Black-Hole Lattices

Eloisa Bentivegna

Abstract The construction of black-hole lattices, first attempted by Richard Lindquist and John Wheeler in 1957, has recently been tackled with renewed interest, as a test bed for studying the behavior of inhomogeneities in the context of the backreaction problem. In this contribution, I discuss how black-hole lattices can help shed light on two important issues, and illustrate the conclusions reached so far in the study of these systems.

1 Introduction

The first appearance of the concept of a periodic arrangement of black holes can be found in [1]. There, the authors discuss a strategy to stitch together patches of the Schwarzschild solution so as to construct a space with a discrete translational symmetry but some degree of spatial inhomogeneity.

In their work, the stitching prescription does not lead to a global solution of Einstein's equation. Accepting the constraint violations, however, buys one some freedom in the specification of such prescription, which the authors use to impose that the time evolution of a suitably-defined scale factor in this space follows that of a universe filled with dust of the same total mass. One then has a simple, analytical test bed in which to measure the effect of inhomogeneities in, say, the optical properties of a cosmological model. In this work and in subsequent ones [2], it was pointed out how an exact initial-data construction could be obtained.

A few years ago, Clifton and Ferreira extended this model, originally limited to the positive-curvature case, to zero and negative curvature [3]. Again, the junction

E. Bentivegna (✉)
Max-Planck-Institut für Gravitationsphysik (Albert-Einstein-Institut), Am Mühlenberg 1, D-14476 Golm, Germany
e-mail: eloisa.bentivegna@aei.mpg.de

A. García-Parrado et al. (eds.), *Progress in Mathematical Relativity, Gravitation and Cosmology*, Springer Proceedings in Mathematics & Statistics 60,
DOI 10.1007/978-3-642-40157-2_13,

conditions were designed to reproduce an assigned time evolution, and the models were used to explore the propagation of null rays in an inhomogeneous universe.

Recently, the first exact initial data describing a black-hole lattice have been analyzed [4] and evolved in full numerical relativity [5]. This has helped make progress on two fronts:

1. From a conceptual point of view, it has clarified some of the conditions under which black-hole solutions can be glued together; this gives some insight into the requirements for constructing a metric tensor for the universe starting from the basic building block of a spherically-symmetric, isolated object. It turns out that these conditions are remarkably close to the conditions for the existence of homogeneous, periodic solutions of Einstein's equation. The requirements that periodic boundary conditions impose on the Hamiltonian constraint are likely at the root of this correspondence.
2. From a practical standpoint, the time evolution of a lattice gives one example of the behavior of inhomogeneities in a cosmological setting and in the non-linear regime, thereby serving as a nice complement to perturbative studies and the averaging framework. Surprisingly, even the time development of these lattices remains in some sense close to the counterpart model in the dust Friedmann–Lemaître–Robertson–Walker (FLRW) class.

In the following two sections, I will discuss the initial-data construction and illustrate the time evolution of a black-hole lattice with positive conformal curvature.

2 The Construction of Exact Black-Hole-Lattice Initial Data

As pointed out in [1], in order to construct an exact black-hole lattice one should directly tackle the Einstein constraints. Working in the *conformal transverse-traceless* decomposition, these read:

$$\tilde{\Delta}\psi - \frac{\tilde{R}}{8}\,\psi - \frac{K^2}{12}\,\psi^5 + \frac{1}{8}\tilde{A}_{ij}\tilde{A}^{ij}\psi^{-7} = -2\pi\,\psi^5\,\rho \tag{1}$$

$$\tilde{D}_i\tilde{A}^{ij} - \frac{2}{3}\psi^6\tilde{\gamma}^{ij}\tilde{D}_i K = 0 \tag{2}$$

Let us focus on the hamiltonian constraint first. If one integrates this equation over one of the cells of the black-hole lattice, the following condition is obtained:

$$\int_D \left(\frac{\tilde{R}}{8}\,\psi + \frac{1}{12}\,K^2\,\psi^5 - \frac{1}{8}\psi^{-7}\,\tilde{A}^{ij}\,\tilde{A}_{ij}\right)\sqrt{\tilde{\gamma}}\,d^3x = 2\pi\Sigma_{i=1}^N m_i \tag{3}$$

where m_i represent the masses of the black holes contained in the cell. This condition implies that $\tilde{R}$ and K_{ij} cannot both be zero. In other words, conformally-flat lattices do not admit a time-symmetric spatial hypersurface; vice versa, lattices with

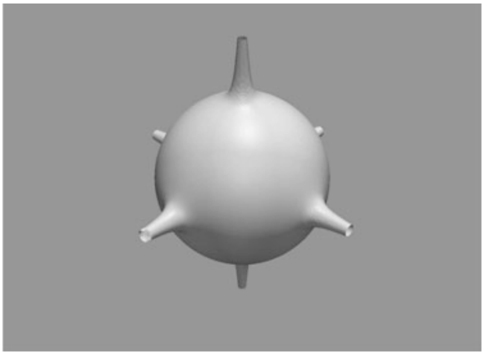

Fig. 1 A two-dimensional section of the $N = 8$ S^3 black-hole lattice, embedded in three dimensions

a $K = 0$ spatial hypersurface must be conformally curved. This mirrors the identical property of the FLRW class.

The two simplest roads to the construction of a periodic black-hole lattice are thus the following:

- Choosing $K = 0$, and solving:

$$\tilde{\Delta}\psi - \frac{\tilde{R}}{8}\,\psi = 0 \tag{4}$$

  Equation (3) implies that $R > 0$, so that the spacetime can be foliated by conformally-S^3 hypersurfaces. As shown in [2], this equation can be solved exactly. Furthermore, it is linear, so one can simply superimpose known solutions to generate new ones. Notice, however, that if one is interested in regular lattices, only six possible arrangements of black holes are possible, corresponding to the six regular tessellations of S^3, which consist of $N = 5, 8, 16, 24, 120$ and 600 cells.
- Choosing $\tilde{R} = 0$, and solving:

$$\tilde{\Delta}\psi - \frac{K^2}{12}\,\psi^5 + \frac{1}{8}\tilde{A}_{ij}\tilde{A}^{ij}\psi^{-7} = 0 \tag{5}$$

$$\tilde{D}_i\tilde{A}^{ij} - \frac{2}{3}\psi^6\tilde{\gamma}^{ij}\tilde{D}_i K = 0 \tag{6}$$

  This system is more difficult to solve, as the Hamiltonian constraint is non-linear and the momentum constraint is not an identity as in the previous case. For a numerical approach to the problem, see [6].

3 The Evolution of an S^3 Lattice of Eight Black Holes

In [5], the initial data for the $N = 8$ S^3 lattice (a section of which is shown in Fig. 1) has been evolved in time for approximately one third of the corresponding FLRW recollapse time. A scale factor can be defined via the proper length of one of

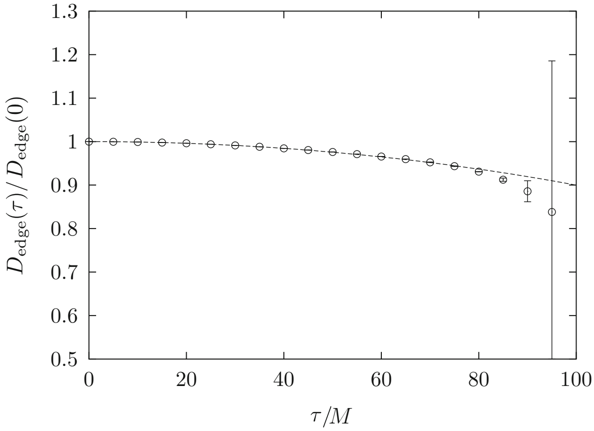

Fig. 2 Proper length of the edge of a lattice cell as a function of proper time τ. The *dashed line* represents a closed FLRW model in which the edge of a cell of the $N = 8$ tessellation is equal to $D_{\mathrm{edge}}(0)$ initially

the cell edges; its evolution is shown in Fig. 2. This scale factor is compatible with the FLRW result in this entire time window; eventually, though, due to the gauge condition used to evolve this system, reaching later and later values of the proper time is subject to an increasing numerical error, and eventually becomes impossible.

Acknowledgement I acknowledge the financial support from a Marie Curie International Reintegration Grant (PIRG05-GA-2009-249290).

References

1. Lindquist, R.W., Wheeler, J.A.: Dynamics of a Lattice Universe by the Schwarzschild-Cell Method. Rev. Mod. Phys. **29**(3), 432–443 (1957). doi: 10.1103/RevModPhys.29.432
2. Wheeler, J.A.: The geometrostatic lattice cell. Foundations of Physics **13**, 161–173 (1983). doi: 10.1007/BF01889418
3. Clifton, T., Ferreira, P.G.: Archipelagian Cosmology: Dynamics and observables in a universe with discretized matter content. Phys. Rev. **D80**, 103503 (2009). doi: 10.1103/PhysRevD.80.103503
4. Clifton, T., Rosquist, K., Tavakol, R.: An exact quantification of backreaction in relativistic cosmology. Phys.Rev. **D86**, 043506 (2012). doi: 10.1103/PhysRevD.86.043506
5. Bentivegna, E., Korzynski, M.: Evolution of a periodic eight-black-hole lattice in numerical relativity. Class.Quant.Grav. **29**, 165007 (2012). doi: 10.1088/0264-9381/29/16/165007
6. Yoo, C., Abe, H., Nakao, K., Takamori, Y.: Black hole universe: Construction and analysis of initial data. Phys.Rev. **D86**, 044027 (2012). doi: 10.1103/PhysRevD.86.044027