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On the higher-spin charges of conical defects

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Abstract

The conical defect solutions in higher-spin gauge theories on 2+1 dimensional space-times with AdS-asymptotics are conjectured to correspond to certain primary fields in the dual conformal field theory on the boundary. In this note we prove that indeed all higher-spin charges match.

1 Introduction

When the dimension of space-time is equal to three, one can build a variety of higher-spin gauge theories. A notable class admits a Chern-Simons formulation with gauge algebra $sl(N,\mathbb{R}) \oplus sl(N,\mathbb{R})$ [1], and models the interactions of a set of tensors of rank $2, 3, \ldots, N$. In this context we investigate the solutions introduced in [2], whose metric displays a conical singularity in a particular gauge. These backgrounds approach asymptotically AdS_3 in the sense described in [3, 4], and play an important role in the conjectured holographic duality between W_N minimal models and Vasiliev theories [5].[1] The latter are extensions of the Chern-Simons setup that include matter couplings [8], but the conical defects can be considered as solutions of the Vasiliev equations in which the scalars vanish. They are conjectured to correspond to specific primary states in the minimal model dual [2, 9, 7].

A strong evidence in support of this identification is the matching of higher-spin charges on both sides of the duality. However, up to present this check was performed explicitly only for the first few charges. In the following we fill this gap, and show that all charges match in the semi-classical regime.

[1] See also [6] for a recent review, although the proposal for the boundary dual of the solutions we consider was later refined in [7].

Our proof goes as follows: in the higher-spin gauge theory, one can compute straightforwardly the charges in the so-called u-basis. On the other hand, it is precisely the u-basis for which one has a free-field construction in the boundary conformal field theory by the (quantum) Miura transformation. To compare to the higher-spin theory, we have to map the standard Miura transformation from the plane to the cylinder by a conformal transformation, because in the higher-spin theory we consider solutions defined on manifolds with the same topology as AdS_3, whose boundary is a cylinder. The free-field construction of the u-currents on the cylinder then makes it possible to compute their zero-mode eigenvalues on ground states, which precisely match the u-charges of the conical solutions.

2 Conical defects and their higher-spin charges

We actually focus on the Euclidean counterparts of $sl(N,\mathbb{R}) \oplus sl(N,\mathbb{R})$ higher-spin theories, which are described by Chern-Simons actions with gauge algebra $sl(N,\mathbb{C})$ (analogously to the description of three-dimensional Euclidean gravity by a $sl(2,\mathbb{C})$ Chern-Simons theory as reviewed e.g. in [10]). This is indeed the setup that simplifies the comparison with the minimal model spectrum [2].

The bulk field equations are solved by any flat $sl(N,\mathbb{C})$-valued connection. However, as the Chern-Simons theory is defined on a manifold with boundary, one also has to impose suitable boundary conditions on these connections. To describe them it is convenient to introduce a radial coordinate r and to parameterise a cylinder at surfaces of constant r by the complex coordinates $w, \bar{w}$. They are defined by $w = \phi + it_E$, where t_E is the Euclidean time and ϕ is an angular coordinate with periodicity $\phi \sim \phi + 2\pi$. In [3,4,11,12] it was proposed to select asymptotically AdS solutions by requiring that one can cast their connections in the form

$$A(r,w,\bar{w}) \;=\; b^{-1}a(w)b\,dw + b^{-1}db\;, \tag{2.1}$$

$$a(w) \;=\; J_1 + \sum_{j=2}^{N} \big(-\sqrt{k}\big)^{-j}\, u_j(w)\, e_{1,j}\;, \tag{2.2}$$

where $b = b(r)$ is a generic group-valued element depending only on the radial coordinate, J_1 is the matrix $(J_1)_a{}^b = -\,\delta_a{}^{b+1}$ and $e_{i,j}$ is the matrix with one entry 1 in the i^{th} row and j^{th} column, and zeroes otherwise.[2] We denote by k the level of the Chern-Simons theory.[3]

[2] In [3,4] the connections were actually presented in a different form that is related to (2.2) by a gauge transformation (see e.g. sect. 4 of [12]).

[3] We follow the conventions of [2] where the trace is normalised such that it satisfies $\mathrm{tr}\, J_a J_b = \binom{N+1}{3}\kappa_{ab}$ for the $sl(2)$ generators J_a and the standard $sl(2)$ Killing form κ_{ab}. In this convention the central charge of the asymptotic Virasoro subalgebra is $c = N(N^2-1)k$.

The conical defect solutions of [2] further preserve time-translation and rotational invariance, and this is achieved by considering only constant u_j. Moreover, the solutions are smooth in the sense that the holonomy around the contractible ϕ-circle is trivial. This signals the absence of singularities in the gauge configuration. When this condition is satisfied, the matrix (2.2) can be diagonalised with imaginary eigenvalues

$$i\, n'_j \;=\; i\Big(m_j - \frac{\sum_j m_j}{N}\Big)\;, \tag{2.3}$$

where m_j are integers satisfying $m_1 < \cdots < m_N$ [2]. Comparing the characteristic polynomial for a in the u-basis (2.2) and in the diagonal basis one concludes

$$u_j \;=\; \big(-i\sqrt{k}\big)^j P_j(n'_1,\ldots,n'_N)\;, \tag{2.4}$$

where $P_j(x_1,\ldots,x_N)$ denotes the j^{th} elementary symmetric polynomial in the variables $x_1,\ldots,x_N$:

$$P_j(x_1,\ldots,x_N) \;=\; \sum_{1\le k_1<\cdots<k_j\le N} x_{k_1}x_{k_2}\cdots x_{k_j}\;. \tag{2.5}$$

Each conical solution is thus specified by a list of N distinct integers, and its u-charges are easily expressed in terms of them through (2.4).

3 Charges from Miura transformation

When one restricts the connections of the Chern-Simons theory to those that correspond to asymptotically AdS solutions of the form (2.2), one finds an asymptotic W_N symmetry algebra on the boundary that is generated by the boundary currents u_j [3,4,11,12]: this is the classical Drinfeld-Sokolov reduction in the 'u-gauge'. The quantum counterparts U_j of the fields u_j can be obtained from the quantum Drinfeld-Sokolov reduction of $sl(N)$ (see e.g. [13]) which defines these higher-spin currents in terms of normal-ordered products of free fields by the so-called quantum Miura transformation [14],

$$(i\alpha_0\partial)^N + \sum_{j=2}^{N} U_j(z)(i\alpha_0\partial)^{N-j} \;=\; \Big((i\alpha_0\partial - i\epsilon_1\cdot J(z))\cdots(i\alpha_0\partial - i\epsilon_N\cdot J(z))\Big)\;, \tag{3.1}$$

where $J(z) = i\partial\varphi(z)$ is a spin-1 current taking values in the weight space of $sl(N)$ (see e.g. sect. 6.3.3 of [13]). The ϵ_i are the weights of the fundamental representation; $\epsilon_1 = \omega_1$ is the first fundamental weight, and $\epsilon_{i+1} = \epsilon_i - \alpha_i$, where α_i are the simple roots. We use the standard normalisation where

$$\epsilon_i\cdot\epsilon_j \;=\; \delta_{ij} - \frac{1}{N}\;. \tag{3.2}$$

The parameter α_0 is related to the central charge of the W_N algebra by

$$c(N,\alpha_0) \;=\; (N-1)\big(1 - N(N+1)\,\alpha_0^2\big)\;. \tag{3.3}$$

The U_j are holomorphic fields of scaling weight j. They are not primary fields, so they transform non-trivially under conformal transformations. On the other hand, this transformation can be derived from the transformation property of the free spin-1 current J.

In the above free-field construction there is a background charge for the free field. In particular the energy-momentum tensor has a shift proportional to the derivative of the current J,

$$U_2(z) \;=\; T(z) \;=\; \frac{1}{2}(J\cdot J)(z) - \alpha_0 \rho\cdot\partial J(z)\ , \tag{3.4}$$

where $\rho \;=\; -\sum_j j\,\epsilon_j$ is the Weyl vector. Under a conformal transformation $z\mapsto w(z)$ the free field transforms as

$$J(z)\to \tilde{J}(w) \;=\; \frac{1}{w'(z)}\left(J(z)-\alpha_0\rho\frac{w''(z)}{w'(z)}\right)\ . \tag{3.5}$$

From this rule one can now deduce the transformation property of the fields U_k. There is however one subtlety: the U_k are determined in terms of normal-ordered products. Under a general conformal transformation the notion of normal-ordering changes, and this can lead to additional terms in the transformation as we will briefly discuss at the end of this section.

We are primarily interested in the transformation from the plane to the cylinder where we can compare the results to the higher-spin computation in the bulk. If z is the coordinate on the plane, the relation to the cylinder with coordinate w is $z=e^{-iw}$. The spin-1 currents then transform as

$$J(z)\to\tilde{J}(w) \;=\; -\,i\big(zJ(z)+\alpha_0\rho\big)\ . \tag{3.6}$$

If we ignore the normal-ordering subtleties (which do not play a role in the semi-classical limit in which we compare the charges), the transformed fields $\tilde{U}_j$ on the cylinder are determined by

$$\begin{aligned}(i\alpha_0\partial_w)^N &+ \sum_{j=2}^{N}\tilde{U}_j(w)(i\alpha_0\partial_w)^{N-j}\\ &= \Big(\big(i\alpha_0\partial_w-\epsilon_1\cdot(zJ(z)+\alpha_0\rho)\big)\cdots\big(i\alpha_0\partial_w-\epsilon_N\cdot(zJ(z)+\alpha_0\rho)\big)\Big)\ . \end{aligned}\tag{3.7}$$

In particular we can now determine the eigenvalues of the zero modes of $\tilde{U}_j$ on a highest-weight representation: let $|\Lambda\rangle$ be a highest-weight state for the free field with

$$J_0|\Lambda\rangle \;=\; \Lambda|\Lambda\rangle\ . \tag{3.8}$$

We then have

$$(i\alpha_0\partial_w)^N+\sum_{j=2}^{N}\tilde{U}_{j,0}(i\alpha_0\partial_w)^{N-j} \;=\; \Big(\big(i\alpha_0\partial_w-\epsilon_1\cdot(\Lambda+\alpha_0\rho)\big)\cdots\big(i\alpha_0\partial_w-\epsilon_N\cdot(\Lambda+\alpha_0\rho)\big)\Big)\ , \tag{3.9}$$

and hence

$$\tilde{U}_{j,0} \;=\; (-1)^j\, P_j\big(\epsilon_1\cdot(\Lambda+\alpha_0\rho),\ldots,\epsilon_N\cdot(\Lambda+\alpha_0\rho)\big)\ , \tag{3.10}$$

where P_j is the j^{th} elementary symmetric polynomial, see (2.5).

Before we compare these values to the charges of the conical solutions in the next section, we want to briefly discuss the quantum corrections to the charge formula (3.10). Let us illustrate this with the well-known example of the energy-momentum tensor $U_2(z) = T(z)$ (see (3.4)). Going to the cylinder we obtain the transformed field

$$\tilde{U}_2(w) \;=\; \frac{1}{2}\big(\tilde{J}\cdot\tilde{J}\big)_w(w) - \alpha_0\rho\cdot\partial_w\tilde{J}(w)\ , \tag{3.11}$$

where the subscript $_w$ on the parentheses signals that the normal ordering is taken on the cylinder. Replacing $\tilde{J}$ according to (3.6) we find

$$\tilde{U}_2(w) \;=\; -z^2\left(\frac{1}{2}\big(J\cdot J\big)_w(z) - \alpha_0\rho\cdot\partial_z J(z)\right) - \frac{1}{2}\alpha_0^2\rho^2\ . \tag{3.12}$$

The normal ordering defined via the subtraction of the singular terms in the operator product expansion gives a different result on the cylinder and on the plane, with the relation

$$\big(J\cdot J\big)_w(z) \;=\; \big(J\cdot J\big)_z(z) - \frac{N-1}{12\,z^2}\ . \tag{3.13}$$

Therefore we find the familiar transformation

$$\tilde{U}_2(w) \;=\; -\,z^2U_2(z) + \frac{N-1}{24} - \frac{1}{2}\alpha_0^2\rho^2 \;=\; -\,z^2U_2(z) + \frac{c}{24}\ . \tag{3.14}$$

In the semi-classical limit where $c\to\infty$ (and N is fixed) we observe that the normal-ordering shift $\frac{N-1}{24}$ (which does not grow with c) is a subleading correction as expected. These shifts become less trivial for higher-spin charges: for example the transformation of the spin-4 field is given by

$$\begin{aligned}\tilde{U}_4 = {}& z^4\,U_4 + 3\,i\,\frac{N-3}{2}\,\alpha_0\,z^3\,U_3 + \frac{(N-2)(N-3)}{24}\left(\alpha_0^2\,(N-13) + \left\{-\frac{1}{N}\right\}\right)z^2\,U_2\\ & - \frac{1}{192}\binom{N-1}{3}\left(\alpha_0^4\,N\,(N+1)(N+\tfrac{7}{5}) + \left\{-\,\alpha_0^2\,\frac{142}{5} + 2\,\alpha_0\,(N-13) + \frac{1}{N}\right\}\right)\ ,\end{aligned} \tag{3.15}$$

where the quantum corrections due to normal ordering are contained in curly brackets. It would be interesting to work out the quantum corrections in general, since they would allow one to test possible proposals for a quantum description of higher-spin theories in the bulk.

4 Comparison and discussion

Degenerate representations of the W_N algebra are labelled by two $sl(N)$ highest weights (Λ_+, Λ_-) [13], and the corresponding weight vector Λ is given by

$$\Lambda = \alpha_+\Lambda_+ + \alpha_-\Lambda_- \ , \tag{4.1}$$

where $\alpha_+\alpha_- = -1$ and $\alpha_+ + \alpha_- = \alpha_0$. According to the proposal in [7] conical solutions correspond in the semi-classical limit ($c \to \infty$ with N fixed) to representations $(0, \Lambda_-)$, whereas the representations (Λ_+, Λ_-) describe excitations of the scalar in the Vasiliev theory on the conical background labelled by $(0, \Lambda_-)$. We can take the semi-classical limit by giving α_- a large imaginary value, $\alpha_- \approx i\frac{c}{N(N^2-1)}$. In this limit α_+ vanishes since it scales as c^{-1}: as a result the charges (3.10) do not depend on Λ_+ and become

$$\tilde{U}_{j,0} = (-\alpha_-)^j P_j\big(\epsilon_1 \cdot (\Lambda_- + \rho), \ldots, \epsilon_N \cdot (\Lambda_- + \rho)\big) \ + \ \cdots \ . \tag{4.2}$$

Upon identifying $\alpha_- \approx i\sqrt{k}$ (which gives the correct matching for the central charge) and $\epsilon_i \cdot (\Lambda_- + \rho) = n'_i$ we find precise agreement with (2.4).

Therefore we have shown that the spectrum of conical solutions exactly matches the spectrum in the conformal field theory in the semi-classical limit, providing an important check of the proposed higher-spin AdS_3/CFT_2 duality.[4]

The construction of smooth asymptotically AdS connections was generalised to higher-spin theories based on the infinite-dimensional gauge algebra hs(λ) in [16]. A class of these solutions can be seen as a continuation of solutions in the $sl(N)$ theories. The charge formulas that we obtained in this note can be straightforwardly continued to this case: for a fixed representation Λ_- the charges stabilise for large N to rational functions in N, and by replacing $N \to \lambda$ we obtain the charge for the hs(λ) theory.

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[4] Further evidence for the correct identification of the conical solutions comes from a matching of the four-point function in the CFT with a two-point function in the background of a conical solution in the bulk [15].

References

[1] E. Bergshoeff, M. P. Blencowe and K. S. Stelle, *Area Preserving Diffeomorphisms And Higher Spin Algebra*, Commun. Math. Phys. **128** (1990) 213.

[2] A. Castro, R. Gopakumar, M. Gutperle and J. Raeymaekers, *Conical Defects in Higher Spin Theories*, JHEP **1202** (2012) 096, [arXiv:1111.3381 [hep-th]].

[3] M. Henneaux and S.-J. Rey, *Nonlinear* W_∞ *as Asymptotic Symmetry of Three-Dimensional Higher Spin Anti-de Sitter Gravity*, JHEP **1012** (2010) 007 [arXiv:1008.4579 [hep-th]].

[4] A. Campoleoni, S. Fredenhagen, S. Pfenninger and S. Theisen, *Asymptotic symmetries of three-dimensional gravity coupled to higher-spin fields*, JHEP **1011** (2010) 007 [arXiv:1008.4744 [hep-th]].

[5] M. R. Gaberdiel and R. Gopakumar, *An* AdS_3 *Dual for Minimal Model CFTs*, Phys. Rev. D **83** (2011) 066007 [arXiv:1011.2986 [hep-th]].

[6] M. R. Gaberdiel and R. Gopakumar, *Minimal Model Holography*, J. Phys. A **46** (2013) 214002 [arXiv:1207.6697 [hep-th]].

[7] E. Perlmutter, T. Procházka and J. Raeymaekers, *The semiclassical limit of* W_N *CFTs and Vasiliev theory*, JHEP **1305** (2013) 007 [arXiv:1210.8452 [hep-th]].

[8] S. F. Prokushkin and M. A. Vasiliev, *Higher spin gauge interactions for massive matter fields in 3-D AdS space-time*, Nucl. Phys. B **545** (1999) 385 [hep-th/9806236].

[9] M. R. Gaberdiel and R. Gopakumar, *Triality in Minimal Model Holography*, JHEP **1207** (2012) 127 [arXiv:1205.2472 [hep-th]].

[10] M. Bañados, *Three-dimensional quantum geometry and black holes*, hep-th/9901148.

[11] M. R. Gaberdiel and T. Hartman, *Symmetries of Holographic Minimal Models*, JHEP **1105** (2011) 031 [arXiv:1101.2910 [hep-th]].

[12] A. Campoleoni, S. Fredenhagen and S. Pfenninger, *Asymptotic W-symmetries in three-dimensional higher-spin gauge theories*, JHEP **1109** (2011) 113 [arXiv:1107.0290 [hep-th]].

[13] P. Bouwknegt and K. Schoutens, *W symmetry in conformal field theory*, Phys. Rept. **223** (1993) 183, [hep-th/9210010].

[14] V. A. Fateev and S. L. Lukyanov, *The Models of Two-Dimensional Conformal Quantum Field Theory with Z(n) Symmetry*, Int. J. Mod. Phys. A **3** (1988) 507.

[15] E. Hijano, P. Kraus and E. Perlmutter, *Matching four-point functions in higher spin* AdS_3/CFT_2, JHEP **1305** (2013) 163 [arXiv:1302.6113 [hep-th]].

[16] A. Campoleoni, T. Procházka and J. Raeymaekers, *A note on conical solutions in 3D Vasiliev theory*, JHEP **1305** (2013) 052, [arXiv:1303.0880 [hep-th]].