About the Triangle Inequality in Perceptual Spaces

Frank Jäkel, Bernhard Schölkopf, and Felix A. Wichmann

Max Planck Institute for Biological Cybernetics

Perceptual similarity is often formalized as a metric in a multi-dimensional space. Stimuli are points in the space and stimuli that are similar are close to each other in this space. A large distance separates stimuli that are very different from each other. This conception of similarity prevails in studies from color perception and face perception to studies of categorization. While this notion of similarity is intuitively plausible there has been an intense debate in cognitive psychology whether perceived dissimilarity satisfies the metric axioms. In a seminal series of papers, Tversky and colleagues have challenged all of the metric axioms [1,2,3].

The triangle inequality has been the hardest of the metric axioms to test experimentally. The reason for this is that measurements of perceived dissimilarity are usually only on an ordinal scale, on an interval scale at most. Hence, the triangle inequality on a finite set of points can always be satisfied, trivially, by adding a big enough constant to the measurements. Tversky and Gati [3] found a way to test the triangle inequality in conjunction with a second, very common assumption. This assumption is segmental additivity [1]: The distance from A to C equals the distance from A to B plus the distance from B to C, if B is “on the way”. All of the metrics that had been suggested to model similarity also had this assumption of segmental additivity, be it the Euclidean metric, the L_p-metric, or any Riemannian geometry. Tversky and Gati collected a substantial amount of data using many different stimulus sets, ranging from perceptual to cognitive, and found strong evidence that many human similarity judgments cannot be accounted for by the usual models of similarity. This led them to the conclusion that either the triangle inequality has to be given up or one has to use metric models with subadditive metrics. They favored the first solution. Here, we present a principled subadditive metric based on Shepard’s *universal law of generalization* [4].

Instead of representing each stimulus as a point in a multi-dimensional space our subadditive metric stems from representing each stimulus by its similarity to all other stimuli in the space. This similarity function, as for example given by Shepard’s law, will usually be a radial basis function and also a positive definite kernel. Hence, there is a natural inner product defined by the kernel and a metric that is induced by the inner product. This metric is subadditive. In addition, this metric has the psychologically desirable property that the distance between stimuli is bounded.

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References
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