All order α'-expansion of superstring trees from the Drinfeld associator

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(Received 10 July 2013; published 24 March 2014)

We derive a recursive formula for the α' expansion of superstring tree amplitudes involving any number N of massless open string states. String corrections to Yang-Mills field theory are shown to enter through the Drinfeld associator, a generating series for multiple zeta values. Our results apply to any number of spacetime dimensions or supersymmetries and chosen helicity configurations.

DOI: 10.1103/PhysRevD.89.066014 PACS numbers: 11.25.-w, 02.10.De

I. INTRODUCTION

Scattering amplitudes are the most fundamental observables in both quantum field theory and string theory. In recent years, numerous hidden structures underlying the S-matrix have been revealed in both disciplines. Several of these discoveries can be attributed to and have benefited from the close interplay between amplitudes of string theory in the low-energy limit and supersymmetric Yang-Mills (YM) field theory.

A main challenge in the study of field theory amplitudes originates from the transcendental functions in their quantum corrections. Novel mathematical techniques such as the symbol [1] helped to streamline the polylogarithms and multiple zeta values (MZVs) in loop amplitudes of (super-) YM theory. In string theory, MZVs appear in the α' corrections already at tree level due to the exchange of infinitely many heavy vibrational modes. These effects are encoded in integrals over world sheets of genus zero.

The study of α' expansions in the superstring tree-level amplitude is interesting from both a mathematical and a physical point of view. On the one hand, the pattern of MZVs appearing therein can be understood from an underlying Hopf algebra structure [2]. On the other hand, explicit knowledge of the associated string corrections is crucial for the classification of candidate counterterms in field theories with unsettled questions about their UV properties [3].

In spite of technical advances to evaluate α' expansions for any multiplicity [4], compact and straightforwardly applicable formulas for string corrections are still lacking. This paper closes this gap by describing a novel method to recursively determine the α' dependence of N-point trees through the generating function of MZVs—the Drinfeld associator. Its connection with superstring amplitudes—in particular the common pattern of MZV appearance—was firstly pointed out in [5]. Our techniques are based on the Knizhnik-Zamolodchikov (KZ) equation [6] obeyed by world-sheet integrals and thereby resemble ideas in field theory to determine loop integrals [7]. Along the lines of [8], the associator is shown to connect boundary values, given by N-point and $(N-1)$-point disk amplitudes, respectively. The method presented in this article bypasses the cumbersome direct evaluation of world-sheet integrals and reduces their α' expansions to simple matrix multiplications. Apart from its conceptual accessibility, it substantially reduces the computational effort in deriving the explicit form[1] of α' corrections.

A. The structure of disk amplitudes

The color-ordered N-point disk amplitude $A_{\text{open}}(\alpha') \coloneqq A_{\text{open}}(1,2,\ldots,N;\alpha')$ was computed in [10,11] based on pure spinor cohomology methods [12]. Its entire polarization dependence was found to enter through color-ordered tree amplitudes A_{YM} of the underlying YM field theory which emerges in the point particle limit $\alpha' \to 0$:

$$A_{\text{open}}(\alpha') = \sum_{\sigma \in S_{N-3}} F^{\sigma}(\alpha') A^{\sigma}_{\text{YM}}. \tag{1}$$

The $(N-3)!$ linearly independent [13] subamplitudes[2] $A_{\text{YM}}(1,\sigma(2,3,\ldots,N-2),N-1,N)$ are grouped into a vector A^{σ}_{YM}. The objects $F^{\sigma}(\alpha')$ describe string corrections to YM amplitudes and will be recursively determined as the main result of this paper. They are generalized Selberg integrals [14] over the boundary of the open string world sheet of disk topology,

$$F^{\sigma} = (-1)^{N-3} \prod_{i=2}^{N-2} \int_{z_i<z_{i+1}} \mathrm{d}z_i \mathcal{I} \sigma \left\{ \prod_{k=2}^{N-2} \sum_{j=1}^{k-1} \frac{s_{jk}}{z_{jk}} \right\}, \tag{2}$$

[1]The web site [9] provides expressions for string corrections to five- to seven-point amplitudes as well as material to apply the presented method up to nine points.

[2]Labels $1,2,\ldots,N$ in the subamplitude Eq. (1) denote any state in the gauge supermultiplet.

$$\mathcal{I} = \prod_{i<j}^{N-1} |z_{ij}|^{s_{ij}}, \qquad (z_1, z_{N-1}, z_N) = (0, 1, \infty). \quad (3)$$

The S_{N-3} permutation σ acts on labels $2, 3, \ldots, N-2$ of $z_{ij} := z_i - z_j$ and of the dimensionless Mandelstam invariants

$$s_{i_1 i_2 \ldots i_p} = \alpha'(k_{i_1} + k_{i_2} + \cdots + k_{i_p})^2, \quad (4)$$

which carry the α' dependence of the string amplitude (1). The k_i denote external on-shell momenta. Hence, the s_{ij} expansion of the integrals (2) encodes the low-energy behavior of superstring tree amplitudes.

B. Multiple zeta values

As discussed in both mathematics [8,15,16] and physics [2,11,17] literature, the α' expansion of Selberg integrals involves (products of) MZVs. They can be defined by iterated integrals over differential forms $\omega_0 := \frac{dz}{z}$ and $\omega_1 := \frac{dz}{1-z}$

$$\zeta_{n_1,\ldots,n_r} = \int_{0<z_i<z_{i+1}<1} \omega_1 \underbrace{\omega_0 \ldots \omega_0}_{n_1-1} \omega_1 \underbrace{\omega_0 \ldots \omega_0}_{n_2-1} \ldots \omega_1 \underbrace{\omega_0 \ldots \omega_0}_{n_r-1}, \quad (5)$$

where $n_j \in \mathbb{N}$ and $n_r \geq 2$. The overall weights $\sum_{j=1}^{r} n_j$ of MZV factors match the power of α' in the string amplitudes' expansion. Instead of labeling MZVs by the set of n_j, one can equivalently encode the integrand of Eq. (5) in a word w in the alphabet $\{0,1\}$ (i.e. $w \in \{0,1\}^\times$) where the function $w[\omega_0, \omega_1]$ translates this word into sequences of $\{\omega_0, \omega_1\}$ [5]:

$$\zeta_{(w)} := \int_{0<z_i<z_{i+1}<1} w[\omega_0, \omega_1]. \quad (6)$$

The pattern of MZVs in the α' expansion of (2) has been revealed in [2] on the basis of a Hopf algebra structure.

C. The Drinfeld associator

Consider the KZ equation with $z_0 \in \mathbb{C}\backslash\{0,1\}$ and Lie-algebra generators e_0, e_1:

$$\frac{d\hat{\mathbf{F}}(z_0)}{dz_0} = \left(\frac{e_0}{z_0} + \frac{e_1}{1-z_0}\right)\hat{\mathbf{F}}(z_0). \quad (7)$$

The solution $\hat{\mathbf{F}}(z_0)$ of the KZ equation takes values in the vector space the representation of e_0 and e_1 is acting upon. The regularized boundary values

$$C_0 := \lim_{z_0\to 0} z_0^{-e_0}\hat{\mathbf{F}}(z_0), \qquad C_1 := \lim_{z_0\to 1}(1-z_0)^{e_1}\hat{\mathbf{F}}(z_0) \quad (8)$$

are related by the Drinfeld associator [18,19]

$$C_1 = \Phi(e_0, e_1)C_0, \quad (9)$$

where C_0, C_1 and Φ take values in the universal enveloping algebra of the Lie algebra generated by e_0 and e_1. The regularizing factors $z_0^{-e_0}$ and $(1-z_0)^{e_1}$ are included into Eq. (8) as to render the $z_0 \to 0$, 1 regime of $\hat{\mathbf{F}}(z_0)$ real-single-valued. In the notation of Eq. (6), the Drinfeld associator can be represented as a generating series of MZVs [20],

$$\Phi(e_0, e_1) = \sum_{w\in\{0,1\}^\times} \tilde{w}[e_0, e_1]\zeta_{(w)}, \quad (10)$$

where $\tilde{w}$ denotes the reversal of the word w. The series expansion of Eq. (10) in a basis of MZVs starts with the following commutators $[\cdot,\cdot]$:

$$\begin{aligned}\Phi(e_0, e_1) = 1 &+ \zeta_2[e_0, e_1] + \zeta_3[e_0 - e_1, [e_0, e_1]] \\ &+ \zeta_4\bigg([e_0, [e_0, [e_0, e_1]]] + \frac{1}{4}[e_1, [e_0, [e_1, e_0]]] \\ &- [e_1, [e_1, [e_1, e_0]]] + \frac{5}{4}[e_0, e_1]^2\bigg) + \ldots. \end{aligned} \quad (11)$$

D. Main result

In this paper, we identify the Drinfeld associator Φ as the link between N-point string amplitudes and those of multiplicity $N-1$. Thus, starting from the α'-independent three-point level, one can build up any tree-level string amplitude recursively.

We will construct a matrix representation for the associator arguments e_0 and e_1 in Sec. I C for each multiplicity. Starting with a boundary value C_0 containing the world-sheet integrals for the $(N-1)$-point amplitude, Eq. (9) yields a vector C_1, which we will show to encode the integrals Eq. (2) for multiplicity N. Consequently, one can express the N-point world-sheet integrals F^σ in terms of those at $(N-1)$-points,

$$F^{\sigma_i} = \sum_{j=1}^{(N-3)!} [\Phi(e_0, e_1)]_{ij} F^{\sigma_j}|_{k_{N-1}=0}, \quad (12)$$

where the soft limit $k_{N-1} = 0$ gives rise to $(N-1)$-point integrals on the right-hand side,

$$F^{\sigma(23\ldots N-2)}|_{k_{N-1}=0} = \begin{cases} F^{\sigma(23\ldots N-3)} & \text{if } \sigma(N-2) = N-2 \\ 0 & \text{otherwise.} \end{cases} \quad (13)$$

The permutations σ_i are canonically ordered in Eq. (12).

II. THE METHOD

The backbone of the recursion Eq. (12) is a vector $\hat{\mathbf{F}}$ of auxiliary functions and a corresponding matrix representation of e_0, e_1 such that the KZ equation (7) holds. Moreover, the boundary values C_0 and C_1 derived from $\hat{\mathbf{F}}$ via Eq. (8) need to reproduce basis functions Eq. (2) of multiplicity $N-1$ and N, respectively. As we will see, these requirements are met by components

$$\hat{F}^{\sigma}_{\nu}(z_0, s_{0k}) = (-1)^{N-3} \int_0^{z_0} \mathrm{d}z_{N-2} \prod_{i=2}^{N-3} \int_0^{z_{i+1}} \mathrm{d}z_i \mathcal{I} \times \prod_{k=2}^{N-2} z_{0k}^{s_{0k}} \sigma \left\{ \prod_{k=2}^{\nu} \sum_{j=1}^{k-1} \frac{s_{jk}}{z_{jk}} \prod_{m=\nu+1}^{N-2} \sum_{n=m+1}^{N-1} \frac{s_{mn}}{z_{mn}} \right\}. \tag{14}$$

The vector $\hat{\mathbf{F}}$ is composed from $N-2$ subvectors $\hat{F}_\nu$ of length $(N-3)!$. Numbered by $\nu = 1, 2, \ldots, N-2$, they appear in decreasing order, that is, $\hat{\mathbf{F}} = (\hat{F}_{N-2}, \hat{F}_{N-3}, \ldots, \hat{F}_1)$. Entries of the $\hat{F}_\nu$ are labeled by permutations $\sigma \in S_{N-3}$.

The integrals in (14) generalize the functions Eq. (2) through an auxiliary world-sheet position z_0 and auxiliary Mandelstam variables s_{0k}.[3] This z_0 enters in the integration limit of the outermost integral as well as in the deformation $\prod_{k=2}^{N-2} (z_{0k})^{s_{0k}}$ of the Koba-Nielsen factor $\mathcal{I}$ and serves as the differentiation variable for the KZ equation (7). As visualized in the Fig. 1, the position z_0 downscales the integration domain on the disk boundary and thus interpolates between world-sheet configurations of an N-point and $(N-1)$-point tree amplitude.

At $z_0 = 1$ and $s_{0k} = 0$—in absence of the augmentation—the functions $\hat{F}^\sigma_\nu$ in Eq. (14) approach the integrals F^σ in the amplitude for any ν. In this regime, ν labels different equivalent representations [4] of the integrals Eq. (2).

Matching the length of the auxiliary vector, e_0 and e_1 in Eq. (7) are $(N-2)! \times (N-2)!$ matrices. It is known [8] that their entries are linear forms on s_{ij}. They can be determined by matching the z_0 derivatives[4] of $\hat{F}^\sigma_\nu$ with the right hand side of the KZ equation (7). Once the resulting matrices e_0 and e_1 are available, one can calculate the Drinfeld associator to any desired order employing its series expansion Eq. (10). Having set up the KZ equation (7) for the auxiliary function $\hat{\mathbf{F}}$, we will now relate its regularized boundary terms Eq. (8) to the integrals Eq. (2) in the string amplitude.

A. The $z_0 \to 0$ boundary value C_0

The boundary term C_0 is determined by taking the limit $z_0 \to 0$ of $z_0^{-e_0} \hat{\mathbf{F}}(z_0)$. This amounts to squeezing the world-sheet positions $z_2, \ldots, z_{N-2}$ into an interval $[0, z_0]$ of vanishing size, see Fig. 1. This effectively removes one of the $N-3$ integrations and makes contact with the $(N-1)$-point problem. Let us make this more precise: the first $(N-3)!$ components of $\hat{\mathbf{F}}(z_0 \to 0)$ at $\nu = N-2$,

$$\hat{F}^\sigma_{N-2}(z_0 \to 0, s_{0i}) = z_0^{s_{\max}} F^\sigma |_{s_{i,N-1} = s_{0i}} + \mathcal{O}(s_{0i}), \tag{15}$$

involve the eigenvalue $s_{\max} = s_{12\ldots N-2} + \sum_{j=2}^{N-2} s_{0j}$ of e_0 [8]. The remaining subvectors of $\hat{\mathbf{F}}(z_0 \to 0)$ at $\nu \le N-3$ are suppressed by $N-2-\nu$ powers of z_0[5] and do not contribute to C_0. The action of $z_0^{-e_0}$ compensates the z_0 dependence of the resulting vector $(z_0^{s_{\max}} F^\sigma, \mathbf{0}_{(N-3)(N-3)!})$.

The desired $(N-1)$-point integrals can be achieved through a soft limit $k_{N-1} \to 0$, see (13). This can be realized by setting $s_{0i} = s_{i,N-1} = 0$ in Eq. (15) which converts the subvector $\hat{F}^\sigma_{N-2}$ into $(N-1)$-point data,

$$C_0 = (F^\sigma |_{k_{N-1}=0}, \mathbf{0}_{(N-3)(N-3)!}). \tag{16}$$

B. The $z_0 \to 1$ boundary value C_1

The $z_0 \to 1$ regime of $(1-z_0)^{e_1} \hat{\mathbf{F}}(z_0)$ underlying C_1 restores the integration domain of the N-point functions Eq. (2). Considering the schematic form of the first $(N-3)!$ rows in

$$(1-z_0)^{e_1} = \begin{pmatrix} \mathbf{1}_{(N-3)! \times (N-3)!} & \mathbf{0}_{(N-3)! \times (N-3)(N-3)!} \\ \vdots & \vdots \end{pmatrix}, \tag{17}$$

we can neglect all components of $\hat{\mathbf{F}}(z_0 \to 1)$ except

$$\hat{F}^\sigma_{N-2}(z_0 \to 1, s_{0i}) = F^\sigma + \mathcal{O}(s_{0i}). \tag{18}$$

Setting $s_{0i} = 0$ as motivated in Sec. II A leads to

$$C_1 = (F^\sigma, \ldots). \tag{19}$$

Our setup does not require the delicate evaluation of the remaining components in the ellipsis.

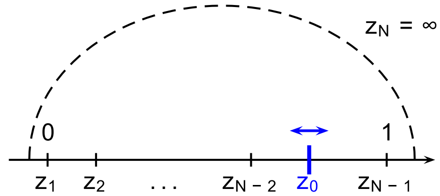

FIG. 1 (color online). World sheet with an auxiliary position z_0.

[3] As will be explained below, we will eventually set $s_{0k} \to 0$ and therefore do not display them as arguments of $\hat{\mathbf{F}}(z_0)$.

[4] The boundary term from acting with $\frac{\mathrm{d}}{\mathrm{d}z_0}$ on the integration limit does not contribute as can be seen by analytic continuation of $(z_{0,N-2})^{s_{0,N-2}}|_{z_{N-2}=z_0} = 0 \; \forall \; s_{0,N-2} \in \mathbb{R}^+$.

[5] This can be seen by a change of integration variables $z_i = z_0 w_i$ rescaling the integration region to $0 \le w_2 \le w_3 \le \ldots \le w_{N-2} \le 1$.

C. Summary

Our main result Eq. (12) follows by specializing the central property Eq. (9) of the associator to the representations of C_i, e_i extracted from the auxiliary vector $\hat{\mathbf{F}}(z_0)$ defined in Eq. (14). In Eqs. (16) and (19), we have identified C_0 and C_1 with $(N-1)$- and N-point world-sheet integrals Eq. (2), respectively. This turns Eq. (9) into a recursion in N where the arguments e_0, e_1 of the connecting associator can be straightforwardly read off from the KZ equation (7) satisfied by $\hat{\mathbf{F}}(z_0)$. Starting from the trivial three-point amplitude, this allows to determine the complete α' expansion to any order and for any multiplicity.

III. EXAMPLES

A. From $N = 3$ to $N = 4$

Any four-point disk integral is proportional to

$$F^{(2)} = \int_0^1 \mathrm{d}z_2 |z_{12}|^{s_{12}} |z_{23}|^{s_{23}} \frac{s_{12}}{z_{21}} = \frac{\Gamma(1+s_{12})\Gamma(1+s_{23})}{\Gamma(1+s_{12}+s_{23})}.$$

We will rederive its α' expansion from the Drinfeld associator along the lines of Sec. II. The auxiliary vector Eq. (14) contains two subvectors of length one:

$$\begin{pmatrix} \hat{F}_2^{(2)} \\ \hat{F}_1^{(2)} \end{pmatrix} = \int_0^{z_0} \mathrm{d}z_2 |z_{12}|^{s_{12}} |z_{23}|^{s_{23}} z_{02}^{s_{02}} \begin{pmatrix} s_{12}/z_{21} \\ s_{23}/z_{32} \end{pmatrix}. \quad (20)$$

Partial fraction decomposition $(z_{12}z_{02})^{-1} = (z_{12}z_{01})^{-1} - (z_{01}z_{02})^{-1}$ followed by discarding a z_2 derivative,

$$0 = \int \mathrm{d}z_2 |z_{12}|^{s_{12}} |z_{23}|^{s_{23}} z_{02}^{s_{02}} \left(\frac{s_{02}}{z_{02}} + \frac{s_{12}}{z_{12}} - \frac{s_{23}}{z_{23}} \right), \quad (21)$$

leads to the following KZ equation after setting $s_{02} = 0$:

$$\frac{\mathrm{d}}{\mathrm{d}z_0} \begin{pmatrix} \hat{F}_2^{(2)} \\ \hat{F}_1^{(2)} \end{pmatrix} = \left(\frac{e_0}{z_{01}} - \frac{e_1}{z_{03}} \right) \begin{pmatrix} \hat{F}_2^{(2)} \\ \hat{F}_1^{(2)} \end{pmatrix}, \quad (22)$$

$$e_0 = \begin{pmatrix} s_{12} & -s_{12} \\ 0 & 0 \end{pmatrix}, \qquad e_1 = \begin{pmatrix} 0 & 0 \\ s_{23} & -s_{23} \end{pmatrix}. \quad (23)$$

The regularized boundary values (8) read

$$C_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad C_1 = \begin{pmatrix} F^{(2)} \\ \ldots \end{pmatrix}, \quad (24)$$

and Eq. (12) becomes

$$\begin{pmatrix} F^{(2)} \\ \ldots \end{pmatrix} = [\Phi(e_0, e_1)]_{2\times 2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (25)$$

with e_0, e_1 given in Eq. (23). Their particular form implies that products of any two matrices $\mathrm{ad}_0^k \mathrm{ad}_1^l [e_0, e_1]$ with $k, l \in \mathbb{N}_0$ vanish, where $\mathrm{ad}_i x := [e_i, x]$. According to [5], this allows to express the four-point disk amplitude exclusively in terms of single ζ's [$r = 1$ in Eq. (5)].

B. From $N = 4$ to $N = 5$

Next we shall derive a closed formula expression for the five-point versions $F^{(23)}$ and $F^{(32)}$ of Eq. (2) by applying the associator method to the auxiliary functions Eq. (14) at $N = 5$,

$$\begin{pmatrix} \hat{F}_3^{(23)} \\ \hat{F}_3^{(32)} \\ \hat{F}_2^{(23)} \\ \hat{F}_2^{(32)} \\ \hat{F}_1^{(23)} \\ \hat{F}_1^{(32)} \end{pmatrix} = \int_0^{z_0} \mathrm{d}z_3 \int_0^{z_3} \mathrm{d}z_2 \mathcal{I} z_{02}^{s_{02}} z_{03}^{s_{03}} \begin{pmatrix} X_{12}(X_{13}+X_{23}) \\ X_{13}(X_{12}+X_{32}) \\ X_{12}X_{34} \\ X_{13}X_{24} \\ (X_{23}+X_{24})X_{34} \\ (X_{32}+X_{34})X_{24} \end{pmatrix},$$

where $X_{ij} := \frac{s_{ij}}{z_{ij}}$. Partial fraction and integration by parts analogous to (21) leads to the (6×6) matrices,

$$e_0 = \begin{pmatrix} s_{123} & 0 & -s_{13}-s_{23} & -s_{12} & -s_{12} & s_{12} \\ 0 & s_{123} & -s_{13} & -s_{12}-s_{23} & s_{13} & -s_{13} \\ 0 & 0 & s_{12} & 0 & -s_{12} & 0 \\ 0 & 0 & 0 & s_{13} & 0 & -s_{13} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$e_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ s_{34} & 0 & -s_{34} & 0 & 0 & 0 \\ 0 & s_{24} & 0 & -s_{24} & 0 & 0 \\ s_{34} & -s_{34} & s_{23}+s_{24} & s_{34} & -s_{234} & 0 \\ -s_{24} & s_{24} & s_{24} & s_{23}+s_{34} & 0 & -s_{234} \end{pmatrix}$$

for which the KZ equation (7) is satisfied after setting $s_{02} = s_{03} = 0$. The corresponding (6×6) associator connects the boundary values C_0 and C_1,

$$C_0 = \begin{pmatrix} F^{(2)} \\ 0 \\ \mathbf{0}_4 \end{pmatrix}, \qquad C_1 = \begin{pmatrix} F^{(23)} \\ F^{(32)} \\ \vdots \end{pmatrix}, \quad (26)$$

via Eq. (9); i.e., we recursively obtain the desired $F^{(23)}$ and $F^{(32)}$ from

$$\begin{pmatrix} F^{(23)} \\ F^{(32)} \\ \vdots \end{pmatrix} = [\Phi(e_0, e_1)]_{6\times 6} \begin{pmatrix} F^{(2)} \\ 0 \\ \mathbf{0}_4 \end{pmatrix}. \tag{27}$$

Given that the four-point amplitude $\sim F^{(2)}$ only involves simple zeta values ζ_n, all the MZVs (5) of depth $r \geq 2$ occurring in the five-point integrals $F^{(23)}$ and $F^{(32)}$ (see [2] for their appearance at weights $w \leq 16$) emerge from the associator in Eq. (27).

C. Higher multiplicity

The techniques to simplify derivatives of $\hat{\mathbf{F}}(z_0)$ and to identify the matrices e_0, e_1 in the KZ equation (7) are universal to all multiplicities. Expressions for e_0, e_1 up to nine points are provided at [9], and the resulting α' corrections at $N = 8$, 9 have been unknown before. Higher N representations of e_0, e_1 are not only straightforward to compute but also suggested by the explicit form of their lower multiplicity cousins. The efficiency of the associator-based recursion Eq. (12) becomes particularly apparent at large multiplicities: The straightforward derivation of e_0, e_1 avoids the growing manual effort (such as pole treatment) required by the method of [4].

IV. CONCLUSIONS AND OUTLOOK

In our main result, Eq. (12), we relate the world-sheet integrals Eq. (2) carrying the α' dependence of N-point disk amplitudes to $(N-1)$-point results by the Drinfeld associator $\Phi(e_0, e_1)$. The challenge of evaluating world-sheet integrals is converted to elementary matrix multiplications among N-dependent representations of e_0, e_1.

The construction works for any multiplicity and—in principle—to any order in α'. It produces previously inaccessible results, e.g. through the explicit form of e_0, e_1 for $N \leq 9$ available from [9]. At lowest orders in α', the new results at $N = 8$, 9 have been checked to preserve the amplitudes' collinear limits, cyclicity and monodromy relations [21,22].

The different origin of α' corrections therein from either the associator or the lower point integrals might shed light on the arrangement of reducible and irreducible diagrams in the underlying low energy effective action.

The string corrections are universal to massless open superstring tree amplitudes in any number of spacetime dimensions, independent on the amount of supersymmetry or chosen helicity configurations. Their α' expansion in terms of MZVs can be directly carried over to closed string trees which are expressed in terms of a specific subsector of the open string's expansion [2]. It would be desirable to extend this analysis to a higher genus such as the maximally supersymmetric one-loop amplitudes calculated in [23].

ACKNOWLEDGMENTS

We are grateful to the organizers (in particular Herbert Gangl) of the workshop "Grothendieck-Teichmueller Groups, Deformation and Operads" held at the Newton Institute from January until April 2013 for creating a fruitful framework to initiate this work. We would like to thank Claude Duhr, Herbert Gangl and Carlos Mafra for stimulating discussions and helpful comments on the draft. The work of O. S. is supported by Michael Green and the European Research Council Advanced Grant No. 247252.

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