A_{\rm open}(1,2,\ldots,N;\alpha')$ was computed in [7, 8] based on pure spinor cohomology methods [9]. Its entire polarization dependence was found to enter through color-ordered tree amplitudes $A_{\rm YM}$ of the underlying YM field theory which emerges in the point particle limit $\alpha' \to 0$:

$$A_{\rm open}(\alpha') = \sum_{\sigma \in S_{N-3}} F^\sigma(\alpha')\, A^\sigma_{\rm YM}\,. \tag{1}$$

The $(N-3)!$ linearly independent [10] subamplitudes $A_{\rm YM}(1,\sigma(2,3,\ldots,N-2),N-1,N)$ are grouped into a vector $A^\sigma_{\rm YM}$. Labels $1,2,\ldots,N$ in the subamplitude eq. (1) denote any state in the gauge supermultiplet. The objects $F^\sigma(\alpha')$ are iterated integrals over the boundary of the string world-sheet and describe string theory modifications to field theory amplitudes. They can be mathematically classified as generalized Selberg integrals [11]:

$$F^\sigma = \prod_{i=2}^{N-2} \int_{z_i<z_{i+1}} {\rm d}z_i\; \mathcal{I}\, \sigma \left\{ \prod_{k=2}^{N-2} \sum_{j=1}^{k-1} \frac{s_{jk}}{z_{jk}} \right\}, \tag{2}$$

$$\mathcal{I} = \prod_{i<j}^{N-1} |z_{ij}|^{s_{ij}}\;, \quad (z_1, z_{N-1}, z_N) = (0,1,\infty)\,. \tag{3}$$

The $S_{N-3}$ permutation $\sigma$ acts on labels $2,3,\ldots,N-2$ of $z_{ij} \equiv z_i - z_j$ and on dimensionless Mandelstam invariants

$$s_{i_1 i_2 \ldots i_p} = \alpha'(k_{i_1} + k_{i_2} + \ldots + k_{i_p})^2\;, \tag{4}$$

which introduce an implicit $\alpha'$-dependence into the string amplitude (1). The $k_i$ denote external on-shell momenta. Hence, the $s_{ij}$-expansion of the integrals (2) encodes the low energy behaviour of superstring tree amplitudes.

### B. Multiple zeta values

It has been discussed in both mathematics [6, 12, 13] and physics [2, 8, 14] literature that the $\alpha'$-expansion of Selberg integrals involves MZVs

$$\zeta_{n_1,\ldots,n_r} = \sum_{0<k_1<\ldots<k_r} \prod_{j=1}^{r} k_j^{-n_j}\;, \quad n_j \in \mathbb{N}\;, \quad n_r \geq 2 \tag{5}$$

as well as products thereof. The overall weights $\sum_{j=1}^{r} n_j$ of MZV factors match the corresponding power of $\alpha'$. Equivalently, MZVs can be defined by iterated integrals

$$\zeta_{n_1,\dots,n_r} = \int\limits_{0<z_i<z_{i+1}<1} \omega_1 \underbrace{\omega_0\dots\omega_0}_{n_1-1}\omega_1 \underbrace{\omega_0\dots\omega_0}_{n_2-1}\dots\omega_1 \underbrace{\omega_0\dots\omega_0}_{n_r-1} \quad (6)$$

with differential forms $\omega_0 \equiv \frac{\mathrm{d}z}{z}$ and $\omega_1 \equiv \frac{\mathrm{d}z}{1-z}$. Defining by $w[\omega_0,\omega_1]$ a function translating sequences of $\{0,1\}$ into sequences of $\{\omega_0,\omega_1\}$, one can assign a (shuffle-regularized) MZV to each word $w \in \{0,1\}^\times$:

$$\zeta_{(w)} \equiv \int\limits_{0<z_i<z_{i+1}<1} w[\omega_0,\omega_1]\,. \quad (7)$$

The pattern of MZVs in the $\alpha'$-expansion of (2) has been revealed in [2] on the basis of a Hopf algebra structure.

### C. The Drinfeld associator

Consider the KZ equation

$$\frac{\mathrm{d}f(t)}{\mathrm{d}t} = \left(\frac{e_0}{t} + \frac{e_1}{1-t}\right) f(t)\ , \quad (8)$$

with $t \in \mathbb{C}\backslash\{0,1\}$ and Lie-algebra generators $e_0, e_1$. The solution $f(t)$ of the KZ equation takes values in the vector space the representation of $e_0$ and $e_1$ is acting upon. The regularized boundary values

$$C_0 \equiv \lim_{t\to 0} t^{-e_0} f(t)\ , \quad C_1 \equiv \lim_{t\to 1}(1-t)^{e_1} f(t) \quad (9)$$

are related by the Drinfeld associator [15, 16]

$$C_1 = \Phi(e_0,e_1)\, C_0\ , \quad (10)$$

where $C_0$, $C_1$ and $\Phi$ take values in the universal enveloping algebra of the Lie algebra generated by $e_0$ and $e_1$. The regularizing factors $t^{-e_0}$ and $(1-t)^{e_1}$ are included into eq. (9) as to render the $t \to 0,1$ regime of $f(t)$ real-single-valued. In the notation of eq. (7), the Drinfeld associator can be represented as a generating series of MZVs [17]:

$$\Phi(e_0,e_1) = \sum_{w\in\{0,1\}^\times} \tilde{w}[e_0,e_1]\zeta_{(w)}\,, \quad (11)$$

where the operation $\tilde{}$ reverses words. The series expansion of eq. (11) in a basis of MZVs starts with

$$\begin{aligned}\Phi(e_0,e_1) = 1 &+ \zeta_2[e_0,e_1] + \zeta_3[e_0-e_1,[e_0,e_1]] \\ &+ \zeta_4([e_0,[e_0,[e_0,e_1]]] + \tfrac{1}{4}[e_1,[e_0,[e_1,e_0]]] \\ &- [e_1,[e_1,[e_1,e_0]]] + \tfrac{5}{4}[e_0,e_1]^2) + \dots \quad , \quad (12)\end{aligned}$$

where $[\cdot,\cdot]$ denotes the usual commutator.

### D. Main result

In this letter, we identify the Drinfeld associator $\Phi$ as the link between $N$-point string amplitudes and those of multiplicity $N-1$. Thus, starting from the $\alpha'$-independent three-point level, one can build up any tree-level string amplitude recursively.

In order to apply eq. (10), we will construct a matrix representation for $e_0$ and $e_1$ for each multiplicity. Starting with a boundary value $C_0$ containing the world-sheet integrals for the $(N-1)$-point amplitude, eq. (10) yields a vector $C_1$, which in turn encodes the integrals eq. (2) for multiplicity $N$. Consequently, one can explicitly express the $N$-point world-sheet integrals $F^\sigma$ in terms of those at $(N-1)$-points

$$F^{\sigma_i} = \sum_{j=1}^{(N-3)!} \big[\Phi(e_0,e_1)\big]_{ij}\ F^{\sigma_j}\big|_{k_{N-1}=0}, \quad (13)$$

where the soft limit $k_{N-1}=0$ gives rise to $(N-1)$-point integrals on the right hand side

$$F^{\sigma(23\dots N-2)}\big|_{k_{N-1}=0} = \begin{cases} F^{\sigma(23\dots N-3)} & ,\sigma(N-2)=N-2 \\ 0 & ,\text{otherwise}\ , \end{cases} \quad (14)$$

and the $\sigma_i$ are canonically ordered in eq. (13).

## II. THE METHOD

In this section, we construct a vector $\hat{\mathbf{F}}$ of auxiliary functions and a corresponding matrix representation of $e_0, e_1$ such that the following KZ equation holds for $z_0 \in \mathbb{C}\backslash\{0,1\}$:

$$\frac{\mathrm{d}\hat{\mathbf{F}}(z_0)}{\mathrm{d}z_0} = \left(\frac{e_0}{z_{01}} - \frac{e_1}{z_{0,N-1}}\right)\hat{\mathbf{F}}(z_0)\,. \quad (15)$$

As will be shown below, the regularized boundary value $C_0$ derived from $\hat{\mathbf{F}}$ via eq. (9) is determined by basis functions eq. (2) of multiplicity $N-1$ and the final data $C_1$ contains their $N$-point analogues. The vector $\hat{\mathbf{F}}$ is composed from $N-2$ subvectors $\hat{F}_\nu$ of length $(N-3)!$. Numbered by $\nu = 1, 2, .., N-2$, they appear in decreasing order, that is, $\hat{\mathbf{F}} = (\hat{F}_{N-2}, \hat{F}_{N-3}, \dots, \hat{F}_1)$. Labeling the entries in each subvector by permutations $\sigma \in S_{N-3}$, the elements of $\hat{\mathbf{F}}$ read:

$$\begin{aligned}\hat{F}^\sigma_\nu(z_0) = \int_0^{z_0} \mathrm{d}z_{N-2} \prod_{i=2}^{N-3} \int_0^{z_{i+1}} \mathrm{d}z_i\ \mathcal{I} \prod_{k=2}^{N-2} (z_{0k})^{s_{0k}} \\ \times\,\sigma\left\{\prod_{k=2}^{\nu}\sum_{j=1}^{k-1}\frac{s_{jk}}{z_{jk}} \prod_{m=\nu+1}^{N-2}\sum_{n=m+1}^{N-1}\frac{s_{mn}}{z_{mn}}\right\}. \quad (16)\end{aligned}$$

These integrals generalize the functions eq. (2) through an auxiliary world-sheet position $z_0$. It enters in the integration limit of the outermost integral as well as in the deformation $\prod_{k=2}^{N-2}(z_{0k})^{s_{0k}}$ of the Koba-Nielsen factor $\mathcal{I}$ and serves as the differentiation variable for the KZ equation (15). Similarly, $s_{0k}$ are auxiliary Mandelstam variables. At $z_0=1$ and $s_{0k}=0$ – in absence of the augmentation – they reduce to $\hat{F}^\sigma_\nu = F^\sigma$ for any $\nu = 1,2,\ldots,N-2$. In this regime, $\nu$ labels different equivalent representations of the integrals eq. (2) [4]. At generic values of $z_0$ and $s_{0k}$, however, the subvectors $\hat{F}_\nu$ no longer agree for different values of $\nu$.

It is known [6] that all the elements of $(N–2)!\times(N–2)!$-matrices $e_0$ and $e_1$ are linear forms on $s_{ij}$. They can be determined by matching the $z_0$ derivatives[1] of $F^\sigma_\nu$ with the right hand side of the KZ equation (15). Once the resulting matrices $e_0$ and $e_1$ are available, one can calculate the Drinfeld associator to any desired order employing eq. (11). Having set up the KZ equation (15) for the auxiliary function $\hat{\mathbf{F}}$, we will now determine its regularized boundary terms eq. (9).

### A. The $z_0 \to 0$ boundary value $C_0$

The boundary term $C_0$ is determined by taking the limit $z_0 \to 0$ of $z_0^{-e_0}\hat{\mathbf{F}}(z_0)$. The first $(N-3)!$ components of $\hat{\mathbf{F}}(z_0 \to 0)$ at $\nu = N-2$ are

$$\hat{F}^\sigma_{N-2}(z_0 \to 0) = z_0^{s_{\max}} F^\sigma\big|_{s_{i,N-1}=s_{0i}} + \mathcal{O}(s_{0i})\,, \qquad (17)$$

with eigenvalue $s_{\max} = s_{12\ldots N-2} + \sum_{j=2}^{N-2} s_{0j}$ of $e_0$ [6]. The remaining subvectors of $\hat{\mathbf{F}}(z_0 \to 0)$ at $\nu \le N-3$ are suppressed by $N–2–\nu$ powers of $z_0$ and do not contribute to $C_0$, regardless of $e_0$ eigenvalues[2]. The action of $z_0^{-e_0}$ compensates the $z_0$ dependence of the resulting vector $(z_0^{s_{\max}}F^\sigma, \mathbf{0}_{(N-3)(N-3)!})$. Setting $s_{0i} = s_{i,N-1} = 0$ in eq. (17) is equivalent to the soft limit $k_{N-1} \to 0$. This reduces the $F^\sigma$ to $(N–1)$-point integrals by virtue of (14):

$$C_0 = (F^\sigma\big|_{k_{N-1}=0}, \mathbf{0}_{(N-3)(N-3)!})\ . \qquad (18)$$

### B. The $z_0 \to 1$ boundary value $C_1$

For the purpose of setting up a recursion in $F^\sigma$, it is sufficient to extract the first $(N-3)!$ components of $C_1$ from the $z_0 \to 1$ regime of $(1-z_0)^{e_1}\hat{\mathbf{F}}(z_0)$. Considering the schematic form of the first $(N-3)!$ rows

$$(1-z_0)^{e_1} = \begin{pmatrix} \mathbf{1}_{(N-3)!\times(N-3)!} & \mathbf{0}_{(N-3)!\times(N-3)(N-3)!} \\ \vdots & \vdots \end{pmatrix} \qquad (19)$$

we can neglect all components of $\hat{\mathbf{F}}(z_0 \to 1)$ except

$$\hat{F}^\sigma_{N-2}(z_0 \to 1) = F^\sigma + \mathcal{O}(s_{0i})\ . \qquad (20)$$

Setting $s_{0i}=0$ as discussed in section II A leads to[3]

$$C_1 = (F^\sigma, \ldots) \qquad (21)$$

by virtue of eq. (19). Our setup does not require the evaluation of the remaining components in the ellipsis.

With the matrices $e_0$ and $e_1$ as well as the boundary value $C_0$ on our disposal, we are now prepared to infer $C_1$ from eq. (10). Our main result eq. (13) follows from the first $(N-3)!$ components of $C_1$: They express the integrals eq. (2) for the $N$-point amplitude in terms of the $(N-1)$-point world-sheet integrals $F^\sigma|_{k_{N-1}=0}$, see eq. (14). This allows to determine the complete $\alpha'$-expansion to any order and for any multiplicity. The remainder of this letter illustrates the method and its result by the simplest non-trivial examples.

## III. EXAMPLES

### A. From $N=3$ to $N=4$

Any four–point disk integral is proportional to

$$F^{(2)} = \int_0^1 \mathrm{d}z_2\, |z_{12}|^{s_{12}}|z_{23}|^{s_{23}}\, \frac{s_{12}}{z_{12}} = \frac{\Gamma(1+s_{12})\Gamma(1+s_{23})}{\Gamma(1+s_{12}+s_{23})}\,.$$

We will rederive its $\alpha'$-expansion from the Drinfeld associator along the lines of section II. The auxiliary vector eq. (16) contains two subvectors of length one:

$$\begin{pmatrix} \hat{F}^{(2)}_2 \\ \hat{F}^{(2)}_1 \end{pmatrix} = \int_0^{z_0} \mathrm{d}z_2\, |z_{12}|^{s_{12}}|z_{23}|^{s_{23}}\, z_{02}^{s_{02}} \begin{pmatrix} s_{12}/z_{12} \\ s_{23}/z_{23} \end{pmatrix}\,. \qquad (22)$$

Partial fraction decomposition $(z_{12}z_{02})^{-1} = (z_{12}z_{01})^{-1} - (z_{01}z_{02})^{-1}$ followed by discarding a $z_2$-derivative

$$0 = \int \mathrm{d}z_2\, |z_{12}|^{s_{12}}|z_{23}|^{s_{23}}\, z_{02}^{s_{02}} \left(\frac{s_{02}}{z_{02}} + \frac{s_{12}}{z_{12}} - \frac{s_{23}}{z_{23}}\right) \qquad (23)$$

leads to the following KZ equation after setting $s_{02}=0$:

$$\frac{\mathrm{d}}{\mathrm{d}z_0}\begin{pmatrix} \hat{F}^{(2)}_2 \\ \hat{F}^{(2)}_1 \end{pmatrix} = \left(\frac{e_0}{z_{01}} - \frac{e_1}{z_{03}}\right)\begin{pmatrix} \hat{F}^{(2)}_2 \\ \hat{F}^{(2)}_1 \end{pmatrix}\,, \qquad (24)$$

$$e_0 = \begin{pmatrix} s_{12} & -s_{12} \\ 0 & 0 \end{pmatrix}\,, \quad e_1 = \begin{pmatrix} 0 & 0 \\ s_{23} & -s_{23} \end{pmatrix}\,. \qquad (25)$$

[1] The boundary term from acting with $\frac{\mathrm{d}}{\mathrm{d}z_0}$ on the integration limit does not contribute as can be seen by analytic continuation of $|z_{0,N-2}|^{s_{0,N-2}}|_{z_{N-2}=z_0} = 0\ \forall\ s_{0,N-2} \in \mathbb{R}^+$.

[2] This can be seen by a change of integration variables $z_i = z_0 w_i$ for $i = 2,\ldots,N-2$ which rescales the integration domain to $0 \le w_2 \le w_3 \le \ldots \le w_{N-2} \le 1$.

[3] The $(N-3)(N-3)!$ components in the ellipsis are sensitive to the interval $(z_0,1)$ absent in the $\mathrm{d}z_{N-2}$ integration range. We can safely omit their evaluation because the first subvector of eq. (21) contains the complete $N$-point information.

The regularized boundary values (9) are found to be

$$C_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ , \quad C_1 = \begin{pmatrix} F^{(2)} \\ F^{(2)} - 1 \end{pmatrix} \tag{26}$$

where the subtraction in the second component of $C_1$ stems from the absence of the $(z_0, 1)$ integration range in (22). This subtlety motivates to neglect the $C_1$ components beyond the first $(N-3)!$ in the following. The KZ equation (24) subject to boundary values eq. (26) allows to extract $F^{(2)}$ from

$$\begin{pmatrix} F^{(2)} \\ F^{(2)} - 1 \end{pmatrix} = \big[\Phi(e_0, e_1)\big]_{2\times 2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{27}$$

with $e_0, e_1$ given in eq. (25). Their particular form implies that products of any two matrices $\mathrm{ad}_0^k \mathrm{ad}_1^l [e_0, e_1]$ with $k, l \in \mathbb{N}_0$ vanish, where $\mathrm{ad}_i x \equiv [e_i, x]$. According to [18], this allows to express the four-point disk amplitude exclusively in terms of $\zeta_{n_1}$ with a single argument $n_1 \geq 2$, i.e. depth $r = 1$ cases of eq. (5).

### B. From $N = 4$ to $N = 5$

Next we shall derive a closed formula expression for the five-point versions $F^{(23)}$ and $F^{(32)}$ of eq. (2) by applying the associator method to the auxiliary functions eq. (16)

$$\begin{pmatrix} \hat F_3^{(23)} \\ \hat F_3^{(32)} \\ \hat F_2^{(23)} \\ \hat F_2^{(32)} \\ \hat F_1^{(23)} \\ \hat F_1^{(32)} \end{pmatrix} = \int_0^{z_0} \mathrm{d}z_3 \int_0^{z_3} \mathrm{d}z_2 \, \mathcal{I} \, z_{02}^{s_{02}} z_{03}^{s_{03}} \begin{pmatrix} X_{12}(X_{13}+X_{23}) \\ X_{13}(X_{12}+X_{32}) \\ X_{12}X_{34} \\ X_{13}X_{24} \\ (X_{23}+X_{24})X_{34} \\ (X_{32}+X_{34})X_{24} \end{pmatrix}$$

where $X_{ij} \equiv \frac{s_{ij}}{z_{ij}}$. Partial fraction and integration by parts analogous to (23) leads to the $(6\times 6)$-matrices

$$e_0 = \begin{pmatrix} s_{123} & 0 & -s_{13}-s_{23} & -s_{12} & -s_{12} & s_{12} \\ 0 & s_{123} & -s_{13} & -s_{12}-s_{23} & s_{13} & -s_{13} \\ 0 & 0 & s_{12} & 0 & -s_{12} & 0 \\ 0 & 0 & 0 & s_{13} & 0 & -s_{13} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$e_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ s_{34} & 0 & -s_{34} & 0 & 0 & 0 \\ 0 & s_{24} & 0 & -s_{24} & 0 & 0 \\ s_{34} & -s_{34} & s_{23}+s_{24} & s_{34} & -s_{234} & 0 \\ -s_{24} & s_{24} & s_{24} & s_{23}+s_{34} & 0 & -s_{234} \end{pmatrix}$$

for which the KZ equation (15) is satisfied after setting $s_{02} = s_{03} = 0$. Their associator connects the boundary values

$$C_0 = \begin{pmatrix} F^{(2)} \\ 0 \\ \mathbf{0}_4 \end{pmatrix} \ , \quad C_1 = \begin{pmatrix} F^{(23)} \\ F^{(32)} \\ \vdots \end{pmatrix} \tag{28}$$

via eq. (10), i.e. we recursively obtain the desired $F^{(23)}$ and $F^{(32)}$ from

$$\begin{pmatrix} F^{(23)} \\ F^{(32)} \\ \vdots \end{pmatrix} = \big[\Phi(e_0, e_1)\big]_{6\times 6} \begin{pmatrix} F^{(2)} \\ 0 \\ \mathbf{0}_4 \end{pmatrix} . \tag{29}$$

Given that the four-point amplitude $\sim F^{(2)}$ only involves simple zeta values $\zeta_n$, all the MZVs (5) of depth $r \geq 2$ occurring in the five-point integrals $F^{(23)}$ and $F^{(32)}$ (see [2] for their appearance at weights $w \leq 16$) emerge from the associator in eq. (29).

### C. From $N = 5$ to $N = 6$

The recursion from five- to six-point integrals requires a system of 24 auxiliary functions, aligned into four subvectors of length six (where, again, $X_{ij} \equiv \frac{s_{ij}}{z_{ij}}$):

$$\begin{pmatrix} \hat F_4^{\sigma(234)} \\ \hat F_3^{\sigma(234)} \\ \hat F_2^{\sigma(234)} \\ \hat F_1^{\sigma(234)} \end{pmatrix} = \int_0^{z_0} \mathrm{d}z_4 \int_0^{z_4} \mathrm{d}z_3 \int_0^{z_3} \mathrm{d}z_2 \, \mathcal{I} \, z_{02}^{s_{02}} z_{03}^{s_{03}} z_{04}^{s_{04}} \times \sigma \begin{pmatrix} X_{12}(X_{13}+X_{23})(X_{14}+X_{24}+X_{34}) \\ X_{12}(X_{13}+X_{23})X_{45} \\ X_{12}(X_{34}+X_{35})X_{45} \\ (X_{23}+X_{24}+X_{25})(X_{34}+X_{35})X_{45} \end{pmatrix} \tag{30}$$

The corresponding $24\times 24$ matrices $e_0$ and $e_1$ are determined by the KZ equation built from appropriate relabelings of

$$\begin{aligned} \frac{\mathrm{d}\hat F_3^{(234)}}{\mathrm{d}z_0} &= \frac{1}{z_{01}}\big\{ s_{123}\hat F_3^{(234)} - (s_{13}+s_{23})\hat F_2^{(234)} - s_{12}(\hat F_2^{(324)} + \hat F_1^{(234)} - \hat F_1^{(324)})\big\} + \frac{1}{z_{05}}\big\{ s_{45}(\hat F_3^{(234)} - \hat F_4^{(234)})\big\} \\ \frac{\mathrm{d}\hat F_4^{(234)}}{\mathrm{d}z_0} &= \frac{1}{z_{01}}\big\{ s_{1234}\hat F_4^{(234)} - (s_{14}+s_{24}+s_{34})\hat F_3^{(234)} - (s_{13}+s_{23})(\hat F_3^{(243)} + \hat F_2^{(234)} - \hat F_2^{(243)}) \\ &\quad + s_{12}(-\hat F_3^{(342)} - \hat F_2^{(324)} - \hat F_2^{(423)} + \hat F_2^{(342)} + \hat F_2^{(432)} + \hat F_1^{(423)} - \hat F_1^{(432)} + \hat F_1^{(324)} - \hat F_1^{(234)})\big\} . \end{aligned} \tag{31}$$

The six-point realization of the boundary terms eq. (18) and eq. (21) leads to the recursion

$$\begin{pmatrix} F^{(234)} \\ F^{(243)} \\ F^{(324)} \\ F^{(342)} \\ F^{(423)} \\ F^{(432)} \\ \vdots \end{pmatrix} = \big[\Phi(e_0,e_1)\big]_{24\times 24} \begin{pmatrix} F^{(23)} \\ 0 \\ F^{(32)} \\ 0 \\ 0 \\ 0 \\ \mathbf{0}_{18} \end{pmatrix} . \qquad (32)$$

## IV. CONCLUSIONS AND OUTLOOK

In our main result, eq. (13), we relate the world-sheet integrals eq. (2) carrying the $\alpha'$-dependence of $N$-point disk amplitudes to $(N-1)$-point results by the Drinfeld associator. The construction works for any multiplicity and - in principle - to any order in $\alpha'$.

The different origin of $\alpha'$-corrections therein from either the associator or the lower point integrals might shed light on the arrangement of reducible and irreducible diagrams in the underlying low energy effective action [19].

The string corrections are universal to massless open superstring tree amplitudes in any number of spacetime dimension, independent on the amount of supersymmetry or chosen helicity configurations. Their $\alpha'$-expansion in terms of MZVs can be directly carried over to closed string trees which are expressed in terms of a specific subsector of the open string's expansion [2]. It would be desirable to extend this analysis to higher genus such as the maximally supersymmetric one loop amplitudes calculated in [20].

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