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The Taylor Expansion at Past Time-like Infinity

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Abstract: We construct initial data for the conformal vacuum field equations on a cone $\mathcal{N}_p$ with vertex p so that for the prospective vacuum solution, the point p will represent past time-like infinity i^-, the set $\mathcal{N}_p \setminus \{p\}$ will represent past null infinity $\mathcal{J}^-$, and the freely prescribed (suitably smooth) data will acquire the meaning of the incoming *radiation field*. It is shown that: (i) On some coordinate neighbourhood of p there exist smooth fields *which satisfy at the point* p the conformal vacuum field equations at all orders and induce the given data at all orders. The Taylor coefficients of these fields at p are uniquely determined by the free data. (ii) On the cone $\mathcal{N}_p$ there exists a unique set of fields which induce the given free data and satisfy the transport equations and the inner constraints induced on $\mathcal{N}_p$ by the conformal field equations. These fields are *smooth at* p in the sense that they coincide there at all orders with the fields which are obtained by restricting to $\mathcal{N}_p$ the functions considered in (i) and they are smooth on the smooth three-manifold $\mathcal{N}_p \setminus \{p\}$ in the standard sense.

1. Introduction

A purely radiative, asymptotically flat space-time should be generated solely by gravitational radiation coming in from past null infinity. Extraneous information entering the space-time at past time-like infinity should be excluded. A natural problem to study then is the asymptotic characteristic initial value problem for the conformal vacuum field equations where data are prescribed on a cone $\mathcal{N}_p$ with vertex p, similar to the cone $\{x_\mu \, x^\mu = 0, \ x^0 \geq 0\}$ in Minkowski space with vertex at $x^\mu = 0$. It is to be arranged such that the prospective vacuum solution admits a smooth conformal extension in which the point p acquires the meaning of past time-like infinity i^-, and the set $\mathcal{N}_p \setminus \{p\}$, swept out by the future directed null geodesics through p, represents past null infinity $\mathcal{J}^-$.

As in any other initial value problem for Einstein's field equations, two different subproblems must be analysed here: (i) one needs to analyse which part of the initial data can be prescribed freely and how the remaining data are determined on the initial set $\mathcal{N}_p$ by the field equations, (ii) for suitably given data one has to show the existence of a smooth solution inducing these data on the initial set. In the situation indicated above both tasks are complicated by the fact that the initial set $\mathcal{N}_p$ is a smooth hypersurface only away from the vertex p. The notion of smoothness and the way data are given thus require particular considerations.

The present article is concerned with the first problem. It shows the existence of initial data on the cone $\mathcal{N}_p$ which satisfy the inner equations induced on $\mathcal{N}_p$ by the conformal field equations, which are smooth in the standard sense on the three-manifold $\mathcal{N}_p \setminus \{p\}$, and which behave at the point p at all orders like fields induced on $\mathcal{N}_p$ by smooth fields that are defined on some smooth coordinate neighbourhood of p. Data with these properties provide a necessary input for the analysis of the second problem with the arguments used in the article [2] by Chruściel and Paetz.

From the point of view of the physical/geometrical interpretation, one would like to construct the space-times from a minimal set of *free data* on $\mathcal{N}_p$ which admit a physical interpretation. There are various ways to prescribe data for Einstein's field equations in characteristic initial value problems (cf. [1]), the specific choice usually depending on technical considerations and the particular situation at hand. A natural datum to prescribe *at null infinity* is the *radiation field*, a complex-valued function that encodes asymptotic information on the two components of the conformal Weyl tensor with the slowest fall-off behaviour at past null infinity. It is thought to represent the two polarization states of the incoming gravitational radiation.

That the radiation field is convenient from the technical point of view has been shown in the proof of J. Kánnár's existence results on the characteristic asymptotic initial value problem, where data are prescribed on an incoming null hypersurface $\mathcal{C}$ which intersects past null infinity in a space-like slice $\Sigma = \mathcal{C} \cap \mathcal{J}^-$ and on the future $\mathcal{J}'^-$ of that slice in past null infinity [8]. A basic step in that proof consists in showing that given the radiation field on $\mathcal{J}'^-$, the solution and its derivatives of any order can be determined on $\mathcal{J}'^-$ by solving ODE's along the null generators of $\mathcal{J}'^-$, where the initial data for the integration are derived from the data prescribed on $\mathcal{C}$ and Σ.

In the problem to be considered here the analysis is complicated by the fact that an analogue of Σ on the hypersurface representing past null infinity necessarily shrinks to a point when it approaches past time-like infinity, leaving no space for a hypersurface like $\mathcal{C}$. The information for the integration of the solution along the null generators of $\mathcal{N}_p$ has thus to be extracted completely from the radiation field on $\mathcal{N}_p$. Together with the need for a careful discussion of the smoothness requirements near the vertex p, this leads to various algebraic subtleties.

This problem was studied for the first time in [5], where it was shown that for a suitably smooth prescribed radiation field on $\mathcal{N}_p$ and a gauge involving a null coordinate adapted to $\mathcal{N}_p$ the prospective solution to the conformal field equations is determined uniquely at all orders along the cone $\mathcal{N}_p$. However, even under the most convenient assumptions, such a null coordinate is singular at and near p. To show that any smooth solution is *determined uniquely in the future of the cone $\mathcal{N}_p$ by its radiation field*, there has been performed in [5] a transformation into a gauge which is regular up to an order sufficient for the argument. An *existence result for smooth solutions* would require, however, a smooth gauge and thus, due to the quasi-linearity of the equations, a transformation which enters the solutions at all orders.

To simplify this tedious problem (in Sect. 9 it will be seen that the analysis of the transport equations on $\mathcal{N}_p$ requires a discussion of singular equations in any case), the analysis in the present article will be based on a smooth gauge right from the outset.

The basic setting of our analysis is described in Sects. 2, 3 and 4, where the field equations are discussed; a suitable gauge for the conformal factor, the coordinates, and the frame field is introduced; and those features of the gauge are pointed out which will be important in the following. This discussion is completed in Sect. 5 by introducing a certain type of expansion of the fields at the point p, referred to as *normal expansion at p*, and by considering its relation to the concept of *null data* and, in particular, to the notion of the *radiation field on $\mathcal{N}_p$*.

In Sect. 6, 7, and 8, we analyse in detail the Taylor expansion at p of the prospective solution to the conformal field equations. It is shown in particular how the Taylor coefficients are related to the freely prescribed radiation field on $\mathcal{N}_p$ and that these coefficients, which to begin with are determined only on a formal level, can indeed be realized as Taylor coefficients of smooth fields near p. This property is important in establishing the existence theorem given in [2].

In Sect. 6 an argument by Penrose ([10, 11]) is adapted to the present situation and it is shown (Lemma 6.1) that the covariant derivatives at p of the curvature fields, the conformal factor, and a further scalar field are determined on a formal level uniquely at all orders by the radiation field and that the latter is not subject to any restriction. To relate these data to a space-time metric, we consider in Sect. 7 the structural equations, written as equations for the metric coefficients and connection coefficients. It turns out that a subset of these equations already determines the formal Taylor expansions of these fields uniquely and that the expansion coefficients so obtained encode the information on the chosen gauge (Lemma 7.1).

Borel's theorem then ensures the existence of smooth fields near p whose Taylor expansion coefficients at p are given precisely by the (symmetric parts of the) coefficients determined by the formal calculations. However, because only the symmetric parts of the covariant derivatives enter the definition of these functions and only a subset of the structural equation has been formally solved in the calculations, it is far from obvious that the functions so defined do indeed satisfy at p the conformal field equations at all orders in the standard sense of the covariant differential calculus. That this is in fact the case (Proposition 8.1) follows by a somewhat involved induction argument.

In Sect. 9 it is finally shown how the data on $\mathcal{N}_p$ are determined in terms of the radiation field. The conformal field equations induce a system of inner equations on $\mathcal{N}_p$ which splits naturally into two subsets. The equations in the first set, referred to as *transport equations*, allow us to determine all unknown fields entering the conformal field equations once the radiation field is given (Proposition 9.1). The equations in the second set are *inner constraints* on the fields so obtained. It is shown that they are satisfied by a solution to the transport equations without imposing any restriction on the prescribed radiation field.

To identify and analyse the inner equations on $\mathcal{N}_p$, one needs to express the equations in terms of a frame adapted to the cone, which is necessarily singular at p. If the resulting equations are solved and the fields are then transformed back into the regular gauge underlying Proposition 8.1, they coincide with the field discussed in that Proposition at all orders at p and thus satisfy a necessary smoothness requirement (Sect. 9.1).

The fields so obtained constitute a complete set of initial data on $\mathcal{N}_p$ for the conformal field equations and any solution to the vacuum field equations which admits a smooth conformal extension at past null and time-like infinity induces there data of this type. They are the starting point for an existence proof in the category of smooth functions as given in [2].

Besides supplying these initial data, the discussion of this article is of independent interest because it provides a large set of detailed information, in particular on the asymptotic behaviour of the various field components. Following the considerations of [5] in the present setting, it is also possible to determine along $\mathcal{N}_p$ higher order derivatives of the fields in directions transverse to $\mathcal{N}_p$. This is not done here because this information is not needed in [2]. At lowest order the following observation is of interest, however. As shown in [6], the Newman-Penrose constants ([9]) of the solutions considered here are given by the values of the components W_{ijkl} of the rescaled conformal Weyl tensor at the point p. It follows from the calculations below that these are encoded in the radiation field. The discussion of this article and the existence results of [2] thus show that there exist asymptotically flat (at null infinity) solutions to the vacuum field equations with arbitrarily prescribed Newman-Penrose constants.

As pointed out at various places, the analysis presented in this paper also applies to the *finite* characteristic initial value problem where data are given on a finite cone which is thought of as being generated by the (future directed) null geodesics through a point which is an inner point of a smooth vacuum space-time. In fact, the analogues of the arguments used in Sects. 2–8 considerably simplify in that case. In Sect. 9, however, we take advantage of the fact that the conformal Weyl tensor vanishes at null infinity. This allows us to obtain explicit expression for various fields. The analogue of Proposition 9.1 has to be established in the finite problem by an abstract discussion of the transport equations, which is not given here.

2. The Metric Conformal Field Equations

Let g denote a Lorentzian metric on a four dimensional manifold and ∇ a connection which is metric compatible so that $\nabla g = 0$. In the following we shall make use of a frame $\{e_k\}_{k=0,\dots,3}$ which is orthonormal so that $g_{ij} = g(e_i, e_j) = \eta_{ij}$. With the directional covariant derivative operators $\nabla_i \equiv \nabla_{e_i}$ the connection coefficients $\Gamma_i{}^k{}_j$ are defined, by the equation $\nabla_i\, e_j = \Gamma_i{}^k{}_j\, e_k$. The relation $\nabla g = 0$ is then equivalent to the anti-symmetry $\Gamma_{i\,l\,j} = -\Gamma_{i\,j\,l}$, where $\Gamma_{i\,l\,j} = \Gamma_i{}^k{}_j\, g_{kl}$. All tensors (except the frame fields) will be given in the following in terms of the frame e_k.

For a vector field Z the commutator of the covariant derivatives satisfies

$$(\nabla_i \nabla_j - \nabla_j \nabla_i) Z^l = r^l{}_{kij}\, Z^k - t_i{}^k{}_j\, \nabla_k Z^l, \tag{2.1}$$

where $t_k{}^i{}_l$ denotes the torsion tensor, given in terms of coordinates x^μ and the frame coefficients $e^\mu{}_k = < e_k, dx^\mu >$ by the relation

$$t_k{}^i{}_l\, e^\mu{}_i = e^\mu{}_{k,\,\nu}\, e^\nu{}_l - e^\mu{}_{l,\,\nu}\, e^\nu{}_k - (\Gamma_l{}^i{}_k - \Gamma_k{}^i{}_l)\, e^\mu{}_i, \tag{2.2}$$

and $r^i{}_{jkl}$ is the curvature tensor, given by

$$\begin{aligned} r^i{}_{jkl} \equiv\; & \Gamma_l{}^i{}_{j,\,\mu}\, e^\mu{}_k - \Gamma_k{}^i{}_{j,\,\mu}\, e^\mu{}_l + \Gamma_k{}^i{}_p\, \Gamma_l{}^p{}_j - \Gamma_l{}^i{}_p\, \Gamma_k{}^p{}_j \\ & -(\Gamma_k{}^p{}_l - \Gamma_l{}^p{}_k - t_k{}^p{}_l)\, \Gamma_p{}^i{}_j. \end{aligned} \tag{2.3}$$

The last term on the right hand side of the equation above can also be expressed in terms of the commutator of the frame fields because $[e_k, e_l] = (\Gamma_k{}^p{}_l - \Gamma_l{}^p{}_k - t_k{}^p{}_l)\, e_p$ by (2.2). The metric is torsion free if and only if the torsion tensor vanishes, which is the case if and only if

$$(\nabla_j \nabla_k - \nabla_k \nabla_j)\, f = 0, \tag{2.4}$$

for any C^2-function f.

The torsion and the curvature tensor satisfy in general the Bianchi identities

$$\sum_{cycl(ijl)} \nabla_i \, t_j{}^k{}_l = \sum_{cycl(ijl)} (r^k{}_{ijl} - t_i{}^m{}_j \, t_m{}^k{}_l), \tag{2.5}$$

$$\sum_{cycl(ijl)} \nabla_i \, r^h{}_{kjl} = \sum_{cycl(ijl)} t_j{}^m{}_i \, r^h{}_{kml}, \tag{2.6}$$

where the sums are performed after a cyclic permutation of the indices i, j, l.

Assume now that the metric g is torsion free and related by a conformal rescaling $g = \Omega^2 \, \tilde{g}$ with a conformal factor Ω to a 'physical' metric $\tilde{g}$ which satisfies Einstein's vacuum field equations. These equations can then be expressed in terms of g and Ω and derived fields as follows. We write

$$R_{ijkl} = C_{ijkl} + 2\,\{g_{i[k}\, L_{l]j} + L_{i[k}\, g_{l]j}\},$$

where C_{ijkl} is the conformal Weyl tensor and

$$L_{ij} = \frac{1}{2}\,(S_{ij} + \frac{1}{12}\,R\,g_{ij}) \quad \text{with} \quad S_{ij} = R_{ij} - \frac{1}{4}\,R\,g_{ij},$$

denotes the Schouten tensor of g with Ricci tensor R_{kl} and Ricci scalar R. In terms of the tensor fields

$$\Omega, \quad g_{ij} = \eta_{ij}, \quad L_{ij}, \quad W^i{}_{jkl} = \Omega^{-1}\, C^i{}_{jkl}, \quad \Pi = \frac{1}{4}\,\nabla_i \nabla^i\,\Omega + \frac{1}{24}\,R\,\Omega,$$

the (metric) *conformal field equations* read ([3,4])

$$6\,\Omega\,\Pi - 3\,\nabla_i\Omega\,\nabla^i\Omega = 0,$$
$$\nabla_j\,\nabla_k\,\Omega = -\Omega\,L_{jk} + \Pi\,g_{jk},$$
$$\nabla_l\,\Pi = -\nabla^k\Omega\,L_{kl},$$
$$\nabla_i\,L_{jk} - \nabla_j\,L_{ik} = \nabla_l\Omega\,W^l{}_{kij},$$
$$\nabla_i\,W^i{}_{jkl} = 0.$$

These equations must be complemented by the structural equations, namely the *torsion-free condition*

$$t_k{}^i{}_l = 0, \tag{2.7}$$

and the equation

$$r^i{}_{jkl} = R^i{}_{jkl}, \tag{2.8}$$

which will be referred to as the *Ricci identity*.

We note that with the choice $\Omega \equiv 1$ the conformal field equations reduce to the vacuum field equations. The only non-trivial fields are then $e^\mu{}_k$, $\Gamma_i{}^j{}_k$, and $W^i{}_{jkl} = C^i{}_{jkl}$ and the only non-trivial equations are the vacuum Bianchi identity $\nabla_i\,W^i{}_{jkl} = 0$ and the structural equations.

In the case of a more general conformal factor the equation $6\,\Omega\,\Pi - 3\,\nabla_i\Omega\,\nabla^i\Omega = 0$ will be satisfied on the connected component C_q of a point q if it holds at q and the other equations are satisfied on C_q. This is a consequence of the fact that the other equations imply the relation

$$\nabla_k(6\,\Omega\,\Pi - 3\,\nabla_j\Omega\,\nabla^j\Omega) = 0.$$

In the situations considered here, in which either $\Omega = 0$, $\nabla_i\Omega = 0$ or $\Omega \equiv 1$ at the point p, the equation $6\,\Omega\,\Pi - 3\,\nabla_i\Omega\,\nabla^i\Omega = 0$ need not be considered any longer.

3. The 2-Index Spinor Representation

The 2-index spin frame formalism is well adapted to the null geometry and will simplify our algebraic task considerably. It amounts essentially to taking complex linear combinations of various expressions in terms of maps of the form

$$T_{ijk\ldots} \to T_{AA'BB'CC'\ldots} = T_{ijk\ldots}\,\alpha^i{}_{AA'}\,\alpha^j{}_{BB'}\,\alpha^k{}_{CC'}\,\ldots, \tag{3.1}$$

where the α's denote the constant van der Waerden symbols

$$\alpha^i{}_{AA'} = \frac{1}{\sqrt{2}}\begin{pmatrix} \delta^i_0 + \delta^i_3 & \delta^i_1 - i\,\delta^i_2 \\ \delta^i_1 + i\,\delta^i_2 & \delta^i_0 - \delta^i_3 \end{pmatrix},$$

which are hermitian matrices so that $\overline{\alpha^k{}_{AB'}} = \alpha^k{}_{BA'}$. Frame indices $k, l, \ldots$ are thus replaced by pairs of indices AA', BB', ..., where $A, B, \ldots, A', B', \ldots$ take values 0 and 1. None of the operations applied in the following to spinor fields mix primed and unprimed indices. Therefore we shall write $T_{ABC\ldots A'B'C'\ldots}$ instead of $T_{AA'BB'CC'\ldots}$ if convenient. There is an operation of complex conjugation under which unprimed indices are converted into primed indices and vice versa. Because of the hermiticity of the α's the reality of a tensor $T_{ijk\ldots}$ is then expressed by the relation

$$\overline{T_{AA'BB'CC'\ldots}} = T_{AA'BB'CC'\ldots}.$$

These tensor fields are considered as members of a tensor algebra which is generated by a 2-dimensional complex vector space and its primed version, both being related to each other by an operation of complex conjugation. The members of these spaces are called spinors. For more details (not in all cases employing the same curvature conventions as used here) we refer to [11].

The e_k are also replaced by $e_{AA'} = \alpha^k{}_{AA'}\,e_k$ so that the indices A, A' specify in this case the frame vector fields. Then $e_{00'}, e_{11'}$ are real and $e_{01'}, e_{10'}$ are complex (conjugate) null vector fields with scalar products

$$g(e_{AA'}, e_{BB'}) = \eta_{jk}\,\alpha^j{}_{AA'}\,\alpha^k{}_{BB'} = \epsilon_{AB}\,\epsilon_{A'B'}, \tag{3.2}$$

where $\epsilon_{AC}, \epsilon_{A'C'}, \epsilon^{AC}, \epsilon^{A'C'}$ denote the anti-symmetric spinor fields with $\epsilon_{01} = \epsilon_{0'1'} = \epsilon^{01} = \epsilon^{0'1'} = 1$, so that, assuming the summation rule for primed and unprimed indices separately, $\epsilon_A{}^B = \epsilon_{AC}\,\epsilon^{BC}$ and $\epsilon_{A'}{}^{B'} = \epsilon_{A'C'}\,\epsilon^{B'C'}$ denote Kronecker spinors. The ϵ's are use to raise and lower indices according to the rules

$$\kappa^A = \epsilon^{AB}\,\kappa_B, \quad \kappa_B = \kappa^A\,\epsilon_{AB},$$

and similar rules apply to primed indices. Upper frame indices can be converted into spinor indices by the van der Waerden symbols $\alpha_i{}^{AA'} = \eta_{ij}\,\epsilon^{AB}\,\epsilon^{A'B'}\,\alpha^j{}_{BB'}$.

Though it will occasionally be convenient to go back to the standard frame notation (or to employ a hybrid notation as discussed below), we shall assume most of the time the fields (except the frame and the spin frame) to be given by their components with respect to a suitably chosen spin frame field $\{\iota_A\}_{A=0,1}$ which is normalized such that

$$\epsilon(\iota_A, \iota_B) = \epsilon_{AB}, \tag{3.3}$$

where ϵ denotes the antisymmetric form on spinor space. As discussed in detail in [11], the fields $e_{00'} = \iota_0\,\bar{\iota}_{0'}$ and $e_{11'} = \iota_1\,\bar{\iota}_{1'}$ correspond to real null vector fields while

$e_{01'} = \iota_0\,\bar{\iota}_{1'}$ and $e_{10'} = \iota_1\,\bar{\iota}_{0'}$ correspond to complex (conjugate) null vector fields which have the scalar products (3.2) as a consequence of (3.3).

We set $\Gamma_{AA'}{}^{BB'}{}_{CC'} = \Gamma_i{}^j{}_k\,\alpha^i{}_{AA'}\,\alpha_j^{BB'}\,\alpha^k{}_{CC'}$. As a consequence of the anti-symmetry $\Gamma_{ijk} = -\Gamma_{ikj}$ these connection coefficients can be decomposed in the form

$$\Gamma_{AA'}{}^{BB'}{}_{CC'} = \Gamma_{AA'}{}^B{}_C\,\epsilon_{C'}{}^{B'} + \bar{\Gamma}_{AA'}{}^{B'}{}_{C'}\,\epsilon_C{}^B,$$

with spin connection coefficients $\Gamma_{AA'}{}^B{}_C = \frac{1}{2}\,\Gamma_{AA'}{}^{BE'}{}_{CE'}$ that satisfy $\Gamma_{AA'BC} = \Gamma_{AA'(BC)}$. Covariant derivatives of spinor fields κ^A resp. $\pi^{A'}$ are then defined by

$$\nabla_{AA'}\kappa^B = e^\mu_{AA'}\partial_\mu\,\kappa^B + \Gamma_{AA'}{}^B{}_C\,\kappa^C, \quad \nabla_{AA'}\pi^B = e^\mu_{AA'}\partial_\mu\,\pi^{B'} + \bar{\Gamma}_{AA'}{}^{B'}{}_{C'}\,\pi^{C'},$$

and the definition of the covariant derivative is extended to arbitrary spinor fields by requiring the Leibniz rule for spinor products. For the commutators of covariant derivatives we get

$$(\nabla_{CC'}\nabla_{DD'} - \nabla_{DD'}\nabla_{CC'})\,\kappa^A = R^A{}_{BCC'DD'}\,\kappa^B, \tag{3.4}$$

and its complex conjugate, where $R_{ABCC'DD'} = R_{(AB)CC'DD'}$ denotes the curvature spinor. The usual curvature tensor describing the commutator of covariant derivatives acting of vector field is then given by

$$\begin{aligned} R^{AA'}{}_{BB'CC'DD'} &= R^i{}_{jkl}\,\alpha_i{}^{AA'}\,\alpha^j{}_{BB'}\,\alpha^k{}_{CC'}\,\alpha^l{}_{DD'} \\ &= R^A{}_{BCC'DD'}\,\epsilon_{B'}{}^{A'} + \bar{R}^{A'}{}_{B'CC'DD'}\,\epsilon_B{}^A. \end{aligned} \tag{3.5}$$

The curvature spinor admits a decomposition of the form

$$R_{ABCC'DD'} = \Psi_{ABCD}\,\epsilon_{C'D'} + \Phi_{ABC'D'}\,\epsilon_{CD} + 2\,\Lambda\;\epsilon_{A(C}\,\epsilon_{D)B}\,\epsilon_{C'D'}. \tag{3.6}$$

The different components are the Weyl spinor

$$\Psi_{ABCD} = \Psi_{(ABCD)} = -C_{ijkl}\,\alpha^i{}_{AE'}\,\alpha^j{}_B{}^{E'}\,\alpha^k{}_{CF'}\,\alpha^l{}_D{}^{F'},$$

which contains the information on the conformal Weyl tensor, given by

$$C_{AA'BB'CC'DD'} = -\Psi_{ABCD}\,\epsilon_{A'B'}\,\epsilon_{C'D'} - \bar{\Psi}_{A'B'C'D'}\,\epsilon_{AB}\,\epsilon_{CD},$$

and the spinor

$$\Phi_{ABA'B'} = \Phi_{(AB)(A'B')} = \bar{\Phi}_{ABA'B'} = \frac{1}{2}\,(R_{jk} - \frac{1}{4}\,R\,\eta_{jk})\,\alpha^j{}_{AA'}\,\alpha^k{}_{BB'},$$

which represents the trace free part of the Ricci tensor, and

$$\Lambda = \bar{\Lambda} = \frac{1}{24}\,R.$$

It holds then

$$L_{ABA'B'} = \Phi_{ABA'B'} + \Lambda\,\epsilon_{AB}\,\epsilon_{A'B'},$$

and the rescaled conformal Weyl tensor $W^i{}_{jkl} = \Omega^{-1}\,C^i{}_{jkl}$ is represented by the rescaled Weyl spinor

$$\psi_{ABCD} = \Omega^{-1}\,\Psi_{ABCD}.$$

With this notation the conformal field equations read

$$\nabla_{AA'}\,\Pi = -\nabla^{BB'}\Omega\,(\Phi_{ABA'B'} + \Lambda\,\epsilon_{AB}\,\epsilon_{A'B'}),$$
$$\nabla_{AA'}\,\nabla_{BB'}\,\Omega = -\Omega\,(\Phi_{ABA'B'} + \Lambda\,\epsilon_{AB}\,\epsilon_{A'B'}) + \Pi\,\epsilon_{AB}\,\epsilon_{A'B'},$$
$$\nabla_A{}^{D'}\,\Phi_{BCB'D'} + 2\,\epsilon_{A(B}\,\nabla_{C)B'}\,\Lambda = \psi_{ABCD}\,\nabla^D{}_{B'}\Omega,$$
$$\nabla^D{}_{B'}\,\psi_{ABCD} = 0.$$

and the structural equations take the form

$$0 = e^\mu{}_{AA',\,\nu}\,e^\nu{}_{BB'} - e^\mu{}_{BB',\,\nu}\,e^\nu{}_{AA'} - (\Gamma_{BB'}{}^{CC'}{}_{AA'} - \Gamma_{AA'}{}^{CC'}{}_{BB'})\,e^\mu{}_{CC'},$$
$$r^A{}_{BCC'DD'} = \Omega\,\psi^A{}_{BCD}\,\epsilon_{C'D'} + \Phi^A{}_{BC'D'}\,\epsilon_{CD} + 2\,\Lambda\,\epsilon^A{}_{(C}\,\epsilon_{D)B}\,\epsilon_{C'D'}.$$

where

$$r^A{}_{BCC'DD'} = \Gamma_{DD'}{}^A{}_{B,\,\mu}\,e^\mu{}_{CC'} - \Gamma_{CC'}{}^A{}_{B,\,\mu}\,e^\mu{}_{DD'} + \Gamma_{CC'}{}^A{}_F\,\Gamma_{DD'}{}^F{}_B$$
$$-\Gamma_{DD'}{}^A{}_F\,\Gamma_{CC'}{}^F{}_B - (\Gamma_{CC'}{}^{FF'}{}_{DD'} - \Gamma_{DD'}{}^{FF'}{}_{CC'})\,\Gamma_{FF'}{}^A{}_B. \quad (3.7)$$

In the case of the vacuum field equations, in which $\Omega \equiv 1$, the non-trivial unknowns are given by $e^\mu{}_{AA'}$, $\Gamma_{AA'}{}^B{}_C$, ψ_{ABCD} and the field equations reduce to $\nabla^D{}_{B'}\,\psi_{ABCD} = 0$ and the structural equations.

The following observations will become important later. Forget the meaning of the fields considered above and let the spinor field $R_{ABCC'DD'}$ in (3.6) be given by spinor fields Ψ_{ABCD}, $\Phi_{ABC'D'}\,\epsilon_{CD}$, and Λ which satisfy the symmetries and reality conditions stated above. The tensor $R^{AA'}{}_{BB'CC'DD'}$ defined by (3.5) then satisfies the analogue of the first Bianchi identity $R^i{}_{[jkl]} = 0$ as a consequence of the symmetries and reality conditions. In fact, the anti-symmetric tensor $\epsilon_{ijkl} = \epsilon_{[ijkl]}$ with $\epsilon_{0123} = 1$ has the spinor representation

$$\epsilon_{AA'BB'CC'DD'} = i\,(\epsilon_{AC}\,\epsilon_{BD}\,\epsilon_{A'D'}\,\epsilon_{B'C'} - \epsilon_{AD}\,\epsilon_{BC}\,\epsilon_{A'C'}\,\epsilon_{B'D'}),$$

which implies

$$R_{AA'BB'CC'DD'}\,\epsilon_{EE'}{}^{BB'CC'DD'} = 2\,i\,(R_{AH\;EA'}{}^H{}_{E'} - \bar{R}_{A'H'\;AE'E}{}^{H'}) = 0, \quad (3.8)$$

because

$$R_{AH\;EA'}{}^H{}_{E'} = \Phi_{AEA'E'} - 3\,\Lambda\,\epsilon_{AE}\,\epsilon_{A'E'} = \bar{R}_{A'H'\;AE'E}{}^{H'}.$$

An analogue of the second Bianchi identity $\nabla_{[m}R^{ij}{}_{kl]} = 0$ follows under suitable assumptions. It holds

$$\nabla_{EE'}R_{AA'BB'CC'DD'}\,\epsilon^{EE'}{}_{FF'}{}^{CC'DD'}$$
$$= 2\,i\,(\epsilon_{B'A'}\,\nabla^{CD'}R_{ABCF'FD'} + \epsilon_{BA}\,\nabla^{CD'}\bar{R}_{A'B'CF'FD'}), \quad (3.9)$$

and, with $\Psi_{ABCD} = \Omega\,\psi_{ABCD}$,

$$\nabla^{CD'}R_{ABCF'FD'} = -\Omega\;\nabla^C{}_{F'}\psi_{ABCF}$$
$$+\left\{\nabla_F{}^{D'}\Phi_{ABF'D'} + 2\,\epsilon_{F(A}\,\nabla_{B)F'}\Lambda - \nabla^C{}_{F'}\Omega\;\psi_{ABCF}\right\}, \quad (3.10)$$

which will vanish if the conformal field equations are satisfied. These relations are not surprising, because the Bianchi identities have in fact been used to derive the symmetry properties of the curvature spinors and also the conformal field equations. Later on we shall need to consider the last two relations, however, under circumstances in which it is not clear, whether the conformal field equations hold.

To shorten the following expressions it will be convenient to introduce some additional notation. In the case of spinor fields which carry pairs of spinor indices like AA' which correspond to a standard frame indices j we shall occasionally employ a hybrid notation by using the index j, so that equation (3.7) takes for instance the form

$$r^A{}_{Bij} = \Gamma_j{}^A{}_{B,\mu}\, e^\mu{}_i - \Gamma_i{}^A{}_{B,\mu}\, e^\mu{}_j + \Gamma_i{}^A{}_F\, \Gamma_j{}^F{}_B \\ - \Gamma_j{}^A{}_F\, \Gamma_i{}^F{}_B - (\Gamma_i{}^k{}_j - \Gamma_j{}^k{}_i)\, \Gamma_k{}^A{}_B.$$

The symmetric part of a spinor field $S_{AB\ldots EF}$ is denoted by $S_{(AB\ldots EF)}$. The *totally symmetric part* of a spinor field $T_{A_1\ldots A_k B'_1\ldots B'_j}$ is then given by $T_{(A_1\ldots A_k)(B'_1\ldots B'_j)}$. If T is a spinor field and $\mathbf{n} = (i_1, \ldots, i_n)$ a multi-index of order $|\mathbf{n}| = n$ we write $\nabla_{\mathbf{n}} T = \nabla_{i_1} \ldots \nabla_{i_n} T$ and $\nabla_{(\mathbf{n})} T = \nabla_{(i_1} \ldots \nabla_{i_n)} T$. If X^i is a vector field we set $X^{\mathbf{n}} = X^{i_1} \ldots X^{i_n}$ and write $X^{i_1} \ldots X^{i_n}\, \nabla_{i_1} \ldots \nabla_{i_n} T = X^{\mathbf{n}}\, \nabla_{\mathbf{n}} T = X^{\mathbf{n}}\, \nabla_{(\mathbf{n})} T$.

4. Gauge Conditions

Unless stated otherwise the connection ∇ will be assumed in the following to be g-compatible and torsion free. We need to restrict the gauge freedom for the conformal factor, the frame, and the coordinates.

The conformal gauge near i^-.

The data for the conformal field equations are to be prescribed on the cone $\mathcal{N}_p = \mathcal{J}^- \cup \{i^-\}$. The vertex $p = i^-$ is to represent past time-like infinity and $\mathcal{N}_p$ is thought to be generated by the future directed null geodesics starting at p. Thus one must assume that

$$\Omega = 0, \quad \nabla_{AA'}\Omega = 0, \quad \Pi \neq 0 \quad \text{at} \quad p.$$

The equations $\nabla_j\, \nabla_k\, \Omega = -\Omega\, L_{jk} + \Pi\, g_{jk}$ and $\nabla_l\, \Pi = -\nabla^k \Omega\, L_{kl}$ suitably transvected with the geodesic null vectors tangent to the null generators of $\mathcal{N}_p$ imply then that

$$\Omega = 0 \quad \text{and} \quad \Pi \neq 0 \quad \text{on} \quad \mathcal{N}_p, \qquad \nabla_j \Omega \neq 0 \quad \text{on} \quad \mathcal{N}_p \setminus \{p\}.$$

(Note that the assumption $\Pi|_p = 0$ would imply that $\nabla_j \Omega = 0$ on $\mathcal{N}_p$).

The sign of Π depends on the signature of g. The equation $\nabla_\mu\, \nabla_\nu\, \Omega = -\Omega\, L_{\mu\nu} + \Pi\, g_{\mu\nu}$ implies for a future directed time-like geodesics γ starting at p the relation $\Pi\, g(\gamma', \gamma')|_p = \nabla_{\gamma'} \nabla_{\gamma'} \Omega|_p$. If we want this to be positive we must assume that $sign(\Pi) = sign(g(\gamma', \gamma')) = sign(\eta_{00})$ at i^-. This discussion shows that with the assumptions above on Ω and Π at p the field equations themselves will take care for the conformal factor Ω to evolve so that it will show near p the desired behaviour on $\mathcal{N}_p$ and on the physical space-time region $I^+(\mathcal{N}_p)$.

Under a rescaling $g_{\mu\nu} \to \hat{g}_{\mu\nu} = \theta^2\, g_{\mu\nu}$, $\Omega \to \hat{\Omega} = \theta\, \Omega$ with some function $\theta > 0$ it follows

$$\Pi|_p \to \hat{\Pi}|_p = (\Pi\, \theta^{-1})|_p.$$

The transformation laws

$$R_{\mu\nu}[g] \to R_{\mu\nu}[\hat{g}] = R_{\mu\nu}[g] - 2\,\theta^{-1}\,\nabla_\mu\,\nabla_\nu\,\theta + 4\,\theta^{-2}\,\nabla_\mu\,\theta\,\nabla_\nu\,\theta$$
$$-\{\theta^{-1}\,\nabla_\lambda\,\nabla^\lambda\,\theta + \theta^{-2}\,\nabla_\lambda\,\theta\,\nabla^\lambda\,\theta\}\,g_{\mu\nu},$$

and

$$R[g] \to R[\hat{g}] = \theta^{-2}\,\{R[g] - 6\,\theta^{-1}\,\nabla_\lambda\,\nabla^\lambda\,\theta\}, \tag{4.1}$$

of the Ricci tensor and the Ricci scalar imply the transformation behaviour

$$S_{\mu\nu}[g] \to S_{\mu\nu}[\hat{g}] = S_{\mu\nu}[g] - 2\,\theta^{-1}\,\nabla_\mu\,\nabla_\nu\,\theta + 4\,\theta^{-2}\,\nabla_\mu\,\theta\,\nabla_\nu\,\theta$$
$$+\frac{1}{2}\,\{\theta^{-1}\,\nabla_\lambda\,\nabla^\lambda\,\theta - 2\,\theta^{-2}\,\nabla_\lambda\,\theta\,\nabla^\lambda\,\theta\}\,g_{\mu\nu}.$$

Let $l^\mu \neq 0$ denote the tangent vector of a future directed null geodesics $\gamma(\tau)$ on $\mathcal{J}_p$ with $\gamma(0) = p$, so that $\nabla_l l = 0$. Then $\hat{l} = \theta^{-2}\,l$ satisfies $\hat{g}(\hat{l}, \hat{l}) = 0$, $\hat{\nabla}_{\hat{l}}\hat{l} = 0$. This gives

$$\theta^4\,\hat{l}^\mu\,\hat{l}^\nu\,S_{\mu\nu}[\hat{g}] = l^\mu\,l^\nu S_{\mu\nu}[g] - 2\,\theta^{-1}\,(l^\mu\nabla_\mu)^2\,\theta + 4\,\theta^{-2}\,(l^\mu\,\nabla_\mu\,\theta)^2,$$

or equivalently

$$\hat{l}^\mu\,\hat{l}^\nu\,S_{\mu\nu}[\hat{g}]\,\theta^3 = l^\mu\,l^\nu\,S_{\mu\nu}[g]\,\theta^{-1} + 2\,(l^\mu\nabla_\mu)^2\,(\theta^{-1}). \tag{4.2}$$

For prescribed value of $\hat{l}^\mu\,\hat{l}^\nu\,S_{\mu\nu}[\hat{g}]$ this represents an ODE for θ along the null generator tangent to l. While the value of θ can be fixed at p by specifying there the value of $|\Pi|$, there remains the freedom to specify the value of $\nabla_\mu\theta$ at p. The equations above suggest that a convenient conformal gauge can be defined in a neighbourhood of p in $J^+(\mathcal{N}_p)$ by requiring

$$\Omega = 0, \quad \nabla_\mu\Omega = 0, \quad \Pi = 2\,\eta_{00} \quad \text{at} \quad p, \tag{4.3}$$

and

$$l^\mu\,l^\nu\,S_{\mu\nu}[g] = 0 \;\text{ on }\; \mathcal{N}_p \;\text{ near } p, \quad R[g] = 0 \;\text{ on }\; J^+(\mathcal{N}_p) \text{ near } p. \tag{4.4}$$

This conformal gauge will be assumed in the following without any problem. When this type of conformal gauge is used in a wider context, however, it is important to know that for a *given* smooth background g equation (4.2) with $\hat{l}^\mu\,\hat{l}^\nu\,S_{\mu\nu}[\hat{g}] = 0$ yields a rescaling factor θ on $\mathcal{N}_p$ which has the appropriate smoothness behaviour on $\mathcal{N}_p$ near the vertex p so that the wave equation obtained on the right hand side of (4.1) by setting $R[\hat{g}] = 0$ can be solved with these data on $\mathcal{N}_p$ for a smooth function θ near p. This question will be discussed in the article [2].

The choice of the coordinates near i^-.

We shall consider p-centered g-normal coordinates x^μ near p. These are determined by the requirements that $x^\mu(p) = 0$, that $g_{\mu\nu}(0) = \eta_{\mu\nu}$, and that for given $x^\mu \neq 0$ and a real parameter τ with $|\tau|$ small enough, the curve $\gamma : \tau \to \tau\,x^\mu$ is a geodesic through the point p. If $g_{\mu\nu}$ and $\Gamma_\mu{}^\rho{}_\nu$ denote the metric coefficients and the Christoffel symbols in the coordinates x^μ, the latter condition is equivalent to

$$0 = 2\,g_{\mu\rho}\,(\nabla_{\gamma'}\gamma')^\rho = 2\,g_{\mu\rho}\,x^\nu\,\Gamma_\nu{}^\rho{}_\lambda(\tau\,x)\,x^\lambda = 2\,x^\nu\,g_{\nu\mu,\lambda}(\tau\,x)\,x^\lambda - x^\nu\,x^\lambda\,g_{\nu\lambda,\mu}(\tau, x),$$

which gives in particular that

$$x^{\nu}\,\Gamma_{\nu}{}^{\rho}{}_{\lambda}(\tau\,x)\,x^{\lambda}=0, \tag{4.5}$$

for small enough $|\tau|$. The first equation above implies further $0=x^{\nu}\,x^{\mu}\,g_{\nu\mu,\lambda}(\tau\,x)\,x^{\lambda}=\frac{d}{d\tau}\left(x^{\nu}\,x^{\mu}\,g_{\nu\mu}(\tau\,x)\right)$ and thus $x^{\nu}\,x^{\mu}\,g_{\nu\mu}(\tau\,x)=x^{\nu}\,x^{\mu}\,g_{\nu\mu}(0)$, whence

$$2\,x^{\nu}\,g_{\nu\mu}(\tau\,x)+\tau\,x^{\nu}\,x^{\lambda}\,g_{\nu\lambda,\mu}(\tau\,x)=2\,x^{\nu}\,g_{\nu\mu}(0).$$

With the first equations it follows then

$$0=\tau\,2\,x^{\nu}\,g_{\nu\mu,\lambda}(\tau\,x)\,x^{\lambda}-\tau\,x^{\nu}\,x^{\lambda}\,g_{\nu\lambda,\mu}(\tau,x)=2\,\frac{d}{d\tau}\left\{\tau\left(x^{\nu}\,g_{\nu\mu}(\tau\,x)-x^{\nu}\,g_{\nu\mu}(0)\right)\right\},$$

and thus

$$x^{\nu}\,g_{\nu\mu}(\tau\,x)=x^{\nu}\,g_{\nu\mu}(0). \tag{4.6}$$

This equation implies in turn $x^{\nu}\,x^{\mu}\,g_{\nu\mu}(\tau\,x)=x^{\nu}\,x^{\mu}\,g_{\nu\mu}(0)$ which gives by differentiation $\tau\,x^{\nu}\,x^{\lambda}\,g_{\nu\lambda,\mu}(\tau\,x)=-2\,x^{\nu}\,g_{\nu\mu}(\tau\,x)+2\,x^{\nu}\,g_{\nu\mu}(0)=0$. Because differentiation of (4.6) with respect to τ gives $0=x^{\nu}\,g_{\nu\mu,\lambda}(\tau\,x)\,x^{\lambda}$ we see that (4.6) implies that the curves γ considered above are in fact geodesics. The relation (4.6) thus completely characterizes normal coordinates in terms of algebraic conditions on the metric coefficients. It follows from the equations above that $g_{\mu\nu,\,\rho}(p)=0$, $\Gamma_{\mu}{}^{\rho}{}_{\nu}(p)=0$.

In this gauge $\mathcal{N}_p$ is now given by the set $\{x^{\mu}\in\mathbb{R}^4|\ \eta_{\mu\nu}\,x^{\mu}\,x^{\nu},\ \ x^0\geq 0\}$.

The choice of the frame near i^-.

Assume now that p-centered g-normal coordinates x^{μ} are given on a convex normal neighbourhood U' of p and take their values in a neighbourhood U of the origin of $\mathbb{R}^4$. A frame $\{e_k\}_{k=0,1,2,3}$ is called a *normal frame centered at p* if it satisfies on U'

$$g(e_j,\,e_k)=\eta_{jk},\quad \text{and}\quad \nabla_{\gamma'}e_k=0,$$

for any geodesic γ passing through p. The frame coefficients satisfying $e_k=e^{\mu}{}_k\,\partial_{\mu}$ are assumed to satisfy

$$e^{\mu}{}_k(0)=\delta^{\mu}_k.$$

The 1-forms dual to e_k will be denoted by σ^j. Then $\sigma^j=\sigma^j{}_{\nu}\,dx^{\nu}$ with $\sigma^j{}_{\mu}\,e^{\mu}{}_k=\delta^j_k$. That the frame field depends in fact smoothly on the coordinates x^{μ} follows by arguments known from the discussion of the exponential function.

The equation $x^{\nu}\,g_{\nu\mu}(\tau\,x)\,e^{\mu}{}_k(\tau\,x)=x^{\nu}\,\eta_{\nu\mu}\,\delta^{\mu}_k$ expresses that the scalar product $g(\gamma',e_k)$ is constant along the geodesic γ. The representation $g_{\mu\nu}=\eta_{ij}\,\sigma^i{}_{\mu}\,\sigma^j{}_{\nu}$ allows us to rewrite it in the form

$$x^{\mu}\,\sigma^j{}_{\mu}(\tau\,x)=x^{\mu}\,\delta^j_{\mu}\quad \text{resp.}\quad x^{\mu}\,\delta^j_{\mu}\,e^{\nu}{}_j(\tau\,x)=x^{\nu}. \tag{4.7}$$

With this relation equation (4.6) implies

$$x^{\mu}\,\eta_{\mu\rho}\,\delta^{\rho}_j\,\sigma^j{}_{\nu}(\tau\,x)=x^{\mu}\,\eta_{\mu\nu}\quad \text{resp.}\quad x^{\mu}\,\eta_{\mu\nu}\,e^{\nu}{}_k(\tau\,x)=x^{\mu}\,\eta_{\mu\nu}\,\delta^{\nu}{}_k. \tag{4.8}$$

If the fields $\sigma^j{}_{\mu}$ and the coordinates x^{μ} satisfy the last two relations, it follows without further assumptions that the metric $g_{\mu\nu}=\eta_{ij}\,\sigma^i{}_{\mu}\,\sigma^j{}_{\nu}$ satisfies (4.6). In terms of the frame field, the information that the x^μ are normal coordinates is thus encoded in (4.7), (4.8).

Writing $\nabla_i \equiv \nabla_{e_i}$, the connection coefficients $\Gamma_i{}^j{}_k$ with respect to the frame e_j are defined by the relations $\nabla_i e_k = \Gamma_i{}^j{}_k e_j$. They satisfy $\Gamma_{ijk} = -\Gamma_{ikj}$, where $\Gamma_{ijk} = \Gamma_i{}^l{}_k \eta_{lj}$.

The tensor field $X(x) = x^\mu \partial_\mu$ tangential to the geodesics through p is characterized uniquely by the conditions

$$X(p) = 0, \quad \nabla_\mu X^\nu(p) = g_\mu{}^\nu(p), \quad \nabla_X X = X. \tag{4.9}$$

By (4.7) it can be written $X = X^k e_k$ with $X^k(x) = \delta^k_\nu x^\nu$. The relation $\nabla_X e_j = 0$ is equivalent to

$$X^k(x)\,\Gamma_k{}^i{}_j(x) = \delta^k_\nu x^\nu\,\Gamma_k{}^i{}_j(x) = 0, \quad x^\mu \in U, \tag{4.10}$$

or

$$X^{AA'}(x)\,\Gamma_{AA'}{}^B{}_C(x) = 0, \quad x^\mu \in U. \tag{4.11}$$

This is the characterizing property of the normal frame.

In the following we shall refer to coordinates x^μ and a frame e_k (resp. $e_{AA'}$) which satisfy the conditions above as to a *normal gauge*. We shall always assume this to be supplemented by a normalized spin-frame $\{\iota_A\}_{A=0,1}$ which satisfies $e_{AA'} = \iota_A\,\bar{\iota}_{A'}$ and $\nabla_X \iota_A = 0$. All spinor fields will be assumed to be given in this frame.

5. Normal Expansions

Let x^μ and $e_{AA'}$ be given in a normal gauge and let X be the vector field defined by (4.9) so that $X = X^i e_i = X^{AA'} e_{AA'}$ with $X^{AA'}(x) = x^\mu\,\alpha_\mu{}^{AA'}$, where we set $\alpha^{AA'}_\mu = \delta^i{}_\mu\,\alpha_i{}^{AA'}$.

Let T denote a smooth spinor field and $T_{A_1\ldots A_j B'_1\ldots B'_k}$ its components in the normal frame. If $x^\mu_* \neq 0$, then we get with (4.7) and (4.11) along the geodesic $\gamma : \tau \to \tau\, x^\mu_*$

$$\begin{aligned}
\frac{d}{d\tau} T_{A_1\ldots A_j B'_1\ldots B'_k}(\tau\, x_*) &= T_{A_1\ldots A_j B'_1\ldots B'_k,\,\mu}(\tau\, x_*)\, x^\mu_* \\
&= x_*^{CC'} \Big\{ T_{A_1\ldots A_j B'_1\ldots B'_k,\,\mu}(\tau\, x_*)\, e^\mu{}_{CC'}(\tau\, x_*) \\
&\quad - \Gamma_{CC'}{}^{D_1}{}_{A_1}\, T_{D_1\ldots A_j B'_1\ldots B'_k}(\tau\, x_*) \ldots - \bar{\Gamma}_{CC'}{}^{E'_k}{}_{B'_k}\, T_{A_1\ldots A_j B'_1\ldots B'_k}(\tau\, x_*) \Big\} \\
&= x_*^{CC'} \nabla_{CC'} T_{A_1\ldots A_j B'_1\ldots B'_k}(\tau\, x_*)
\end{aligned}$$

with $x_*^{CC'} = x^\mu_*\,\delta^i{}_\mu\,\alpha_i{}^{CC'}$. Applying the argument repeatedly gives

$$\frac{d^n}{d\tau^n} T_{A_1\ldots A_j B'_1\ldots B'_k}(\tau\, x_*) = x_*^{C_1 C'_1} \ldots x_*^{C_n C'_n} \nabla_{C_1 C'_1} \ldots \nabla_{C_n C'_n} T_{A_1\ldots A_j B'_1\ldots B'_k}(\tau\, x_*).$$

Setting $x^\mu = \tau\, x^\mu_*$ in the Taylor expansion

$$T_{A_1\ldots A_j B'_1\ldots B'_k}(\tau\, x_*) = \sum_{n=0}^{N} \frac{1}{n!}\,\tau^n\,\frac{d^n}{d\tau^n} T_{A_1\ldots A_j B'_1\ldots B'_k}(0) + O(|\tau|^{N+1}),$$

the Taylor expansion of $T_{A_1\dots A_j B'_1\dots B'_k}$ at p is obtained in the form

$$T_{A_1\dots A_j B'_1\dots B'_k}(x) = \sum_{|\mathbf{n}|=0}^{N} \frac{1}{|\mathbf{n}|!} X^{\mathbf{n}} \nabla_{\mathbf{n}} T_{A_1\dots A_j B'_1\dots B'_k}(0) + O(|x|^{N+1})$$

$$= \sum_{|\mathbf{n}|=0}^{N} \frac{1}{|\mathbf{n}|!} X^{\mathbf{n}} \nabla_{(\mathbf{n})} T_{A_1\dots A_j B'_1\dots B'_k}(0) + O(|x|^{N+1}). \quad (5.1)$$

This will be referred to as the *normal expansion* of T at p. It will be known once the *symmetrized* covariant derivatives $\nabla_{(\mathbf{n})} T_{A_1\dots A_j B'_1\dots B'_k}(p)$, $|\mathbf{n}| > 0$, are given.

5.1. The null data. The set $C_p \sim S^2$ of future directed null vectors at p satisfying $g(l,l) = 0$ and $g(l, e_0) = \eta_{00}/\sqrt{2}$ defines a parametrization of the null generators of $\mathcal{N}_p$, which are given in the normal gauge by the curves $\tau \to \tau\, l^\mu$, $l^\mu \in C_p$, $0 \le \tau < a$ for some suitable $a > 0$. Denote by $\mathcal{W}_p$ the subset of $\mathcal{N}_p$ which is generated by the null generators parametrized by a proper open subset W of C_p.

Let $\kappa^A(x)$ be a smooth spinor field on $\mathcal{W}_p \setminus \{p\}$ which is parallely propagated along the null generators and such that $\kappa^A\,\bar{\kappa}^{A'}$ is tangent to the null generators of $\mathcal{W}_p$. Because the components κ^A are given in the normal frame they are constant along the null generators. Thus, κ^A assumes a limit as $\tau \to 0$ along the curve $\tau \to \tau\, l^\mu$ and it can be assumed that $\kappa^A\,\bar{\kappa}^{A'} = l^{AA'}$ along that curve. The field κ^A is then determined uniquely up to phase transformations $\kappa^A \to e^{i\,\phi}\,\kappa^A$ with smooth phase factors which are constant along the null generators.

For a given tensor field T with spin frame components $T_{A_1\,\dots\,A_j B'_1\,\dots\,B'_k}$ we define its *null datum* on $\mathcal{W}_p$ as the spin weighted function

$$T_0(x) = \kappa^{A_1}(x)\dots\kappa^{A_j}(x)\bar{\kappa}^{B'_1}(x)\dots\bar{\kappa}^{B'_k}(x)\, T_{A_1\,\dots\,A_j B'_1\,\dots\,B'_k}(x), \quad x^\mu \in \mathcal{W}_p \setminus \{p\}.$$

With the normal expansion for T given above this gives at p the asymptotic representation

$$T_0(\tau\, x) = \sum_{n=0}^{N} \frac{\tau^n}{n!}\,\kappa^{C_1}\,\dots\,\bar{\kappa}^{C'_n}\,\kappa^{A_1}\dots\bar{\kappa}^{B'_k}\,\nabla_{C_1C'_1}\,\dots\,\nabla_{C_nC'_n}\,T_{A_1\dots A_j B'_1\dots B'_k}(0) + O(|\tau|^{N+1}).$$

for $\tau > 0$. The sum is determined uniquely by the coefficients

$$\tilde{T}_n(\kappa) = \kappa^{C_1}\,\dots\,\bar{\kappa}^{C'_n}\,\kappa^{A_1}\,\dots\,\bar{\kappa}^{B'_k}\,\nabla_{C_1C'_1}\,\dots\,\nabla_{C_nC'_n}\,T_{A_1\dots A_j B'_1\dots B'_k}(0).$$

Because the directions $\kappa^A\,\bar{\kappa}^{A'} = l^{AA'}$ are allowed to vary in the open subset W of C_p, knowing these coefficients is equivalent to knowing the symmetrized derivatives

$$T_{(A_1\dots A_j)}{}^{(B'_1\dots B'_k)}(0), \quad \nabla_{(C_1}{}^{(C'_1}\dots\nabla_{C_n}{}^{C'_n}\,T_{A_1\dots A_j)}{}^{B'_1\dots B'_k)}(0), \quad n = 1, 2, \dots. \quad (5.2)$$

In fact, let $S_{A_1\dots A_p\,A'_1\dots A'_q} = S_{(A_1\dots A_p)\,(A'_1\dots A'_q)}$ be a symmetric spinor. It will be known once its 'essential components', denoted by $S_{ij} = S_{(A_1\dots A_p)_i\,(A'_1\dots A'_q)_j}$, are known, which are obtained by setting for given integers i, j, with $0 \le i \le p$, $0 \le j \le q$, precisely i unprimed resp. j primed indices to equal to one. Choose $(\kappa^0, \kappa^1) = \beta\,(1, z)$ with $z \in \mathbb{C}$

and the factor $\beta = (1 + |z|^2)^{-1/2}$ which ensures the normalization condition on l^μ. If the function $S(\kappa) = \kappa^{A_1} \dots \bar{\kappa}^{A'_q} \, S_{A_1 \dots A_p \, A'_1 \dots A'_q}$ is known then also the function

$$\beta^{-p-q} \, S(\kappa) = \sum_{i=0}^{p} \sum_{j=0}^{q} \binom{p}{i} \binom{q}{j} \, S_{ij} \, z^i \, \bar{z}^j,$$

and the essential components are given by $S_{ij} = \frac{(p-i)!}{p!} \frac{(q-j)!}{q!} \, \partial_z^i \, \partial_{\bar{z}}^j \, (\beta^{-p-q} \, S(\kappa))|_{z=0}$.

While the *null datum on* $\mathcal{W}_p$ is a spin weighted function which depends on the choice of κ^A, the spinors (5.2) at p are given with respect to the spin-frame ι_A and are independent of any phase factors. They will be referred to as to the *null data of* T *at* p.

Of particular importance will be for us the null datum

$$\psi_0 = \kappa^A \, \kappa^B \, \kappa^C \, \kappa^D \, \psi_{ABCD}, \tag{5.3}$$

associated with the rescaled conformal Weyl spinor ψ_{ABCD}. It is referred to as the *radiation field*.

To illustrate some of its properties it will be convenient to proceed as follows. Let $SU(2, \mathbb{C})$ denote the subgroup of transformations $(s^A{}_B)_{A,B=0,1} \in sl(2, \mathbb{C})$ which satisfy $\epsilon_{AC} \, s^A{}_B \, s^C{}_D = \epsilon_{BD}$ and $s^A{}_B \, \bar{s}^{A'}{}_{B'} \, \alpha_0^{BB'} = \alpha_0^{AA'}$. Then the null vectors $l^\mu = l^\mu(s)$ at p with spinor components $l^{AA'} = s^A{}_0 \, \bar{s}^{A'}{}_{0'}$ sweep out the null directions at p and the $m^{AA'} = m^{AA'}(s) = s^A{}_0 \, \bar{s}^{A'}{}_{1'}$ are complex null vectors orthogonal to $l^{AA'}$. By requiring them to be constant along the null generators tangent to $l^{AA'}$ they will be parallely transported and tangent to $\mathcal{N}_p$ along the generators.

The information on the radiation field is equivalent to the information contained in the pull back of the tensor $W_{ijkl} \, l^i \, l^k$ to $\mathcal{N}_p$. In fact, the latter can be specified by the contractions of the symmetric tensor $W_{ijkl} \, l^i \, l^k$ with the field m and $\bar{m}$. Because $W_{ijkl} \, l^i \, m^j \, l^k \, m^l$ and $W_{ijkl} \, l^i \, \bar{m}^j \, l^k \, \bar{m}^l$ are complex conjugates of each other and the trace-freeness of W_{ijkl} implies that $W_{ijkl} \, l^i \, m^j \, l^k \, \bar{m}^l = 0$, the information is stored in $W_{ijkl} \, l^i \, m^j \, l^k \, m^l = s^A{}_0 \, s^B{}_0 \, s^C{}_0 \, s^D{}_0 \, \psi_{ABCD} = \psi_0$. Note that this description includes the complete freedom to perform phase transformations. If this is to be removed, one has to restrict the choice of s to a local section of the Hopf map $SU(2) \ni s \to l^{AA'}(s) \in S^2$, where S^2 is identified with the set of future directed null directions at p.

The *null data of* ψ *at* p can be extracted from the *null datum* ψ *on* $\mathcal{N}_p$ as follows. By taking derivatives with respect to τ at $\tau = 0$ one gets from the null datum the quantities

$$\tilde{\psi}_n(s) = s^{C_1}{}_0 \, \bar{s}^{C'_1}{}_{0'} \dots s^{C_n}{}_0 \, \bar{s}^{C'_n}{}_{0'} s^A{}_0 \, s^B{}_0 \, s^C{}_0 \, s^D{}_0 \, \nabla_{C_1 C'_1} \dots \nabla_{C_n C'_n} \psi_{ABCD}(0).$$

As discussed in detail in [5], these functions on $SU(2, \mathbb{C})$ translate naturally into expansions in terms of the coefficients $T_m{}^i{}_j(s)$ of certain finite unitary representations of the group $SU(2, \mathbb{C})$. With this understanding the essential components of the null data $\nabla_{(C_1}{}^{(C'_1} \dots \nabla_{C_n}{}^{C'_n)} \psi_{ABCD)}(0)$ can be obtained by performing integrals of $\tilde{\psi}_n(s) \, \bar{T}_m{}^i{}_j(s)$ with respect to the Haar measure on $SU(2, \mathbb{C})$. Any ambiguities related to choices of phase factors as indicated above are cancelled out by the integration.

To prescribe the null datum in a way which ensures the necessary smoothness properties, we start with some symmetric spinor field $\psi^*_{ABCD} = \psi^*_{ABCD}(x^\mu)$ which is defined and smooth in a suitable neighbourhood of the origin p of $\mathbb{R}^4$ (so that $x^\mu(p) = 0$).

This field will be thought of as being given in a conformal and normal gauge as described in Sect. 4. Assuming $s^A{}_B$ as above, one can then consider on the cone $\mathcal{N}_p = \{\eta_{\mu\nu}\, x^\mu\, x^\nu = 0,\ x^0 \geq 0\}$ (or more precisely on the bundle $\tilde{\mathcal{N}}_p \sim \mathbb{R}^+_0 \times SU(2)$ over $\mathcal{N}_p$, see Sect. 9) the complex-valued function

$$\psi_0(\tau, s) = s^A{}_0\, s^B{}_0\, s^C{}_0\, s^D{}_0\, \psi^*_{ABCD}(\tau\, \alpha^\mu_{EE'}\, s^E{}_0\, \bar{s}^{E'}{}_{0'}), \tag{5.4}$$

as a 'smooth' radiation field.

The gauge conditions give control on the null data at p for some of the unknowns in the conformal field equations. It follows immediately from the discussion above and the first of conditions (4.4) that the conformal gauge implies

$$\Phi_{AB}{}^{A'B'}(0) = 0, \quad \nabla_{(C_1}{}^{(C'_1} \ldots \nabla_{C_n}{}^{C'_n}\, \Phi_{AB)}{}^{A'B')}(0) = 0, \quad n = 1, 2, \ldots. \tag{5.5}$$

6. Formal Expansions at i^-

In a conformal gauge satisfying (4.4) the conformal field equations read

$$\nabla_{AA'}\, \nabla_{BB'}\, \Omega = -\Omega\, \Phi_{ABA'B'} + \Pi\, \epsilon_{AB}\, \epsilon_{A'B'}, \tag{6.1}$$

$$\nabla_{AA'}\, \Pi = -\nabla^{BB'} \Omega\, \Phi_{ABA'B'}, \tag{6.2}$$

$$\nabla_A{}^{D'}\, \Phi_{BCB'D'} = \psi_{ABCD}\, \nabla^D{}_{B'} \Omega, \tag{6.3}$$

$$\nabla^D{}_{B'}\, \psi_{ABCD} = 0, \tag{6.4}$$

and the curvature spinor (3.6) takes the form

$$R_{ABCC'DD'} = \Omega\, \psi_{ABCD}\, \epsilon_{C'D'} + \Phi_{ABC'D'}\, \epsilon_{CD}. \tag{6.5}$$

The following algebraic considerations will be simplified by rewriting equations (6.3) and (6.4). The symmetry of ψ_{ABCD} and the fact that vanishing spinor contractions indicate index symmetries imply that equation (6.4) is equivalent to

$$\nabla_E{}^{E'}\, \psi_{ABCD} = \nabla_{(E}{}^{E'}\, \psi_{ABCD)}. \tag{6.6}$$

If (6.4) holds, equation (6.3) and its complex conjugate are equivalent to the equations

$$\nabla_A{}^{A'}\, \Phi_{BC}{}^{B'C'} - \nabla_B{}^{A'}\, \Phi_{AC}{}^{B'C'} = -\, \epsilon_{AB}\, \nabla_C{}^{H'} \Omega\, \bar{\psi}^{A'B'C'}{}_{H'}, \tag{6.7}$$

$$\nabla_A{}^{A'}\, \Phi_{BC}{}^{B'C'} - \nabla_A{}^{B'}\, \Phi_{BC}{}^{A'C'} = -\, \epsilon^{A'B'}\, \nabla^{HC'} \Omega\, \psi_{ABCH}. \tag{6.8}$$

With the identity

$$\nabla_A{}^{A'}\, \Phi_{BC}{}^{B'C'} = \nabla_{(A}{}^{(A'}\, \Phi_{BC)}{}^{B'C')} + \frac{2}{3}\, \nabla_{(A}{}^{H'}\, \Phi_{BC)H'}{}^{(B'}\, \epsilon^{C')A'}$$
$$-\frac{2}{3}\, \epsilon_{A(B}\, \nabla^{H(A'}\, \Phi_{C)H}{}^{B'C')} - \frac{4}{9}\, \epsilon_{A(B}\, \nabla^{HH'}\, \Phi_{C)HH'}{}^{(B'}\, \epsilon^{C')A'},$$

these two equations are seen to be equivalent to the equation

$$\nabla_A{}^{A'}\, \Phi_{BC}{}^{B'C'} = \nabla_{(A}{}^{(A'}\, \Phi_{BC)}{}^{B'C')}$$
$$+\frac{2}{3}\, \psi_{ABCH}\, \nabla^{H(B'}\, \Omega\, \epsilon^{C')A'} + \frac{2}{3}\, \epsilon_{A(B}\, \nabla_{C)H'} \Omega\, \bar{\psi}^{A'B'C'H'}. \tag{6.9}$$

We note that

$$\psi_{ABCD}(0), \qquad \nabla_E{}^{E'}\,\psi_{ABCD}(0) = \nabla_{(E}{}^{E'}\,\psi_{ABCD)}(0), \tag{6.10}$$

represent null data of ψ_{ABCD} and that the conformal gauge (4.3), (4.4) implies by (5.5) and (6.9) that

$$\Phi_{BC}{}^{B'C'}(0) = 0, \quad \nabla_A{}^{A'}\,\Phi_{BC}{}^{B'C'}(0) = \nabla_{(A}{}^{(A'}\,\Phi_{BC)}{}^{B'C')}(0) = 0. \tag{6.11}$$

With this it follows from equations (6.2), (6.1) and the gauge conditions that

$$\nabla_{AA'}\nabla_{BB'}\Omega(0) = \Pi(0)\,\epsilon_{AB}\,\epsilon_{A'B'}, \quad \nabla_{\mathbf{k}}\,\Omega(0) = 0 \quad \text{for} \quad |\mathbf{k}| = 0, 1, 3, 4, 5, \tag{6.12}$$

$$\nabla_{\mathbf{k}}\Pi(0) = 0 \quad \text{for} \quad |\mathbf{k}| = 1, 2, 3. \tag{6.13}$$

The relations above imply furthermore that

$$R_{ABCC'DD'}(0) = 0, \qquad \nabla_{EE'}R_{ABCC'DD'}(0) = 0. \tag{6.14}$$

The following result, which relates the formal expansion of the curvature fields at a given point p to the null data of ψ_{ABCD} at p, applies and extends arguments of the theory of *exact sets of fields* discussed in [10,11].

Lemma 6.1. *In a neighbourhood of the point p let the fields Ω, Π, $\Phi_{ABA'B'}$, ψ_{ABCD}, $e^{\mu}{}_{AA'}$, $\Gamma_{AA'}{}^{B}{}_{C}$ be smooth and be given in a p-centered normal gauge for the coordinates and the frame and in a conformal gauge satisfying (4.3), (4.4). Then, if they satisfy the structural equations and the conformal field equations the covariant derivatives of the fields Ω, Π, $\Phi_{ABA'B'}$, ψ_{ABCD} at all orders are determined uniquely at p by the null data $\nabla_{(E_1}{}^{(E'_1} \ldots \nabla_{E_n}{}^{E'_n)}\,\psi_{ABCD)}(p)$, $n \in \mathbb{N}_0$, at p.*

The resulting map, which relates to the null data of ψ at p the covariant derivatives of the fields Ω, Π, $\Phi_{ABA'B'}$, ψ_{ABCD} at p, extends in a unique way so that it associates with any freely specified sequence of totally symmetric spinors

$$\xi_{ABCD}, \quad \xi^{E'_1 \ldots E'_n}_{E_1 \ldots E_n ABCD}, \quad n = 1, 2, 3, \ldots$$

at p formally 'covariant derivatives' the of fields Ω, Π, $\Phi_{ABA'B'}$, ψ_{ABCD} of any order at p such that

$$\psi_{ABCD}(p) = \xi_{ABCD}, \quad \nabla_{(E_1}{}^{(E'_1} \ldots \nabla_{E_n}{}^{E'_n)}\,\psi_{ABCD)}(p) = \xi^{E'_1 \ldots E'_n}_{E_1 \ldots E_n ABCD}. \tag{6.15}$$

Remark. The coefficients $e^{\mu}{}_{AA'}$ and $\Gamma_{AA'}{}^{B}{}_{C}$ have been listened in the first statement because the field equations involve covariant derivatives of tensor fields and thus require the frame and connection coefficients for their formulation. The following argument will, however, never make use of explicit expressions of covariant derivatives in terms of these coefficients and partial derivatives of the fields. It only uses formal expressions of covariant derivatives and the standard rules for covariant derivatives such as commutation relations and the Leibniz rule. Therefore the coefficients are not mentioned in the second part of the Lemma. How they are determined will be discussed in the following section.

Proof. At lowest order the first assertion of the Lemma follows from (6.10), (6.11), (6.12) and (6.13). That it is true at higher orders will be shown by an induction argument. In this we shall repeatedly make use of (3.4) and (3.6) with $\Lambda = 0$. With the identity

$$\nabla_{CC'}\nabla_{DD'} - \nabla_{DD'}\nabla_{CC'} = \epsilon_{CD}\,\nabla_{H(C'}\,\nabla^{H}{}_{D')} + \epsilon_{C'D'}\,\nabla_{(C|H'|}\,\nabla_{D)}{}^{H'},$$

it is seen that (3.4) and its complex conjugate are with our assumptions equivalent to the relations

$$\epsilon_{C'D'}\,\nabla_{(C}{}^{C'}\nabla_{D)}{}^{D'}\kappa_A = \Omega\,\psi_{ABCD}\,\kappa^B, \qquad \epsilon_{C'D'}\,\nabla_{(C}{}^{C'}\nabla_{D)}{}^{D'}\bar{\kappa}_{A'} = \Phi_{CDA'B'}\,\kappa^{B'},$$
$$\epsilon^{CD}\,\nabla_{C}{}^{(C'}\nabla_{D}{}^{D')}\kappa_A = \Phi_{AB}{}^{C'D'}\,\kappa^B, \qquad \epsilon^{CD}\,\nabla_{C}{}^{(C'}\nabla_{D}{}^{D')}\bar{\kappa}_{A'} = \Omega\,\bar{\psi}_{A'B'}{}^{C'D'}\,\bar{\kappa}^{B'}.$$

While the induction argument is fairly obvious for the fields Ω, Π, it is more involved in the case of $\Phi_{ABA'B'}$ and ψ_{ABCD}. The following observations are important. Consider the quantities $\nabla_{E_1}{}^{E_1'} \ldots \nabla_{E_n}{}^{E_n'}\,\psi_{ABCD}$ with $n \geq 2$. If the covariant derivatives would commute it would follow that

$$\nabla_{E_1}{}^{E_1'} \ldots \nabla_{E_n}{}^{E_n'}\,\psi_{ABCD} = \nabla_{(E_1}{}^{(E_1'} \ldots \nabla_{E_n}{}^{E_n')}\,\psi_{ABCD)}. \tag{6.16}$$

In fact, any order of the upper indices can be achieved by commuting the covariant derivatives. If can be shown that the lower indices can be brought into any order without changing the position of the upper indices, the assertion will follow. Consider, for instance, the index positions given on the left hand side of the equation above. To interchange the indices E_k and A (say) we commute $\nabla_{E_k}{}^{E_k'}$ to the right until we can use (6.6) to swap E_k and A, then we commute again to bring $\nabla_{A}{}^{E_k'}$ back to the k-th position. To show that indices E_k, E_j can be interchanged we operate with $\nabla_{E_k}{}^{E_k'}$ as before to get $\nabla_{A}{}^{E_k'}$, then commute $\nabla_{E_j}{}^{E_j'}$ to the right and use (6.6) again to get $\nabla_{E_k}{}^{E_j'}$, then commute $\nabla_{A}{}^{E_k'}$ to the right to get $\nabla_{E_j}{}^{E_k'}$ by using again (6.6). Finally, commute $\nabla_{E_j}{}^{E_k'}$ and $\nabla_{E_k}{}^{E_j'}$ into the k-th and j-th position respectively so that the order of the upper indices remains unchanged.

If the covariant derivatives do not commute one can still operate as above but use (3.4) and (3.6) with $\Lambda = 0$ each time we commute derivatives. By this procedure the curvature spinor $R^{A}{}_{BCC'DD'}$ and its derivatives enter the expressions and (6.16) is replaced by an equation of the form

$$\nabla_{E_1}{}^{E_1'} \ldots \nabla_{E_n}{}^{E_n'}\,\psi_{ABCD} = \nabla_{(E_1}{}^{(E_1'} \ldots \nabla_{E_n}{}^{E_n')}\,\psi_{ABCD)} + \ldots, \tag{6.17}$$

where the dots indicate terms which depend on the curvature tensor and its derivatives and thus via the field equations on the fields Ω, s, $\Phi_{ABA'B'}$, ψ_{ABCD} and their covariant derivatives of order $\leq n-2$. Restriction to p then implies with the induction hypothesis that the $\nabla_{\mathbf{n}}\,\psi_{ABCD}(0)$with $|\mathbf{n}| \geq 2$ can be expressed in terms of $s(0)$ and the null data of ψ_{ABCD} of order $\leq n$.

Using (6.7) and (6.8) to interchange unprimed as well as primed indices we conclude by similar arguments that for $n \geq 2$

$$\nabla_{E_1}{}^{E_1'} \ldots \nabla_{E_n}{}^{E_n'}\,\Phi_{BC}{}^{B'C'} = \nabla_{(E_1}{}^{(E_1'} \ldots \nabla_{E_n}{}^{E_n'}\,\Phi_{BC)}{}^{B'C')} + \ldots, \tag{6.18}$$

where the dots indicate the terms of order $\leq n-2$, which are generated by commutating covariant derivatives and the terms which arise from the right hand sides of equations (6.7) and (6.8). These terms and the commutators contain expressions $\nabla_{\mathbf{k}}\psi_{ABCD}$, $\nabla_{\mathbf{j}}\bar{\psi}_{A'B'C'D'}$ with $|\mathbf{k}|, |\mathbf{j}| \leq n-1$ and derivatives $\nabla_{\mathbf{l}}\,\Omega$ with $|\mathbf{l}| \leq n$. Equation (6.1) allows us to express the latter in terms of $\nabla_{\mathbf{m}}\,\Omega$, $\nabla_{\mathbf{p}}\,s$ and $\nabla_{\mathbf{q}}\,\Phi_{ABCD}$ with $|\mathbf{m}|, |\mathbf{p}|, |\mathbf{q}| \leq n-2$. Restricting to $x^\mu = 0$ and observing that the right hand side of (6.7), (6.8) vanish at p, we conclude with our induction hypothesis that $\nabla_{\mathbf{n}}\,\Phi_{BCB'C'}(0)$ is obtained as an expression of $s(0)$ and the null data of ψ_{ABCD} of order $\leq n-2$.

For the quantities $\nabla_{\mathbf{n}}\Omega(0)$ the induction step follows immediately from (6.1) and for the quantities $\nabla_{\mathbf{n}}s(0)$ it follows with (6.2) by using (6.1) again.

This proves the first part of the Lemma. The second statement follows because equation (6.17) shows that no restrictions are imposed by the field equations on the quantities $\nabla_{(E_1}{}^{(E'_1} \ldots \nabla_{E_n}{}^{E'_n)}\,\psi_{ABCD)}(0)$. By the argument given above, all formal covariant derivatives are given by algebraic expressions of the null data of ψ at p and these expressions impose no restrictions on the null data. □

By (5.1) the symmetric parts of the covariant derivatives determined in Lemma 6.1 can be regarded as Taylor coefficients of corresponding tensor fields. By Borel's theorem ([7]) we can then find smooth fields $\hat{\Omega}$, $\hat{\Pi}$, $\hat{\psi}_{ABCD}$, $\hat{\Phi}_{ABA'B'}$ near p whose Taylor coefficients at p coincide with the Taylor coefficients determined by the procedure above (but fairly arbitrary away from p). We can assume that these fields satisfy near p the symmetry and the reality properties discussed in Sect. 3. With these fields we set $\hat{R}_{ABCC'DD'} = \hat{\Omega}\,\hat{\psi}_{ABCD}\,\epsilon_{C'D'} + \hat{\Phi}_{ABC'D'}\,\epsilon_{CD}$, which corresponds to the curvature spinor whose Taylor coefficients entered the discussion above, and define the 'curvature tensor' $\hat{R}^i{}_{jkl}$ by following (3.5).

To decide whether these smooth fields do in fact satisfy the field equations at all orders at p we first need to determine frame and connection coefficients consistent with the curvature tensor.

7. The Structural Equations

The frame and the connection coefficients which we want to satisfy the structural equations with the 'curvature spinor' $\hat{R}_{ABCC'DD'}$ will be denoted in the following by $\hat{e}^\mu{}_i$ and $\hat{\Gamma}_{AA'}{}^C{}_B$. It turns out that these functions are determined already by the subsystem

$$\hat{t}_k{}^i{}_l\,\hat{e}^\mu{}_i\,X^l = 0, \qquad (\hat{r}^A{}_{Bkl} - \hat{R}^A{}_{Bkl})\,X^k = 0, \tag{7.1}$$

of the structural equations, where the fields $\hat{t}_k{}^i{}_l$ and $\hat{r}^A{}_{Bkl}$ are given by the right hand sides of (2.2), (2.3) with e and Γ replaced by $\hat{e}$ and $\hat{\Gamma}$ and where $X^i = \delta^i{}_\mu\,x^\mu$. Assuming (4.7) and (4.10) to be satisfied by $\hat{e}^\mu{}_i$ and $\hat{\Gamma}_{AA'}{}^C{}_B$, these equations can be written

$$\hat{e}^\mu{}_{k,\,\nu}\,x^\nu + \hat{e}^\mu{}_l\,(\delta^l_\nu\,\hat{e}^\nu{}_k - \delta^l_k) + \hat{\Gamma}_k{}^i{}_l\,X^l\,\hat{e}^\mu{}_i = 0, \tag{7.2}$$

$$\hat{\Gamma}_l{}^A{}_{B,\,\mu}\,x^\mu + \hat{\Gamma}_k{}^A{}_B\,\delta^k_\mu\,\hat{e}^\mu{}_l + \hat{\Gamma}_l{}^j{}_k\,X^k\,\hat{\Gamma}_j{}^A{}_B = \hat{R}^A{}_{Bkl}\,X^k, \tag{7.3}$$

where the $\hat{\Gamma}_k{}^i{}_l$ are given in spinor notation by

$$\hat{\Gamma}_{AA'}{}^{CC'}{}_{BB'} = \hat{\Gamma}_{AA'}{}^C{}_B\,\epsilon_{B'}{}^{C'} + \bar{\hat{\Gamma}}_{AA'}{}^{C'}{}_{B'}\,\epsilon_B{}^C,$$

so that they are real and satisfy $\hat{\Gamma}_{k\,i\,l} = -\hat{\Gamma}_{k\,l\,i}$ as a consequence of $\hat{\Gamma}_{l\,AB} = \hat{\Gamma}_{l\,(AB)}$. Equations (7.2), (7.3) imply that a smooth solution $\hat{e}^\mu{}_i(x^\mu)$, $\hat{\Gamma}_{AA'}{}^C{}_B(x^\mu)$ near $x^\mu = 0$ with $\det(\hat{e}^\mu{}_i) \neq 0$ must satisfy

$$\hat{e}^\mu{}_k(0) = \delta^\mu{}_k, \qquad \hat{\Gamma}_l{}^A{}_B(0) = 0. \tag{7.4}$$

Equations (7.2), (7.3) can be discussed by analysing the ODE's which are implied by them along the curves $\tau \to \tau\, x_*^\mu$, $x_*^\mu \neq 0$. These ODE's will be considered in Sect. 9, for our present purpose a more direct approach will be sufficient. To simplify the algebra we rewrite the equations in terms of the unknowns

$$\hat{c}^\mu{}_k \equiv \hat{e}^\mu{}_k - \delta^\mu{}_k, \quad \hat{\Gamma}_{AA'}{}^C{}_B,$$

to obtain them in the form

$$\hat{c}^\mu{}_{k,\,\nu}\, x^\nu + \hat{c}^\mu{}_k + \hat{c}^\mu{}_l\, \delta^l_\nu\, \hat{c}^\nu{}_k + \hat{\Gamma}_k{}^i{}_l\, X^l\, \hat{c}^\mu{}_i + \hat{\Gamma}_k{}^i{}_l\, X^l\, \delta^\mu{}_i = 0, \tag{7.5}$$

$$\hat{\Gamma}_l{}^A{}_{B,\,\nu}\, x^\nu + \hat{\Gamma}_l{}^A{}_B + \hat{\Gamma}_k{}^A{}_B\, \delta^k_\mu\, \hat{c}^\mu{}_l + \hat{\Gamma}_l{}^j{}_k\, X^k\, \hat{\Gamma}_j{}^A{}_B - \hat{R}^A{}_{Bkl}\, X^k = 0. \tag{7.6}$$

By taking formally partial derivatives, observing (7.4), and evaluating at $x^\mu = 0$ one obtains unique sequences of derivatives

$$\hat{c}^\mu{}_{k,\,\nu_1 \ldots \nu_k}(0), \quad \hat{\Gamma}_l{}^A{}_{B,\,\nu_1 \ldots \nu_k}(0), \quad k \in \mathbb{N},$$

which are symmetric in the indices $\nu_1 \ldots \nu_k$ and are determined by the partial derivatives of the field $\hat{R}^A{}_{Bkl}$ at the origin. By Borel's theorem ([7]) we can then find smooth fields $\hat{c}^\mu{}_k$ and $\hat{\Gamma}_l{}^A{}_B$ near $x^\mu = 0$ whose Taylor coefficients coincide with the coefficients given above. Because of $\hat{R}_{ABkl} = \hat{R}_{(AB)kl}$ and the structure of the equations, these fields can be chosen such that $\hat{c}^\mu{}_k$ is real and $\hat{\Gamma}_{l\,AB} = \hat{\Gamma}_{l\,(AB)}$. While the choice of the fields is rather arbitrary away from $x^\mu = 0$ they satisfy the structural equations at all orders at $x^\mu = 0$ so that

$$\hat{c}^\mu{}_{k,\,\nu}\, x^\nu + \hat{c}^\mu{}_k + \hat{c}^\mu{}_l\, \delta^l_\nu\, \hat{c}^\nu{}_k + \hat{\Gamma}_k{}^i{}_l\, X^l\, \hat{c}^\mu{}_i + \hat{\Gamma}_k{}^i{}_l\, X^l\, \delta^\mu{}_i = O(|x|^\infty), \tag{7.7}$$

$$\hat{\Gamma}_l{}^A{}_{B,\,\nu}\, x^\nu + \hat{\Gamma}_l{}^A{}_B + \hat{\Gamma}_k{}^A{}_B\, \delta^k_\mu\, \hat{c}^\mu{}_l + \hat{\Gamma}_l{}^j{}_k\, X^k\, \hat{\Gamma}_j{}^A{}_B - \hat{R}^A{}_{Bkl}\, X^k = O(|x|^\infty), \tag{7.8}$$

where the symbols $O(|x|^\infty)$ on the right hand sides indicate that the quantities on the left hand side are for all $n \in \mathbb{N}$ of the order $O(|x|^n)$ as $x^\mu \to 0$.

With (7.4) it follows that

$$\hat{c}^\mu{}_k(0) = 0, \quad \hat{c}^\mu{}_{k,\nu}(0) = 0, \quad \hat{\Gamma}_l{}^A{}_B(0) = 0. \tag{7.9}$$

We restrict the following discussion to some neighbourhood of the origin on which the smooth field $\hat{e}^\mu{}_k \equiv \delta^\mu{}_k + \hat{c}^\mu{}_k$ satisfies $\det(\hat{e}^\mu{}_k) \neq 0$. It is there orthonormal for the metric $\hat{g}_{\mu\nu} \equiv \eta_{ij}\, \hat{\sigma}^i{}_\mu\, \hat{\sigma}^j{}_\nu$, where the $\hat{\sigma}^i{}_\mu$ denote the 1-forms dual to the $\hat{e}^\mu{}_k$. Because $\hat{\Gamma}_{i\,AB} = \hat{\Gamma}_{i\,BA}$, whence $\hat{\Gamma}_{i\,j\,k} = -\hat{\Gamma}_{i\,k\,j}$, the connection $\hat{\nabla}$ defined by $\hat{e}^\mu{}_k$ and $\hat{\Gamma}_i{}^j{}_k$ resp. $\hat{\Gamma}_i{}^A{}_B$, which satisfies for instance $\hat{\nabla}_i\, \hat{e}_k = \hat{\Gamma}_i{}^j{}_k\, \hat{e}_k$ with $\hat{\nabla}_i \equiv \hat{\nabla}_{\hat{e}_i}$, is $\hat{g}$-metric compatible in the sense that $\hat{\nabla}\, \hat{g} = 0$.

The symmetries of the fields $\hat{\Gamma}_{k\,AB}$ and $\hat{R}_{AB\,jk}$ imply the following results.

Lemma 7.1. *(i) The coordinates x^μ and the frame coefficients $\hat{e}^\mu{}_k$ satisfy the requirements (4.7), (4.8), (4.10) of a normal gauge at all orders at $x^\mu = 0$, so that*

$$(\hat{e}^\nu{}_j(x) - \delta^\nu{}_j)\, \delta^j_\mu\, x^\mu = O(|x|^\infty), \tag{7.10}$$

$$x^\mu\, \eta_{\mu\nu}\, (\hat{e}^\nu{}_k(x) - \delta^\nu{}_k) = O(|x|^\infty), \tag{7.11}$$

$$\delta^k_\nu\, x^\nu\, \hat{\Gamma}_k{}^i{}_j(x) = O(|x|^\infty). \tag{7.12}$$

(ii) Consider the curve $\tau \to x^\mu(\tau) = \tau\, x_*^\mu$, $x_*^\mu \neq 0$, through the origin. The components of its tangent vectors $\dot{x}^\mu = x_*^\mu$ in the frame $\hat{e}^\mu{}_k$, given by $z^k(\tau) = x_*^\mu\, \hat{\sigma}^k{}_\mu(\tau\, x_*)$, satisfies

$$z^k(\tau) - \delta^k{}_\mu\, x_*^\mu = O(|\tau\, x_*|^\infty), \tag{7.13}$$

the curve satisfies the geodesic equation at all orders at $\tau = 0$,

$$\hat{\nabla}_{\dot{x}}\, \dot{x} = O(|\tau\, x_*|^\infty), \tag{7.14}$$

and the frame $\hat{e}_k = \hat{e}^\mu{}_k\, \partial_{x^\mu}$ satisfies the equation of parallel transport along these curves at all orders at $\tau = 0$,

$$\hat{\nabla}_{\dot{x}}\, \hat{e}_k = O(|\tau\, x_*|^\infty). \tag{7.15}$$

Proof. To obtain the relations (7.10), (7.11), (7.12) we contract (7.7) and (7.8) with $\delta^k_\mu\, x^\mu$ and $\delta^l_\mu\, x^\mu$ respectively to obtain the relations

$$\hat{c}^\mu{}_{,\nu}\, x^\nu + \hat{c}^\mu{}_l\, \delta^l_\nu\, \hat{c}^\nu + \hat{\Gamma}^i{}_l\, X^l\, (\hat{c}^\mu{}_i + \delta^\mu{}_i) = O(|x|^\infty), \tag{7.16}$$

$$\hat{\Gamma}^A{}_{B,\nu}\, x^\nu + \hat{\Gamma}_k{}^A{}_B\, \delta^k_\mu\, \hat{c}^\mu + \hat{\Gamma}^j{}_k\, X^k\, \hat{\Gamma}_j{}^A{}_B = O(|x|^\infty). \tag{7.17}$$

for the quantities

$$\hat{c}^\nu \equiv \hat{c}^\nu{}_j(x)\, \delta^j_\mu\, x^\mu, \quad \hat{c}^\nu{}_k \equiv x^\mu\, \eta_{\mu\nu}\, \hat{c}^\nu{}_k(x), \quad \hat{\Gamma}^A{}_B \equiv \delta^k_\nu\, x^\nu\, \hat{\Gamma}_k{}^A{}_B(x),$$
$$\hat{\Gamma}^i{}_j \equiv \delta^k_\nu\, x^\nu\, \hat{\Gamma}_k{}^i{}_j(x).$$

If $\hat{c}^\mu = O(|x|^p)$ and $\hat{\Gamma}^A{}_B = O(|x|^q)$ with some $p, q \in \mathbb{N}$, these relations imply with (7.9) relations of the form

$$\hat{c}^\mu{}_{,\nu}\, x^\nu = O(|x|^{p+2}) + O(|x|^{q+1}), \quad \hat{\Gamma}^A{}_{B,\nu}\, x^\nu = O(|x|^{p+1}) + O(|x|^{q+2}).$$

Because $\hat{c}^\mu = O(|x|^3)$ and $\hat{\Gamma}^A{}_B = O(|x|^2)$ by (7.9), the second relation implies that $\hat{\Gamma}^A{}_{B,\nu}\, x^\nu = O(|x|^4)$ whence also $\hat{\Gamma}^A{}_B = O(|x|^4)$ and the first relation gives then $\hat{c}^\mu{}_{,\nu}\, x^\nu = O(|x|^5)$ whence $\hat{c}^\mu = O(|x|^5)$. Repeating the argument we conclude that $\hat{c}^\mu = O(|x|^\infty)$ and $\hat{\Gamma}^A{}_B = O(|x|^\infty)$, which are the relations (7.10) and (7.12).

Observing that

$$\hat{\Gamma}_k{}^i{}_l\, X^l\, \delta^\mu{}_i\, x^\lambda\, \eta_{\lambda\mu} = \hat{\Gamma}_{k\,j\,l}\, X^l\, \eta^{ji}\, \delta^\mu{}_i\, x^\lambda\, \eta_{\lambda\mu} = \hat{\Gamma}_{k\,j\,l}\, X^j\, X^l = 0,$$

the contraction of (7.7) with $x^\lambda\, \eta_{\lambda\mu}$ gives

$$\hat{c}_{k,\nu}\, x^\nu + \hat{c}_l\, \delta^l_\nu\, \hat{c}^\nu{}_k + \hat{\Gamma}_k{}^i{}_l\, X^l\, \hat{c}_i + \hat{\Gamma}_k{}^i{}_l\, X^l\, \delta^\mu{}_i = O(|x|^\infty), \tag{7.18}$$

which implies with the previous result that $\hat{c}_k = O(|x|^\infty)$, which is in fact (7.11).

Contraction of the relation $(\hat{e}^\nu{}_j(\tau\, x_*) - \delta^\nu{}_j)\, \delta^j_\mu\, x_*^\mu = O(|\tau\, x_*|^\infty)$, which holds by (7.10), with $-\hat{\sigma}^k{}_\nu(\tau\, x_*)$ gives (7.13). In terms of the frame one has

$$(\hat{\nabla}_{\dot{x}}\, \dot{x})^k = \frac{d}{d\tau} z^k + z^j\, \hat{\Gamma}_j{}^k{}_l(\tau\, x_*)\, z^l = O(|\tau\, x_*|^\infty),$$

and

$$\hat{\nabla}_{\dot{x}}\, \hat{e}_k = z^j\;\hat{\Gamma}_j{}^l{}_k(\tau\, x_*)\, \hat{e}_l = O(|\tau\, x_*|^\infty),$$

by (7.13) and (7.12). □

8. Formal and Factual Derivatives

The subsystem (7.1) of the structural equations determines the functions $\hat{e}^{\mu}{}_{k}$ and $\hat{\Gamma}_{i}{}^{j}{}_{k}$ uniquely and implies that

$$\hat{t}_{i}{}^{j}{}_{k} = O(|x|^{n}), \qquad \hat{r}^{h}{}_{kjl} - \hat{R}^{h}{}_{kjl} = O(|x|^{n}) \tag{8.1}$$

with $n = 1$. Moreover, direct calculations involving (7.4), (6.10), (6.11), (6.12), (6.13) show that

$$\hat{\nabla}_{\mathbf{k}}\, \hat{\Omega}(0) = \nabla_{\mathbf{k}}\, \Omega(0), \quad \hat{\nabla}_{\mathbf{k}}\, \hat{\Pi}(0) = \nabla_{\mathbf{k}}\, \Pi(0), \tag{8.2}$$

for $|\mathbf{k}| \leq 3$ and

$$\hat{\nabla}_{\mathbf{k}} \hat{\Phi}_{ABA'B'}(0) = \nabla_{\mathbf{k}} \Phi_{ABA'B'}(0), \quad \hat{\nabla}_{\mathbf{k}} \hat{\psi}_{ABCD}(0) = \nabla_{\mathbf{k}} \psi_{ABCD}(0), \tag{8.3}$$

whence

$$\hat{\nabla}_{\mathbf{k}} \hat{R}^{i}{}_{jkl}(0) = \nabla_{\mathbf{k}} R^{i}{}_{jkl}(0), \tag{8.4}$$

for $|\mathbf{k}| \leq 1$ where on the right hand sides are given the formal expressions derived in the previous section and on the left hand sides the factual covariant derivatives of the smooth fields $\hat{\Omega}$, $\hat{s}$, $\hat{\Phi}_{ABA'B'}$, $\hat{\psi}_{ABCD}$, $\hat{R}^{i}{}_{jkl}$ at the point $x^{\mu} = 0$ with respect to the connection $\hat{\nabla}$. These relations imply that

$$\hat{\nabla}_{AA'}\, \hat{\nabla}_{BB'}\, \hat{\Omega} + \hat{\Omega}\, \hat{\Phi}_{ABA'B'} - \hat{\Pi}\, \epsilon_{AB}\, \epsilon_{A'B'} = O(|x|^{n}), \tag{8.5}$$

$$\hat{\nabla}_{AA'}\, \hat{\Pi} + \hat{\nabla}^{BB'} \hat{\Omega}\, \hat{\Phi}_{ABA'B'} = O(|x|^{n}), \tag{8.6}$$

$$\hat{\nabla}_{F}{}^{D'} \hat{\Phi}_{ABF'D'} - \hat{\nabla}^{C}{}_{F'} \hat{\Omega}\, \hat{\psi}_{ABCF} = O(|x|^{n}), \tag{8.7}$$

$$\hat{\nabla}^{C}{}_{F'} \hat{\psi}_{ABCF} = O(|x|^{n}), \tag{8.8}$$

hold with $n = 1$. Because the quantities $\nabla_{\mathbf{k}} R^{i}{}_{jkl}(0)$ have been determined by invoking the Bianchi identities (see the discussion of (3.9), (3.10)) it follows from (8.4) that

$$\sum_{cycl(ijl)} \hat{\nabla}_{i}\, \hat{R}^{h}{}_{kjl} = O(|x|^{n}), \tag{8.9}$$

with $n = 1$. The purpose of this section is to derive the following result.

Proposition 8.1. *Relations (8.1) to (8.9) hold true for all integers $n \in \mathbb{N}$ resp. multi-indices* $\mathbf{k}$.

Remark 8.2. The following argument covers in particular the vacuum case in which Ω is set equal to 1 and the only non-trivial fields are given by $e^{\mu}{}_{k}$, $\Gamma_{i}{}^{j}{}_{k}$ and ψ_{ABCD}.

Before we begin with the proof we need to make a few observations. Because only a subsystem of the structural equations has been used so far, it is not clear whether the order relations (8.1) hold for all $n \in \mathbb{N}$. The following result shows in particular how this question is related to the Bianchi identity (8.9).

Lemma 8.3. *Denote by* $\hat{\gamma}_k{}^i{}_l$ *the connection coefficients of the Levi-Civita connection of the metric* $\hat{g}_{\mu\nu} = \eta_{jk}\,\hat{\sigma}^j{}_\mu\,\hat{\sigma}^k{}_\nu$ *with respect to the frame* $\hat{e}_k$*. If the torsion tensor* $\hat{t}_i{}^j{}_k$ *of the connection* $\hat{\nabla}$ *behaves as* $\hat{t}_i{}^j{}_k = O(|x|^N)$ *for some* $N \in \mathbb{N}$, $N \geq 1$, *then* $\hat{\Gamma}_k{}^i{}_l - \hat{\gamma}_k{}^i{}_l = O(|x|^N)$.

If $N \in \mathbb{N}$, $N \geq 1$, *and*

$$\sum_{cycl(ijl)} \hat{\nabla}_i\,\hat{R}^h{}_{kjl} = O(|x|^N), \tag{8.10}$$

then $\hat{t}_j{}^k{}_l = O(|x|^{N+2})$, $\hat{r}^h{}_{kjl} - \hat{R}^h{}_{kjl} = O(|x|^{N+1})$.

Proof. Denote by $\hat{c}_l{}^j{}_k$ the commutator coefficients satisfying $[\hat{e}_l\,\hat{e}_k] = \hat{c}_l{}^j{}_k\,\hat{e}_j$. With $\hat{c}_{l\,i\,k} = \hat{c}_l{}^j{}_k\,\eta_{ji}$ and $\hat{t}_{l\,i\,k} = \hat{t}_l{}^j{}_k\,\eta_{ji}$ the torsion free relation can be written $\hat{\Gamma}_{l\,i\,k} - \hat{\Gamma}_{k\,i\,l} - \hat{c}_{l\,i\,k} = \hat{t}_{k\,i\,l}$. It is well known that this implies

$$2\,\hat{\Gamma}_{kli} - \{\hat{c}_{l\,i\,k} + \hat{c}_{k\,l\,i} - \hat{c}_{i\,k\,l}\} = \hat{t}_{l\,i\,k} + \hat{t}_{k\,l\,i} - \hat{t}_{i\,k\,l}.$$

The same relations hold with $\hat{t}_{l\,i\,k} = 0$ if $\hat{\Gamma}_{l\,i\,k}$ is replaced by $\hat{\gamma}_{l\,i\,k}$. This gives

$$2\,(\hat{\Gamma}_{kli} - \hat{\gamma}_{kli}) = \hat{t}_{l\,i\,k} + \hat{t}_{k\,l\,i} - \hat{t}_{i\,k\,l},$$

which implies the desired result.

The connection $\hat{\nabla}$ defined by $\hat{e}_k$ and $\hat{\Gamma}_i{}^j{}_k$ is metric compatible but at this stage not known to be torsion free. As pointed out in Sect. 2, the Bianchi identities for the torsion tensor $\hat{t}_i{}^j{}_k$ and the curvature tensor $\hat{r}^i{}_{jkl}$ then take the form

$$\sum_{cycl(ijl)} \hat{\nabla}_i\,\hat{t}_j{}^k{}_l = \sum_{cycl(ijl)} (\hat{r}^k{}_{ijl} - \hat{t}_i{}^m{}_j\,\hat{t}_m{}^k{}_l), \tag{8.11}$$

$$\sum_{cycl(ijl)} \hat{\nabla}_i\,\hat{r}^h{}_{kjl} = \sum_{cycl(ijl)} \hat{t}_j{}^m{}_i\,\hat{r}^h{}_{kml}. \tag{8.12}$$

By the symmetries and reality conditions of the fields defining $\hat{R}^k{}_{ijl}$ the arguments which led to (3.8) imply $\sum_{cycl(ijl)} \hat{R}^k{}_{ijl} = 0$ near p. Equation (8.11) can thus be written

$$\sum_{cycl(ijl)} \hat{\nabla}_i\,\hat{t}_j{}^k{}_l = \sum_{cycl(ijl)} (\hat{r}^k{}_{ijl} - \hat{R}^k{}_{ijl} - \hat{t}_i{}^m{}_j\,\hat{t}_m{}^k{}_l).$$

Transvecting this equation with X^i, observing (7.1) and the anti-symmetry of the torsion tensor gives

$$X^i\,\hat{\nabla}_i\,\hat{t}_j{}^k{}_l + \hat{t}_j{}^k{}_i\,\hat{\nabla}_l X^i + \hat{\nabla}_j X^i\,\hat{t}_i{}^k{}_l = x^i\,(\hat{r}^k{}_{ijl} - \hat{R}^k{}_{ijl}).$$

Similarly, transvecting the rewrite

$$\sum_{cycl(ijl)} \hat{\nabla}_i\,(\hat{r}^h{}_{kjl} - \hat{R}^h{}_{kjl}) = \sum_{cycl(ijl)} \hat{t}_j{}^m{}_i\,\hat{r}^h{}_{kml} - \sum_{cycl(ijl)} \hat{\nabla}_i\,\hat{R}^h{}_{kjl},$$

of (8.12) with X^i gives

$$X^i\,\hat{\nabla}_i\,(\hat{r}^h{}_{kjl} - \hat{R}^h{}_{kjl}) + (\hat{r}^h{}_{kji} - \hat{R}^h{}_{kji})\,\hat{\nabla}_l X^i + \hat{\nabla}_j X^i\,(\hat{r}^h{}_{kil} - \hat{R}^h{}_{kil})$$
$$= \hat{t}_l{}^m{}_j\,\hat{r}^h{}_{kmi}\,X^i - X^i\left(\sum_{cycl(ijl)} \hat{\nabla}_i\,\hat{R}^h{}_{kjl}\right).$$

The result follows now with (4.9) by taking derivatives and evaluating at $x^\mu = 0$. □

Assume that there exists a smooth solution to the field equations in the given gauge which induces the prescribed null data at p. By the arguments given above the ∞-jet of the solution at p must then coincide with the expressions on the right hand sides of (8.2), (8.3), (8.4). It is not obvious, however, that it must also coincide with the ∞-jets of the functions $\hat{e}^\mu{}_k$, $\hat{\Gamma}_i{}^j{}_k$, $\hat{\Omega}$, $\hat{\Pi}$, $\hat{\Phi}_{ABA'B'}$, $\hat{\psi}_{ABCD}$ at p. The reason is, that, following (5.1), these functions have been defined so that their Taylor coefficients at $x^\mu = 0$ coincide with the *symmetrized* derivatives $\nabla_{(\mathbf{k})}\,\Omega(0)$, $\nabla_{(\mathbf{k})}\,\Pi(0)$, $\nabla_{(\mathbf{k})}\,\Phi_{ABA'B'}(0)$, $\nabla_{(\mathbf{k})}\,\psi_{ABCD}(0)$ and it is not clear how much of the information encoded in the unsymmetrized derivatives is transported by the symmetrized derivatives. In particular, while the Bianchi identities are by (3.9), (3.10) part of the conformal field equations and the coefficients on the right hand sides of (8.2), (8.3), (8.4) have been determined so as to satisfy these identities, it is not obvious at this stage that relation (8.10) should be satisfied for integers $N > 1$.

Proof of Proposition 8.1. The induction argument to be given below will make use of the following general considerations. Let $T_{A_1\ldots A_j\,B'_1\ldots B'_k}$ denote a smooth spinor field and ∇ a metric compatible connection with curvature tensor $r^i{}_{jkl}$ and torsion tensor $t_i{}^j{}_k$. To begin with assume that $t_i{}^j{}_k = 0$. If the derivatives on the right hand side of the symmetrization formula

$$\nabla_{(i_1}\ldots\nabla_{i_n)}T_{A_1\ldots A_j\,B'_1\ldots B'_k} = \frac{1}{n!}\sum_{\pi\in\mathcal{S}_n}\nabla_{i_{\pi(1)}}\ldots\nabla_{i_{\pi(n)}}T_{A_1\ldots A_j\,B'_1\ldots B'_k}$$

are then commuted to bring them into their natural order, one obtains an equation of the form

$$\nabla_{\mathbf{n}}\,T_{A_1\ldots A_j\,B'_1\ldots B'_k} = \nabla_{(\mathbf{n})}\,T_{A_1\ldots A_j\,B'_1\ldots B'_k} + C^*_{\mathbf{n}\;A_1\ldots A_j\,B'_1\ldots B'_k},$$

where the spinor field C^* is a sum of terms which depend on the covariant derivatives of T and $r^i{}_{jkl}$ of order $\leq |\mathbf{n}| - 2$. Using these formulas to substitute successively in the formulas for $n = 3, 4, \ldots$ the covariant derivatives of T of lower order by their symmetric parts one obtains formulas

$$\nabla_{\mathbf{n}}\,T_{A_1\ldots A_j\,B'_1\ldots B'_k} = \nabla_{(\mathbf{n})}\,T_{A_1\ldots A_j\,B'_1\ldots B'_k} + C_{\mathbf{n}\;A_1\ldots A_j\,B'_1\ldots B'_k}, \quad |\mathbf{n}| \geq 0, \qquad (8.13)$$

with spinor valued functions

$$C_{\mathbf{n}} = C_{\mathbf{n}}(\nabla_{(\mathbf{p})}T, \nabla_{\mathbf{q}}\,r) \quad \text{where} \quad |\mathbf{p}|, |\mathbf{q}| \leq |\mathbf{n}| - 2,$$

which satisfy $C_{(\mathbf{n})} = 0$. These formulas show how the covariant derivatives of T at the point $x = 0$ are determined from the Taylor coefficients in (5.1) and the derivatives of the curvature tensor at $x = 0$.

Formulas (8.13) represent universal relations. The functions $C_{\mathbf{n}}$ depend on the connection ∇ only via the derivatives $\nabla_{\mathbf{q}}\, r$ of its curvature tensor. (We ignore the fact that the explicit dependence of $C_{\mathbf{n}}$ on the $\nabla_{\mathbf{q}}\, r$ may be written in different forms by using the symmetries and the differential identities satisfied by the curvature tensor). The full index notation of (8.13) emphasizes that the explicit structure of the functions $C_{\mathbf{n}}$ does depend on the index type of the spinor field T and in following equations we shall write out the appropriate indices.

With the notation of Sect. 6 the unknowns in the field equations must have representations of the form

$$\nabla_{\mathbf{n}}\, \Omega = \nabla_{(\mathbf{n})}\, \Omega + C_{\mathbf{n}}(\nabla_{(\mathbf{p})}\, \Omega,\, \nabla_{\mathbf{q}}\, R), \tag{8.14}$$

$$\nabla_{\mathbf{n}}\, \Pi = \nabla_{(\mathbf{n})}\, \Pi + C_{\mathbf{n}}(\nabla_{(\mathbf{p})}\, \Pi,\, \nabla_{\mathbf{q}}\, R), \tag{8.15}$$

$$\nabla_{\mathbf{n}}\, \Phi_{ABA'B'} = \nabla_{(\mathbf{n})}\, \Phi_{ABA'B'} + C_{\mathbf{n}\ ABA'B'}(\nabla_{(\mathbf{p})}\, \Phi,\, \nabla_{\mathbf{q}}\, R), \tag{8.16}$$

$$\nabla_{\mathbf{n}}\, \psi_{ABCD} = \nabla_{(\mathbf{n})}\, \psi_{ABCD} + C_{\mathbf{n}\ ABCD}(\nabla_{(\mathbf{p})}\, \psi,\, \nabla_{\mathbf{q}}\, R), \tag{8.17}$$

with $|\mathbf{p}|, |\mathbf{q}| \leq |\mathbf{n}| - 2$ and quantities $\nabla_{\mathbf{q}}\, R$ which are understood as derivatives of the curvature defined by ∇. Using these as a starting point we can impose equations (6.1) to (6.5) and proceed, as in Sect. 6, to derive for all multi-indices $\mathbf{k}$ expressions for the quantities $\nabla_{\mathbf{k}}\, \Omega(0)$, $\nabla_{\mathbf{k}}\, \Pi(0)$, $\nabla_{\mathbf{k}} \Phi_{ABA'B'}(0)$, $\nabla_{\mathbf{k}} \psi_{ABCD}(0)$, and thus also for $\nabla_{\mathbf{k}} R^i{}_{jkl}(0)$ in terms of the null data, which are given by the totally symmetric part of $\hat{\nabla}_{\mathbf{n}}\, \hat{\psi}_{ABCD}(0)$. These expressions could be inserted into the equations above but for the sake of comparison it will be better not to do this here.

Formulas (8.13) do not immediately apply to the functions $\hat{\Omega}$, $\hat{\Pi}$, $\hat{\Phi}_{ABA'B'}$, $\hat{\psi}_{ABCD}$ with the connection $\hat{\nabla}$ and the curvature tensor $\hat{r}^i{}_{jkl}$. They can be generalized, however, to the case where the connection ∇ is not torsion free by observing (2.1). The functions $C_{\mathbf{n}}$ will then depend on the symmetrized derivatives $\nabla_{(\mathbf{k})} T_{A_1 \ldots A_j\, B'_1 \ldots B'_k}$ of order $|\mathbf{k}| \leq |\mathbf{n}| - 1$ and on the derivatives of the torsion as well as on those of the curvature tensor.

Consider now the point p with $x^\mu(p) = 0$ and assume that $t_i{}^j{}_k(x^\mu) = O(|x|^{N'})$ with some integer $N' \geq 1$ as $x^\mu \to 0$. It follows then with (2.1) that the restriction of (8.13) to the point $x^\mu = 0$ is valid as it stands if $|\mathbf{n}| \leq N' + 1$. *At that point* we thus get for $|\mathbf{n}| \leq N' + 1$ the relations

$$\hat{\nabla}_{\mathbf{n}}\, \hat{\Omega} = \hat{\nabla}_{(\mathbf{n})}\, \hat{\Omega} + C_{\mathbf{n}}(\hat{\nabla}_{(\mathbf{p})}\, \hat{\Omega},\, \hat{\nabla}_{\mathbf{q}}\, \hat{r}), \tag{8.18}$$

$$\hat{\nabla}_{\mathbf{n}}\, \hat{\Pi} = \hat{\nabla}_{(\mathbf{n})}\, \hat{\Pi} + C_{\mathbf{n}}(\hat{\nabla}_{(\mathbf{p})}\, \hat{\Pi},\, \hat{\nabla}_{\mathbf{q}}\, \hat{r}), \tag{8.19}$$

$$\hat{\nabla}_{\mathbf{n}}\, \hat{\Phi}_{ABA'B'} = \hat{\nabla}_{(\mathbf{n})}\, \hat{\Phi}_{ABA'B'} + C_{\mathbf{n}\ ABA'B'}(\hat{\nabla}_{(\mathbf{p})}\, \hat{\Phi},\, \nabla_{\mathbf{q}}\, \hat{r}), \tag{8.20}$$

$$\hat{\nabla}_{\mathbf{n}}\, \hat{\psi}_{ABCD} = \hat{\nabla}_{(\mathbf{n})}\, \hat{\psi}_{ABCD} + C_{\mathbf{n}\ ABCD}(\hat{\nabla}_{(\mathbf{p})}\, \hat{\psi},\, \hat{\nabla}_{\mathbf{q}}\, \hat{r}), \tag{8.21}$$

with $|\mathbf{p}|, |\mathbf{q}| \leq |\mathbf{n}| - 2$ and functions $C_{\mathbf{n}}$ which are identical with those appearing in the corresponding equation in (8.14) to (8.17).

To compare these two sets of equations we observe that only the properties (4.7) and (4.10) of the frame and the connection coefficients have been used to derive the normal expansion (5.1). Because these are satisfied by Lemma 7.1 also by the coefficients $\hat{e}^\mu{}_k$ and $\hat{\Gamma}_i{}^j{}_k$, the normal expansions of the fields $\hat{\Omega}$, $\hat{s}$, $\hat{\Phi}_{ABA'B'}$, $\hat{\psi}_{ABCD}$ can thus be expressed in terms of the derivatives with respect to the connection $\hat{\nabla}$. This implies $\hat{\nabla}_{(\mathbf{k})}\hat{\Omega} = \nabla_{(\mathbf{k})}\Omega$, $\hat{\nabla}_{(\mathbf{k})}\hat{\Pi} = \nabla_{(\mathbf{k})}\Pi$, $\hat{\nabla}_{(\mathbf{k})}\hat{\Phi}_{ABA'B'} = \nabla_{(\mathbf{k})}\Phi_{ABA'B'}$, $\hat{\nabla}_{(\mathbf{k})}\hat{\psi}_{ABCD} = \nabla_{(\mathbf{k})}\psi_{ABCD}$ for all multi-indices $\mathbf{k}$ (here and in the following all spinors are thought to be taken at the point $x^\mu = 0$). It follows that the right hand sides of the two sets of equations are distinguished now only by the occurrence of the spinors $\nabla_{\mathbf{q}} R$ in the first set and the spinors $\hat{\nabla}_{\mathbf{q}} \hat{r}$ in the second set.

Consider now as an induction hypothesis the relations (8.2), (8.3) (8.4) with multi-indices $\mathbf{k}$ such that $|\mathbf{k}| \leq N'$. Because the formal derivatives of the tensor $R^i{}_{jkl}$ have been determined such that the Bianchi identities are satisfied at all orders, relations (8.4) imply that (8.10) holds with $N = N'$. It follows then from Lemma 8.3 that the assumption above on the torsion tensor is satisfied and (8.4) implies with the Lemma that $\hat{\nabla}_{\mathbf{q}} \hat{r}^h{}_{kjl} = \hat{\nabla}_{\mathbf{q}} \hat{R}^h{}_{kjl} = \nabla_{\mathbf{q}} R^h{}_{kjl}$ with $|\mathbf{q}| \leq N'$. Comparing the two sets of equations above we can obtain relations (8.2), (8.3) (8.4) with multi-indices $\mathbf{k}$ such that $|\mathbf{k}| = N'+1$.

With the properties noted in the beginning of this section this implies that (8.2), (8.3) (8.4) hold true for multi-indices $\mathbf{k}$ of all orders. It follows that the order relations (8.1) and (8.5) to (8.9) are true for all integers $n \in \mathbb{N}$. $\square$

9. Transport Equations and Inner Constraints

We have prescribed the radiation field, read off the null data at the vertex p, and constructed sequences of expansion coefficients at p which can be realized as ∞-jets at p of smooth fields which satisfy the (conformal) field equations at all orders at p. We want to discuss now which information can be derived from the radiation field in some neighbourhood of p on $\mathcal{N}_p$.

By definition, the characteristics of any hyperbolic system of first order are those hypersurfaces on which the system induces inner equations on (combinations of) the dependent variables. On the other hand, the (conformal) Einstein equations induce as a consequence of their gauge freedom constraints on their Cauchy data on any hypersurface. On null hypersurfaces, which represent the characteristics of the (conformal) Einstein equations, these facts combine and result in a particular set of inner equations. This set splits into two subsets. There are equations which involve in particular derivatives in the direction of the null generators of the null hypersurface. These will be referred to as *transport equations*. The remaining equations only involve derivatives in directions which are still tangent to the null hypersurface but transverse to the null generators. These will be referred to as *inner constraints*.

At most points of $\mathcal{N}_p$ none of the frame vectors e_k in the normal gauge is tangent to $\mathcal{N}_p$. To derive from the complete set of equations subsystems which only contain derivatives in directions tangent to $\mathcal{N}_p$, one thus needs to take (point dependent) linear combinations of the equations and the dependent variables. Whatever one does to obtain the maximal number of transport equations will amount in the end to expressing the equations in terms of a new frame field on $\mathcal{N}_p \setminus \{p\}$ which is such that three of the new frame vectors will be tangent to $\mathcal{N}_p \setminus \{p\}$.

We shall describe the procedure and the resulting equations and derive the information which will be needed to construct the desired fields on $\mathcal{N}_p$ near p. The following discussion, which works out some of the considerations at the end of Sect. 5 in a systematic way, makes use of the analysis in [5], to which we refer for more details. Let $\{\kappa_a\}_{a=0,1}$ denote the new spin frame field. If it is chosen such that the null vector $\kappa_0 \bar{\kappa}_{0'}$ is tangent to the null generators on $\mathcal{N}_p \setminus \{p\}$, the vectors $\kappa_0 \bar{\kappa}_{1'}$ and $\kappa_1 \bar{\kappa}_{0'}$ will be tangent to $\mathcal{N}_p \setminus \{p\}$ as well. Because such a frame field cannot have a direction independent limit at p, particular care has to be taken to construct this frame so near p that the resulting equations will still admit a convenient analysis near p. It will be required that the frame assumes regular limits at the point p if p is approached along the null generators of $\mathcal{N}_p \setminus \{p\}$. Let κ_a denote such a limit frame at p. It can be expanded in terms of the normal spin frame ι_A underlying our earlier analysis in the form $\kappa_a = \kappa^A{}_a \, \iota_A$. It will be convenient and implies no restriction to assume the spinors $\kappa^A{}_a$, $a = 0, 1$ or, in other words, the frame transformation matrix $(\kappa^A{}_a)_{A,a=0,1}$ to be normalized such that

$$\kappa^A{}_a \, \epsilon_{AB} \, \kappa^B{}_b = \epsilon_{ab}, \qquad \kappa^A{}_a \, \tau_{AB'} \, \bar{\kappa}^{B'}{}_{b'} = \tau_{ab'}. \tag{9.1}$$

Here $\tau_{AB'} = \sqrt{2} \, \alpha^0{}_{AA'} = \epsilon_A{}^0 \, \epsilon_{A'}{}^{0'} + \epsilon_A{}^1 \, \epsilon_{A'}{}^{1'}$, the quantities ϵ_{ab}, $\tau_{ab'}$ and $\alpha^\mu{}_{aa'}$ referring to the new frame take the same numerical values as ϵ_{AB}, $\tau_{AB'}$ and $\alpha^\mu{}_{AA'}$, and the small letter indices are treated in the same way as the large letter indices.

Because we did not specify the null generator along which the limit was taken, the conditions above characterize in fact a family of frames at p. To describe them in detail, denote by $SU(2)$ the Lie group given by the set of complex 2×2-matrices $(s^a{}_b)_{a,b=0,1}$ satisfying the conditions

$$s^a{}_c \, \epsilon_{ab} \, s^b{}_d = \epsilon_{bd}, \qquad s^a{}_c \, \tau_{ab'} \, \bar{s}^{b'}{}_{d'} = \tau_{cd'}. \tag{9.2}$$

Any $s \in SU(2)$ can be written in the form

$$s = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}, \qquad \alpha, \beta \in \mathbb{C}, \quad |\alpha|^2 + |\beta|^2 = 1, \tag{9.3}$$

and a basis of its Lie-algebra is given by the matrices

$$h = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad u_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad u_2 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{9.4}$$

The subgroup of $SU(2)$ consisting of the matrices

$$exp(\phi \, h) = \frac{1}{2}\begin{pmatrix} e^{i\,\frac{\phi}{2}} & 0 \\ 0 & e^{-i\,\frac{\phi}{2}} \end{pmatrix}, \qquad \phi \in \mathbb{R}, \tag{9.5}$$

will be denoted by $U(1)$. Comparing (9.1) with (9.2) shows that a complete parametrization of the transformation matrices $\kappa^A{}_a$ is obtained by setting $\kappa^A{}_a(s) = \delta^A{}_b \, s^b{}_a$ with $s \in SU(2)$. The corresponding frame spinors will be denote by $\kappa_a(s)$.

We shall make use of the left invariant vector fields Z_{u_1}, Z_{u_2}, Z_h generated by u_1, u_2, h and define the operators

$$Z_+ = -(Z_{u_2} + i \, Z_{u_1}), \quad Z_- = -(Z_{u_2} - i \, Z_{u_1}),$$

which satisfy the commutation relation $[Z_+, Z_-] = 2 \, i \, Z_h$. It should be noted that $SU(2)$ is a real but not a complex analytic Lie group and Z_{u_1}, Z_{u_2} must be considered as real vector fields while Z_+ and Z_- take values in the complexifications of the tangent spaces of $SU(2)$ and are complex conjugate to each other. If f is a complex-valued function on $SU(2)$ with complex conjugate $\bar{f}$ it holds thus $Z_\pm \, \bar{f} = \overline{Z_\mp \, f}$. In particular, if the $\kappa^A{}_a$ are considered as complex-valued functions on $SU(2)$ as indicated above we get

$$Z_+ \, \kappa^A{}_0 = 0, \qquad Z_+ \, \kappa^A{}_1 = \kappa^A{}_0, \qquad Z_- \, \kappa^A{}_0 = -\kappa^a{}_1, \qquad Z_- \, \kappa^A{}_1 = 0, \tag{9.6}$$

and if $\bar{\kappa}^{A'}{}_{a'}$ is its spinor complex conjugate we find with the rule above

$$Z_+ \, \bar{\kappa}^{A'}{}_{0'} = -\bar{\kappa}^{A'}{}_{1'}, \qquad Z_+ \, \bar{\kappa}^{A'}{}_{1'} = 0, \qquad Z_- \, \bar{\kappa}^{A'}{}_{0'} = 0, \qquad Z_- \, \bar{\kappa}^{A'}{}_{1'} = \bar{\kappa}^{A'}{}_{0'}. \tag{9.7}$$

Let $c^{\mu}{}_{aa'}(s) = e^{\mu}{}_{AA'}\,\kappa^{A}{}_{a}(s)\,\bar{\kappa}^{A'}{}_{a'}(s)$ be the frame field associated with κ_a at p and denote by S^2 the sphere $\{x^{\mu} \in T_pM / x_{\mu}\,x^{\mu} = 0, \sqrt{2}\,x^0 = 1\}$ in the tangent space of p. It holds

$$
\begin{aligned}
x_*^{\mu}(s) &\equiv c^{\mu}{}_{00'}(s) = \alpha^{\mu}_{aa'}\,s^{a}{}_{0}\,\bar{s}^{a'}{}_{0'} \\
&= \frac{1}{\sqrt{2}}\,(\delta^{\mu}{}_{0} + 2\,Re(\alpha\,\bar{\beta})\,\delta^{\mu}{}_{1} + 2\,Im(\alpha\,\bar{\beta})\,\delta^{\mu}{}_{2} + (|\alpha|^2 - |\beta|^2)\,\delta^{\mu}{}_{3}), \qquad (9.8)
\end{aligned}
$$

and $x_*^{\mu}(s\cdot t) = x_*^{\mu}(s)$ for all $t \in U(1)$. The Hopf map

$$
S^3 \sim SU(2) \ni s \rightarrow x_*^{\mu}(s) \in S^2,
$$

thus associates with the left cosets $s \cdot U(1)$, $s \in SU(2)$ the null directions $x_*^{\mu}(s)$. It will be assumed that the frame $\kappa_a(s)$ (resp. $c_{aa'}(s)$) is parallelly propagated along the null geodesic $\tau \rightarrow \tau\,x_*^{\mu}(s)$, $\tau \geq 0$, of $\mathcal{N}_p$. Because ι_A (resp. $e_{AA'}$) is a p-centered normal frame, it is related to the frame κ_a (resp. $c_{aa'}(s)$) along this curve by the τ-independent transformation $\kappa^{A}{}_{a}(s)$ (resp. $\kappa^{A}{}_{a}\,\bar{\kappa}^{A'}{}_{a'}(s)$, which corresponds to a rotation in $SO(3,\mathbb{R}) \sim SU(2))/\{1,-1\}$ that leaves the direction e_0 invariant). While the null directions $x_*^{\mu}(s)$ are invariant under the action of $U(1)$ the frames ι_a resp. $c_{aa'}$ are not and our prescription defines in fact a smooth bundle of frames $\iota_a(\tau, s)$ (resp. $c_{aa'}(\tau, s)$) over $\mathcal{N}_p\backslash\{p\}$ with projection $\pi : \iota_a(\tau, s) \rightarrow \tau\,x_*^{\mu}(s)$ (resp. $c_{aa'}(\tau, s) \rightarrow \tau\,x_*^{\mu}(s)$) and structure group $U(1)$ (resp. $U(1)/\{1,-1\}$). For simplicity we will concentrate in the following on the bundle of spin frames, the discussion of the bundle of vector frames being very similar. The parallel transport of the frames defines lifts of the null geodesics $\tau \rightarrow \tau\,x_*^{\mu}(s)$ to this bundle ('horizontal curves'). The tangent vector field defined by the lifts will be denoted by ∂_{τ} and τ will be considered as a coordinate on $\tilde{\mathcal{N}}_p$. In the limit as $\tau \rightarrow 0$ everything extends smoothly with the limits of the fibers corresponding to the left cosets of $SU(2)$ (in this sense the limit is even preserving the bundle structure). However, while the projection π has rank three over points of $\mathcal{N}_p\backslash\{p\}$, its rank drops to one in the limit to $\pi^{-1}(p)$. In the new setting this fact will be reflected by the singular behaviour at $\pi^{-1}(p)$ of the frame and the connection coefficients defined below. We denote the bundle in the following by $\tilde{\mathcal{N}}_p$ and consider it as a four dimensional smooth manifold with boundary $\pi^{-1}(p)$, the set of frames $c_{aa'}(s)$ at p, diffeomorphic to $\mathbb{R}^+_0 \times SU(2)$.

To discuss the field equations one could choose a local section of the Hopf fibration at p and push it forward with the flow of ∂_{τ} to generate a section of $\tilde{\mathcal{N}}_p$. Because the restriction of the projection π will then be a $1:1$ map away from $\pi^{-1}(p)$, it will then be obvious how to lift the frame field. However, apart from a subtlety which will be discussed in the proof of the second part of Proposition 9.1, it will in fact be more convenient to formulate the transport equations as equations on $\tilde{\mathcal{N}}_p$, as has been done in [5].

A suitable lift of the frame field can conveniently be discussed by introducing on $\tilde{\mathcal{N}}_p$ besides ∂_{τ} vector fields $X_{\pm}$ and S. Because the set $\pi^{-1}(p)$ is parametrized by $SU(2)$, the field $Z_{\pm}$ transfer naturally to this set. We set

$$
X_{\pm} = Z_{\pm}, \quad S = -2\,i\,Z_h \quad \text{on} \quad \pi^{-1}(p),
$$

and extend these fields to $\tilde{\mathcal{N}}_p$ by Lie transport so that

$$
[\partial_{\tau}, X_{\pm}] = 0, \quad [\partial_{\tau}, S] = 0.
$$

It follows then that S is tangent to the fibers of $\tilde{\mathcal{N}}_p$ and

$$X_{\pm}\,\tau = 0, \quad [X_+, X_-] = -S.$$

In fact, the first result follows from $0 = [\partial_\tau, X_\pm]\,\tau = \partial_\tau(X_\pm\,\tau) - X_\pm\,1 = \partial_\tau(X_\pm\,\tau)$ and the observation that $\lim_{\tau\to 0} X_\pm\,\tau = 0$ because the fields $X_\pm$ become in this limit tangent to the set $\pi^{-1}(p)$ on which τ vanishes. The second result follows because it is satisfied in the limit as $\tau \to 0$ and because the definitions imply that $[\partial_\tau, [X_+, X_-] + S] = 0$ on $\tilde{\mathcal{N}}_p$. Because the images of the fields $Z_\pm$ under the Hopf map are linearly independent, the images of the fields $X_\pm$ under the projection π will be linearly independent for $\tau > 0$ (and sufficiently small that no caustic points will be met).

The scalar fields Ω and s lift from $\mathcal{N}_p$ to $\tilde{\mathcal{N}}_p$ by simple pull-back under the projection map. The fields ψ_{ABCD} and $\Phi_{ABA'B'}$ are in addition subject to a frame transformation so that they are related to the lifted fields by $\psi_{abcd}(\tau, s) = \psi_{ABCD}(\tau\, x^\mu_*(s))\,\kappa^A{}_a(s) \ldots \kappa^D{}_d(s)$ and $\Phi_{aba'b'}(\tau, s) = \psi_{ABA'B'}(\tau\, x^\mu_*(s))\,\kappa^A{}_a(s) \ldots \bar{\kappa}^{B'}{}_{b'}(s)$.

Only the fields $c_{aa'} = e_{AA'}\,\kappa^A{}_a\,\kappa^{A'}{}_{a'}$ with $aa' \neq 11'$ are tangent to $\mathcal{N}_p$ at the points $\tau\, x^\mu_*(s)$ with $\tau > 0$. Lifts of these tangent vector fields on $\mathcal{N}_p$ to points of $\tilde{\mathcal{N}}_p$ are not immediately well defined because the kernel of the projection π is one-dimensional. For $\tau > 0$ there exist, however, unique lifts $\tilde{c}_{aa'}$ for $aa' \neq 11'$, i.e. fields satisfying $T\pi(\tilde{c}_{aa'}) = c_{aa'}$, that can be expanded in terms of the vector fields ∂_τ, X_+, X_-. Because $c_{00'}$ is tangent to the null geodesics of $\mathcal{N}_p$, it follows then immediately that $\tilde{c}_{aa'} = \partial_\tau$. To analyse the precise behaviour of $\tilde{c}_{aa'}$ as $\tau \to 0$, we observe that by our earlier discussions $e^\mu{}_{AA'} = \alpha^\mu{}_{AA'} + O(|x|)$ as $x^\mu \to 0$, which gives $c^\mu{}_{aa'} = \alpha^\mu{}_{AA'}\,\kappa^A{}_a\,\kappa^{A'}{}_{a'} + O(|\tau|)$ in this limit. For any smooth function $f = f(x^\mu)$ we find thus with $x^\mu = \tau\, x^\mu_*(s)$ and (9.3)

$$f_{,\mu}\, c^\mu{}_{aa'} = f_{,\mu}\, \alpha^\mu{}_{AA'}\,\kappa^A{}_a\,\kappa^{A'}{}_{a'} + O(|\tau|) \quad \text{for} \quad aa' \neq 11'.$$

To see how this is related to the action of the vector field X_+ on the lift of this function to $\tilde{\mathcal{N}}_p$, we observe that the vector fields $X_\pm$ inherit properties of the fields $Z_\pm$ such as (9.6), (9.7) and find with (9.8)

$$X_+ f = \tau\, f_{,\mu}\, X_+(\alpha^\mu{}_{AA'}\,\kappa^A{}_0\,\bar{\kappa}^{A'}{}_{0'}) = -\tau\, f_{,\mu}\,\alpha^\mu{}_{AA'}\,\kappa^A{}_0\,\bar{\kappa}^{A'}{}_{1'},$$

and similarly

$$X_- f = -\tau\, f_{,\mu}\,\alpha^\mu{}_{AA'}\,\kappa^A{}_1\,\bar{\kappa}^{A'}{}_{0'},$$

so that we can write

$$f_{,\mu}\, c^\mu{}_{01'} = -\frac{1}{\tau}\, X_+ f + O(|\tau|), \qquad f_{,\mu}\, c^\mu{}_{10'} = -\frac{1}{\tau}\, X_- f + O(|\tau|).$$

It follows that the lifted fields with $aa' \neq 11'$ must have expansions of the form

$$\tilde{c}_{aa'} = \epsilon_a{}^0\,\epsilon_{a'}{}^{0'}\,\partial_\tau - \frac{1}{\tau}\,(\epsilon_a{}^0\,\epsilon_{a'}{}^{1'}\,X_+ + \epsilon_a{}^1\,\epsilon_{a'}{}^{0'}\,X_-) + c^*_{aa'}, \tag{9.9}$$

with

$$c^*_{aa'} = b_{aa'}\,X_+ + \bar{b}_{aa'}\,X_- + r_{aa'}\,\partial_\tau, \tag{9.10}$$

and complex fields $b_{aa'}$ and $r_{aa'}$ satisfying

$$\bar{r}_{aa'} = r_{aa'}, \quad b_{00'} = 0, \quad r_{00'} = 0, \quad b_{aa'} = O(|\tau|), \quad r_{aa'} = O(|\tau|). \tag{9.11}$$

Because there has not been specified a rule how to extend the new coordinates and the fields $\tilde{c}_{aa'}$ off $\tilde{\mathcal{N}}_p$, there cannot be given an explicit coordinate expression for the field $\tilde{c}_{11'}$. It should be noted, however, that the field $\tilde{c}_{11'}$ is determined on $\tilde{\mathcal{N}}_p$ once the fields $\tilde{c}_{aa'}$, $aa' \neq 11'$, are known there.

If it is assumed that the relation $c_{aa'} = e_{AA'}\,\kappa^A{}_a\,\kappa^{A'}{}_{a'}$ holds in a full neighbourhood of the point p with an x^μ-dependent transformation matrix $\kappa^A{}_a$ and it is used that $\kappa^a{}_A \equiv \epsilon^{ab}\,\kappa^B{}_b\,\epsilon_{BA}$ satisfies $\kappa^A{}_a\,\kappa^a{}_B = -\epsilon_B{}^A$, the well known transformation law which relates the connection coefficients $\tilde{\Gamma}_{aa'bc}$ with respect to the frame $c_{aa'}$ to the connection coefficients $\Gamma_{AA'BC}$ with respect to the frame $e_{AA'}$ is obtained in the form

$$\tilde{\Gamma}_{aa'bc} = -\kappa^B{}_b\,\epsilon_{BC}\,\kappa^C{}_{c,\mu}\,c^\mu{}_{aa'} + \Gamma_{AA'BC}\,\kappa^A{}_a\,\kappa^{A'}{}_{a'}\kappa^B{}_b\,\kappa^C{}_c.$$

Under our assumptions the derivatives $\kappa^C{}_{c,\mu}\,c^\mu{}_{aa'}$ are defined on $\tilde{\mathcal{N}}_p$ only for $aa' \neq 11'$ so that the formula above can only be used under this restriction. With (9.6) and (9.9) it follows then that

$$\tilde{\Gamma}_{aa'bc} = -\frac{1}{\tau}(\epsilon_a{}^0\,\epsilon_{a'}{}^{1'}\,\epsilon_b{}^1\,\epsilon_c{}^1 + \epsilon_a{}^1\,\epsilon_{a'}{}^{0'}\,\epsilon_b{}^0\,\epsilon_c{}^0) + \Gamma_{aa'bc} \quad \text{for} \quad aa' \neq 11', \tag{9.12}$$

with a complex-valued field $\Gamma_{aa'bc}$ that satisfies

$$\Gamma_{00'bc} = 0, \qquad \Gamma_{aa'bc} = O(|\tau|), \tag{9.13}$$

so that

$$\tilde{\Gamma}_{00'bc} = 0.$$

In this form the coefficients lift to $\tilde{\mathcal{N}}_p$. As discussed in [5], the coefficients $\tilde{\Gamma}_{aa'bc}$ are in fact obtained by contracting the connection form on the bundle of frames with the frame field $\tilde{c}_{aa'}$.

On $\tilde{\mathcal{N}}_p$ the covariant derivative in the direction of $\tilde{c}_{aa'}$, $aa' \neq 11'$, which will be denoted by $\tilde{\nabla}_{aa'}$, is now given with (9.9), (9.12) by the same rule as known on the base space so that e.g.

$$\tilde{\nabla}^d{}_{0'}\,\psi_{abcd} = \epsilon^{de}\,(\tilde{c}_{e0'}(\psi_{abcd}) - \tilde{\Gamma}_{e0'}{}^f{}_{(a}\,\psi_{bcd)f}).$$

It will be convenient to introduce $\Sigma_{AA'} = \nabla_{AA'}\Omega$ as an additional unknown tensor field. Because no rule has been specified to extend the new coordinates and the fields $\tilde{c}_{aa'}$ away from $\tilde{\mathcal{N}}_p$, there cannot be given an explicit coordinate expression for the derivative of Ω in the direction of $\tilde{c}_{11'}$. Because the field $\tilde{c}_{11'}$ is determined on $\tilde{\mathcal{N}}_p$ once the fields $\tilde{c}_{aa'}$, $aa' \neq 11'$, are known there, the field $\Sigma_{aa'}(\tau, s) = \Sigma_{AA'}\,\kappa^A{}_a\,\bar{\kappa}^{B'}{}_{a'}$ can still be discussed as a tensor field on $\tilde{\mathcal{N}}_p$.

We are in a position now to obtain the expressions for the transport equations induced on $\tilde{\mathcal{N}}_p$ in the new gauge and to prove the following result.

Proposition 9.1. *In the conformal gauge (4.3), (4.4) the transport equations induced on $\tilde{\mathcal{N}}_p$ by the conformal field equations and the structural equations uniquely determine the fields Ω, Π, $\Phi_{aba'b'}$ and ψ_{abcd} on $\tilde{\mathcal{N}}_p$ once the radiation field*

$$\psi_0(\tau, s) = \kappa^A{}_0\,\kappa^B{}_0\,\kappa^C{}_0\,\kappa^D{}_0\,\psi_{ABCD}|_{x^\mu = \tau\,\alpha^\mu_{EE'}\,\kappa^E{}_0\,\bar{\kappa}^{E'}{}_{0'}}, \tag{9.14}$$

is prescribed there.

The fields so obtained also satisfy the inner constraint equations on $\tilde{\mathcal{N}}_p$.

Remark 9.2. A similar result can be obtained in the vacuum case $\Omega = 1$. The discussion of that case is more complicated than the one below because then the conformal Weyl tensor does not necessarily vanish on $\tilde{\mathcal{N}}_p$. We do not work out the details here.

Proof. The gauge conditions (4.3), (4.4) read in the present setting

$$\Omega = 0, \quad \Sigma_{aa'} = 0, \quad \Pi = \Pi_* \equiv 2\,\eta_{00} \quad \text{on} \quad \pi^{-1}(p), \tag{9.15}$$

$$\Phi_{000'0'} = 0, \quad \Lambda = 0 \quad \text{on} \quad \tilde{\mathcal{N}}_p. \tag{9.16}$$

The transport equations induced by (6.1), i.e. the equations which involve the directional derivative $\tilde{c}_{00'}$ imply in particular

$$\partial_\tau\, \Omega = \Sigma_{00'}, \quad \partial_\tau\, \Sigma_{00'} = 0,$$

and thus $\Omega = 0$, $\Sigma_{00'} = 0$ on $\tilde{\mathcal{N}}_p$. With this it follows further

$$\partial_\tau\, \Sigma_{01'} = 0, \quad \partial_\tau\, \Sigma_{10'} = 0,$$

whence $\Sigma_{01'} = 0$, $\Sigma_{10'} = 0$ on $\tilde{\mathcal{N}}_p$. The transport equations induced by (6.1), (6.2) then finally imply

$$\partial_\tau\, \Sigma_{11'} = \Pi, \quad \partial_\tau\, \Pi = 0,$$

and thus $\Sigma_{11'} = \tau\, \Pi_*$, $\Pi = \Pi_*$ on $\tilde{\mathcal{N}}_p$. Collecting results we find

$$\Omega = 0, \quad \Sigma_{aa'} = \tau\, \Pi_*\, \epsilon_a{}^1\, \epsilon_{a'}{}^{1'}, \quad \Pi = \Pi_* \quad \text{on} \quad \tilde{\mathcal{N}}_p. \tag{9.17}$$

The transport equations induced by the torsion free conditions are given by

$$0 = \tilde{t}_{bb'\,aa'} = [\tilde{c}_{bb'}, \tilde{c}_{aa'}] - (\tilde{\Gamma}_{bb'}{}^{ee'}{}_{aa'} - \tilde{\Gamma}_{aa'}{}^{ee'}{}_{bb'})\, \tilde{c}_{ee'},$$

with $bb' = 00'$ and $aa' \neq 11'$. Inserting here expressions (9.9), (9.12) and setting the factors of ∂_τ, X_+, X_- in the resulting equation separately equal to zero shows that the content of this equation is equivalent to the conditions

$$\partial_\tau b_{aa'} + \frac{1}{\tau}\, b_{aa'} + \frac{1}{\tau}\, \bar{\Gamma}_{aa'0'0'} = \Gamma_{aa'00}\, b_{10'} + \bar{\Gamma}_{aa'0'0'}\, b_{01'}, \tag{9.18}$$

$$\partial_\tau r_{aa'} + \frac{1}{\tau}\, r_{aa'} = \Gamma_{aa'00}\, r_{10'} + \bar{\Gamma}_{aa'0'0'}\, r_{01'} - \Gamma_{aa'01} - \bar{\Gamma}_{aa'0'1'}, \tag{9.19}$$

(which are satisfied identically for $aa' = 00'$).

The Ricci identity is given for $cc', dd' \neq 11'$ on $\tilde{\mathcal{N}}_p$ by

$$\tilde{c}_{cc'}(\tilde{\Gamma}_{dd'ab}) - \tilde{c}_{dd'}(\tilde{\Gamma}_{cc'ab}) + \tilde{\Gamma}_{cc'af}\, \tilde{\Gamma}_{dd'}{}^f{}_b - \tilde{\Gamma}_{dd'af}\, \tilde{\Gamma}_{cc'}{}^f{}_b$$
$$-(\tilde{\Gamma}_{cc'}{}^{ff'}{}_{dd'} - \tilde{\Gamma}_{dd'}{}^{ff'}{}_{cc'})\, \tilde{\Gamma}_{ff'ab} = \Omega\, \psi_{abcd}\epsilon_{c'd'} + \Phi_{abc'd'}\, \epsilon_{cd}.$$

With (9.17), $\tilde{\Gamma}_{00'ab} = 0$ and $\tilde{c}_{00'} = \partial_\tau$ it follows

$$\partial_\tau \tilde{\Gamma}_{10'ab} - \tilde{\Gamma}_{10'00}\, \tilde{\Gamma}_{10'ab} - \bar{\tilde{\Gamma}}_{10'0'0'}\, \tilde{\Gamma}_{01'ab} = \Phi_{ab0'0'},$$
$$\partial_\tau \tilde{\Gamma}_{01'ab} - \tilde{\Gamma}_{01'00}\, \tilde{\Gamma}_{10'ab} - \bar{\tilde{\Gamma}}_{01'0'0'}\, \tilde{\Gamma}_{01'ab} = 0,$$

and thus with (9.9), (9.12)

$$\partial_\tau \Gamma_{10'ab} + \frac{1}{\tau}\left\{\Gamma_{10'ab} - \Gamma_{10'00}\,\epsilon_a{}^0\,\epsilon_b{}^0 + \bar{\Gamma}_{10'0'0'}\,\epsilon_a{}^1\,\epsilon_b{}^1\right\}$$
$$= \Gamma_{10'00}\,\Gamma_{10'ab} + \bar{\Gamma}_{10'0'0'}\,\Gamma_{01'ab} + \Phi_{ab0'0'}, \tag{9.20}$$
$$\partial_\tau \Gamma_{01'ab} + \frac{1}{\tau}\left\{\Gamma_{01'ab} + \Gamma_{01'00}\,\epsilon_a{}^0\,\epsilon_b{}^0 + \bar{\Gamma}_{01'0'0'}\,\epsilon_a{}^1\,\epsilon_b{}^1\right\}$$
$$= \Gamma_{01'00}\,\Gamma_{10'ab} + \bar{\Gamma}_{01'0'0'}\,\Gamma_{01'ab}. \tag{9.21}$$

The transport equations induced by (6.3) are $\tilde{\nabla}_0{}^{c'}\,\Phi_{bcb'c'} = \psi_{bcd0}\,\Sigma^d{}_{b'}$ or, more explicitly,

$$\partial_\tau \Phi_{bcb'1'} + \frac{1}{\tau}\left\{X_+\Phi_{bcb'0'} - 2\,\epsilon_{(b}{}^1\,\Phi_{c)0b'0'} + \epsilon_{b'}{}^{0'}\,\Phi_{bc1'0'} + \Phi_{bcb'1'}\right\} - c^*_{01'}(\Phi_{bcb'0'})$$
$$= -2\,\Gamma_{01'}{}^f{}_{(b}\,\Phi_{c)fb'0'} - \bar{\Gamma}_{01'}{}^{f'}{}_{b'}\,\Phi_{bcf'0'} - \bar{\Gamma}_{01'}{}^{f'}{}_{0'}\,\Phi_{bcb'f'} - \tau\,\Pi_*\,\psi_{bc00}\,\epsilon_{b'}{}^{1'}, \tag{9.22}$$

while the transport equations induced by (6.4) are $\tilde{\nabla}^d{}_{0'}\,\psi_{abcd} = 0$, or, more explicitly,

$$\partial_\tau\,\psi_{abc1} + \frac{1}{\tau}\left\{X_-\psi_{abc0} + 3\,\epsilon_{(a}{}^0\,\psi_{bc)01} + \psi_{abc1}\right\} - c^*_{10'}(\psi_{abc0})$$
$$= -3\,\Gamma_{10'}{}^f{}_{(a}\,\psi_{bc)f0} - \Gamma_{10'}{}^f{}_0\,\psi_{abcf}. \tag{9.23}$$

While the initial data at $\tau = 0$ are given for $b_{aa'}$, $r_{aa'}$, $\Gamma_{aa'bc}$ by (9.11) and (9.13), they still have to be specified for $\Phi_{aba'b'}$, ψ_{abcd}. In principle they can be read off from the formal expansions determined earlier but we give a different argument because it sheds some light on the content of the equations. It is convenient here to use the 'essential components' $\psi_k = \kappa^A{}_{(a}\,\kappa^B{}_b\,\kappa^C{}_c\,\kappa^D{}_{d)_k}\,\psi_{ABCD}(0))$ which are obtained by setting k of the lower indices in brackets equal to 1 and the remaining ones equal to 0. Because the vector fields $X_\pm$ approach in the limit $\tau \to 0$ the vector fields $Z_\pm$, it follows with (9.6) and (9.14)

$$\lim_{\tau\to 0} X_-\psi_0 = Z_-(\kappa^A{}_0\,\kappa^B{}_0\,\kappa^C{}_0\,\kappa^D{}_0)\,\psi_{ABCD}(0) = -4\,\lim_{\tau\to 0}\psi_1,$$

and, more generally,

$$\lim_{\tau\to 0} X_-\psi_k = -(4-k)\,\lim_{\tau\to 0}\psi_{k+1}, \qquad k = 0, \ldots, 4.$$

In the notation of (9.23) this is precisely the relation

$$\lim_{\tau\to 0}(X_-\psi_{abc0} + 3\,\epsilon_{(a}{}^0\,\psi_{bc)01} + \psi_{abc1}) = 0.$$

It allows one to determine the initial data $\psi_{abcd}(0)$ from the radiation field and at the same time ensures that the formally singular term in (9.23) admits a limit as $\tau \to 0$ along any given given null generator of $\tilde{\mathcal{N}}_p$. Similarly one can determine by X_+ and X_- operations the values of $\lim_{\tau\to 0}\Phi_{aba'b'}$ from $\Phi_{000'0'}$ with the result that

$$\lim_{\tau\to 0}(X_+\Phi_{bcb'0'} - 2\,\epsilon_{(b}{}^1\,\Phi_{c)0b'0'} + \epsilon_{b'}{}^{0'}\,\Phi_{bc1'0'} + \Phi_{bcb'1'}) = 0,$$

so that the formally singular term in (9.22) admits a limit along a fixed null generator. However, because $\Phi_{000'0'} = 0$ on $\tilde{\mathcal{N}}_p$ by (9.16), it follows that

$$\lim_{\tau \to 0} \Phi_{aba'b'} = 0.$$

The gauge condition (9.16) and the vanishing of the Weyl tensor on $\tilde{\mathcal{N}}_p$ lead to simplifications. With this (9.22) implies

$$\partial_\tau \Phi_{000'1'} + \frac{2}{\tau} \Phi_{000'1'} = 2\, \Gamma_{01'00}\, \Phi_{010'0'} + 2\, \bar{\Gamma}_{01'0'0'}\, \Phi_{000'1'}.$$

Because $\Phi_{010'0'}$ is by assumption the complex conjugate of $\Phi_{000'1'}$ it follows that

$$\Phi_{000'1'} = 0, \quad \Phi_{010'0'} = 0 \quad \text{on} \quad \tilde{\mathcal{N}}_p. \tag{9.24}$$

Equation (9.21) implies the coupled system

$$\partial_\tau \Gamma_{01'00} + \frac{2}{\tau} \Gamma_{01'00} = (\Gamma_{10'00} + \bar{\Gamma}_{01'0'0'})\, \Gamma_{01'00},$$
$$\partial_\tau \Gamma_{01'01} + \frac{1}{\tau} \Gamma_{01'01} = \Gamma_{10'01}\, \Gamma_{01'00} + \bar{\Gamma}_{01'0'0'}\, \Gamma_{01'01},$$

for $\Gamma_{01'00}$ and $\Gamma_{01'01}$ whence

$$\Gamma_{01'00} = 0, \quad \Gamma_{01'01} = 0 \quad \text{on} \quad \tilde{\mathcal{N}}_p. \tag{9.25}$$

With (9.16), (9.24), (9.25) equation (9.20) implies

$$\partial_\tau \Gamma_{10'00} = \Gamma_{10'00}\, \Gamma_{10'00},$$
$$\partial_\tau \Gamma_{10'01} + \frac{1}{\tau} \Gamma_{10'01} = \Gamma_{10'00}\, \Gamma_{10'01},$$

from which we conclude that

$$\Gamma_{10'00} = 0, \quad \Gamma_{10'01} = 0 \quad \text{on} \quad \tilde{\mathcal{N}}_p. \tag{9.26}$$

With this the remaining equations of (9.21) and (9.20) read

$$\begin{aligned} \partial_\tau \Gamma_{01'11} + \frac{1}{\tau} \Gamma_{01'11} &= 0, \\ \partial_\tau \Gamma_{10'11} + \frac{1}{\tau} \Gamma_{10'11} &= \Phi_{110'0'}, \end{aligned} \tag{9.27}$$

which give

$$\Gamma_{01'11} = 0, \quad \Gamma_{10'11} = \frac{1}{\tau} \int_0^\tau \tau'\, \Phi_{110'0'}\, d\tau' \quad \text{on} \quad \tilde{\mathcal{N}}_p. \tag{9.28}$$

With these results it follows from (9.18), (9.19) that

$$b_{aa'} = 0, \quad r_{aa'} = 0, \quad c^*_{aa'} = 0 \quad \text{on} \quad \tilde{\mathcal{N}}_p \quad \text{for} \quad aa' \neq 11'. \tag{9.29}$$

With the resulting simplifications equations (9.22) read

$$\partial_\tau \Phi_{010'1'} + \frac{2}{\tau}\,\Phi_{010'1'} = 0,$$

$$\partial_\tau \Phi_{001'1'} + \frac{1}{\tau}\,\Phi_{001'1'} = -\tau\,\Pi_*\,\psi_{0000},$$

$$\partial_\tau \Phi_{011'1'} + \frac{1}{\tau}\,\{X_+\Phi_{010'1'} + \Phi_{011'1'}\} = -\tau\,\Pi_*\,\psi_{0001},$$

$$\partial_\tau \Phi_{110'1'} + \frac{1}{\tau}\,\{X_+\Phi_{110'0'} + 2\,\Phi_{110'1'}\} = 0,$$

$$\partial_\tau \Phi_{111'1'} + \frac{1}{\tau}\,\{X_+\Phi_{110'1'} + \Phi_{111'1'}\} = -2\,\Gamma_{01'11}\,\Phi_{010'1'} - \bar{\Gamma}_{01'1'1'}\,\Phi_{110'0'} - \tau\,\Pi_*\,\psi_{1100}.$$

The first three of these equations imply

$$\Phi_{010'1'} = 0, \quad \Phi_{001'1'} = -\frac{\Pi_*}{\tau}\int_0^\tau \tau'^2\,\psi_{0000}\,d\tau',$$
$$\Phi_{011'1'} = -\frac{\Pi_*}{\tau}\int_0^\tau \tau'^2\,\psi_{0001}\,d\tau' \quad \text{on} \quad \tilde{\mathcal{N}}_p. \tag{9.30}$$

Explicit expressions can also be obtained for the solutions of the remaining equations. In particular, imposing the reality conditions, using in the forth equation the expression for $\Phi_{001'1'}$ given by (9.30), and observing that $X_\pm\tau = 0$ gives for $\Phi_{011'1'}$ the alternative expression

$$\Phi_{011'1'} = \frac{\Pi_*}{\tau^2}\int_0^\tau \left(\int_0^{\tau'} \tau''^2 X_-\psi_{0000}\,d\tau''\right) d\tau'. \tag{9.31}$$

Comparing this with the expression in (9.30), it is seen that consistency requires

$$\partial_\tau \psi_{0001} + \frac{1}{\tau}\,\{X_-\,\psi_{0000} + 4\,\psi_{0001}\} = 0,$$

which is in fact the first of the equations which follow.

With the results obtained so far the transport equations (9.23) read

$$\partial_\tau\,\psi_{0001} + \frac{1}{\tau}\,\{X_-\psi_{0000} + 4\,\psi_{0001}\} = 0, \tag{9.32}$$

$$\partial_\tau\,\psi_{0011} + \frac{1}{\tau}\,\{X_-\psi_{0010} + 3\,\psi_{0011}\} = -\Gamma_{10'11}\,\psi_{0000}, \tag{9.33}$$

$$\partial_\tau\,\psi_{0111} + \frac{1}{\tau}\,\{X_-\psi_{0011} + 2\,\psi_{0111}\} = -2\,\Gamma_{10'11}\,\psi_{0001}, \tag{9.34}$$

$$\partial_\tau\,\psi_{1111} + \frac{1}{\tau}\,\{X_-\psi_{0111} + \psi_{1111}\} = -3\,\Gamma_{10'11}\,\psi_{0011}. \tag{9.35}$$

Equation (9.32) has the regular solution

$$\psi_{0001} = -\frac{1}{\tau^4}\int_0^\tau \tau'^3\,X_-\psi_{0000}\,d\tau'.$$

With (9.28), (9.30) one obtains

$$\Gamma_{10'11} = -\frac{\Pi_*}{\tau} \int_0^\tau \left(\int_0^{\tau'} \tau''^2 \, \bar{\psi}_{0'0'0'0'} \, d\tau'' \right) d\tau', \tag{9.36}$$

which allows one to obtain successively integral expressions for the remaining components of ψ_{abcd} on $\tilde{\mathcal{N}}_p$. This completes the proof of the first part of the Proposition.

Equations (6.1) and (6.2) imply the inner constraints

$$0 = \tilde{c}_{01'}(\Sigma_{bb'}) - \tilde{\Gamma}_{01'}{}^{f}{}_{b} \Sigma_{fb'} - \bar{\tilde{\Gamma}}_{01'}{}^{f'}{}_{b'} \Sigma_{bf'} + \Omega \, \Phi_{0b1'b'} - \Pi \, \epsilon_{0b} \, \epsilon_{1'b'},$$
$$0 = \tilde{c}_{01'}(\Pi) + \Sigma^{bb'} \, \Phi_{0b1'b'},$$

and their complex conjugates. A direct calculation using (9.17), (9.25), (9.26) shows that they are indeed satisfied on $\tilde{\mathcal{N}}_p$.

There do not arise inner constraints from (6.3), (6.4). Those which have not been discussed yet contain the operator $\tilde{c}_{11'}$ and thus differentiations in directions transverse to $\mathcal{N}_p$.

Inner constraints are implied by the torsion-free condition and the Ricci identity. Formula (2.2) suggests that the torsion free condition should read on $\tilde{\mathcal{N}}_p$

$$0 = \left\{ [\tilde{c}_{01'}, \tilde{c}_{10'}] - (\tilde{\Gamma}_{01'}{}^{ee'}{}_{10'} - \tilde{\Gamma}_{10'}{}^{ee'}{}_{01'}) \, \tilde{c}_{ee'} \right\} \tag{9.37}$$

There arises, however, a subtlety because the commutator of the fields $\tilde{c}_{01'}$ and $\tilde{c}_{10'}$ contributes a component which is tangential to the fibers of $\tilde{\mathcal{N}}_p$. One way to deal this problem is to follow the torsion-free condition in the form (2.4) and test whether the operator above applied to a function f vanishes if this function is the lift of a scalar function on $\mathcal{N}_p$, whence constant on the fibres. For reasons which become clear when we discuss the Ricci identity, we prefer a different procedure. If the operator (2.2) is lifted according to our rules, it should not contain a vertical part and therefore the formula above should be corrected by subtracting the vertical part supplied by the commutator. By (9.29) the commutator is, however, totally vertical,

$$[\tilde{c}_{01'}, \tilde{c}_{10'}] = \frac{1}{\tau^2} \, [X_+, X_-] = -\frac{1}{\tau^2} \, S,$$

and thus drops out after the correction altogether (as it does if applied to the lift of a scalar function). A second subtlety arises because the relation above appears to involve the operator $\tilde{c}_{11'}$ which suggests that it is not an inner condition on $\tilde{\mathcal{N}}_p$. With (9.25), (9.26) and with (9.28), which states that $\Gamma_{01'11}$ as well as its complex conjugate $\bar{\Gamma}_{10'1'1'}$ vanishes, it follows, however that not only the factor of $\tilde{c}_{11'}$ vanishes but that $(\tilde{\Gamma}_{01'}{}^{ee'}{}_{10'} - \tilde{\Gamma}_{10'}{}^{ee'}{}_{01'}) = 0$ for arbitrary indices ee'. The inner constraint induced by the torsion free conditions is thus indeed satisfied on $\tilde{\mathcal{N}}_p$.

The problem arising from the commutator of $\tilde{c}_{01'}$ and $\tilde{c}_{10'}$ also affects the discussion of the inner constraints induced by the Ricci identity. If one calculates the spinor analogue of (2.1), which reads for the components of interest here

$$(\tilde{\nabla}_{01'} \tilde{\nabla}_{10'} - \tilde{\nabla}_{10'} \tilde{\nabla}_{01'}) \lambda^a = r^a{}_{b01'10'} \, \lambda^b - t_{01'}{}^{ee'}{}_{10'} \, \nabla_{ee'} \lambda^a,$$

one finds that the second term on the right hand side contains a term of the form

$$[\tilde{c}_{01'}, \tilde{c}_{10'}](\lambda^a) - (\tilde{\Gamma}_{01'}{}^{ee'}{}_{10'} - \tilde{\Gamma}_{10'}{}^{ee'}{}_{01'}) \, \tilde{c}_{ee'}(\lambda^a).$$

Performing here the replacement $[\tilde{c}_{01'}, \tilde{c}_{10'}] \to [\tilde{c}_{01'}, \tilde{c}_{10'}](\lambda^a) + \frac{1}{\tau^2} S(\lambda^a)$ and then ignoring the torsion term as suggested above, has to be compensated by the replacement

$$r^a{}_{b01'10'}\,\lambda^b \to r^a{}_{b01'10'}\,\lambda^b - \frac{1}{\tau^2}\,S\,\lambda^a,$$

of the curvature term. To show that the inner constraint induced by the Ricci identity vanishes, we have to take into account the corrected curvature term.

Under the action of the group $U(1)$ the frame κ_a transforms as $\kappa_a \to \kappa_b\,(exp(\phi\,h))^b{}_a$ and the components of a spinor field $\lambda = \lambda^a\,\kappa_a$ transform thus as $\lambda^a \to (exp(-\phi\,h))^a{}_b\,\lambda^b$. This implies that

$$S\,\lambda^a = -2\,i\,\frac{d}{d\phi}((exp(-\phi\,h)^a{}_b\,\lambda^b)|_{\phi=0} = 2\,i\,h^a{}_b\lambda^b,$$

with $(h^a{}_b)_{a,b=0,1}$ denoting the matrix h in (9.4). The equation which should be checked thus reads

$$0 = \tilde{c}_{01'}(\tilde{\Gamma}_{10'ab}) - \tilde{c}_{10'}(\tilde{\Gamma}_{01'ab}) + \tilde{\Gamma}_{01'af}\,\tilde{\Gamma}_{10'}{}^f{}_b - \tilde{\Gamma}_{10'af}\,\tilde{\Gamma}_{01'}{}^f{}_b$$
$$-(\tilde{\Gamma}_{01'}{}^{ff'}{}_{10'} - \tilde{\Gamma}_{10'}{}^{ff'}{}_{01'})\,\tilde{\Gamma}_{ff'ab} - \frac{2\,i}{\tau^2}\,h_{ab} - \Omega\,\psi_{ab01}\epsilon_{1'0'} - \Phi_{ab1'0'}\,\epsilon_{01},$$

where we set $h_{ab} = h^c{}_b\,\epsilon_{ca}$. In the cases $ab = 00$ and $ab = 01$ a direct calculation using the results obtained above shows that this condition is indeed satisfied on $\tilde{\mathcal{N}}_p$. The case $ab = 11$ is slightly more difficult. With the given results it readily reduces to the condition

$$0 = -\frac{1}{\tau}\,X_+\tilde{\Gamma}_{10'11} - \Phi_{1101'}.$$

Observing (9.36), taking the complex conjugate, and using (9.31) shows that the condition is indeed satisfied. This proves the second assertion of the Proposition. □

9.1. The fields on $\mathcal{N}_p$ in the normal gauge. In the first part of this section, 4 has been shown that there is associated with the radiation field (5.4), which reads in the present notation

$$\psi_0(\tau, s) = \kappa^A{}_0\,\kappa^B{}_0\,\kappa^C{}_0\,\kappa^D{}_0\,\psi^*_{ABCD}(\tau\,\alpha^\mu_{EE'}\,\kappa^E{}_0\,\bar{\kappa}^{E'}{}_{0'}),$$

a unique set of fields

$$\Omega,\ \Sigma_{aa'},\ \Pi,\ \Phi_{aba'b'},\ \psi_{abcd},\quad \text{and}\quad \tilde{c}_{aa'},\ \tilde{\Gamma}_{aa'bc},\quad aa' \neq 11', \tag{9.38}$$

on $\tilde{\mathcal{N}}_p$ which satisfy the transport equations and the inner constraints induced by the conformal field equations so that the 0000 components of ψ_{abcd} coincides with $\psi_0(\tau, s)$. Apart from the explicitly described singular terms of $\tilde{c}_{aa'}$ and $\tilde{\Gamma}_{aa'bc}$ these fields are smooth functions of τ and $s \in SU(2)$. On the other hand, it has been shown in Sects. 6 to 8 that with the *null data derived from ψ_0 at p* can be associated fields

$$\hat{\Omega},\ \hat{\Sigma}_{AA'},\ \hat{\Pi},\ \hat{\Phi}_{ABA'B'},\ \hat{\psi}_{ABCD},\ \hat{e}^\mu{}_{AA'},\ \hat{\Gamma}_{AA'BC}, \tag{9.39}$$

which are defined and smooth on a neighbourhood of p, satisfy at p the conformal field equations at all orders, and which have ∞-jets at p which are uniquely determined by this property and the requirement that null data derived from ψ_0 at p coincide with null data at p derived from

$$\hat{\psi}_0(\tau, s) = \kappa^A{}_0 \kappa^B{}_0 \kappa^C{}_0 \kappa^D{}_0 \hat{\psi}_{ABCD}(\tau\, \alpha^\mu_{EE'} \kappa^E{}_0 \bar{\kappa}^{E'}{}_{0'}).$$

While the Taylor expansions of these functions at p are fixed uniquely, they are fairly arbitrary away from p.

To understand the relations between these two sets of fields, we consider the fields (9.39) at the points $x^\mu = \tau\, \alpha^\mu_{EE'} \kappa^E{}_0 \bar{\kappa}^{E'}{}_{0'}$ of $\mathcal{N}_p$ and use the τ-independent frame transformation $\kappa^A{}_a$ employed in Sect. 9 to express the fields (9.39) in terms of the adapted frame to obtain on $\mathbb{R}^+_0 \times SU(2) \sim \tilde{\mathcal{N}}_p$ the fields

$$\hat{\Omega}(\tau, s) = \hat{\Omega}(\tau\, \alpha^\mu_{EE'} \kappa^E{}_0 \bar{\kappa}^{E'}{}_{0'}), \qquad \hat{\Pi}(\tau, s) = \hat{\Pi}(\tau\, \alpha^\mu_{EE'} \kappa^E{}_0 \bar{\kappa}^{E'}{}_{0'}), \tag{9.40}$$

$$\hat{\Sigma}_{aa'}(\tau, s) = \hat{\Sigma}_{AA'}(\tau\, \alpha^\mu_{EE'} \kappa^E{}_0 \bar{\kappa}^{E'}{}_{0'})\, \kappa^A{}_a \bar{\kappa}^{A'}{}_{a'}, \tag{9.41}$$

$$\hat{\Phi}_{aba'b'}(\tau, s) = \hat{\Phi}_{ABA'B'}(\tau\, \alpha^\mu_{EE'} \kappa^E{}_0 \bar{\kappa}^{E'}{}_{0'})\, \kappa^A{}_a \kappa^B{}_b \bar{\kappa}^{A'}{}_{a'} \bar{\kappa}^{B'}{}_{b'}, \tag{9.42}$$

$$\hat{\psi}_{abcd}(\tau, s) = \hat{\psi}_{ABCD}(\tau\, \alpha^\mu_{EE'} \kappa^E{}_0 \bar{\kappa}^{E'}{}_{0'})\, \kappa^A{}_a \kappa^B{}_b \kappa^C{}_c \kappa^D{}_d. \tag{9.43}$$

Further, we use the considerations of Sect. 9 to derive fields $\hat{c}_{aa'}$, $\hat{\Gamma}_{aa'bc}$, $aa' \neq 11'$, on $\mathbb{R}^+_0 \times SU(2)$ from $\hat{e}^\mu{}_{AA'}$, $\hat{\Gamma}_{AA'BC}$ which have the meaning and the singularity/regularity structure described in (9.9), (9.12).

Because the fields (9.39) satisfy the field equations at all orders at p and have only been subject to a coordinate and frame transformation, the new fields (9.40)–(9.43) must satisfy, together with the transformed frame and connection coefficients, the transport equations and inner constraints induced on $\tilde{\mathcal{N}}_p$ at all orders at p. The uniqueness property stated in Proposition 9.1 thus implies that the Taylor expansion of the fields (9.40)–(9.43) in terms of τ at $\tau = 0$ must coincide with the corresponding Taylor expansion of the fields (9.38) at $\tau = 0$.

This fact can be expressed in the following way. If the curvature fields given by (9.38) are transformed into the normal gauge of Sect. 4 by setting

$$\Phi_{ABA'B'} = \Phi_{aba'b'}\, \kappa^a{}_A \kappa^b{}_B \bar{\kappa}^{a'}{}_{A'} \bar{\kappa}^{b'}{}_{B'}, \quad \psi_{ABCD} = \psi_{abcd}\, \kappa^a{}_A \kappa^b{}_B \kappa^c{}_C \kappa^d{}_D, \tag{9.44}$$

on $\mathcal{N}_p$, then

$$\Phi_{ABA'B'} = \sum_{n=0}^{N} \frac{1}{n!}\, \tau^n \kappa^{E_1}{}_0 \bar{\kappa}^{E'_1}{}_{0'} \ldots \kappa^{E_n}{}_0 \bar{\kappa}^{E'_n}{}_{0'} \nabla_{E_1 E'_1} \ldots \nabla_{E_n E'_n} \Phi_{ABA'B'}(0) + O(|\tau|^{N+1}),$$

$$\psi_{ABCD} = \sum_{n=0}^{N} \frac{1}{n!}\, \tau^n \kappa^{E_1}{}_0 \bar{\kappa}^{E'_1}{}_{0'} \ldots \kappa^{E_n}{}_0 \bar{\kappa}^{E'_n}{}_{0'} \nabla_{E_1 E'_1} \ldots \nabla_{E_n E'_n} \psi_{ABCD}(0) + O(|\tau|^{N+1}),$$

for given $N \in \mathbb{N}$, where the coefficients on the right hand sides are the expansion coefficients associated with the null data derived from ϕ_0 at p as described in Sects. 5 and 6.

One can also transform the frame vector fields and the connection coefficients given by (9.38) into the normal gauge but more complete information is obtained by using the curvature spinor

$$R_{ABCC'DD'} = \Omega\, \psi_{ABCD}\, \epsilon_{C'D'} + \Phi_{ABC'D'}\, \epsilon_{CD},$$

supplied on $\mathcal{N}_p$ by (9.44) to integrate the analogues of equations (7.5) and (7.6) on $\mathcal{N}_p$ along the curves $\tau \to x^\mu(\tau) = \tau\, x^\mu_*$, where $x^\mu_* = \alpha^\mu{}_{AA'}\,\kappa^A{}_0\,\bar{\kappa}^{A'}{}_{0'}$ is constant along these curves. Let $e^\mu{}_k$ and $\Gamma_i{}^A{}_B$ denote the frame and connection coefficients which constitute in the normal gauge together with the fields Ω, Π $\Phi_{ABC'D'}$, ψ_{ABCD} supplied by (9.38) initial data on $\mathcal{N}_p$ for the conformal vacuum equations and set $c^\mu{}_k = e^\mu{}_k - \delta^\mu{}_k$. The restriction of equations (7.5) and (7.6) to the curves $x^\mu(\tau)$ can then be written in the form

$$\tau\,\frac{d}{d\tau} c^\mu{}_k + c^\mu{}_k + c^\mu{}_l\,\delta^l_\nu\,c^\nu{}_k + \Gamma_k{}^i{}_l\,\tau\,X^l_*\,(c^\mu{}_i + \delta^\mu{}_i) = 0, \tag{9.45}$$

$$\tau\,\frac{d}{d\tau}\Gamma_k{}^A{}_B + \Gamma_k{}^A{}_B + \Gamma_l{}^A{}_B\,\delta^l_\mu\,c^\mu{}_k + \Gamma_k{}^i{}_l\,\tau\,X^l_*\,\Gamma_i{}^A{}_B - R^A{}_{Bik}\,\tau\,X^i_* = 0, \tag{9.46}$$

with $X^l_* = \delta^l{}_\mu\,x^\mu_*$. We are interested here in the solutions which are C^1 in τ and satisfy

$$c^\mu{}_k|_{\tau=0} = 0, \quad \Gamma_k{}^A{}_B|_{\tau=0} = 0.$$

If the left hand sides of the equations are contracted with X^l_*, the curvature term drops out and one gets for $c^\mu \equiv c^\mu{}_k\,X^k_*$ and $\Gamma^A{}_B \equiv X^k_*\,\Gamma_k{}^A{}_B$ equations which can be written

$$\tau\,\frac{d}{d\tau}(\tau\,c^\mu) + (\tau^{-1}\,c^\mu{}_l)\,\delta^l_\nu\,(\tau\,c^\nu) + (\tau\,\Gamma^i{}_l)\,X^l_*\,(c^\mu{}_i + \delta^\mu{}_i) = 0,$$
$$\tau\,\frac{d}{d\tau}(\tau\,\Gamma^A{}_B) + (\tau^{-1}\,\Gamma_l{}^A{}_B)\,\delta^l_\mu\,(\tau\,c^\mu) + (\tau\,\Gamma^i{}_l)\,X^l_*\,\Gamma_i{}^A{}_B = 0.$$

Because of the smoothness assumption and the initial conditions we can assume that $\tau^{-1}\,c^\mu{}_l$ and $\tau^{-1}\,\Gamma_l{}^A{}_B$ extend as continuous functions to $\tau = 0$. This allows us to conclude that

$$(e^\mu{}_k - \delta^\mu{}_k)\,\delta^k{}_\mu\,x^\mu = 0, \quad \delta^k{}_\mu\,x^\mu\,\Gamma_k{}^A{}_B = 0 \quad \text{along} \quad x^\mu(\tau).$$

By contracting (9.45) with $x^\nu_*\,\eta_{\nu\mu}$ and observing that $\Gamma_k{}^i{}_l\,X^l_*\,\delta^\mu{}_i\,x^\nu_*\,\eta_{\nu\mu} = \Gamma_{k\,i\,l}\;X^i_*\,X^l_* = 0$, one gets for $c_k = x^\nu_*\,\eta_{\nu\mu}\,c^\mu{}_k$ the equation

$$\frac{d}{d\tau}(\tau\,c_k) + (\tau\,c_l)\,\delta^l_\nu\,(\tau^{-1}\,c^\nu{}_k) + \Gamma_k{}^i{}_l\,X^l_*\,(\tau\,c_i) = 0,$$

which implies

$$x^\nu\,\eta_{\nu\mu}\,(e^\mu{}_k - \delta^\mu{}_k) = 0 \quad \text{along} \quad x^\mu(\tau).$$

This shows that the gauge conditions (4.7), (4.8), (4.11) will be satisfied on $\mathcal{N}_p$ by any C^1 solution to (9.45), (9.46).

We know from the explicit calculations above that $\Phi_{ADA'D'}\kappa^A{}_a\,\kappa^D{}_0\,\bar{\kappa}^{A'}{}_0\,\bar{\kappa}^{D'}{}_{d'} = \Phi_{a0\,0'd'} = 0$ on $\mathcal{N}_p$. This implies that

$$R^A{}_{B\,CC'DD'}\,\kappa^B{}_0\,X^{CC'}_* = \Phi^A{}_{B\,C'D'}\,\kappa^B{}_0\,\bar{\kappa}^{C'}{}_0\,\kappa_{D\,0} = 0 \quad \text{along} \quad x^\mu(\tau).$$

The contraction of (9.46) with $\kappa^B{}_0$ thus gives

$$\frac{d}{d\tau}(\tau\,\Gamma_k{}^A{}_B\,\kappa^B{}_0) + (\tau\,\Gamma_l{}^A{}_B\,\kappa^B{}_0)\,\delta^l_\mu\,(\tau^{-1}\,c^\mu{}_k) + \Gamma_k{}^i{}_l\,X^l_*\,(\tau\,\Gamma_i{}^A{}_B\,\kappa^B{}_0) = 0,$$

whence

$$\Gamma_k{}^A{}_B\,\kappa^B{}_0 = 0 \quad \text{along} \quad x^\mu(\tau).$$

Consequently, $\Gamma_k{}^{CC'}{}_{DD'}\,X_*^{DD'} = \Gamma_k{}^C{}_D\,\kappa^D{}_0\,\bar{\kappa}^{C'}{}_{0'} + \bar{\Gamma}_k{}^{C'}{}_{D'}\,\kappa^C{}_0\,\bar{\kappa}^{D'}{}_{0'} = 0$ along $x^\mu(\tau)$ and equation (9.45) reduces to

$$\tau\,\frac{d}{d\tau}c^\mu{}_k + c^\mu{}_k + c^\mu{}_l\,\delta^l_\nu\,c^\nu{}_k + \Gamma_k{}^i{}_l\,X^l_*\,\tau\,c^\mu{}_i = 0$$

The only C^1 solution vanishing at $\tau = 0$ is given by $c^\mu{}_k = 0$ and thus

$$e^\mu{}_k = \delta^\mu{}_k \quad \text{whence} \quad g_{\mu\nu} = \eta_{\mu\nu} \quad \text{along} \quad x^\mu(\tau).$$

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