Proceedings of the 7th IFAC Symposium on Robust Control Design
The International Federation of Automatic Control
Aalborg, Denmark, June 20-22, 2012

Robust and Efficient Algorithms for $\mathcal{L}_\infty$-Norm Computation for Descriptor Systems ⋆

P. Benner * **V. Sima** ** **M. Voigt** *

* *Max Planck Institute for Dynamics of Complex Technical Systems, 39106 Magdeburg, Germany (e-mails: benner@mpi-magdeburg.mpg.de, voigtm@mpi-magdeburg.mpg.de)*
** *National Institute for Research and Development in Informatics, 011455 Bucharest, Romania (e-mail: vsima@ici.ro)*

Abstract: In this paper we discuss algorithms for the computation of the $\mathcal{L}_\infty$-norm of transfer functions related to descriptor systems, both in the continuous- and discrete-time context. We show how one can achieve this goal by computing the eigenvalues of certain structured matrix pencils. These pencils can be transformed to skew-Hamiltonian/Hamiltonian matrix pencils which are constructed by only using the original data. Furthermore, we apply a structure-preserving algorithm to compute the desired eigenvalues. In this way we increase robustness and efficiency of the method. Finally, we present numerical results in order to illustrate the advantages of our approach.

Keywords: Computer-aided control systems design, descriptor systems, eigenvalue problems, $\mathcal{H}_\infty$ control, linear control systems, norms, numerical algorithms, robust control

1. INTRODUCTION

For the analysis of linear dynamical systems, system norms play a great role. One of the most popular norms is the $\mathcal{L}_\infty$-norm. It has important applications in model order reduction as an error measure (Mehrmann and Stykel (2005)). Another field of application is robust control where it takes the role of a robustness measure for dynamical systems (Zhou and Doyle (1998); Losse et al. (2008)). This paper is devoted to the computation of this norm for the special case of descriptor systems which represents a more general concept than that of standard state-space systems.

In this paper we consider *continuous-time linear time-invariant (LTI) systems*

$$\Sigma_\mathrm{c}: \quad \begin{cases} E\dot{x}(t) = Ax(t) + Bu(t), \\ \;\; y(t) = Cx(t) + Du(t), \end{cases} \tag{1}$$

and *discrete-time LTI systems* of the form

$$\Sigma_\mathrm{d}: \quad \begin{cases} Ex(t+1) = Ax(t) + Bu(t), \\ \qquad\; y(t) = Cx(t) + Du(t), \end{cases} \tag{2}$$

with $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$, descriptor vector $x(t) \in \mathbb{R}^n$, control vector $u(t) \in \mathbb{R}^m$, and output vector $y(t) \in \mathbb{R}^p$. We allow the matrix E to be singular. In this case we speak about *descriptor systems or singular systems*. However, throughout this work, we always assume that the matrix pencil $\lambda E - A$ is *regular*, i.e., $\det(\lambda E - A) \not\equiv 0$.

By taking the *Laplace transform* of the equations in (1) or the *Z-transform* of the equations in (2) we obtain the *transfer function* of the corresponding system, given by

$$G(\lambda) = C\left(\lambda E - A\right)^{-1} B + D, \tag{3}$$

where λ is replacing the Laplace variable s for a continuous-time system, and the Z-transform variable z for a discrete-time system. By $\mathcal{L}_\infty^{p\times m}(\mathrm{i}\omega)$ and $\mathcal{L}_\infty^{p\times m}\left(e^{\mathrm{i}\omega}\right)$ we denote the Banach spaces of all $p \times m$ matrix-valued functions that are bounded on the imaginary axis, or the unit circle, respectively. In this work we consider the corresponding rational subspaces $\mathcal{RL}_\infty^{p\times m}(\mathrm{i}\omega)$ and $\mathcal{RL}_\infty^{p\times m}\left(e^{\mathrm{i}\omega}\right)$. It can be shown that each $G \in \mathcal{RL}_\infty^{p\times m}(\mathrm{i}\omega)$ has a realization of the form (1) and that each $G \in \mathcal{RL}_\infty^{p\times m}\left(e^{\mathrm{i}\omega}\right)$ has a realization of the form (2). For $G \in \mathcal{RL}_\infty^{p\times m}(\mathrm{i}\omega)$, the $\mathcal{L}_\infty$-*norm* is defined by

$$\|G\|_{\mathcal{L}_\infty} := \sup_{\omega\in\mathbb{R}} \sigma_{\max}\left(G\left(\mathrm{i}\omega\right)\right),$$

and for $G \in \mathcal{RL}_\infty^{p\times m}\left(e^{\mathrm{i}\omega}\right)$ it is given by

$$\|G\|_{\mathcal{L}_\infty} := \sup_{\omega\in[-\pi,\pi)} \sigma_{\max}\left(G\left(e^{\mathrm{i}\omega}\right)\right),$$

where $\sigma_{\max}(\cdot)$ denotes the largest singular value. As an agreement, we set $\|G\|_{\mathcal{L}_\infty} = \infty$ if G is not in the corresponding space. For continuous-time systems this is the case when G has purely imaginary poles, or when it is improper, that is $\lim_{\omega\to\infty} G(\mathrm{i}\omega) = \infty$. For discrete-time systems, $G \notin \mathcal{RL}_\infty^{p\times m}\left(e^{\mathrm{i}\omega}\right)$ when G has unitary poles, i.e., poles on the unit circle. The poles of G are the controllable and observable eigenvalues of $\lambda E - A$. The implementation of the algorithms can detect whether G is not an element of the corresponding $\mathcal{L}_\infty$-space and returns the norm value ∞ in this case.

⋆ This work was supported by the *Deutsche Forschungsgemeinschaft* under grant BE-2174/6-1.

Note that for stable systems, the $\mathcal{L}_\infty$-norm is equivalent to the well-known $\mathcal{H}_\infty$-norm.

2. COMPUTATION OF THE $\mathcal{L}_\infty$-NORM

For the computation of the $\mathcal{L}_\infty$-norm we make use of the following matrix pencils

$$H_{\mathrm{c}}(\gamma) := \begin{bmatrix} \lambda E - A & 0 \\ 0 & \lambda E^T + A^T \end{bmatrix} - \begin{bmatrix} B & 0 \\ 0 & -C^T \end{bmatrix} \times \begin{bmatrix} -D & \gamma I_p \\ \gamma I_m & -D^T \end{bmatrix}^{-1} \begin{bmatrix} C & 0 \\ 0 & B^T \end{bmatrix}$$
$$= \begin{bmatrix} \lambda E - A + BR_\gamma^{-1} D^T C & \gamma B R_\gamma^{-1} B^T \\ -\gamma C^T S_\gamma^{-1} C & \lambda E^T + A^T - C^T D R_\gamma^{-1} B^T \end{bmatrix},$$

$$H_{\mathrm{d}}(\gamma) := \begin{bmatrix} \lambda E - A & 0 \\ 0 & \lambda A^T - E^T \end{bmatrix} - \begin{bmatrix} B & 0 \\ 0 & -\lambda C^T \end{bmatrix} \times \begin{bmatrix} -D & \gamma I_p \\ \gamma I_m & -D^T \end{bmatrix}^{-1} \begin{bmatrix} C & 0 \\ 0 & B^T \end{bmatrix}$$
$$= \begin{bmatrix} \lambda E - A + BR_\gamma^{-1} D^T C & \gamma B R_\gamma^{-1} B^T \\ -\gamma \lambda C^T S_\gamma^{-1} C & \lambda A^T - E^T - \lambda C^T D R_\gamma^{-1} B^T \end{bmatrix},$$

with $R_\gamma := D^T D - \gamma^2 I_m$ and $S_\gamma := DD^T - \gamma^2 I_p$ (Genin et al. (1998); Voigt (2010)). The following theorem connects the singular values of $G(\mathrm{i}\omega)$ and $G\left(e^{\mathrm{i}\omega}\right)$ with the finite, purely imaginary eigenvalues of $H_{\mathrm{c}}(\gamma)$ and the unitary eigenvalues of $H_{\mathrm{d}}(\gamma)$, respectively (Benner et al. (2012); Benner and Voigt (2011)).

Theorem 1. (a) Assume that $G \in \mathcal{RL}_\infty^{p \times m}(\mathrm{i}\omega)$, $\gamma > 0$ is not a singular value of D and $\omega_0 \in \mathbb{R}$. Then, γ is a singular value of $G(\mathrm{i}\omega_0)$ if and only if $H_{\mathrm{c}}(\gamma)$ has the eigenvalue $\mathrm{i}\omega_0$.
(b) Assume that $G \in \mathcal{RL}_\infty^{p \times m}\left(e^{\mathrm{i}\omega}\right)$, $\gamma > 0$ is not a singular value of D and $\omega_0 \in [-\pi, \pi)$. Then, γ is a singular value of $G\left(e^{\mathrm{i}\omega_0}\right)$ if and only if $H_{\mathrm{d}}(\gamma)$ has the eigenvalue $e^{\mathrm{i}\omega_0}$.

The following statement is a direct consequence of Theorem 1 (Benner et al. (2012); Benner and Voigt (2011)).

Theorem 2. (a) Assume that $G \in \mathcal{RL}_\infty^{p \times m}(\mathrm{i}\omega)$ and let $\gamma > \min_{\omega \in \mathbb{R}} \sigma_{\max}(G(\mathrm{i}\omega))$ be not a singular value of D. Then $\|G\|_{\mathcal{L}_\infty} \geq \gamma$ if and only if $H_{\mathrm{c}}(\gamma)$ has finite, purely imaginary eigenvalues.
(b) Assume that $G \in \mathcal{RL}_\infty^{p \times m}\left(e^{\mathrm{i}\omega}\right)$ and let $\gamma > \min_{\omega \in [-\pi, \pi)} \sigma_{\max}(G(e^{\mathrm{i}\omega}))$ be not a singular value of D. Then $\|G\|_{\mathcal{L}_\infty} \geq \gamma$ if and only if $H_{\mathrm{d}}(\gamma)$ has unitary eigenvalues.

Algorithm 1 summarizes the generalization of the method presented in Bruinsma and Steinbuch (1990); Boyd and Balakrishnan (1990) to the descriptor system case. It is monotonically converging with a quadratic rate of convergence, and the relative error is at most ε (assuming exact arithmetic). The computing time is affected by the number of frequency points in each step. A good choice of initial value γ_{lb} can reduce the CPU time drastically. The value γ_{lb} is determined by evaluating $\sigma_{\max}(G(\mathrm{i}\omega))$, or $\sigma_{\max}\left(G(e^{\mathrm{i}\omega})\right)$ at the boundary of the frequency intervals $[0, \infty)$, or $[0, \pi)$, respectively, and further inner test frequencies, see Sima (2006) for further details. Care must be taken of the eigenvalue computation, as missing only one of the desired eigenvalues could force the algorithm to fail. Within the next section we show how one can exploit the structure of the pencils $H_{\mathrm{c}}(\gamma)$ and $H_{\mathrm{d}}(\gamma)$ for the robust determination of the eigenvalues.

Algorithm 1 Computation of the $\mathcal{L}_\infty$-Norm

Input: LTI descriptor system with transfer function $G \in \mathcal{RL}_\infty^{p \times m}(\mathrm{i}\omega)$ or $G \in \mathcal{RL}_\infty^{p \times m}\left(e^{\mathrm{i}\omega}\right)$, tolerance ε.
Output: $\|G\|_{\mathcal{L}_\infty}$.

```
Compute an initial value γ_lb < ||G||_L∞.
if continuous-time system then
  repeat
    Set γ := (1 + ε)γ_lb.
    Compute the finite, purely imaginary eigenvalues of the matrix pencil H_c(γ).
    if no finite, purely imaginary eigenvalues then
      break.
    else
      Set {iω_1, ..., iω_k} = finite, purely imaginary eigenvalues with ω_j ∈ [0, ∞), j = 1, ..., k.
      Set m_j = sqrt(ω_j ω_{j+1}), j = 1, ..., k − 1.
      Compute the largest singular value of G(im_j), j = 1, ..., k − 1.
      Set γ_lb = max_j σ_max(G(im_j)).
    end if
  until break
else
  repeat
    Set γ := (1 + ε)γ_lb.
    Compute the unitary eigenvalues of the matrix pencil H_d(γ).
    if no unitary eigenvalues then
      break.
    else
      Set {e^{iω_1}, ..., e^{iω_k}} = unitary eigenvalues with ω_j ∈ [0, π), j = 1, ..., k.
      Set m_j = 1/2 (ω_j + ω_{j+1}), j = 1, ..., k − 1.
      Compute the largest singular value of G(e^{im_j}), j = 1, ..., k − 1.
      Set γ_lb = max_j σ_max(G(e^{im_j})).
    end if
  until break
end if
Set ||G||_L∞ = γ_lb.
```

3. ROBUST COMPUTATION OF THE DESIRED EIGENVALUES

In this section we further analyze the matrix pencils $H_{\mathrm{c}}(\gamma)$ and $H_{\mathrm{d}}(\gamma)$ which both have certain structures that we want to exploit and preserve. First, note that $H_{\mathrm{c}}(\gamma)$ is a *skew-Hamiltonian/Hamiltonian matrix pencil* (Benner et al. (1999)) and that $H_{\mathrm{d}}(\gamma)$ is equivalent to a so-called *reduced BVD pencil* which can be seen as a generalization of symplectic pencils (Byers et al. (2009)). Define the matrix $\mathcal{J}_n := \left[\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right]$. For a better readability, throughout this paper, we always assume that $\mathcal{J}$ is a copy of $\mathcal{J}_n$ of appropriate dimension. Let $\mathbb{R}[\lambda]^{k \times k}$ be the set of all polynomials with coefficients in $\mathbb{R}^{k \times k}$. A matrix pencil $\lambda \mathcal{S} - \mathcal{H} \in \mathbb{R}[\lambda]^{2n \times 2n}$ is called skew-Hamiltonian/Hamiltonian if $\mathcal{S}$ is *skew-Hamiltonian*, i.e., $(\mathcal{S}\mathcal{J})^T = -\mathcal{S}\mathcal{J}$, and $\mathcal{H}$ is *Hamiltonian*, i.e., $(\mathcal{H}\mathcal{J})^T = \mathcal{H}\mathcal{J}$. Pencils of this structure have many nice structural properties. The most important one is the Hamiltonian eigensymmetry, i.e., the eigenvalues are symmetric with respect to the real and the imaginary axis. This means that the eigenvalues occur in pairs $(\lambda, -\lambda)$ if they are real, or in pairs $\left(\lambda, \overline{\lambda}\right)$ if they are imaginary, or otherwise in quadruples $\left(\lambda, -\lambda, \overline{\lambda}, -\overline{\lambda}\right)$. Reduced BVD pencils have a particular block structure described in Byers et al. (2009). They satisfy the symplectic eigensymmetry, i.e., symmetry with respect to the unit circle. In other words, eigenvalues occur in pairs $\left(\lambda, \overline{\lambda}\right)$ if they are unitary, or otherwise in quadruples $\left(\lambda, \lambda^{-1}, \overline{\lambda}, \overline{\lambda}^{-1}\right)$.

Naively constructing the matrix pencils $H_{\mathrm{c}}(\gamma)$ and $H_{\mathrm{d}}(\gamma)$ could be very ill-advised because they contain a lot of matrix products and inverses. If γ is close to a singular value of D, then the matrices R_γ and S_γ are ill-conditioned. And even if they are not, forming "matrix-times-its-transpose"-like products like $BR_\gamma^{-1}B^T$ suffers from numerical instability (Benner et al. (1999)). Therefore, explicitly forming the matrix pencils must be avoided, if possible. Luckily, it is possible to formulate related matrix pencils which can be directly constructed by only using the original data. This can be achieved by applying an extension strategy similar to the one described in Benner et al. (1999). Within the next two subsections we describe the transformations applied to $H_{\mathrm{c}}(\gamma)$ and $H_{\mathrm{d}}(\gamma)$ in order to get the pencils that were used in our implementation (Benner et al. (2012); Benner and Voigt (2011)). For both, the continuous-time and the discrete-time case, we first construct so-called *even matrix pencils* whose finite, purely imaginary eigenvalues we are interested in. However, for the actual computations we use a structure-preserving algorithm on related skew-Hamiltonian/Hamiltonian matrix pencils (Benner et al. (1999)). In this way we obtain very accurate and reliable results.

3.1 The Continuous-Time Case

By reverting the Schur complement structure of $H_{\mathrm{c}}(\gamma)$ we obtain an extended matrix pencil

$$\mathcal{H}_{\mathrm{c}}^{(1)}(\gamma) = \left[\begin{array}{cc|cc} \lambda E - A & 0 & -B & 0 \\ 0 & \lambda E^T + A^T & 0 & C^T \\ \hline -C & 0 & -D & \gamma I_p \\ 0 & -B^T & \gamma I_m & -D^T \end{array}\right],$$

which has the same finite eigenvalues as $H_{\mathrm{c}}(\gamma)$ (Voigt (2010)). However, we loose the skew-Hamiltonian/Hamiltonian structure by this operation. Luckily, we can transform $\mathcal{H}_{\mathrm{c}}^{(1)}(\gamma)$ to an even pencil. A matrix pencil $\lambda\mathcal{S} - \mathcal{H} \in \mathbb{R}[\lambda]^{n\times n}$ is called even, if $\mathcal{S} = -\mathcal{S}^T$ and $\mathcal{H} = \mathcal{H}^T$. Even and skew-Hamiltonian/Hamiltonian matrix pencils are closely related to each other which we will explain in detail in Subsection 3.3. In particular, even pencils also have a Hamiltonian spectrum (Schröder (2008)). Now, by performing some block permutations in $\mathcal{H}_{\mathrm{c}}^{(1)}(\gamma)$ we obtain the even matrix pencil

$$\mathcal{H}_{\mathrm{c}}^{(2)}(\gamma) = \left[\begin{array}{cc|cc} 0 & \lambda E - A & 0 & -B \\ -\lambda E^T - A^T & 0 & -C^T & 0 \\ \hline 0 & -C & \gamma I_p & -D \\ -B^T & 0 & -D^T & \gamma I_m \end{array}\right].$$

3.2 The Discrete-Time Case

The discrete-time case is more involved than the continuous-time one. Similar to the considerations above we can exploit the Schur complement structure of $H_{\mathrm{d}}(\gamma)$ and obtain the extended matrix pencil

$$\mathcal{H}_{\mathrm{d}}^{(1)}(\gamma) = \left[\begin{array}{cc|cc} \lambda E - A & 0 & -B & 0 \\ 0 & \lambda A^T - E^T & 0 & \lambda C^T \\ \hline -C & 0 & -D & \gamma I_p \\ 0 & -B^T & \gamma I_m & -D^T \end{array}\right],$$

with the same finite eigenvalues as $H_{\mathrm{d}}(\gamma)$. By some block permutations and transposing the pencil we obtain the following *D-type matrix pencil* (Xu (2006))

$$\mathcal{H}_{\mathrm{d}}^{(2)}(\gamma) = \left[\begin{array}{c|ccc} 0 & -\lambda E^T + A^T & C^T & 0 \\ \hline \lambda A - E & 0 & 0 & -B \\ \lambda C & 0 & \gamma I_p & -D \\ 0 & -B^T & -D^T & \gamma I_m \end{array}\right].$$

D-type matrix pencils have the general form

$$\lambda\mathcal{E}_{\mathrm{D}} - \mathcal{A}_{\mathrm{D}} = \lambda \begin{bmatrix} 0 & F \\ -G^T & 0 \end{bmatrix} - \begin{bmatrix} 0 & G \\ -F^T & H \end{bmatrix}$$

with symmetric H, and have symplectic eigenstructure with possible additional infinite eigenvalues. We consider the *generalized Cayley transform*

$$\mathbf{c}(\mathcal{E}, \mathcal{A}) := \lambda\,(\mathcal{A} + \mathcal{E}) - (\mathcal{A} - \mathcal{E})\,.$$

Applying $\mathbf{c}(\cdot)$ to a D-type matrix pencil yields a matrix pencil (Xu (2006)) of the form

$$\begin{aligned}\lambda\tilde{\mathcal{E}} - \tilde{\mathcal{A}} &:= \mathbf{c}(\mathcal{E}_{\mathrm{D}}, \mathcal{A}_{\mathrm{D}}) \\ &= \lambda \begin{bmatrix} 0 & G + F \\ -G^T - F^T & H \end{bmatrix} - \begin{bmatrix} 0 & G - F \\ G^T - F^T & H \end{bmatrix}.\end{aligned}$$

Pencils of this structure have Hamiltonian eigensymmetry and might have additional eigenvalues 1. In particular, unitary eigenvalues of $\lambda\mathcal{E}_{\mathrm{D}} - \mathcal{A}_{\mathrm{D}}$ are mapped to the purely imaginary eigenvalues of $\lambda\tilde{\mathcal{E}} - \tilde{\mathcal{A}}$. Unfortunately, $\lambda\tilde{\mathcal{E}} - \tilde{\mathcal{A}}$ still has a structure that we cannot exploit. Therefore we apply an additional drop/add transformation to obtain a *C-type matrix pencil* (Xu (2006))

$$\begin{aligned}\lambda\mathcal{E}_{\mathrm{C}} - \mathcal{A}_{\mathrm{C}} &:= \mathbf{d}\left(\tilde{\mathcal{E}}, \tilde{\mathcal{A}}\right) \\ &= \begin{bmatrix} (1-\lambda)I & 0 \\ 0 & I \end{bmatrix} \left(\lambda\tilde{\mathcal{E}} - \tilde{\mathcal{A}}\right) \begin{bmatrix} I & 0 \\ 0 & (1-\lambda)^{-1}I \end{bmatrix} \\ &= \lambda \begin{bmatrix} 0 & G + F \\ -G^T - F^T & 0 \end{bmatrix} - \begin{bmatrix} 0 & G - F \\ G^T - F^T & H \end{bmatrix}.\end{aligned}$$

This pencil has even structure and therefore satisfies the Hamiltonian eigensymmetry. Note that the transformation is singular if $\lambda = 1, \infty$. Therefore, the multiplicities of these eigenvalues may have changed. However, this does not affect our problem, since we are only interested in the finite, purely imaginary eigenvalues. Applied to our problem we obtain the even matrix pencil

$$\begin{aligned}\mathcal{H}_{\mathrm{d}}^{(3)}(\gamma) = \lambda &\left[\begin{array}{c|ccc} 0 & -A^T - E^T & -C^T & 0 \\ \hline A + E & 0 & 0 & 0 \\ C & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ &- \left[\begin{array}{c|ccc} 0 & -A^T + E^T & -C^T & 0 \\ \hline -A + E & 0 & 0 & B \\ -C & 0 & -\gamma I_p & D \\ 0 & B^T & D^T & -\gamma I_m \end{array}\right].\end{aligned}$$

3.3 Even and skew-Hamiltonian/Hamiltonian Pencils

In this subsection we briefly describe the relations between even and skew-Hamiltonian/Hamiltonian matrix pencils and explain how we can get skew-Hamiltonian/Hamiltonian matrix pencils from $\mathcal{H}_{\mathrm{c}}^{(2)}(\gamma)$ and $\mathcal{H}_{\mathrm{d}}^{(3)}(\gamma)$. Let $\lambda\mathcal{S}-\mathcal{H}\in\mathbb{R}[\lambda]^{2n\times 2n}$ be an even matrix pencil. Then, it not difficult to show that $\lambda\mathcal{S}\mathcal{J}-\mathcal{H}\mathcal{J}$ is skew-Hamiltonian/Hamiltonian. However, $\lambda\mathcal{S}-\mathcal{H}$ might also be of odd dimension, whereas every skew-Hamiltonian/Hamiltonian matrix pencil always has an even dimension. In this case we have to inflate the matrix pencil $\lambda\mathcal{S}-\mathcal{H}$ by one dimension. Of course, that inflated pencil must be of even structure, too. Following from these considerations we define the skew-Hamiltonian/Hamiltonian matrix pencils

$$\mathcal{H}_{\mathrm{c}}(\gamma) := \begin{cases} \mathcal{H}_{\mathrm{c}}^{(2)}(\gamma)\mathcal{J} & \text{if } m+p \text{ is even,} \\ \operatorname{diag}\left(\mathcal{H}_{\mathrm{c}}^{(2)}(\gamma),1\right)\mathcal{J} & \text{if } m+p \text{ is odd,} \end{cases}$$

$$\mathcal{H}_{\mathrm{d}}(\gamma) := \begin{cases} \mathcal{H}_{\mathrm{d}}^{(3)}(\gamma)\mathcal{J} & \text{if } m+p \text{ is even,} \\ \operatorname{diag}\left(\mathcal{H}_{\mathrm{d}}^{(3)}(\gamma),1\right)\mathcal{J} & \text{if } m+p \text{ is odd.} \end{cases}$$

In our implementation we use modified versions of $\mathcal{H}_{\mathrm{c}}(\gamma)$ and $\mathcal{H}_{\mathrm{d}}(\gamma)$ which are slightly different from those presented in Benner et al. (2012); Benner and Voigt (2011). Note that the construction of these pencils usually requires to split some of the matrices from the system realization into subblocks. There exist other approaches for the inflation where such a splitting is not necessary but this usually requires a larger extension of the even pencils, see Voigt (2010). However, if $|m-p|$ is small this is usually still feasible.

3.4 Structure-Preserving Eigensolver

In this subsection we briefly describe how we determine the desired, i.e., finite, purely imaginary eigenvalues of the skew-Hamiltonian/Hamiltonian pencils $\mathcal{H}_{\mathrm{c}}(\gamma)$ and $\mathcal{H}_{\mathrm{d}}(\gamma)$ in a reliable and accurate way.

Consider a regular skew-Hamiltonian/Hamiltonian matrix pencil $\lambda\mathcal{S}-\mathcal{H}\in\mathbb{R}[\lambda]^{2n\times 2n}$. First, we recall the fact that transformations of the type $\left(\mathcal{J}\mathcal{P}^T\mathcal{J}^T\right)(\lambda\mathcal{S}-\mathcal{H})\mathcal{P}$ with nonsingular $\mathcal{P}$ preserve the skew-Hamiltonian/Hamiltonian structure (Benner et al. (1999)). Therefore we hope that we can compute an orthogonal matrix $\mathcal{Q}$ such that

$$\left(\mathcal{J}\mathcal{Q}^T\mathcal{J}^T\right)(\lambda\mathcal{S}-\mathcal{H})\mathcal{Q} = \lambda\begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{11}^T \end{bmatrix} - \begin{bmatrix} H_{11} & H_{12} \\ 0 & -H_{11}^T \end{bmatrix},$$

with upper triangular S_{11} and upper quasi triangular H_{11}. This condensed form is called *structured Schur form* (Benner et al. (1999)). Unfortunately, such a structured Schur form does not always exist, for instance if $\lambda\mathcal{S}-\mathcal{H}$ has simple, finite, purely imaginary eigenvalues. This is the generic situation in our application. However, we can use an alternative factorization of the matrix pencil which is called *generalized symplectic URV decomposition* (Benner et al. (1999)).

Theorem 3. Let $\lambda\mathcal{S}-\mathcal{H}$ be a regular skew-Hamiltonian/Hamiltonian matrix pencil. Then there exist orthogonal matrices $\mathcal{Q}_1$ and $\mathcal{Q}_2$ such that

$$\mathcal{Q}_1^T\mathcal{S}\mathcal{J}\mathcal{Q}_1\mathcal{J}^T = \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{11}^T \end{bmatrix},$$

$$\mathcal{J}\mathcal{Q}_2^T\mathcal{J}^T\mathcal{S}\mathcal{Q}_2 = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{11}^T \end{bmatrix},$$

$$\mathcal{Q}_1^T\mathcal{H}\mathcal{Q}_2 = \begin{bmatrix} H_{11} & H_{12} \\ 0 & H_{22} \end{bmatrix},$$

where S_{11}, T_{11}, H_{11} are upper triangular, and H_{22}^T is upper quasi triangular.

The proof of the above theorem is constructive and directly leads to an algorithm for computing the generalized symplectic URV decomposition. As shown in Voigt (2010), the eigenvalues of $\lambda\mathcal{S}-\mathcal{H}$ are the positive and negative square roots of the eigenvalues of generalized matrix product $-S_{11}^{-1}H_{11}T_{11}^{-1}H_{22}^T$. In particular, the finite, purely imaginary eigenvalues of the matrix pencil correspond to the 1×1 diagonal blocks of the generalized matrix product. Therefore, when computing purely imaginary eigenvalues we do not make any error in the real parts, i.e., a robust and reliable detection of the desired eigenvalues is achieved. We note that problems might still occur if we compute finite, purely imaginary eigenvalues which are very close to each other. Then, under certain conditions it can happen that they split on the imaginary axis and form a quadruple of eigenvalues together with their complex conjugate counterparts. This situation typically arises when γ approaches the true value of the $\mathcal{L}_\infty$-norm.

4. NUMERICAL RESULTS

This section presents some numerical results, based on a Fortran implementation of the algorithm — to be included in the SLICOT Library [1] — and a corresponding MATLAB MEX-file. The calculations have been performed on a portable Intel Dual Core computer at 2 GHz, with 2 GB RAM, and relative machine precision $\varepsilon \approx 2.22\times 10^{-16}$, using Windows XP (Service Pack 2) operating system, Intel Visual Fortran 11.1 compiler, MATLAB 7.13.0.564 (R2011b), and the optimized LAPACK and BLAS libraries available in MATLAB. Tolerances have been set to $\sqrt{\varepsilon}\approx 10^{-8}$. The balancing (equilibration) option was not activated for the SLICOT calculations.

Many tests have been performed for random systems with elements chosen from a uniform distribution in the range $(0,1)$ and various dimensions n, m, and p, with nonsingular (including identity) or singular matrices E. The results practically coincided with those delivered by the MATLAB function `norm`.

Other tests have been performed for linear systems from the COMPl$_e$ib collection (Leibfritz and Lipinski (2004)). This collection contains 124 continuous-time examples (put in the standard form), with several variations, giving a total of 168 problems. The matrix D is zero. All but 16 problems (for systems of order larger than 2000, with matrices in sparse format) have been tried.

To generate descriptor systems with singular matrix E, we used $E=\operatorname{diag}(I_{n-1},0)$. With this modification, 22 systems (AC9, HE3, JE2, DIS4, BDT2, PAS, NN1, NN2,

[1] `http://www.slicot.org/`

NN9, NN15, NN16, FS, ROC1-ROC10) became improper, and therefore, these systems have infinite values for the $\mathcal{L}_\infty$-norm and corresponding frequencies. These examples were excluded from the comparison.

The relative errors between the $\mathcal{L}_\infty$-norms computed by the new, structured SLICOT solver and by the MATLAB function `norm` usually had values of the order 10^{-12} or less. Just two examples (CM5 and CM6) had relative errors of order 10^{-6} (the largest relative errors) and two examples (CM4 and CM6_IS) had relative errors of order 10^{-7}. The SLICOT solver found a larger value of $\mathcal{L}_\infty$-norm than `norm` for 58 problems, and the same value for 19 problems.

The sum of the CPU (central processing unit) execution times for SLICOT and MATLAB solvers was 997 and 2280 seconds, respectively, i.e., the SLICOT solver was globally about 2.3 times faster than the MATLAB solver. Figure 1 presents the ratios of the CPU times needed by `norm` and by the new solver.

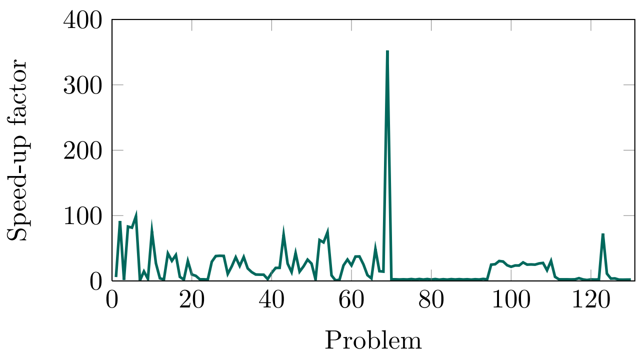

Fig. 1. SLICOT structured solver versus MATLAB `norm` for COMPl$_e$ib modified examples: speed-up factor comparison.

Actually, the speed-up has been lower than 1.5 for just three examples, and lower than 2 for nine examples. Its mean value was 21. The reason is that the SLICOT solver exploits the skew-Hamiltonian/Hamiltonian structure of the matrix pencils involved. Large speed-up factors have been obtained for small order problems. The largest CPU time was about 535 seconds (for example NN18, with $n = 1006$, $m = p = 1$), and the corresponding speed-up factor value was 2.46.

Fig. 2 shows the speed-up factors for the 24 medium-size problems in the HF2D group (with orders between 256 and 576). These are actually 12 pairs of examples with smaller and larger order, which explains the zig-zagging appearance. This figure zooms into the interval with low speed-ups around the problem numbered 80 in Fig. 1. Solving these problems needed about one third of the total execution time.

We have also performed tests with standard systems ($E = I_n$). All COMPl$_e$ib examples, except those with over 2000 states, have been considered in this case. An improvement, based on the HAPACK [2] approach, of the solver in Sima (2006) is used. The new solver found a larger value of $\mathcal{L}_\infty$-norm than `norm` for 55 problems, and the same value for 42 problems. The Euclidean norm of the relative differences between the finite $\mathcal{L}_\infty$-norms computed by the two solvers was of the order of 10^{-10}.

[2] http://www.tu-chemnitz.de/mathematik/hapack/

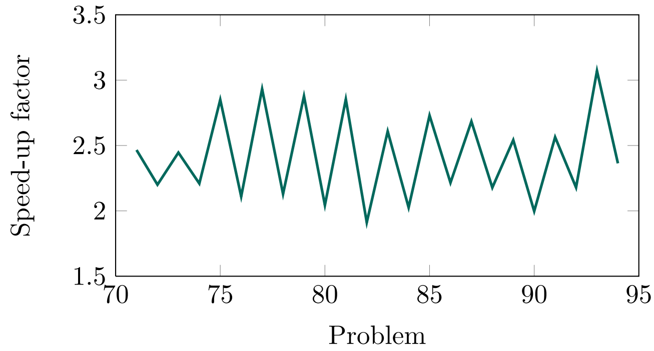

Fig. 2. SLICOT structured solver versus MATLAB `norm` for COMPl$_e$ib modified HF2D examples: speed-up comparison.

The SLICOT solver has been significantly faster than the MATLAB function `norm`. The speed-up factor can be impressive even for large problems (e.g., 3.8 for example NN18). Actually, the speed-up has been lower than 3 for just two examples, and lower than 4 for four examples. Its mean value was about 44. The ratio of the sum of the CPU times needed by `norm` and by the SLICOT solver was about 4.7.

Figure 3 presents the ratios of the CPU times needed by `norm` and by the new solver.

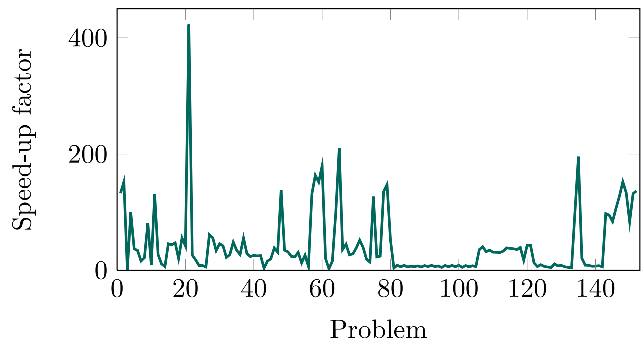

Fig. 3. SLICOT structured solver versus MATLAB `norm` for COMPl$_e$ib examples: speed-up factor comparison.

Fig. 4 shows the speed-up factors for the medium-size problems in the HF2D group.

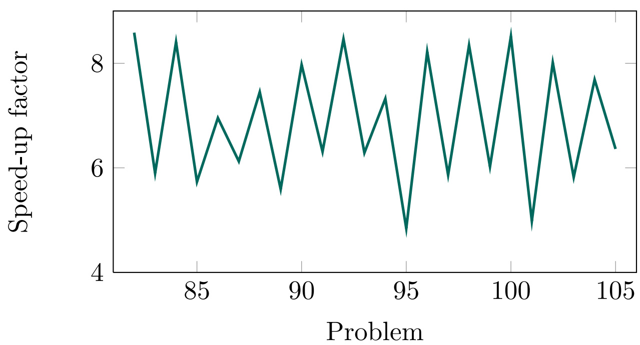

Fig. 4. SLICOT structured solver versus MATLAB `norm` for COMPl$_e$ib HF2D examples: speed-up comparison.

5. CONCLUSIONS

We have presented algorithms for computing the $\mathcal{L}_\infty$-norm of transfer functions related to LTI descriptor systems. The crucial step of the algorithms is the computation of the eigenvalues of certain structured matrix pencils. We have shown how one can transform these matrix pencils to more convenient structures in order to improve accuracy, reliability and efficiency of the eigenvalue computation. Finally, our theoretical considerations have been verified by performing a large test sequence with both random and COMPl$_e$ib benchmark examples.

REFERENCES

Benner, P., Byers, R., Mehrmann, V., and Xu, H. (1999). Numerical computation of deflating subspaces of embedded Hamiltonian pencils. Technical report, Chemnitz University of Technology, Department of Mathematics, Germany. SFB393-Preprint 99-15.

Benner, P., Sima, V., and Voigt, M. (2012). $\mathcal{L}_\infty$-norm computation for continuous-time descriptor systems using structured matrix pencils. *IEEE Trans. Automat. Control*, 57(1), 233–238. doi:10.1109/TAC.2011.2161833.

Benner, P. and Voigt, M. (2011). $\mathcal{L}_\infty$-norm computation for discrete-time descriptor systems. In O. Sawodny and J. Adamy (eds.), *Tagungsband des GMA-Fachausschusses 1.30 "Modellbildung, Identifikation und Simulation in der Automatisierungstechnik"*, 228–240. Technische Universität Darmstadt, Institut für Automatisierungstechnik und Mechatronik.

Boyd, S. and Balakrishnan, V. (1990). A regularity result for the singular values of a transfer matrix and a quadratically convergent algorithm for computing its L_∞-norm. *Syst. Control Lett.*, 15(1), 1–7. doi:10.1016/0167-6911(90)90037-U.

Bruinsma, N.A. and Steinbuch, M. (1990). A fast algorithm to compute the H_∞-norm of a transfer function matrix. *Syst. Control Lett.*, 14(4), 287–293. doi:10.1016/0167-6911(90)90049-Z.

Byers, R., Mackey, D.S., Mehrmann, V., and Xu, H. (2009). Symplectic, BVD, and palindromic approaches to discrete-time control problems. In *Collection of Papers Dedicated to the 60-th Anniversary of Mihail Konstantinov*, 81–102.

Genin, Y., Van Dooren, P., and Vermaut, V. (1998). Convergence of the calculation of $\mathcal{H}_\infty$-norms and related questions. In *Proceedings MTNS-98*, 429–432.

Leibfritz, F. and Lipinski, W. (2004). *COMPl$_e$ib* 1.0 – User manual and quick reference. Technical report, Department of Mathematics, University of Trier, D–54286 Trier, Germany.

Losse, P., Mehrmann, V., Poppe, L., and Reis, T. (2008). The modified optimal $\mathcal{H}_\infty$ control problem for descriptor systems. *SIAM J. Control Optim.*, 47(6), 2795–2811.

Mehrmann, V. and Stykel, T. (2005). Balanced truncation model reduction for large-scale systems in descriptor form. In P. Benner, V. Mehrmann, and D. Sorensen (eds.), *Dimension Reduction of Large-Scale Systems*, volume 45 of *Lecture Notes in Computational Science and Engineering*, chapter 3, 89–116. Springer-Verlag, Berlin, Heidelberg, New York.

Schröder, C. (2008). *Palindromic and Even Eigenvalue Problems — Analysis and Numerical Methods*. Ph.D. thesis, Institut für Mathematik, Technische Universität Berlin, Germany.

Sima, V. (2006). Efficient algorithm for L_∞-norm calculations. Preprints of 5th IFAC Symposium on Robust Control Design, Toulouse, France. Invited Session MD001 - "Advances in numerical algorithms for robust control and its applications".

Voigt, M. (2010). $\mathcal{L}_\infty$-norm computation for descriptor systems. URL `http://nbn-resolving.de/urn:nbn:de:bsz:ch1-201001050`.

Xu, H. (2006). On equivalence of pencils from discrete-time and continuous-time control. *Lin. Alg. Appl.*, 414(1), 97–124.

Zhou, K. and Doyle, J.D. (1998). *Essentials of Robust Control*. Prentice Hall, 1st edition.