FROM TRISTIMULUS COLOR SPACE TO CONE SPACE

Harald J. Teufel[2,3] and Christian Wehrhahn[1]
Max-Planck-Institut für biologische Kybernetik[1], Theoretische Astrophysik, Universität Tübingen[2] and Color Physics GmbH, Tübingen[3]
harald@color-physics.de, christian.wehrhahn@tuebingen.mpg.de

Abstract

A mathematically simple transformation from the Judd (1951) modified CIE space into the space spanned by the cones is derived. The transformation is based on geometrical intuition (Pitt 1945).

Light fluxes that initiate vision are transmitted into electrical signals by the photoreceptors. In day light these are the retinal cones. Most humans have three types of cones with spectral sensitivities in the short (S), middle (M) and long (L) part of the visible spectrum, and hence are called trichromats. Absorption of a photon leads to a structural change of photo pigment, which - through an enzymatic cascade - generates the electrical cone signal. In this process information about the wavelength of the photon is lost. Thus the cone signals processed further in the retina just depend on the numbers of photons that are absorbed in the respective type. Therefore colors can be described in a three-dimensional space formed by the excitations of the three cone types (MacLeod & Boynton, 1973). When color metrics was developed, absorption spectra of cones were not known yet. Color perception had to be quantified using other means. Three primary colors (primaries) were defined, each chosen from a different range of colors. Human observers can match any color by varying the intensities of the primaries and hence define this color within the chosen metric (Judd & Wysecki 1975). In some cases one of the primaries has to be added to the color being analyzed to make it appear less saturated. This procedure is the reason for the appearance of negative color mixture values.
Colors perceived due to occurrence of energy in just one line of the spectrum are called spectral colors. The intensities of the three primaries needed to match spectral colors plotted over the wavelength are called color matching functions (CMF). CMFs can be transformed from one set of primaries to another using linear algebra. Obviously, each primary has a defined overlap with the absorption spectra of the cones. Thus changing the intensities of the primaries leads to proportional changes in the excitations of the cones. This means that we can find a linear transformation between the CMF and the spectral sensitivities of the cones. One way to derive these is based on the specific deficits of dichromats in discriminating between certain colors. Dichromats are naturally occurring mutants in (mostly male) humans, each of whom has only two of the three cone pigments found in trichromats. Though frequently used in color vision, transformations based on these findings still lack a simple interpretation. The aim of this paper is to give a simple geometric interpretation of this crucial step in color metrics from conventional color space (CIE) into cone space. We elaborate an earlier proposal (Pitt, 1945). We avoid technical details and emphasize the rather simple steps of the transformation. Elementary mathematical tools required are provided in the appendix.

Brief describtion of CIE space

Since the starting point of our transformation is the modified version of CIE-xy space (Judd, 1951), denoted x'y'-space, we will mention a few features of it. In 1931 the CIE (Commission Internationale de l'Eclairage) recommended a special set of CMFs $\underline{x}(\lambda)$, $\underline{y}(\lambda)$, $\underline{z}(\lambda)$ as the standard colorimetric system. These functions were transformations of the results of color mixture experiments into a space of primaries which does not have a physical realisation. The reason for this choice was that all these CMFs have positive values and that the $\underline{y}(\lambda)$ function is proportional to V_λ, the photopic luminous-efficiency function (of the 1924 standard human observer). V_λ is the spectral distribution connected to the perception of brightness. The three numbers that characterize a color in CIE space are called the tristimulus values X, Y and Z, where Y corresponds to the luminance of the color. X, Y and Z are calculated by integrating the spectral radiance over the CMF. The CIE chromaticity diagram (CIE-xy space) is a normalized representation of the colors perceived by the standard observer formed by the tristimulus values X, Y and Z. It is calculated by the fractions of X and Y in terms of X+Y+Z. The x- and z-values are called chromaticity coordinates.
The 1924 V_λ is known to be in error in the short wavelength region of the spectrum. Judd (1951) proposed a modified version of the luminous-efficiency function, which also implies the use of a modified set of CMF, denoted $\underline{x}'(\lambda)$, $\underline{y}'(\lambda)$ and $\underline{z}'(\lambda)$.
Color mixture data show a large variability between different observers. The question which set of CMF is to be preferred over the others is discussed in detail by Stockman et al. (1993ab). In psychophysical experiments (see next section) these authors measured the spectral sensitivities of the S-, M- and L-cones in human observers using a transient chromatic adaptation method.

Cone fundamentals

Psychophysical methods yield the cone absorption spectra or cone fundamentals including the spectral properties of the ocular media. Although the psychophysical methods are indirect, they give a more realistic measure for the cone signals actually relevant in human color perception. One of these psychophysical methods is based on chromatic adaptation and tries to isolate one cone system by selectively reducing the sensitivity of the other two systems with an adequately chosen background (Stockmann et al. 1993). Another method uses the specific deficits in the perception of color of the three types of dichromats.
There are, in fact, more than three cone pigments found in humans, but this is not essential to the argument put forward here. Dichromats are usually called color blinds although they do have a well measurable ability to discriminate colors. In the CIE-xy space the locations of the colors confused by each type of dichromat are situated on a specific set of lines (see Wyszecki & Stiles 1982) and therefore these lines are called the confusion lines of the specific type of dichromat. These lines converge into a single point in CIE-xy space which is called the confusion point of that type of dichromat. Obviously dichromats fail to distinguish between the colors situated along the confusion lines because their two remaining cone systems yield a constant fraction of excitation for these colors. Hence at the confusion points the excitations of the two respective systems vanish (Vos & Walraven 1970). Thus the position in CIE-space of the color (perceived by trichromats and) defined by the excitation of a single cone type can be determined and this can be done for all three cone types. This statement is based on the assumption that dichromats are lacking one and only one type of cone pigment (König & Dieterici 1886).

A simple geometric transformation into cone space

What happens to the confusion lines and the confusion points in a color space where the primaries had the same spectral shape as the cone fundamentals (Pitt, 1945)? The left part of figure 1 shows the locations of the confusion points in the x'y'-space. The right part of figure 1 shows a color space spanned by the cone fundamentals. Figure 2 shows the same space together with two colors denoted by P and Q.

Let a normal trichromat view color P. S/he has three relationships from excitations of the three cone systems. So s/he is able to determine the exact location of the color in the three- dimensional color space. Let now a dichromat, say a protanope view the same color. Because

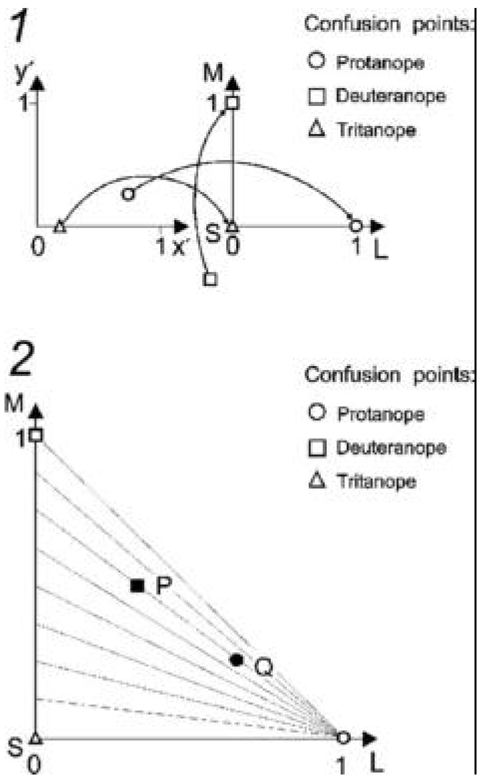

Fig. 1:Left half: x'y'-space together with the locations of the three confusion points of the three types of dichromats. The values of the confusion points are taken from Smith & Pokorny (1975). Right half: Relative representation of the cone space formed by primaries with the same spectral shape as the cone fundamentals. In this space the confusion points must be located at the base vectors. The arrows indicate the transformation from x'y'-space to the cone space.
Fig. 2: Cone space formed by primaries with the same spectral shape as the cone fundamentals. Plotted are two colors P and Q that are situated on a confusion line of the protanope. All confusion lines of the protanope converge into the location of the first base vector.

of her/his missing L-pigment there is no measure for the L-system versus the M-system and the L-system versus the S-system. The only remaining relationship is between the excitations of the S- and the M-system.
Let both, the trichromat and the protanope now view the other color denoted by Q. The trichromat has again three relationships two of them differing from the ones for color P, so s/he is able to determine the exact location of color Q in color space. Both, the protanope and the trichromat yield the same relationship between the excitations of the M- and the S-system as for color P. Color Q looks the same for the protanope as color P, except for a possible luminance difference.
All colors with the same fraction of M- and S-cone excitation must lie on a straight line. The intersection of this line with the abscissa at the point (1,0) represents the base vector with respect to the L coordinate, i. e. the L-cone pigment. There exists an infinity of those lines all converging to that point for the protanope. Corresponding considerations yield the location of the deuteranopic confusion point at position (0,1) and the tritanopic confusion point at position (0,0) in this color space. The confusion lines presented here are projections from the three-dimensional cone space on the plane given by S+M+L=1. This enables us to express the colors in the two-dimensional plane formed by the L- and M-cone excitations. The third coordinate of the cone space would elongate

from the origin to the viewer and is given by S=1-M-L. For a detailed description of how the confusion lines are situated in the three-dimensional space see MacLeod & Boynton (1973). We conclude that in three-dimensional cone space the confusion points of dichromats are situated at the locations of the base vectors, as depicted in both figures. Following this conclusion we obtain the algorithm to transform from the x‘y‘z‘-space, (where the confusion points are determined experimentally) to the cone space (Wysecki & Stiles 1982). Our aim now is to find a matrix that converts the confusion points given in x‘y‘-space to the locations of the base vectors in cone space (Connections between the confusion points of the left and right half of Fig. 1) . The transformation matrix must convert the confusion points of the left half of Fig. 1 to the locations of the base vectors in the right half of Fig. 1. Thus it must fullfil the following three equations

$$[M] \cdot \begin{bmatrix} 0{,}7465 \\ 0{,}2535 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad [M] \cdot \begin{bmatrix} 1{,}4 \\ -0{,}4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad [M] \cdot \begin{bmatrix} 0{,}1748 \\ 0 \\ 0{,}8252 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \text{Eq. (1)}$$

where the vectors on the right side of the equations (1) represent the three base vectors of the cone space (L, M and S), the vectors on the left side are the locations of the confusion points in x‘y‘-space (Smith & Pokorny 1975). The matrix M is a transformation matrix from x‘y‘-space to the cone space. By solving the equations (1) the matrix M is estimated to

$$[M] = \begin{bmatrix} 0{,}61209 & 2{,}14231 & -0{,}12966 \\ 0{,}38791 & -1{,}14231 & -0{,}08217 \\ 0 & 0 & 1{,}21183 \end{bmatrix} \qquad \text{Eq. (2)}$$

Using this matrix we can transform any color from CIE to cone space.

Conclusions

The method of the confusion points depends to a large extent on the precision of the experimentally determined CMF and the confusion points. Derivations of the latter vary considerably between different studies. Another difficulty is to find a reasonable way to average the confusion points of different subjects in x‘y‘-space (Stockman et al. 1993). The cone fundamentals estimated by selective bleaching or the method of the confusion points do not differ very much, at least if broad band stimuli are used. The method of the confusion points assumes that dichromacy is a reduced form of trichromacy. The spectral shape of the cone fundamentals is determined by the locations of the confusion points in x‘y‘-space.

The transformation to the cone space described above does not necessarily use the Judd (1951) modified CIE xy-space as its starting point. Each two-dimensional representation of a tristimulus color space might be used under the restriction that the exact location of the confusion points is known within this space.

Tristimulus values are calculated by integrating the incident spectrum over CMFs. Use of a spectroradiometer allows straightforward calculation of cone excitations by integrating the spectrum directly over the cone fundamentals. In this case it is not necessary to calculate tristimulus X‘Y‘Z‘-values which could be further transformed into cone excitations by the matrix multiplication described above. Most colorimeters express chromaticity coordinates and luminance with respect to the 1931 CIE color matching functions. It is desirable to apply a matrix multiplication that converts these 1931 CIE XYZ-tristimulus values into cone excitations, which could then be calculated from the XYZ output of the colorimeter. There is one caveat still: Judd & Wyszecki (1975) suggested a set of confusion points required for the 1931 CIE chromaticity diagram. Because the underlying color matching functions are calculated partly from the 1924 CIE luminous efficiency function which is seriously in error at short wavelengths the validity of this transformation is questionable. Especially the S-cone excitations obtained by this transformation suffer from uncertainties. A recent set of transformation values to convert 1931 CIE tristimulus values into M- and L-cone excitations was suggested by Stockman et al. (1993). Using these transformation values they obtained the best fit with their cone fundamentals based on the Stiles & Burch (1955) 2 degree CMF.

Our goal in presenting still another transformation from (modified) CIE color space into cone space is to show that this is possible using a mathematically simple way following straightforward geometrical ideas (Pitt, 1945). The method of the confusion points thus provides clear insight into dichromatic and trichromatic color metrics. In spite of this, the amplitude of cone fundamentals remains an unsolved problem. It was shown that a value for the S-cone fundamental of about an eleventh of the L-cone fundamental and about a sixth of the M-cone fundamental leads to circular detection contours (Teufel & Wehrhahn 2000).

Appendix A. The fraction between M- and S-cone excitation is constant along a confusion line of the protanope.

The parameterization of a line through a certain point in a two-dimensional space is given by:

$$y - y_1 = m\,(x - x_1), \qquad \text{Eq. (3)}$$

where m is the slope of the line and (x_1,y_1) is a given point in the xy-plane. If the abscissa is the L-cone excitation and the ordinate is the M-cone excitation, the parameterization of a line through the confusion point of the protanope (1,0) is given by

$$M = m \cdot (L-1) \qquad \text{Eq. (4)}$$

The two points P and Q that are situated on this line

$$M_P = m \cdot (L_P - 1) \quad \text{and} \quad M_Q = m \cdot (L_Q - 1) \qquad \text{Eq. (5)}$$

must both satisfy the relative representation (S+M+L=1), thus

$$L_P = 1 - S_P - M_P \quad \text{and} \quad L_Q = 1 - S_Q - M_Q \quad . \qquad \text{Eq. (6)}$$

Inserting equations (6) into equations (5) yields:

$$\frac{M_P}{S_P} = -\frac{m}{m+1} \quad \text{and} \quad \frac{M_Q}{S_Q} = -\frac{m}{m+1} \qquad \text{Eq. (7)}$$

which means that the fraction of M- and S-cone excitation is constant along the confusion lines of the protanope.

Appendix B. The multiplication of a vector with a matrix

A matrix A of range 3 is given by its 9 coefficients a_{ij}, where i and j are 1, 2 or 3. The vector v of a three-dimensional vector space is given by its 3 components v_j, where j again is 1, 2 or 3. The matrix multiplication

$$\begin{bmatrix} a_{11}\cdot v_1 + a_{12}\cdot v_2 + a_{13}\cdot v_3 \\ a_{21}\cdot v_1 + a_{22}\cdot v_2 + a_{23}\cdot v_3 \\ a_{31}\cdot v_1 + a_{32}\cdot v_2 + a_{33}\cdot v_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \qquad \text{Eq. (8)}$$

is performed by multiplying each component of a row of the matrix with the component of the vector having the same j as the coefficient of the matrix. Adding these three values yields the i'th component of the resulting vector on the left of equation (8).

References:

Boynton, R.M. (1996). History and current status of a physiologically based system of photometry and colorimetry. J.Opt.Soc.Am. A 13:1609-1621.

Derrington, A.M., Krauskopf, J., Lennie, P. (1984). Chromatic mechanisms in lateral geniculate nucleus of macaque. J. Physiol.357:241-265.
De Valois, R.L., De Valois, K.K., Switkes, E., Mahon, L. (1997). Hue scaling of isoluminant and cone- specific lights. Vision Res.37:885-897.
Helmholtz, H. (1852). Über die Theorie der zusammengesetzten Farben. Physiologisch-optische Abhandlung. Habilitation. Gebr. Unger, Berlin.
Judd, D.B., Wyszecki, G. (1975). Color in Business, Science and Industry, third edition. New York: John Wiley & Sons.
Kaiser, K.K., Boynton, R.M. (1996). Human Color Vision 2nd edition. Optical Society of America, Washington, DC.
König, A., Dieterici, C. (1886). Die Grundempfindungen und ihre Intensitäts-Vertheilung im Spectrum. Berlin: Sitz. Akad. Wiss.; p 805-829.
Krauskopf, J., Williams, D.R., Heeley, D.W. (1982). Cardinal directions of color space. Vision Res.22:1123-1131.
Lee, B.B., Kremers, J., Yeh, T. (1998). Receptive fields of primate retinal ganglion cells studied with a novel technique. Visual Neuroscience 15:161-175.
MacAdam, D.L. (1942). Visual sensitivities to color differences in daylight. J. Opt. Soc. Am. 32:247-274.
MacLeod, D.I.A., Boynton, R.M. (1979). Chromaticity diagram showing cone excitation by stimuli of equal luminance. J. Opt. Soc. Am 69:1183-1186.
Pitt, F.H.G. (1945). The nature of normal trichromatic and dichromatic vision. Proc. Roy. Soc. B 132:101-117.
Schnapf
Smith, V.C., Pokorny, J. (1975). Spectral sensitivity of the foveal cone photopigments between 400 and 500 nm. Vision Res. 15:161-171.
Stockman, A., MacLeod, D.I.A., Johnson, D. (1993). Spectral sensitivities of the human cones. J. Opt. Soc. Am. A 10:2491-2521.
Stockman, A., MacLeod, D.I.A., Vivien, J.A. (1993). Isolation of the middle- and long-wavelength-selective cones in normal trichromats. J. Opt. Soc. Am. A 10:2471-2490.
Teufel, H.J., Wehrhahn, C. (2000). Evidence for the contribution of S-cones to the detection of flicker brightness and red-green. J. Opt. Soc. Am. A 17: 994-1006.
Vos, J.J., Walraven, P.L. (1970). On the derivation of the foveal receptor primaries. Vision Res.11:799-818.
Webster, M.A., Mollon, J.D. (1994). The influence of contrast adaptation on color appearance. Vision Res. 34:1993-2020.
Wyszecky, G., Stiles, W.S. (1982). Color Science: Concepts and methods, quantitative data and formulae, second edition. New York: John Wiley & Sons; 1982.