

Lagrangian theory of structure formation in relativistic cosmology. II. Average properties of a generic evolution model

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Kinematical and dynamical properties of a generic inhomogeneous cosmological model, spatially averaged with respect to free-falling (generalized fundamental) observers, are investigated for the matter model “irrotational dust.” Paraphrasing a previous Newtonian investigation, we present a relativistic generalization of a backreaction model based on volume-averaging the “relativistic Zel’dovich approximation.” In this model we investigate the effect of “kinematical backreaction” on the evolution of cosmological parameters as they are defined in an averaged inhomogeneous cosmology, and we show that the backreaction model interpolates between orthogonal symmetry properties by covering subcases of the plane-symmetric solution, the Lemaître-Tolman-Bondi solution and the Szekeres solution. We so obtain a powerful model that lays the foundations for quantitatively addressing curvature inhomogeneities as they would be interpreted as “dark energy” or “dark matter” in a quasi-Newtonian cosmology. The present model, having a limited architecture due to an assumed Friedmann-Lemaître-Robertson-Walker background, is nevertheless capable of replacing 1/4 of the needed amount for dark energy on domains of 200 Mpc in diameter for typical (one-sigma) fluctuations in a cold dark matter initial power spectrum. However, the model is far from explaining dark energy on larger scales (spatially), where a 6% effect on 400 Mpc domains is identified that can be traced back to an on average negative intrinsic curvature today. One drawback of the quantitative results presented is the fact that the epoch when backreaction is effective on large scales and leads to volume acceleration lies in the future. We discuss this issue in relation to the initial spectrum, the dark matter problem, the coincidence problem, and the fact that large-scale dark energy is an effect on the past light cone (not spatial), and we pinpoint key elements of future research.

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I. INTRODUCTION

In a previous work [1] we developed a Lagrangian theory for the evolution of structure in irrotational dust continua in the framework of general relativity. This theory is characterized by employing as a single dynamical variable the set of spatial coframes $\boldsymbol{\eta}^a$, in the comoving-synchronous metric form:

$$^{(4)}\mathbf{g} = -\mathbf{d}t^2 + ^{(3)}\mathbf{g}; \quad ^{(3)}\mathbf{g} = \delta_{ab}\boldsymbol{\eta}^a \otimes \boldsymbol{\eta}^b = g_{ij}\mathbf{d}X^i \otimes \mathbf{d}X^j,$$

with the spatial metric coefficients $g_{ij} = \delta_{ab}\boldsymbol{\eta}^a_i \boldsymbol{\eta}^b_j$, where X^i are Gaussian normal (Lagrangian) coordinates that are constant along flow lines (here geodesics); $i, j, k = 1, 2, 3$ denote coordinate indices, while $a, b, c = 1, 2, 3$ are just counters. Having written Einstein’s equations in terms of the variable $\boldsymbol{\eta}^a$, we are entitled to set up a perturbation scheme that only perturbs this variable, while other variables like the density, the metric or the curvature are expressed as functionals of this perturbation. Such a point of view extrapolates these latter physical quantities beyond a

strict perturbative expansion. In this spirit we defined and evaluated the first-order perturbation scheme on a Friedmann-Lemaître-Robertson-Walker (FLRW) background cosmology as a clear-cut definition of what has been known as the “relativistic Zel’dovich approximation” (henceforth abbreviated by RZA) in previous work (see [1] and references on previous work therein, in particular [2–6]).

In the present work we will model the fluctuations using this approximation and combine the resulting backreaction model with exact average properties of inhomogeneous dust cosmologies. (This framework is reviewed in [7,8] and will be briefly recalled below.) We will so paraphrase a detailed investigation in Newtonian cosmology [9], where we presented the results of an in-depth quantitative study of the *kinematical backreaction effect* in Newtonian cosmology [10]. In order to evaluate the influence of inhomogeneities on effective cosmological parameters we employed in [9] exact solutions with planar and spherical symmetry, as well as perturbative solutions for generic initial conditions in the Eulerian and Lagrangian first-order perturbation schemes. The most general, nonperturbative model studied was based on the first-order Lagrangian deformation that was used as input into the kinematical

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backreaction functional (fluctuation model), but the effective cosmology (the average) was then evaluated within the framework of the exact averaged equations. We will here provide the corresponding model in general relativity.

In [9] our initial data setting was conservative; i.e. we started with a standard *cold dark matter* (CDM) power spectrum without a cosmological constant at the beginning of the matter-dominated era, maintaining the early Universe description as in the standard model. Perhaps the most surprising outcome of this study was that, although the kinematical backreaction term $\mathcal{Q}_{\mathcal{D}}$ itself was quantitatively negligible on large expanding domains, the other cosmological parameters could still experience significant changes; e.g., for the deceleration parameter we found typical fluctuations up to 30% on the scale of 200 Mpc (with $h = 0.5$). In this work we will give the results for two background models: (i) a CDM spectrum with $h = 0.5$ as in [9] to allow for comparison, and (ii) a Λ CDM spectrum with $h = 0.7$ to study the effect of the background. For the interpretation of the emerging curvature inhomogeneities as “dark energy” we refer to the first model. Note that both homogeneous-isotropic models are assumed to have Euclidean space sections. Recall that, in Newtonian cosmology, the kinematical backreaction vanishes by construction on the scale of an imposed torus architecture (i.e. the use of periodic boundary conditions) [10], a restriction that no longer applies to the general relativistic model. The present investigation thus offers a more general view on the backreaction model, notably by allowing for a nontrivial curvature evolution.

We may approach the present work with the following guidelines:

- (i) The *cosmic triangle* [11] of the three parameters of the standard model (related to the homogeneous density, the constant curvature and the cosmological constant) *fluctuates* on a given spatial scale. The measure of deviations from the FLRW time evolution of these parameters is a kinematical backreaction functional $\mathcal{Q}_{\mathcal{D}}$ that encodes averaged fluctuations in kinematical invariants (the rate of expansion and the rate of shear).
- (ii) The influence of backreaction may be twofold: for dominating shear fluctuations the effective density decelerating the expansion is larger than the actual matter density, thus mimicking the effect of a *kinematical dark matter*; for dominating expansion fluctuations backreaction acts accelerating, thus mimicking the effect of a (positive) cosmological constant, i.e. *kinematical dark energy*.
- (iii) In general relativity, backreaction entails an emerging intrinsic curvature. Thus, as soon as we require, on some scale, that the average model is given by a standard FLRW model geometry, then the emerging curvature term allows us to interpret the above-mentioned kinematical effects as “dark matter” or

dark energy, respectively. In Newtonian cosmology, the backreaction term should be interpreted as cosmic variance of velocity fluctuations on a Euclidean space; a general-relativistic model entails deviations from a Euclidean model geometry; we so are led to interpret the dark sources in an assumed Euclidean background as “curvature energies” of the actual average distribution.

- (iv) A genuinely relativistic property is the coupling of the kinematical backreaction functional to the averaged scalar (3-Ricci) curvature. This coupling is absent in weakly perturbed FLRW spacetimes, e.g., in a Newtonian framework where a fluctuating scalar curvature is absent, or in a post-Newtonian framework and standard relativistic perturbation theory at first order, where the cosmological model is required to stay close to the constant-curvature FLRW spacetime, if this latter is taken as the background spacetime.
- (v) If kinematical backreaction is positive, i.e. dominated by expansion fluctuations and by a negative averaged curvature, a small source term is capable of driving the averaged system away from the homogeneous-isotropic FLRW background. We found, with the help of a detailed dynamical systems analysis of scaling solutions [12], generalizing a previous analysis [13], that the standard model is a *repellor* in this situation, and the physical background (the average) is no longer given by the FLRW background. We may say that the physical background lies in the dark energy sector. A similar situation occurs in the case of shear-dominated evolution of small-scale domains with positive averaged curvature, where the average is driven to lie in the dark matter sector. This insight suggests that a perturbation theory has to be set up on a background that interacts with structure formation (a framework for such a perturbation theory has been suggested [14]). This remark implies that the present investigation is limited, since we employ perturbations on a FLRW background. The resulting backreaction model, however, can be formally employed also in such a more general framework, which is the subject of forthcoming work.

We proceed as follows. In Sec. II we briefly review the equations governing the averaged inhomogeneous dust cosmologies in the relativistic framework and provide different representations of the equations that are used in this article. We then move in Sec. III to dynamical models that allow estimating the backreaction terms: we provide backreaction and curvature models for the Lagrangian linear perturbation theory, restricted to a relativistic form of Zel’dovich’s approximation investigated in detail in [1]. Section IV is devoted to consistency checks including the Newtonian limit. The plane, spherical and quasiperpherical

Szekeres solutions serve as exact reference models in Sec. V. We then illustrate the results in Sec. VI in terms of the time evolution of cosmological parameters: we examine the evolution of the volume scale factor for typical initial conditions; then we explore the cosmological density parameters in the averaged model. Here, we also quantify the emerging intrinsic curvature. We discuss the results and conclude in Sec. VII.

II. AVERAGED DUST MODEL IN GENERAL RELATIVITY

A. Averaging

In this section we will briefly set up the averaging framework developed in [15–17].

Given a foliation of spacetime into flow-orthogonal hypersurfaces, we investigate fluid averaging for the matter model “irrotational dust”; the geometry of the dust continuum is described by the spatial metric coefficients g_{ij} in the comoving and synchronous line element:

$$ds^2 = -dt^2 + g_{ij}dX^i dX^j, \quad (1)$$

where X^i are the already introduced (Lagrangian) coordinates that are constant along flow lines (spacetime geodesics of freely falling observers). Proper time derivative along the 4-velocity u^μ , that is here equal to the coordinate time derivative, will be denoted by $\partial_t := u^\mu \partial_\mu$ and sometimes by an overdot (where greek indices label spacetime and latin ones spatial coordinates).

We look at the scalar parts of Einstein’s equations, averaged over a compact spatial domain \mathcal{D} at time t that evolved out of the initial domain \mathcal{D}_i at time t_i , conserving the material rest mass inside the domain. The volume scale factor $a_{\mathcal{D}}$, depending on content, shape and position of the domain \mathcal{D} , is defined via the domain’s volume $V_{\mathcal{D}}(t) = |\mathcal{D}|$, and the initial volume $V_{\mathcal{D}_i} = V_{\mathcal{D}}(t_i) = |\mathcal{D}_i|$:

$$a_{\mathcal{D}}(t) := \left(\frac{V_{\mathcal{D}}(t)}{V_{\mathcal{D}_i}} \right)^{1/3}, \quad (2)$$

where the volume of the domain is given by

$$V_{\mathcal{D}}(t) := \int_{\mathcal{D}} d\mu_g, \quad (3)$$

with the *Riemannian* volume element $d\mu_g = Jd^3X$, and the local volume deformation $J := \sqrt{\det(g_{ij})/\det(G_{ij})}$, $G_{ij} = g_{ij}(t_i)$.

For domains \mathcal{D} , which keep the total material mass $M_{\mathcal{D}}$ constant, as for Lagrangian domains with geodesic motion of their boundaries, the average density is simply given by

$$\langle \varrho \rangle_{\mathcal{D}} = \frac{\langle \varrho(t_i) \rangle_{\mathcal{D}_i}}{a_{\mathcal{D}}^3} = \frac{M_{\mathcal{D}}}{a_{\mathcal{D}}^3 V_{\mathcal{D}_i}}; \quad M_{\mathcal{D}} = M_{\mathcal{D}_i}. \quad (4)$$

In this setting we can derive equations analogous to the averaged equations derived in Newtonian cosmology [10]

(restricting attention to irrotational dust flows). Spatial averaging of a scalar field Ψ is defined by

$$\langle \Psi(t, X^k) \rangle_{\mathcal{D}} := \frac{1}{V_{\mathcal{D}}} \int_{\mathcal{D}} d\mu_g \Psi(t, X^k). \quad (5)$$

The key property of inhomogeneity of the field Ψ is revealed by the *rule of noncommutativity* [10,15]:

$$\partial_t \langle \Psi(t, X^k) \rangle_{\mathcal{D}} - \langle \partial_t \Psi(t, X^k) \rangle_{\mathcal{D}} = \langle \Theta \Psi \rangle_{\mathcal{D}} - \langle \Theta \rangle_{\mathcal{D}} \langle \Psi \rangle_{\mathcal{D}}, \quad (6)$$

where Θ denotes the trace of the expansion tensor. In the present spacetime setting the latter is, up to the sign, the extrinsic curvature tensor $\Theta_{ij} = -K_{ij}$. Θ describes the local rate of volume change of fluid elements along the vector field ∂_t .

It is formally convenient in the following calculations, but not necessary, to introduce the following formal average:

$$\langle \mathcal{A} \rangle_I = \frac{1}{V_{\mathcal{D}_i}} \int_{\mathcal{D}_i} d^3X \mathcal{A}, \quad (7)$$

where $V_{\mathcal{D}_i} = \int_{\mathcal{D}_i} d^3X = \int_{\mathcal{D}_i} J(X^i, t_i) d^3X$ is the volume of the initial domain \mathcal{D}_i for $J(X^i, t_i) = 1$ (we will explain below that this latter property of the local volume deformation holds for non-normalized coframes). This average coincides with the Riemannian volume average, if we consider fields at initial time, $\mathcal{A} = \mathcal{A}(t_i)$ (therefore we label it by I). Rewriting the Riemannian average in terms of this formal average operation, we obtain the useful formula

$$\langle \mathcal{A} \rangle_{\mathcal{D}} = \frac{\langle \mathcal{A} J \rangle_I}{\langle J \rangle_I}; \quad \langle J \rangle_I = a_{\mathcal{D}}^3. \quad (8)$$

It will simplify the following calculations and is useful to discuss the Newtonian limit (see Sec. IVA).

B. Equations for the evolution of average characteristics

1. Generalized Friedmann equations

The spatially averaged equations for the volume scale factor $a_{\mathcal{D}}$, respecting rest mass conservation within the domain \mathcal{D} , read [15]

(i) *averaged Raychaudhuri equation*:

$$3 \frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} + 4\pi G \frac{M_{\mathcal{D}_i}}{V_{\mathcal{D}_i} a_{\mathcal{D}}^3} - \Lambda = \mathcal{Q}_{\mathcal{D}}; \quad (9)$$

(ii) *averaged Hamilton constraint*:

$$\left(\frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} \right)^2 - \frac{8\pi G}{3} \frac{M_{\mathcal{D}_i}}{V_{\mathcal{D}_i} a_{\mathcal{D}}^3} + \frac{\langle \mathcal{R} \rangle_{\mathcal{D}}}{6} - \frac{\Lambda}{3} = -\frac{\mathcal{Q}_{\mathcal{D}}}{6}, \quad (10)$$

where the total rest mass $M_{\mathcal{D}_i}$, the averaged spatial 3-Ricci scalar $\langle \mathcal{R} \rangle_{\mathcal{D}}$ and the *kinematical backreaction* $\mathcal{Q}_{\mathcal{D}}$ are

domain-dependent and, except the mass, time-dependent functions. The kinematical backreaction source term itself is given by

$$\mathcal{Q}_{\mathcal{D}} := 2\langle \text{II} \rangle_{\mathcal{D}} - \frac{2}{3}\langle \text{I} \rangle_{\mathcal{D}}^2 = \frac{2}{3}\langle (\Theta - \langle \Theta \rangle_{\mathcal{D}})^2 \rangle_{\mathcal{D}} - 2\langle \sigma^2 \rangle_{\mathcal{D}}. \quad (11)$$

Here, I and II denote the principal scalar invariants [defined equivalently to Eq. (44) below] of the extrinsic curvature coefficients K_{ij} . The second equality follows by introducing the decomposition of the extrinsic curvature into the kinematical variables, by means of the identity $\Theta^i_j := -K^i_j$. The kinematical variables are the *rate of expansion* $\Theta := -K^k_k$ and the *shear* $\sigma^i_j = -K^i_j - \frac{1}{3}\Theta\delta^i_j$. We also defined the *rate of shear* $\sigma^2 := 1/2\sigma^i_j\sigma^j_i$.

We appreciate a close correspondence of the general relativity (GR) equations (9) and (10) with their Newtonian counterparts (see [9]). The first equation is formally identical to the Newtonian one, while the second delivers an additional relation between the averaged scalar curvature and the kinematical backreaction term that has no Newtonian analogue. This implies an important difference that becomes manifest by looking at the time derivative of Eq. (10). The integrability condition that this time derivative agrees with Eq. (9) is nontrivial in the inhomogeneous GR context and reads

$$\partial_t \mathcal{Q}_{\mathcal{D}} + 6H_{\mathcal{D}}\mathcal{Q}_{\mathcal{D}} + \partial_i \langle \mathcal{R} \rangle_{\mathcal{D}} + 2H_{\mathcal{D}}\langle \mathcal{R} \rangle_{\mathcal{D}} = 0, \quad (12)$$

where $H_{\mathcal{D}} := \dot{a}_{\mathcal{D}}/a_{\mathcal{D}}$ denotes the *volume Hubble functional*.

Equation (12) shows that averaged scalar curvature and the kinematical backreaction term are directly coupled, unlike in the Newtonian case, where the curvature is non-existent. For initially vanishing k in the standard FLRW cosmology, the scalar curvature remains zero. This is not the case in the inhomogeneous GR context, where kinematical backreaction produces averaged curvature in the course of structure formation, even for domains that are on average flat initially. (Note also that the sign of the averaged curvature may change during the evolution contrary to FLRW models.) To express the deviation of cosmic curvature from a constant-curvature (quasi-Friedmannian) behavior, we define the average peculiar-scalar curvature by

$$\mathcal{W}_{\mathcal{D}} := \langle \mathcal{R} \rangle_{\mathcal{D}} - 6k_{\mathcal{D}_i}/a_{\mathcal{D}}^2. \quad (13)$$

Integrating Eq. (12) with this definition inserted, we obtain

$$\frac{1}{3a_{\mathcal{D}}^2} \left(\frac{c_{\mathcal{D}_i}}{2} - \int_{t_i}^t dt' \mathcal{Q}_{\mathcal{D}} \frac{d}{dt'} a_{\mathcal{D}}^2(t') \right) = \frac{1}{6} (\mathcal{W}_{\mathcal{D}} + \mathcal{Q}_{\mathcal{D}}), \quad (14)$$

with $c_{\mathcal{D}_i} = \mathcal{W}_{\mathcal{D}_i} + \mathcal{Q}_{\mathcal{D}_i}$. Inserting it back into Eq. (10) we obtain the *generalized Friedmann equation*:

$$\begin{aligned} \frac{\dot{a}_{\mathcal{D}}^2 + k_{\mathcal{D}_i}}{a_{\mathcal{D}}^2} - \frac{8\pi G \langle \varrho \rangle_{\mathcal{D}}}{3} - \frac{\Lambda}{3} \\ = \frac{1}{3a_{\mathcal{D}}^2} \left(\frac{c_{\mathcal{D}_i}}{2} - \int_{t_i}^t dt' \mathcal{Q}_{\mathcal{D}} \frac{d}{dt'} a_{\mathcal{D}}^2(t') \right). \end{aligned} \quad (15)$$

This equation is formally equivalent to its Newtonian counterpart [10]. It shows that, by eliminating the averaged curvature, the whole history of the averaged kinematical fluctuations acts as a source of a generalized Friedmann equation. This equation was the starting point of our investigations in [9]. Since it is also valid in general relativity, we are in the position to translate many results from [9] into the GR context. In particular, our numerical codes can be accordingly applied.

We now provide a compact form of the averaged equations introduced above, as well as some derived quantities that we will analyze in this paper.

2. Effective Friedmannian framework

The above equations can formally be recast into standard Friedmann equations for an effective perfect fluid energy momentum tensor with new effective sources [16]:

$$\begin{aligned} \varrho_{\text{eff}}^{\mathcal{D}} &= \langle \varrho \rangle_{\mathcal{D}} - \frac{1}{16\pi G} \mathcal{Q}_{\mathcal{D}} - \frac{1}{16\pi G} \mathcal{W}_{\mathcal{D}}; \\ p_{\text{eff}}^{\mathcal{D}} &= -\frac{1}{16\pi G} \mathcal{Q}_{\mathcal{D}} + \frac{1}{48\pi G} \mathcal{W}_{\mathcal{D}}. \end{aligned} \quad (16)$$

$$\begin{aligned} 3\frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} &= \Lambda - 4\pi G(\varrho_{\text{eff}}^{\mathcal{D}} + 3p_{\text{eff}}^{\mathcal{D}}); \\ 3H_{\mathcal{D}}^2 + \frac{3k_{\mathcal{D}_i}}{a_{\mathcal{D}}^2} &= \Lambda + 8\pi G\varrho_{\text{eff}}^{\mathcal{D}}; \\ \dot{\varrho}_{\text{eff}}^{\mathcal{D}} + 3H_{\mathcal{D}}(\varrho_{\text{eff}}^{\mathcal{D}} + p_{\text{eff}}^{\mathcal{D}}) &= 0. \end{aligned} \quad (17)$$

Equations (17) correspond to Eqs. (9), (10), and (12), respectively.

This system of equations does not close unless we impose a model for the inhomogeneities. Note that, if the system would close, this would mean that we solved the scalar parts of the GR equations in general by reducing them to a set of ordinary differential equations on arbitrary scales \mathcal{D} . Closure assumptions have been studied by prescribing a *cosmic equation of state* of the form $p_{\text{eff}}^{\mathcal{D}} = \beta(\varrho_{\text{eff}}^{\mathcal{D}}, a_{\mathcal{D}})$ [7,17], or by prescribing the backreaction terms through *scaling solutions*, e.g., $\mathcal{Q}_{\mathcal{D}} \propto a_{\mathcal{D}}^n$ [12,13]. In this paper we are going to explicitly model $\mathcal{Q}_{\mathcal{D}}$ by a relativistic Lagrangian perturbation scheme.

3. The cosmic quartet and derived parameters

For later convenience we introduce the set of dimensionless average characteristics:

$$\begin{aligned}\Omega_m^{\mathcal{D}} &:= \frac{8\pi G}{3H_{\mathcal{D}}^2} \langle \rho \rangle_{\mathcal{D}}; & \Omega_{\Lambda}^{\mathcal{D}} &:= \frac{\Lambda}{3H_{\mathcal{D}}^2}; \\ \Omega_{\mathcal{R}}^{\mathcal{D}} &:= -\frac{\langle \mathcal{R} \rangle_{\mathcal{D}}}{6H_{\mathcal{D}}^2}; & \Omega_{\mathcal{Q}}^{\mathcal{D}} &:= -\frac{\mathcal{Q}_{\mathcal{D}}}{6H_{\mathcal{D}}^2}.\end{aligned}\quad (18)$$

We will, henceforth, call these characteristics ‘‘parameters,’’ but the reader should keep in mind that these are functionals on \mathcal{D} . Expressed through these parameters the averaged Hamilton constraint (10) assumes the form of a *cosmic quartet* [18]:

$$\Omega_m^{\mathcal{D}} + \Omega_{\Lambda}^{\mathcal{D}} + \Omega_{\mathcal{R}}^{\mathcal{D}} + \Omega_{\mathcal{Q}}^{\mathcal{D}} = 1. \quad (19)$$

In this set, the averaged scalar curvature parameter and the kinematical backreaction parameter are directly expressed through $\langle \mathcal{R} \rangle_{\mathcal{D}}$ and $\mathcal{Q}_{\mathcal{D}}$, respectively.

In order to compare these parameters with the ‘‘Friedmannian curvature parameter’’ that had to be used in Newtonian cosmology [9], and that is employed to interpret observational data, we can alternatively introduce the set of parameters

$$\begin{aligned}\Omega_k^{\mathcal{D}} &:= -\frac{k_{\mathcal{D}_i}}{a_{\mathcal{D}}^2 H_{\mathcal{D}}^2} = \Omega_{\mathcal{R}}^{\mathcal{D}} - \Omega_{\mathcal{W}}^{\mathcal{D}}; & \Omega_{\mathcal{W}}^{\mathcal{D}} &:= \frac{-\mathcal{W}_{\mathcal{D}}}{6H_{\mathcal{D}}^2}; \\ \Omega_{\mathcal{Q}_N}^{\mathcal{D}} &:= \left(\frac{c_{\mathcal{D}_i}}{2} - \int_{t_i}^t dt' \mathcal{Q}_{\mathcal{D}} \frac{d}{dt'} a_{\mathcal{D}}^2(t') \right),\end{aligned}\quad (20)$$

being related to the previous parameters by

$$\Omega_k^{\mathcal{D}} + \Omega_{\mathcal{Q}_N}^{\mathcal{D}} = \Omega_{\mathcal{R}}^{\mathcal{D}} + \Omega_{\mathcal{Q}}^{\mathcal{D}}. \quad (21)$$

Like the volume scale factor $a_{\mathcal{D}}$ and the volume Hubble functional $H_{\mathcal{D}}$, we may introduce ‘‘parameters’’ for higher derivatives of the volume scale factor, e.g., the *volume deceleration functional*

$$q^{\mathcal{D}} := -\frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} \frac{1}{H_{\mathcal{D}}^2} = \frac{1}{2} \Omega_m^{\mathcal{D}} + 2\Omega_{\mathcal{Q}}^{\mathcal{D}} - \Omega_{\Lambda}^{\mathcal{D}}. \quad (22)$$

(For higher derivatives such as the *state finders*, see [7].) In this paper we denote all the parameters evaluated at the initial time by the index \mathcal{D}_i and at the present time by the index \mathcal{D}_0 .

III. QUANTIFYING BACKREACTION

In the last section we saw how the appearance of the backreaction term modifies the standard Friedmannian evolution. In the following we will quantify this departure. Exact inhomogeneous solutions for estimating the amount of backreaction are only available for highly symmetric models like for models with plane symmetry (Sec. VA) and the spherically symmetric solutions (Sec. VB). In generic situations we have to rely on approximations. As the Newtonian result [9] suggests, we will obtain also here an approximation that interpolates in special cases between the above two subclasses of exact GR solutions with orthogonal kinematical properties. However, the classes of

solutions will be more strongly restricted than in the Newtonian case, because of the Hamilton constraint that joins the system of equations in the GR context.

A. Modeling the local deformation

1. Representation of deformations through coframes

To be able to use the relativistic Zel’dovich approximation as formulated in [1], we will first express the kinematical backreaction term of Eq. (11) in terms of the spatial coframe coefficients η^a_i . These latter are defined through the metric form coefficients

$$g_{ij}(t, X^k) := G_{ab} \eta^a_i(t, X^k) \eta^b_j(t, X^k), \quad (23)$$

where $G_{ab} \delta^a_i \delta^b_j = G_{ij} = g_{ij}(t_i, X^k)$ are the initial metric coefficients. The coframes therefore contain the complete time evolution of the metric. We now express the expansion tensor

$$\Theta^i_j = g^{ik} \dot{g}_{kj}$$

with the help of the coframes. Since the inverse metric is analogously decomposable into frames e_a^i (that are inverse to the coframes $e_a^i \eta^a_j = \delta^i_j$; $e_a^i \eta^b_i = \delta_a^b$),

$$g^{ij} = G^{ab} e_a^i e_b^j, \quad (24)$$

we can finally write (for details see [1])

$$\Theta^i_j = e_a^i \dot{\eta}^a_j = \frac{1}{2J} \epsilon_{abc} \epsilon^{ikl} \dot{\eta}^a_j \eta^b_k \eta^c_l, \quad (25)$$

with the local volume deformation (corresponding to the Jacobian in Newtonian theory)

$$J := \det(\eta^a_i) = \frac{1}{6} \epsilon_{abc} \epsilon^{ijk} \eta^a_i \eta^b_j \eta^c_k. \quad (26)$$

This may then be used to calculate the first two invariants of Θ^i_j and to obtain the expression

$$\mathcal{Q}_{\mathcal{D}} = \frac{1}{\langle J \rangle_I} \langle \epsilon_{abc} \epsilon^{ikl} \dot{\eta}^a_i \dot{\eta}^b_k \eta^c_l \rangle_I - \frac{2}{3} \left(\frac{\langle \dot{J} \rangle_I}{\langle J \rangle_I} \right)^2, \quad (27)$$

for $\mathcal{Q}_{\mathcal{D}}$ that solely depends on the coframes η^a_i and their time derivatives.

2. The relativistic Zel’dovich approximation

The equations and parameters introduced in Sec. II can live without introducing a background spacetime. The description is background-free and nonperturbative. Also expressing $\mathcal{Q}_{\mathcal{D}}$ in terms of coframes as sketched in the previous section is still completely general. However, in order to provide a concrete model for the backreaction terms, we employ methods of perturbation theory. We will, however, only model the fluctuations by perturbation theory; by using the exact expressions for the functionals in terms of coframes, we extrapolate the first-order perturbative solution.

We now introduce perturbed coframes analogous to [1]. In perturbation schemes one usually defines a reference frame through a known solution, e.g., a frame *comoving* with the Hubble flow (i.e. a FLRW solution) that we now represent through three homogeneous-isotropic deformation one-forms (labeled by $a, b, c \dots$ and expressed in the local exact coordinate basis labeled by $i, j, k \dots$):

$$\boldsymbol{\eta}_H^a = \eta_H^a \mathbf{d}X^i := a(t) \boldsymbol{\eta}_H^a(t_i), \quad \eta_H^a := a(t) \delta^a_i, \quad (28)$$

where $a(t)$ is a solution of Friedmann's differential equation

$$H^2 = \frac{8\pi G \rho_H(t) + \Lambda}{3} - \frac{k}{a^2(t)}, \quad (29)$$

with $H(t) := \dot{a}/a$, $\rho_H(t) = \rho_{H_i}/a^3(t)$ and $a(t_i) = 1$. For the full deformation one-forms we prescribe the superposition

$$\boldsymbol{\eta}^a = \boldsymbol{\eta}_H^a + a(t) \mathbf{P}^a, \quad (30)$$

with inhomogeneous deformation one-forms $\mathbf{P}^a(t, X^k)$. To first order, they can be restricted to the relativistic generalization of Zel'dovich's approximation (RZA) [1]:

$$\mathbf{P}^a = \xi(t) \dot{P}^a_i \mathbf{d}X^i, \quad (31)$$

with $\dot{P}^a_i := \dot{P}^a_i(t_i, X^k)$. The function $\xi(t)$ is defined by

$$\xi(t) := [q(t) - q(t_i)]/\dot{q}(t_i), \quad (32)$$

where the function $q(t)$ is solution of the equation

$$\ddot{q}(t) + 2\frac{\dot{a}(t)}{a(t)}\dot{q}(t) + \left(3\frac{\ddot{a}(t)}{a(t)} - \Lambda\right)q(t) = 0. \quad (33)$$

Thus, the function ξ satisfies

$$\ddot{\xi}(t) + 2\frac{\dot{a}(t)}{a(t)}\dot{\xi}(t) + \left(3\frac{\ddot{a}(t)}{a(t)} - \Lambda\right)\left(\xi(t) + \frac{q(t_i)}{\dot{q}(t_i)}\right) = 0. \quad (34)$$

Explicit solutions of this equation for different backgrounds may be found in [19], those including a cosmological constant in [20].

Writing the RZA of Eq. (30) in the exact coordinate basis $\mathbf{d}X^i$ yields

$${}^{\text{RZA}}\boldsymbol{\eta}^a_i(t, X^k) := a(t)(N^a_i + \xi(t)\dot{P}^a_i), \quad (35)$$

where $N^a_i := {}^{\text{RZA}}\boldsymbol{\eta}^a_i(t_i, X^k)$ and $\dot{P}^a_i = \dot{P}^a_i(t_i, X^k)$.

Before we use this expression to determine the backreaction term, let us elaborate on a subtlety with the choice of initial conditions. The expression of the metric tensor in terms of nonintegrable coframes,

$$g_{ij} := G_{ab} \boldsymbol{\eta}^a_i \boldsymbol{\eta}^b_j, \quad (36)$$

allows two different treatments of the initial displacements. One can either include them into G_{ab} which means that the noncoordinate basis is orthogonal but not orthonormal, or one can choose G_{ab} to be orthonormal (being the standard

assumption), i.e. $G_{ab} = \delta_{ab}$, but then one has to deal with the initial values of the coframes. To have both at a time, i.e. $N^a_i = \delta^a_i$ and $G_{ab} = \delta_{ab}$, is not possible as this would mean that the RZA initial metric would be Euclidean, and this would disable any time evolution of this metric as pointed out in [3,5]. As a nontrivial time evolution of the metric is what we are interested in, there are only two options:

[O1:] If $\tilde{N}^a_i := \delta^a_i + P^a_i$, with $P^a_i := P^a_i(t_i, X^k)$ and $P^a_i \neq 0$, then \mathbf{G} can be restricted to $\boldsymbol{\delta}$, and the coframes read ${}^{\text{RZA}}\tilde{\boldsymbol{\eta}}^a_i(t, X^k) := a(t)(\delta^a_i + P^a_i + \xi(t)\dot{P}^a_i)$ with the metric $g_{ij} := \delta_{ab} \tilde{\boldsymbol{\eta}}^a_i \tilde{\boldsymbol{\eta}}^b_j$.

[O2:] By appropriate coordinate transformations, one may set N^a_i in Eq. (35) to δ^a_i ; the transformation then sends $\dot{P}^a_i \rightarrow \dot{\mathcal{P}}^a_i = \delta^a_j {}^{\text{RZA}}\tilde{e}^j_b(t_i, X^k) \dot{P}^b_i$, and all information about the initial geometrical inhomogeneities is contained in \mathbf{G} . The coframes become ${}^{\text{RZA}}\boldsymbol{\eta}^a_i(t, X^k) := a(t)(\delta^a_i + \xi(t)\dot{\mathcal{P}}^a_i)$, and the metric $g_{ij} := G_{ab} \boldsymbol{\eta}^a_i \boldsymbol{\eta}^b_j$ with $G_{ab} = \delta_{cd} \tilde{N}^c_a \tilde{N}^d_b$, with \tilde{N}^c_a as defined above.

In order to have a complete formal correspondence with the Newtonian model, we stick to the second option in what follows. (Notice that we have chosen the first option in [1].)

B. Kinematical backreaction and intrinsic curvature models

After the definition of the perturbative setup explained in the previous section, we now turn to the concrete calculation of the backreaction and curvature models. As discussed in the previous section, we use non-normalized coframes (option 2; the corresponding expressions with option 1 will be listed in Appendix B):

$$g_{ij} = G_{ab} \boldsymbol{\eta}^a_i \boldsymbol{\eta}^b_j. \quad (37)$$

As pointed out by Chandrasekhar [21] such a choice can lead to formally simpler expressions in some cases, and here we encounter such a case. Indeed, the expression for the kinematical backreaction term turns out to resemble more closely its Newtonian counterpart, and the formulas become more concise.

The relativistic Zel'dovich approximation is then defined as

$${}^{\text{RZA}}\boldsymbol{\eta}^a_i(t, X^k) := a(t)(\delta^a_i + \xi(t)\dot{\mathcal{P}}^a_i),$$

with

$$\dot{\mathcal{P}}^a_i = \dot{\mathcal{P}}^a_i(t_i, X^k), \quad \xi(t_i) = 0, \quad a(t_i) = 1. \quad (38)$$

By definition (see [1]), the RZA of any field is the evaluation of this field in terms of its functional dependence on the RZA coframes, Eq. (38), without any further approximation or truncation. Therefore, the metrical distances are calculated exactly for the approximated deformation:

$$\begin{aligned} {}^{\text{RZA}}g_{ij}(t, X^k) &= a^2(t)\{G_{ij} + \xi(t)(G_{aj}\dot{P}^a_i + G_{ib}\dot{P}^b_j) \\ &+ \xi^2(t)G_{ab}\dot{P}^a_i\dot{P}^b_j\}, \end{aligned} \quad (39)$$

and so

$${}^{\text{RZA}}g_{ij}(t_i) := G_{ij}. \quad (40)$$

The local volume deformation, Eq. (26), then becomes

$${}^{\text{RZA}}J := a^3(t)\mathcal{J} \quad (41)$$

(implying that for the approximated deformation we exactly conserve mass, which would not be the case in a strictly linearized setting); we have introduced the *peculiar-volume deformation*, following from (38),

$$\mathcal{J}(t, X^k) := 1 + \xi(t)\text{I}_i + \xi^2(t)\text{II}_i + \xi^3(t)\text{III}_i, \quad (42)$$

$$\text{I}_i := \text{I}(\dot{P}^a_i); \quad \text{II}_i := \text{II}(\dot{P}^a_i); \quad \text{III}_i := \text{III}(\dot{P}^a_i), \quad (43)$$

where the principal scalar invariants of the matrix \dot{P}^a_i are given by

$$\begin{aligned} \text{I}(\dot{P}^a_i) &:= \frac{1}{2}\epsilon_{abc}\epsilon^{ijk}\dot{P}^a_i\delta^b_j\delta^c_k, \\ \text{II}(\dot{P}^a_i) &:= \frac{1}{2}\epsilon_{abc}\epsilon^{ijk}\dot{P}^a_i\dot{P}^b_j\delta^c_k, \\ \text{III}(\dot{P}^a_i) &:= \frac{1}{6}\epsilon_{abc}\epsilon^{ijk}\dot{P}^a_i\dot{P}^b_j\dot{P}^c_k. \end{aligned} \quad (44)$$

1. The kinematical backreaction

In addition to the local volume deformation J , we need the invariants of the expansion tensor Θ^i_j . As we have introduced a background by Eq. (30), we can define a *peculiar-expansion tensor* $\theta^i_j := \Theta^i_j - H(t)\delta^i_j$ with respect to this background. The three principal scalar invariants of the full expansion tensor decompose into invariants of the peculiar-expansion tensor as follows:

$$\begin{aligned} \text{I}(\Theta^i_j) &= 3H + \text{I}(\theta^i_j), \\ \text{II}(\Theta^i_j) &= 3H^2 + 2HI(\theta^i_j) + \text{II}(\theta^i_j), \\ \text{III}(\Theta^i_j) &= H^3 + H^2\text{I}(\theta^i_j) + H\text{II}(\theta^i_j) + \text{III}(\theta^i_j). \end{aligned} \quad (45)$$

Inserting Eqs. (45) into Eq. (11), we can write the backreaction variable in terms of invariants of the peculiar-expansion tensor:

$$\mathcal{Q}_{\mathcal{D}} = 2\langle \text{II}(\theta^i_j) \rangle_{\mathcal{D}} - \frac{2}{3}\langle \text{I}(\theta^i_j) \rangle_{\mathcal{D}}^2. \quad (46)$$

This nontrivial result demonstrates that the backreaction effects do not depend on an assumed homogeneous reference background (a Hubble flow): backreaction is only due to inhomogeneities. Now, using the formula, Eq. (8), the backreaction term (46) can be expressed in terms of the formal average (7):

$$\mathcal{Q}_{\mathcal{D}} = \frac{2}{\langle \mathcal{J} \rangle_{\mathcal{I}}^2} \left[\langle \text{II}(\theta^i_j) \mathcal{J} \rangle_{\mathcal{I}} \langle \mathcal{J} \rangle_{\mathcal{I}} - \frac{1}{3} \langle \text{I}(\theta^i_j) \mathcal{J} \rangle_{\mathcal{I}}^2 \right]. \quad (47)$$

Now we have all we need. Plugging Eq. (38) into Eq. (25) and subtracting $H(t)\delta^i_j$, we obtain for the scalar invariants of the peculiar-expansion tensor

$$\begin{aligned} {}^{\text{RZA}}\text{I}(\theta^i_j) &= \frac{\dot{\mathcal{J}}}{\mathcal{J}}, \quad {}^{\text{RZA}}\text{II}(\theta^i_j) = \frac{1}{2} \left(\frac{\ddot{\mathcal{J}}}{\mathcal{J}} - \frac{\dot{\xi}(t)\dot{\mathcal{J}}}{\xi(t)\mathcal{J}} \right), \\ {}^{\text{RZA}}\text{III}(\theta^i_j) &= \frac{1}{6} \left(\frac{\ddot{\mathcal{J}}}{\mathcal{J}} - \frac{\dot{\mathcal{J}}\ddot{\xi}(t)}{J\xi(t)} \right) - \frac{1}{2} \left(\frac{\ddot{\mathcal{J}}}{\mathcal{J}} - \frac{\dot{\mathcal{J}}\dot{\xi}(t)}{J\xi(t)} \right) \frac{\ddot{\xi}(t)}{\xi(t)}, \end{aligned} \quad (48)$$

which implies for the backreaction term

$${}^{\text{RZA}}\mathcal{Q}_{\mathcal{D}} = \frac{\langle \dot{\mathcal{J}} \rangle_{\mathcal{I}}}{\langle \mathcal{J} \rangle_{\mathcal{I}}} - \frac{\dot{\xi}}{\xi} \frac{\langle \dot{\mathcal{J}} \rangle_{\mathcal{I}}}{\langle \mathcal{J} \rangle_{\mathcal{I}}} - \frac{2}{3} \left(\frac{\langle \dot{\mathcal{J}} \rangle_{\mathcal{I}}}{\langle \mathcal{J} \rangle_{\mathcal{I}}} \right)^2, \quad (49)$$

which with Eq. (42) yields a compact form of the RZA backreaction model:

$${}^{\text{RZA}}\mathcal{Q}_{\mathcal{D}} = \frac{\xi^2(\gamma_1 + \xi\gamma_2 + \xi^2\gamma_3)}{(1 + \xi\langle \text{I}_i \rangle_{\mathcal{I}} + \xi^2\langle \text{II}_i \rangle_{\mathcal{I}} + \xi^3\langle \text{III}_i \rangle_{\mathcal{I}})^2},$$

with

$$\begin{aligned} \gamma_1 &:= 2\langle \text{II}_i \rangle_{\mathcal{I}} - \frac{2}{3}\langle \text{I}_i \rangle_{\mathcal{I}}^2, \quad \gamma_2 := 6\langle \text{III}_i \rangle_{\mathcal{I}} - \frac{2}{3}\langle \text{II}_i \rangle_{\mathcal{I}}\langle \text{I}_i \rangle_{\mathcal{I}}, \\ \gamma_3 &:= 2\langle \text{I}_i \rangle_{\mathcal{I}}\langle \text{III}_i \rangle_{\mathcal{I}} - \frac{2}{3}\langle \text{II}_i \rangle_{\mathcal{I}}^2. \end{aligned} \quad (50)$$

We note that the initial metric tensor \mathbf{G} does not explicitly appear in the expression for the backreaction model; the above expression is formally equivalent with the Newtonian expression [9].

2. The intrinsic curvature model

Unlike in the Newtonian case [9], where there exists a flat embedding space, RZA is fully intrinsic (i.e. the structures are not embedded into an external space but propagate within the space given by the coframe deformation), and we have a nonvanishing scalar 3-curvature. Its expression may be found by combining the Hamilton constraint and the Raychaudhuri equation:

$$\mathcal{R} = 6\text{II}(\Theta^i_j) - 4\text{I}^2(\Theta^i_j) - 4\dot{\text{I}}(\Theta^i_j) + 6\Lambda, \quad (51)$$

or in terms of the first two scalar invariants of the peculiar-expansion tensor:

$$\mathcal{R} = 6\text{II}(\theta^i_j) - 4\text{I}^2(\theta^i_j) - 4\dot{\text{I}}(\theta^i_j) - 12HI(\theta^i_j) + \frac{6k}{a^2(t)}. \quad (52)$$

For the averaged peculiar-scalar curvature, Eq. (13), using Eq. (48) and (52), we get

$${}^{\text{RZA}}\mathcal{W}_{\mathcal{D}} = -\frac{\langle \dot{\mathcal{J}} \rangle_{\mathcal{I}}}{\langle \mathcal{J} \rangle_{\mathcal{I}}} + 3 \left(\frac{\dot{\xi}}{\xi} + 4\frac{\dot{a}}{a} \right) \frac{\langle \dot{\mathcal{J}} \rangle_{\mathcal{I}}}{\langle \mathcal{J} \rangle_{\mathcal{I}}} + 6 \left(\frac{k}{a^2} - \frac{k_{\mathcal{D}_i}}{a_{\mathcal{D}}^2} \right). \quad (53)$$

Inserting the result for \mathcal{J} from Eq. (42), we find the explicit time dependence in a form similar to $\mathcal{Q}_{\mathcal{D}}$ in Eq. (50):

$$\text{RZA } \mathcal{W}_{\mathcal{D}} = \frac{\xi^2(\tilde{\gamma}_1 + \xi\tilde{\gamma}_2 + \xi^2\tilde{\gamma}_3)}{1 + \xi\langle\text{I}_i\rangle_I + \xi^2\langle\text{II}_i\rangle_I + \xi^3\langle\text{III}_i\rangle_I} + 6\left(\frac{k}{a^2} - \frac{k_{\mathcal{D}_1}}{a_{\mathcal{D}_1}^2}\right),$$

with

$$\begin{aligned}\tilde{\gamma}_1 &:= -2\langle\text{II}_i\rangle_I - 12\langle\text{I}_i\rangle_I \frac{H}{\xi} - 4\langle\text{I}_i\rangle_I \frac{\ddot{\xi}}{\xi^2}, \\ \tilde{\gamma}_2 &:= -6\langle\text{III}_i\rangle_I - 24\langle\text{II}_i\rangle_I \frac{H}{\xi} - 8\langle\text{II}_i\rangle_I \frac{\ddot{\xi}}{\xi^2}, \\ \tilde{\gamma}_3 &:= -36\langle\text{III}_i\rangle_I \frac{H}{\xi} - 12\langle\text{III}_i\rangle_I \frac{\ddot{\xi}}{\xi^2}.\end{aligned}\quad (54)$$

In this expression there are two important formal differences compared with the functional form of $\mathcal{Q}_{\mathcal{D}}$. First of all, the time derivatives of ξ no longer disappear automatically from the solution. In addition, the curvature term also explicitly depends on the background via H . This leads to a time dependence of the coefficients $\tilde{\gamma}_i$, whereas for the kinematical backreaction functional they were simply constants.

In the case of the Einstein–de Sitter (EdS) background, the growth of the first-order perturbation goes with the scale factor $q = a$, and so we can simplify the expression to

$$\begin{aligned}\text{RZA } \mathcal{W}_{\mathcal{D}} &= \frac{-10H_1\xi^2}{a} \frac{\langle\text{I}_i\rangle_I + 2\langle\text{II}_i\rangle_I\xi + 3\langle\text{III}_i\rangle_I\xi^2}{1 + \xi\langle\text{I}_i\rangle_I + \xi^2\langle\text{II}_i\rangle_I + \xi^3\langle\text{III}_i\rangle_I} \\ &\quad - \xi^2 \frac{2\langle\text{II}_i\rangle_I + 6\langle\text{III}_i\rangle_I\xi}{1 + \xi\langle\text{I}_i\rangle_I + \xi^2\langle\text{II}_i\rangle_I + \xi^3\langle\text{III}_i\rangle_I}.\end{aligned}\quad (55)$$

As $\xi^2 \propto a^{-1}$ in the EdS case, we recover the well-known result that the leading curvature contribution goes as $\propto a^{-2}$. The second term, which is the leading second-order contribution, goes as a^{-1} as expected [22]. We will learn below that this form of the peculiar-curvature functional is not the best choice. As it will turn out in the explicit consideration within the class of exact spherically symmetric solutions, Sec. VB 1, there exists a better approximation for numerical evaluations that employ $\mathcal{Q}_{\mathcal{D}}$ together with the exact integrability constraint. This expression and the comparison with the above expression will be explicitly provided in Sec. IV B.

IV. CONSISTENCY CHECKS

In order to evaluate the goodness of the averaged model based on the RZA deformation, we will perform three consistency checks in this section.

A. Newtonian limit

First we show that the RZA result of Eq. (50) for the backreaction model has the correct Newtonian limit. The procedure to arrive at the Newtonian limit of the relativistic

quantities follows the prescription of [1], that sends the nonintegrable Cartan coframes to integrable ones:

$$\eta^a{}_i \rightarrow {}^N\eta^a{}_i = f^a|_i.$$

As was demonstrated in [1], this leads exactly to the Newtonian system of fluid equations in the Lagrangian description. f^a is the position vector field that maps the Lagrangian coordinates to Eulerian positions, and its Lagrangian gradient encodes the volume deformation of fluid elements. In [9] this has been approximated by the Newtonian form of the Zel'dovich approximation. Comparing the corresponding expressions we find that

$${}^N\mathcal{P}^a{}_i = \psi^{|a}{}_{|i};$$

i.e. the time derivative of the displacement one-forms can now be represented by spatial derivatives of a potential ψ in the Newtonian approximation. The invariants of Eq. (43) similarly become

$$\text{I}_i := \text{I}(\psi^{|i}{}_{|j}), \quad \text{II}_i := \text{II}(\psi^{|i}{}_{|j}), \quad \text{III}_i := \text{III}(\psi^{|i}{}_{|j}). \quad (56)$$

With this spatial geometrical limit we also send the general curved space, on which we defined our average, to Euclidean space. Note that the existence of a vector field f^a implies that the counterindex a becomes a coordinate index of (now existing) global coordinates in an embedding space. The Riemannian volume average automatically corresponds to the Euclidean volume average over a flat domain \mathcal{D}_i . [Note that sending the speed of light to infinity is only needed in order to change the spacetime metric signature; since the backreaction model is spatial, and since we have eliminated all terms that would contain the speed of light (the curvature), we do not have to send c to infinity.]

Taking the spatial geometrical limit is thus sufficient to reduce the result, Eq. (50), to the Newtonian expression:

$$\text{NZA } \mathcal{Q}_{\mathcal{D}} = \frac{\xi^2(Y_1 + \xi Y_2 + \xi^2 Y_3)}{(1 + \xi\langle\text{I}_i\rangle_{\mathcal{D}_i} + \xi^2\langle\text{II}_i\rangle_{\mathcal{D}_i} + \xi^3\langle\text{III}_i\rangle_{\mathcal{D}_i})^2},$$

with

$$\begin{aligned}Y_1 &:= 2\langle\text{II}_i\rangle_{\mathcal{D}_i} - \frac{2}{3}\langle\text{I}_i\rangle_{\mathcal{D}_i}^2, & Y_2 &:= 6\langle\text{III}_i\rangle_{\mathcal{D}_i} - \frac{2}{3}\langle\text{I}_i\rangle_{\mathcal{D}_i}\langle\text{II}_i\rangle_{\mathcal{D}_i}, \\ Y_3 &:= 2\langle\text{I}_i\rangle_{\mathcal{D}_i}\langle\text{III}_i\rangle_{\mathcal{D}_i} - \frac{2}{3}\langle\text{II}_i\rangle_{\mathcal{D}_i}^2.\end{aligned}\quad (57)$$

B. Matching curvature expressions

Another consistency check considers the scalar curvature in the RZA. There are in principle three ways to calculate the intrinsic curvature that all agree for an exact solution. The first way is to use the RZA metric and to calculate the curvature geometrically. The other two ways will be studied below; both relate the scalar curvature to extrinsic curvature invariants (note that these are relevant for this work, since we study kinematical backreaction being related to extrinsic curvature):

- (a) using a combination of the Hamilton constraint and the Raychaudhuri equation,
 (b) using the integrability condition.

Thus the scalar curvature is a good tool to test the RZA: we expect that to the first order the two results obtained by (a) and (b) are the same. The expression for the way (a) has already been obtained as Eq. (53). For the way (b) we express $\mathcal{W}_{\mathcal{D}}$ through Eq. (14) as

$$\mathcal{W}_{\mathcal{D}} = \frac{1}{a_{\mathcal{D}}^2} \left(c_{\mathcal{D}_1} - 6k_{\mathcal{D}_1} - 2 \int_{t_1}^t \mathcal{Q}_{\mathcal{D}} \frac{d}{dt'} a_{\mathcal{D}}^2(t') dt' \right) - \mathcal{Q}_{\mathcal{D}}. \quad (58)$$

Using (33) and (49), the Friedmann equations for the background scale factor, and

$${}^{\text{RZA}}a_{\mathcal{D}} = a(t) \langle \mathcal{J} \rangle_I^{1/3}, \quad (59)$$

one can show after a long but not too technical calculation the following:

$$\begin{aligned} {}^{\text{RZA}}\mathcal{W}_{\mathcal{D}} = & - \left[\frac{\langle \dot{\mathcal{J}} \rangle_I}{\langle \mathcal{J} \rangle_I} + 3 \left(\frac{\ddot{\xi}}{\xi} + 4 \frac{\dot{a}}{a} \right) \frac{\langle \dot{\mathcal{J}} \rangle_I}{\langle \mathcal{J} \rangle_I} \right] \\ & + \frac{2}{a^2 \langle \mathcal{J} \rangle_I^{2/3}} \cdot \int_{t_1}^t \left[\frac{8\pi G \rho_{H_1}}{a \langle \mathcal{J} \rangle_I^{1/3}} q(t') \frac{d}{dt'} \left(\frac{\langle \dot{\mathcal{J}} \rangle_I}{\xi} \right) \right] dt' \\ & + 6 \left(\frac{k}{a^2} - \frac{k_{\mathcal{D}_1}}{a_{\mathcal{D}}^2} \right). \end{aligned} \quad (60)$$

Using this expression for the averaged curvature assures the integrability condition (12) to hold for a given kinematical backreaction model. In other words, this expression respects the conservation law for the combined action of kinematical backreaction and averaged curvature [third equation of Eq. (17); note that the averaged rest mass is individually conserved].

If we now compare Eqs. (53) and (60), we find that the RZA is consistent if and only if the remaining integral vanishes. After a change of integration variables this integral has the form

$$\int_{\xi_1}^{\xi_f} \frac{8\pi G \rho_{H_1}}{a(\xi) \langle \mathcal{J} \rangle_I^{1/3}} \left(\xi + \frac{q(t_1)}{\dot{q}(t_1)} \right) \frac{d^2}{d\xi^2} \langle \mathcal{J} \rangle_I d\xi = 0. \quad (61)$$

As it is zero for all values of ξ_f , already the integrand has to vanish. Its first three factors are nonzero which means

$$\frac{d^2}{d\xi^2} \langle \mathcal{J} \rangle_I = 0 \quad \forall \xi. \quad (62)$$

Using the definition of $\langle \mathcal{J} \rangle_I$ this implies that the RZA approximation is consistent iff

$$a_{\mathcal{D}}^3 = 1 + \xi(t) \langle I_1 \rangle_I \quad \text{and} \quad \langle II_1 \rangle_I = 0 = \langle III_1 \rangle_I, \quad (63)$$

which encodes the fact that it is strictly speaking only a first-order scheme. Its success in the Newtonian case, however, motivates its use in the relativistic case as well, despite this result which otherwise stated reads

$$\int_{\xi_1}^{\xi_f} \xi + \frac{q(t_1)}{\dot{q}(t_1)} \frac{\langle II_1 \rangle_I + 3\xi \langle III_1 \rangle_I}{\langle \mathcal{J} \rangle_I^{1/3}} d\xi = \mathcal{O}(\langle II_1 \rangle_I). \quad (64)$$

This shows that the deviation from Eq. (53) contains only terms of second and higher order in the initial conditions.

An equivalent way to check the consistency is to insert ${}^{\text{RZA}}\mathcal{Q}_{\mathcal{D}}$ [Eq. (50)], ${}^{\text{RZA}}\mathcal{W}_{\mathcal{D}}$ [Eq. (54)] and ${}^{\text{RZA}}a_{\mathcal{D}}$ [Eq. (59)] directly into the integrability condition [Eq. (12)]. The result may be written as

$$- \frac{6H_0 \Omega_m (1 + H_0 \xi)}{a^3(t) \langle \mathcal{J} \rangle_I} \xi \frac{d^2}{d\xi^2} \langle \mathcal{J} \rangle_I = 0.$$

This can be interpreted as the amount by which the ‘‘closure’’ of the integrability condition of ${}^{\text{RZA}}\mathcal{Q}_{\mathcal{D}}$ and ${}^{\text{RZA}}\mathcal{W}_{\mathcal{D}}$ fails. As the prefactors are nonvanishing, we recover the condition of Eq. (62).

C. Self-consistency test in terms of the scale factor

We have already seen above that the RZA delivers only consistent expressions, if it is employed strictly as a first-order scheme. Therefore, an extrapolation as suggested in [1], bearing a number of advantages as discussed there, is only an approximation for the terms at second and higher order. It is therefore not surprising that, in addition to the different curvature expressions, we also have two concurring definitions for the volume scale factor. One is the kinematical volume scale factor defined as the cubic root of the local volume deformation in the RZA, Eq. (41), ${}^{\text{KIN}}a_{\mathcal{D}} = ({}^{\text{RZA}}\mathcal{J})^{1/3}$. The other one is ${}^{\text{RZA}}a_{\mathcal{D}}$, calculated from the RZA of the backreaction term and using this latter as a source of the general equations governing the average evolution, i.e. the prescription of Sec. VIA. Just like the curvature expressions, they do not coincide for the RZA. For a full n th-order calculation they are the same up to the given order. For the curvature we will see in Sec. VB 1 that the expression derived from the backreaction term is more powerful and seems more appropriate to capture the non-linear evolution. With this insight we are going to use ${}^{\text{RZA}}a_{\mathcal{D}}$ in the concrete calculations presented further below. Fortunately, it will turn out that the possible error induced by instead calculating the kinematical volume scale factor is numerically rather small: evaluating the quantity

$$\epsilon_a = \frac{|{}^{\text{RZA}}a_{\mathcal{D}} - {}^{\text{KIN}}a_{\mathcal{D}}|}{{}^{\text{RZA}}a_{\mathcal{D}}}, \quad (65)$$

we found that ϵ_a stayed well below 0.1 at all times, as long as ${}^{\text{RZA}}a_{\mathcal{D}}$ did not approach zero and the domain’s effective radius was larger than 16 Mpc.

V. EXACT SUBCASES

We are now going to study subclasses of exact solutions that are contained in the above approximation. We will

learn that (i) a subclass of the averaged plane-symmetric solutions, (ii) a subclass of the averaged Lemaitre-Tolman-Bondi solutions, as well as (iii) a subclass of the averaged Szekeres solutions are found by suitably restricting the initial data. This fact demonstrates that the investigated approximation interpolates between two kinematically orthogonal exact solutions that are, however, unlike in the corresponding Newtonian model, subjected to constraints. This latter fact suggests that one may be able to construct more general backreaction models that include these cases in full generality. However, we do not expect this to be possible for first-order deformations.

A. The relativistic Zel'dovich approximation and plane collapse models

In Newtonian cosmology the Zel'dovich approximation is an *exact* three-dimensional solution to the Newtonian dynamics of self-gravitating dust matter for initial conditions with $\text{II}(\theta_{ij}) = 0 = \text{III}(\theta_{ij})$ at each trajectory [19]. This ‘‘locally one-dimensional’’ class of motions contains as a subcase the globally plane-symmetric solution (see also [23–25]). In the relativistic case one may study the corresponding model using the plane-symmetric ansatz for the line element,

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + (1 + P(z, t))^2 dz^2), \quad (66)$$

which also has vanishing higher invariants of the peculiar expansion tensor: $\text{II}(\theta_{ij}) = 0 = \text{III}(\theta_{ij})$. The first invariant is nontrivial and reads

$$\text{I}(\theta^i_j) = \frac{\dot{P}(z, t)}{1 + P(z, t)}. \quad (67)$$

The equation determining the time evolution of $P(z, t)$ was in the Newtonian case simply

$$\dot{\Theta} + \Theta^k_l \Theta^l_k = -4\pi G \varrho + \Lambda, \quad (68)$$

which gave

$$\ddot{P}(z, t) + 2\frac{\dot{a}}{a}\dot{P}(z, t) = 4\pi G \varrho_H(P(z, t) - P_0(z)). \quad (69)$$

Hence, the Newtonian plane collapse had two solutions, e.g., for an EdS background,

$$P(z, t) = P_0(z) + aC_1(z) + \frac{C_2(z)}{a^{3/2}}, \quad (70)$$

a growing and a decaying one. In the relativistic case, however, there are more constraints. In the Lagrange-Einstein system of [1], also a link to the scalar curvature comes in:

$$\dot{\Theta}^i_j + \Theta \Theta^i_j = (4\pi G \varrho + \Lambda)\delta^i_j - \mathcal{R}^i_j; \quad (71)$$

$$\Theta^2 - \Theta^k_l \Theta^l_k = 16\pi G \varrho + 2\Lambda - \mathcal{R}. \quad (72)$$

These two equations combined also give Eq. (68) for the relativistic case. Additionally, however, they have to be satisfied individually. As the plane-symmetric metric ansatz Eq. (66) implies that $\mathcal{R}^i_j = 0$, the relativistic solution space is not the same as the Newtonian one. For the Hamilton constraint Eq. (72) we find

$$\frac{\dot{a}}{a}\dot{P}(z, t) = -4\pi G \varrho_H(P(z, t) - P_0(z)), \quad (73)$$

which is now, for $\mathcal{R} = 0$, only a differential equation of first order with the solution

$$P(z, t) = P_0(z) + \frac{C(z)}{a^{3/2}}. \quad (74)$$

This $P(z, t)$ also satisfies Eq. (71), but one of the solutions of the Newtonian case has disappeared.

From (74) it follows that $P(z, t) = P_0(z) + q(t)C(z)$, with q being a solution of (33). Inserting this result into (67), and using Eq. (46), we find for the backreaction term

$$\mathcal{Q}_D^{\text{plane}} = -\frac{2}{3}\langle \text{I}(\theta^i_j) \rangle_D^2 = -\frac{2}{3}\left(\frac{\xi \langle \text{I}_i \rangle_I}{1 + \xi \langle \text{I}_i \rangle_I}\right)^2, \quad (75)$$

where we recover $\xi(t) = (q(t) - q(t_0))/\dot{q}(t_0)$, and where $\text{I}_i = \text{I}(\theta^i_j)(t_i)$. Extracting the coframe from the metric (66) we find also that $\text{I}_i = \text{I}(\dot{\mathcal{P}}^a_i(z, t_i))$, where $\dot{\mathcal{P}}^a_i(z, t_i)$ is the RZA perturbation. This shows that the plane-symmetric metric is, as in the Newtonian case, a particular exact solution that is contained in the solutions of the RZA. Note again, however, that this solution in the RZA as well as for the plane-symmetric metric does not have the growing mode that was present in the Newtonian solution. This is due to the vanishing scalar curvature for cylindrical symmetry and the relation to the Hamilton constraint that did not exist in the Newtonian case. [Note that the integrability condition (12) is satisfied by this solution.] This is another interesting example of a case in which a class of Newtonian solutions may not automatically provide a solution of general relativity.

For negative $\langle \text{I}_i \rangle_I$, corresponding to overdense regions [Eq. (92)], $\mathcal{Q}_D^{\text{plane}}$ is diverging at some time when $1 + \xi(t)\langle \text{I}_i \rangle_I$ approaches zero, even though the solution is decaying. Our special initial conditions imply a one-dimensional symmetry of inhomogeneities on a three-dimensional background (cylindrical symmetry), and the diverging $\mathcal{Q}_D^{\text{plane}}$ is supposed to mimic the highly anisotropic pancake collapse in the three-dimensional situation.

The plane-symmetric case has also been considered in [28], but in the framework of a post-Newtonian approximation. The evolution model discussed here does not need to invoke a post-Newtonian approximation, but this can be useful for a precise calculation of the initial invariants.

B. The relativistic Zel'dovich approximation and spherical collapse models

1. The Lemaître-Tolman-Bondi solutions

We now investigate the Lemaître-Tolman-Bondi (henceforth LTB) solutions [29]. The LTB model, written below in the time-synchronous metric form, is the general inhomogeneous spherically symmetric solution for irrotational dust. The spherical solution can be seen as a superposition of infinitesimally thin homogeneous shells governed by their own dynamics. In such a domain one can show (see [30] for a demonstration but with different notation, and the review [31] for a comprehensive study of backreaction in the LTB solutions) that the line element has the form

$$ds^2 = -dt^2 + \frac{R^2(t, r)}{1 + 2E(r)} dr^2 + R^2(t, r)d\Omega^2, \quad (76)$$

E being a free function of r satisfying $E(r) > -1/2$; the prime denotes partial differentiation with respect to r . In this metric, the scalar parts of Einstein's field equations read

$$4\pi\rho(t, r) = \frac{M'(r)}{R'(t, r)R^2(t, r)} \quad (77)$$

and

$$\frac{1}{2}\dot{R}^2(t, r) - \frac{GM(r)}{R(t, r)} = E(r), \quad (78)$$

M being another free function of r ; the dot denotes partial time derivative. Using the relation between the coefficients of the expansion tensor and the metric tensor,

$$\Theta^i_j := \frac{1}{2}g^{ik}\dot{g}_{kj}, \quad (79)$$

the averaged scalar invariants of the expansion tensor can be calculated:

$$\langle \text{I}(\Theta^i_j) \rangle_{\text{LTB}} = \frac{4\pi}{V_{\text{LTB}}} \int_0^{r_{\mathcal{D}}} \frac{\partial_r(\dot{R}R^2)}{\sqrt{1+2E}} dr; \quad (80)$$

$$\langle \text{II}(\Theta^i_j) \rangle_{\text{LTB}} = \frac{4\pi}{V_{\text{LTB}}} \int_0^{r_{\mathcal{D}}} \frac{\partial_r(\dot{R}^2 R)}{\sqrt{1+2E}} dr; \quad (81)$$

$$\langle \text{III}(\Theta^i_j) \rangle_{\text{LTB}} = \frac{4\pi}{3V_{\text{LTB}}} \int_0^{r_{\mathcal{D}}} \frac{\partial_r(\dot{R}^3)}{\sqrt{1+2E}} dr, \quad (82)$$

where the Riemannian volume of an LTB domain is given by

$$V_{\text{LTB}} = \frac{4\pi}{3} \int_0^{r_{\mathcal{D}}} \frac{\partial_r(R^3)}{\sqrt{1+2E}} dr. \quad (83)$$

The deviation from constant curvature can also be averaged on a LTB domain:

$$\mathcal{W}_{\text{LTB}} = \langle \mathcal{R} \rangle_{\text{LTB}} - \frac{6k_{\mathcal{D}_i} V_{\text{LTB}_i}^{2/3}}{V_{\text{LTB}}^{2/3}}, \quad (84)$$

with

$$\langle \mathcal{R} \rangle_{\text{LTB}} = -\frac{16\pi}{V_{\text{LTB}}} \int_0^{r_{\mathcal{D}}} \frac{\partial_r(ER)}{\sqrt{1+2E}} dr. \quad (85)$$

There are two cases for which we find relations between the invariants without having to solve the system (77) and (78) explicitly: the first one is a separable solution $R(t, r)$ of the form

$$R(t, r) = f(t) \cdot g(r).$$

The second one is the case of an LTB domain where $E(r) = E$. The restriction $E = \text{const}$ corresponds to self-similar LTB solutions if we require at the same time that the function $M(r) \propto r$ [27].

In both cases, one can show for $R(t, 0) = 0$

$$\langle \text{II}(\Theta^i_j) \rangle_{\text{LTB}} = \frac{1}{3} \langle \text{I}(\Theta^i_j) \rangle_{\text{LTB}}^2, \quad (86)$$

$$\langle \text{III}(\Theta^i_j) \rangle_{\text{LTB}} = \frac{1}{27} \langle \text{I}(\Theta^i_j) \rangle_{\text{LTB}}^3.$$

Combining these terms in the backreaction \mathcal{Q}_{LTB} given by Eq. (11), we get for a spherically symmetric $E = \text{const}$ domain or a separable $R(t, r)$

$$\mathcal{Q}_{\text{LTB}} = 0, \quad \mathcal{W}_{\text{LTB}} = 0, \quad (87)$$

where the result $\mathcal{W}_{\text{LTB}} = 0$ follows from $\mathcal{Q}_{\text{LTB}} = 0$ by the integrability condition (14) and its definition (13). We here generalize a result obtained in [32] to a nonflat domain for a special case.

Note that, using the expression for the curvature of Eq. (54), we find that the result is not zero for the invariants Eq. (86). This means that this expression is not yet containing the correct second-order contributions. This is one important reason why we use in the evaluation of the importance of $\mathcal{W}_{\mathcal{D}}$ in Sec. VIB 2 the numerically integrated expression, starting from the $\mathcal{Q}_{\mathcal{D}}$ of Eq. (50).

In the case $E = \text{const}$, Eqs. (84) and (13) give for the averaged scalar curvature

$$\langle \mathcal{R} \rangle_{\text{LTB}} = -\frac{12E}{R^2(r_{\mathcal{D}})}. \quad (88)$$

With $\mathcal{W}_{\text{LTB}} = 0$, one can express $k_{\mathcal{D}_i}$ as a function of E :

$$k_{\mathcal{D}_i} = -\frac{2E}{R^2(t_i, r_{\mathcal{D}})} < \frac{1}{R^2(t_i, r_{\mathcal{D}})}. \quad (89)$$

As R is a growing function of $r_{\mathcal{D}}$, $k_{\mathcal{D}_i}$ becomes smaller when we increase the averaging domain.

2. The Szekeres solutions

A possible generalization of the LTB models is the quasispherical Szekeres model that has additional

anisotropies which destroy the spherical symmetry of the LTB models [33]. The line element reads

$$ds^2 = -dt^2 + X^2 dr^2 + Y^2(dx^2 + dy^2), \quad (90)$$

and with $\zeta = x + iy$; $\bar{\zeta} = x - iy$, we have

$$X = \frac{\mathcal{E}(r, \zeta, \bar{\zeta}) Y'(t, r, \zeta, \bar{\zeta})}{\sqrt{\epsilon - k(r)}}; \quad Y = \frac{R(t, r)}{\mathcal{E}(r, \zeta, \bar{\zeta})};$$

$$\mathcal{E}(r, \zeta, \bar{\zeta}) = a(r)\zeta\bar{\zeta} + b(r)\zeta + c(r)\bar{\zeta} + d(r); \quad \epsilon = 0, \pm 1,$$

where a prime denotes ∂_r . We consider only the quasi-spherical model $\epsilon = 1$ in which case, for $\mathcal{E}' = 0$, it simply reproduces the LTB model. But even for the general case $\mathcal{E}' \neq 0$ it has been shown in [33] that the invariants take the form of Eqs. (80)–(82) with $k(r) = -2E(r)$. Therefore also the flat quasispherical Szekeres model has no backreaction:

$$\mathcal{Q}_{Sz} = 0, \quad \mathcal{W}_{Sz} = 0. \quad (91)$$

We considered only this example for the flat case, but the other flat cases should be accordingly solvable. However, in the really interesting cases, i.e. for arbitrary curvature, it is highly difficult to find a general statement, as it is also the case for the LTB model. For further work on the Szekeres model see [34,35], and in relation to observations [36–38].

3. GR theorems corresponding to Newton's "iron sphere theorem"

An interesting property arises, if we look at situations where the kinematical backreaction term vanishes. From the averaged equations it then follows that, e.g., a flat spherically symmetric but inhomogeneous domain, cut out of a homogeneous FLRW model, has no influence on the kinematical expansion properties. Such a result is known as Newton's iron sphere theorem in a Newtonian model. Let us now look at situations where the backreaction term of the RZA approximation vanishes. Using it in the form of Eq. (50), one can show the following propositions.

Proposition 1.—

$${}^{\text{RZA}}\mathcal{Q}_{\mathcal{D}} = 0 \Leftrightarrow \begin{cases} \langle \text{II}_i \rangle_I = \frac{1}{3} \langle \text{I}_i \rangle_I^2 \\ \langle \text{III}_i \rangle_I = \frac{1}{27} \langle \text{I}_i \rangle_I^3. \end{cases}$$

Proof.—See Appendix A.

Proposition 2.—

$${}^{\text{RZA}}\mathcal{Q}_{\mathcal{D}} = 0 \Leftrightarrow \begin{cases} \langle {}^{\text{RZA}}\text{II}(\Theta^i_j) \rangle_{\mathcal{D}} = \frac{1}{3} \langle {}^{\text{RZA}}\text{I}(\Theta^i_j) \rangle_{\mathcal{D}}^2 \\ \langle {}^{\text{RZA}}\text{III}(\Theta^i_j) \rangle_{\mathcal{D}} = \frac{1}{27} \langle {}^{\text{RZA}}\text{I}(\Theta^i_j) \rangle_{\mathcal{D}}^3. \end{cases}$$

Proof.—See Appendix B.

This means that, for a flat background, the RZA correctly reproduces the average evolution of the LTB matter shells, without knowing the specific distribution of matter $M(r)$. This corresponds to the iron sphere theorem of

Newton in the relativistic case (compare also [32]), and it is in the spirit of Birkhoff's and almost-Birkhoff theorems, but for nonvacuum spacetimes (see the recent papers [39,40], and references therein).

In [9] it was shown that, in the Newtonian case, the averaged Zel'dovich approximation also describes the behavior of a spherical matter distribution. Here we find the analogous property for the flat LTB and flat Szekeres metrics.

To summarize one may say that the RZA approximation interpolates between the exact GR solutions of a plane-symmetric metric, the flat LTB and the flat Szekeres metrics, which adds reliability to the employed extrapolation of a strictly first-order scheme and its use within the exact averaging framework.

VI. THE EVOLUTION OF COSMOLOGICAL PARAMETERS

The backreaction term itself decays in the averaged relativistic Zel'dovich approximation. However, to quantify the deviations from the behavior of the scale factor and especially its time derivatives in the standard model, the different strengths of the sources in the generalized Friedmann equation have to be compared. As the matter source term decays faster than the backreaction term, the influence of this latter grows. To evaluate the importance of this growth in the standard cosmological picture, starting with a nearly homogeneous and isotropic initial state, we only need to determine the magnitude of the three invariants of the perturbation one-forms $\dot{\mathcal{P}}^a_i$. The time evolution of ${}^{\text{RZA}}\mathcal{Q}_{\mathcal{D}}$ is then determined by Eq. (50). It then also determines the evolution of all other cosmic parameters via the averages of the Hamilton constraint, Eq. (10), and Raychaudhuri's equation, Eq. (9). The context of the standard scenario implies that the calculation of the initial values is performed in a universe model that is close to spatially flat, looking back to a history that is identical to the standard model. If we additionally neglect tensor modes, this means that we are in the limit described in Sec. IV A. We therefore assume that we can use the values of $\langle \text{I}_i \rangle_{\mathcal{D}_i}$, $\langle \text{II}_i \rangle_{\mathcal{D}_i}$ and $\langle \text{III}_i \rangle_{\mathcal{D}_i}$ in [9] as our initial conditions for the averaged invariants of $\dot{\mathcal{P}}^a_i$ to a very good approximation. In formal analogy of Eq. (50) to the corresponding expression of [9], this then implies that many results of [9] carry over to the RZA context. There are, however, new phenomena that emerge due to the fact that, unlike in the Newtonian ZA of [9], geometry is a dynamical variable and the RZA develops nonvanishing scalar curvature. In this section, we will therefore comment on which results of [9] remain valid and discuss where the GR description brings in new phenomena. As we emphasized already in the introduction, an emerging curvature is key to the new interpretation of the dark components in the standard model.

A. Calculation of the scale factor $a_{\mathcal{D}}$

We calculate the average scale factor ${}^{\text{RZA}}a_{\mathcal{D}}$ as in [9] directly by integrating the averaged Raychaudhuri equation Eq. (9). The input is the average density

$$\langle \varrho \rangle_{\mathcal{D}} = \frac{a^3}{a_{\mathcal{D}}^3} \varrho_H (1 + \langle \delta(t_i) \rangle_{\mathcal{D}_i}) = \frac{a^3}{a_{\mathcal{D}}^3} \varrho_H (1 - \langle I_i \rangle_{\mathcal{D}_i}), \quad (92)$$

and the RZA backreaction model, Eq. (50). The initial conditions for ${}^{\text{RZA}}a_{\mathcal{D}}$ are $a_{\mathcal{D}_i} = 1$ and $\dot{a}_{\mathcal{D}}(t_i) = \dot{a}(t_i)(1 + \frac{1}{3}\langle I_i \rangle_{\mathcal{D}_i})$.

1. Statistics of initial conditions

As discussed, we use for the numerical evaluation the initial values given in [9]. As was shown there, the expected values of the initial invariants are zero:

$$\mathbb{E}[\langle I_i \rangle_{\mathcal{D}_i}] = \mathbb{E}[\langle II_i \rangle_{\mathcal{D}_i}] = \mathbb{E}[\langle III_i \rangle_{\mathcal{D}_i}] = 0. \quad (93)$$

$\mathbb{E}[\cdot \cdot \cdot]$ denotes the ensemble expectation value over many realizations of universe models. However, for a specific domain, any of the volume-averaged invariants may be positive or negative. These invariants fluctuate with the variance, e.g., $\sigma_1^2(R) = \mathbb{E}[\langle I_i \rangle_{\mathcal{D}_i}^2]$. In our calculation of the time evolution of $a_{\mathcal{D}}(t)$, we consider one- σ fluctuations of the averaged invariants for spherical domains of radius R , e.g., $\langle I_i \rangle_{\mathcal{D}_i} = \pm \sigma_1(R)$. Reference [9] showed how these fluctuations are linked to the matter power spectrum. As indicated, $\sigma_1(R)$ will explicitly depend on the radius of the initial domain but implicitly also on the shape of the power spectrum. Reference [9] used a standard CDM power spectrum normalized to $\sigma_8 = 1$ and an h of $h = 0.5$. In addition to the EdS background considered there, we will also present some results for a standard Λ CDM background with $\Omega_{\Lambda} \approx 0.73$, $h = 0.7$ and $\sigma_8 = 0.8$.

Although the possible initial conditions have a rich structure due to the fact that we can choose any combination of the signs of the three initial invariants, we choose for the presentation equal signs for all the invariants, i.e. for expanding, underdense domains, $\sigma_1 > 0$, all initial invariants are taken to be positive, and for collapsing, overdense domains, $\sigma_1 < 0$, all initial invariants are taken to be negative. (A showcase of mixed signs has been analyzed in [9].)

It is important to recognize that our choice of using one- σ fluctuations in the invariants does not imply that the fluctuations of other parameters constructed from them are then also one- σ fluctuations of these parameters. For the initial value of $\mathcal{Q}_{\mathcal{D}}$, for example, a one- σ fluctuation of the invariants is related to a one- σ fluctuation of $\mathcal{Q}_{\mathcal{D}}$ by

$$\sigma^2[{}^{\text{RZA}}\mathcal{Q}_{\mathcal{D}_i}] = 4\sigma_{\text{II}}^2(R) - \frac{8}{3}\mathbb{E}[\langle I_i \rangle_{\mathcal{D}_i}^2 \langle II_i \rangle_{\mathcal{D}_i}] + \frac{8}{9}\sigma_1^4(R). \quad (94)$$

On the other hand, calculating ${}^{\text{RZA}}\mathcal{Q}_{\mathcal{D}_i}$ with one- σ fluctuations yields

$${}^{\text{RZA}}\mathcal{Q}_{\mathcal{D}_i} = 2\sigma_{\text{II}}(R) - \frac{2}{3}\sigma_1^2(R). \quad (95)$$

In an abuse of language we will nevertheless speak of one- σ fluctuations in the following, but one should keep in mind that we mean those of the initial invariants (except for Fig. 1). For the values of ${}^{\text{RZA}}\mathcal{Q}_{\mathcal{D}_i}$, in the case of large values of R , the two prescriptions coincide as $\sigma_1^2(R)$ drops off faster than $\sigma_{\text{II}}(R)$.

2. Scale dependence of initial conditions

From the very definition of the averaged parameters it seems natural that all averaged quantities would be scale dependent and decay with a growing domain \mathcal{D} . That this is not necessarily the case shows the example of the initial expansion and shear fluctuations. In the RZA they are given by

$$\langle \theta^2 \rangle_{\mathcal{D}} - \langle \theta \rangle_{\mathcal{D}}^2 = \langle I(\dot{\mathcal{P}}^a_i)^2 \rangle_{\mathcal{D}} - \langle I(\dot{\mathcal{P}}^a_i) \rangle_{\mathcal{D}}^2; \quad (96)$$

$$\langle \sigma^2 \rangle_{\mathcal{D}} = \frac{1}{3} \langle I(\dot{\mathcal{P}}^a_i)^2 \rangle_{\mathcal{D}} - \langle II(\dot{\mathcal{P}}^a_i) \rangle_{\mathcal{D}}. \quad (97)$$

Using the acceleration equation in terms of coframes of [1], one can show that for the RZA we have $I(\dot{\mathcal{P}}^a_i) = -\delta$, where δ is the local density contrast. This means that also in the RZA, as already in the Newtonian case, the expectation values of expansion and shear fluctuations are no longer scale dependent. They are rather given by

$$\mathbb{E}[\langle \theta^2 \rangle_{\mathcal{D}} - \langle \theta \rangle_{\mathcal{D}}^2] = H_i^2 \left(\int_{\mathbb{R}^3} d^3k P_i(k) - \sigma_1^2(R) \right); \quad (98)$$

$$\mathbb{E}[\langle \sigma^2 \rangle_{\mathcal{D}}] = \frac{1}{3} H_i^2 \int_{\mathbb{R}^3} d^3k P_i(k), \quad (99)$$

where we again made use of the assumed approximate flatness of the initial universe model, necessary to employ the Fourier transformation. Interestingly, only part of the expected expansion fluctuations still contains the information on the domain \mathcal{D} . The other part and the shear are domain independent. Calculating the value of this integral may be used to estimate the importance of backreaction, if the shear fluctuations were negligible. To this end we calculate

$$\mathbb{E}[{}^{\text{RZA}}\Omega_{\mathcal{Q},\text{trunc}}^{\mathcal{D}_0}] = -\frac{1}{9H_{\mathcal{D}_0}^2} \mathbb{E}[\langle \theta^2 \rangle_{\mathcal{D}_0} - \langle \theta \rangle_{\mathcal{D}_0}^2], \quad (100)$$

where 0 stands for today and we evolved the model from the initial time up to today with the leading a^{-1} mode of Eq. (50) only. A quantitative estimate with an exponential IR cutoff at the Hubble scale and UV cutoff at 1 kpc then yields $\mathbb{E}[{}^{\text{RZA}}\Omega_{\mathcal{Q},\text{trunc}}^{\mathcal{D}_0}] \approx 0.73$. This illustrates the well-known fact that the expansion and shear fluctuations by

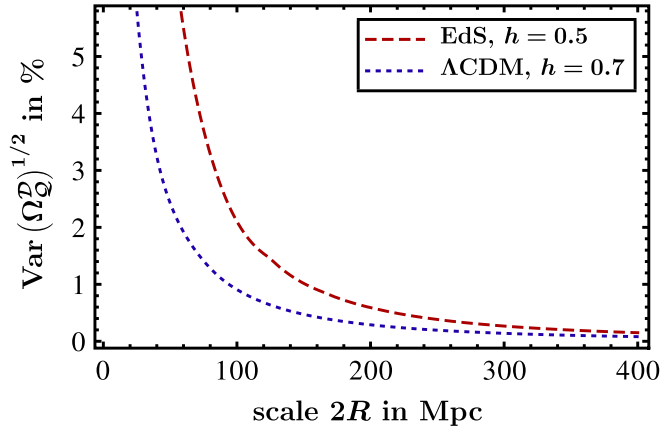


FIG. 1 (color online). One- σ fluctuations of $\Omega_{\mathcal{Q}}^{\mathcal{D}_0}$ as a function of scale for (i) an EdS background ($h = 0.5$, $\sigma_8 = 1$) and (ii) for a Λ CDM background ($\Omega_{\Lambda} \approx 0.73$, $h = 0.7$, $\sigma_8 = 0.8$).

themselves are important even in a perturbative framework. In the backreaction term, however, they combine in a way that leaves only the domain-dependent contribution in the second term of the expansion fluctuations. In the Newtonian framework this is expected by the fact that $\mathcal{Q}_{\mathcal{D}}$ can be written as a surface term. In GR it is not a necessity, but in the linear RZA the cancellation is still effective. For higher orders, however, this is no longer true and [41] reported the survival of a domain-independent contribution to $\mathcal{Q}_{\mathcal{D}}$ at second order.

To close this section we calculate, as an illustration of the scale dependence of the parameters, the backreaction term $\mathcal{Q}_{\mathcal{D}}$. It will imprint its scale dependence on the other parameters and is therefore particularly interesting. To get a feeling for the magnitude of the values, we evolve it again with the a^{-1} mode until today and normalize it with $H_{\mathcal{D}_0}^2$ to get $\Omega_{\mathcal{Q}}^{\mathcal{D}_0}$. The result is shown in Fig. 1. We plot the

one- σ fluctuation of $\Omega_{\mathcal{Q}}^{\mathcal{D}_0}$ with the correct σ interpretation of Eq. (50). The result shows that only below the assumed “almost homogeneity scale” of about 400 Mpc the (with a^{-1} upscaled) backreaction term begins to enter the range of a percent contribution. For larger scales it is clearly below this value, while for smaller scales the nonlinear terms in the backreaction functional count in. This explains that in the following all parameters converge to their background values for large \mathcal{D} . It is here where our model faces its strongest restriction, since we neglect any interaction of structure with the assumed background model.

B. Time evolution of cosmic parameters

Having discussed in the previous section how we fix our initial conditions and determine the scale factor from them, we will now present and discuss the results.

1. Evolution of the scale factor $a_{\mathcal{D}}$, the Hubble and deceleration parameters

As mentioned, the evolution of the dynamical quantities $a_{\mathcal{D}}$, $H_{\mathcal{D}} = \dot{a}_{\mathcal{D}}/a_{\mathcal{D}}$ and $q_{\mathcal{D}} = -(\ddot{a}_{\mathcal{D}}/a_{\mathcal{D}})/H_{\mathcal{D}}^2$ turns out to coincide with those of [9] for our choice of initial conditions. This means that the partially drastic deviation from the background values for these quantities, that has been found in [9], also occurs in the RZA. In the case of the volume deceleration parameter $q_{\mathcal{D}}$ for one- σ fluctuations this leads to deviations of 30% on a scale of 200 Mpc.

The effects on $a_{\mathcal{D}}$ and $H_{\mathcal{D}}$ are smaller but may also become important for special regions that are more than one σ away from the background. In the relativistic framework the deviations of the volume scale factor $a_{\mathcal{D}}$ from the background scale factor $a(t)$ can also be interpreted as

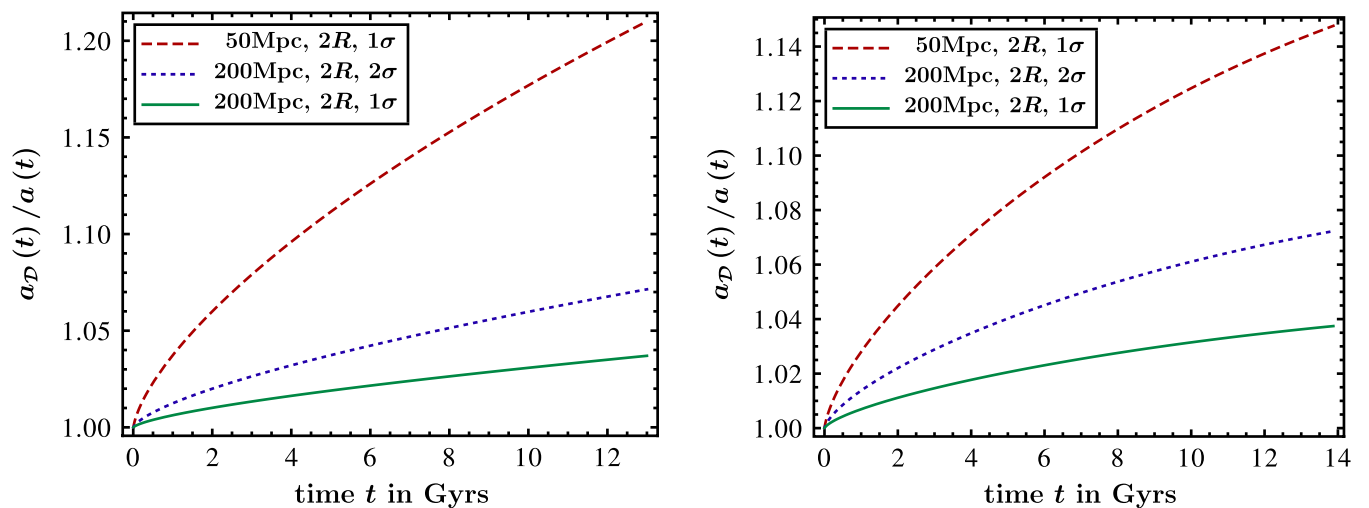


FIG. 2 (color online). Evolution of the volume scale factor, normalized by the background scale factor, on typical domains of 50 and 200 Mpc, and on a domain that is a 2-sigma fluctuation in the initial conditions on 200 Mpc for comparison. Left: EdS background with $\Omega_m = 1$ ($h = 0.5$, $\sigma_8 = 1$). Right: Λ CDM background with $\Omega_m = 0.27$ ($h = 0.7$, $\sigma_8 = 0.8$).

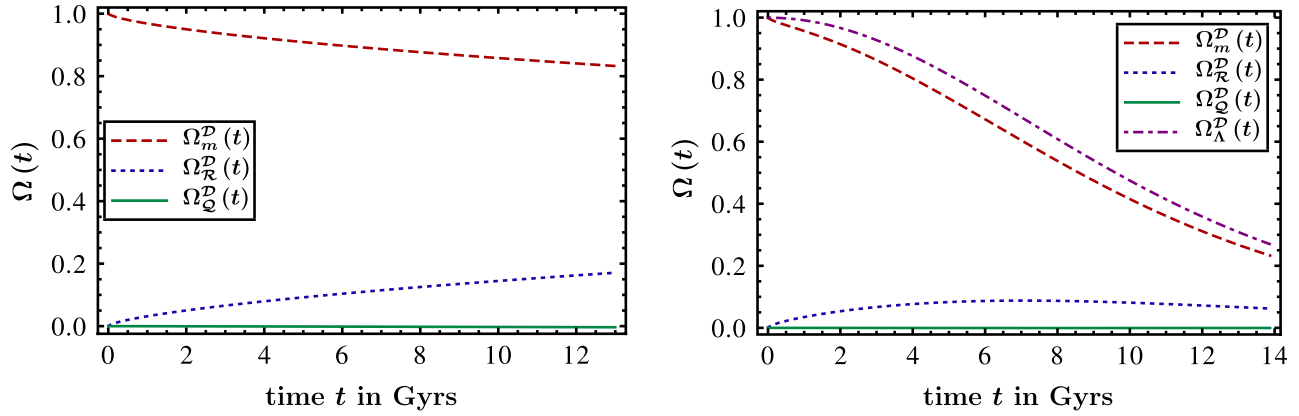


FIG. 3 (color online). Evolution of the domain-dependent cosmological parameters of Eq. (18) with cosmic time. One background is the EdS model with $\Omega_m = 1$ ($h = 0.5$, $\sigma_8 = 1$) (left), the other one the Λ CDM model with $\Omega_m = 0.27$ ($h = 0.7$, $\sigma_8 = 0.8$) (right: the background density parameter is plotted here as the upper curve). The figure shows values for an expanding underdense domain of 200 Mpc effective diameter with one- σ fluctuations of the initial invariants of the perturbation one-form. This figure confirms the findings of [9] reporting substantial deviations from the background of, e.g., the matter density parameter, while the quantitative importance of the backreaction parameter is seemingly negligible. The new interpretation in the GR context is mirrored by the curvature density parameter that has to compensate (in the case of an EdS background without a dark energy component) the large deviations in the matter density parameter.

giving the strength of metrical perturbations, since the calculation of the volume just involves the metric determinant and not higher derivatives of the metric. Taking the cube of this deviation gives us a typical strength for the volume fluctuation; e.g., for one-sigma fluctuations on the scale of 200 Mpc we find a 12% effect—see Fig. 2 for the evolution of the scale factors. This also means that the influence of metrical perturbations is not necessarily small, if their averaged effect is considered (see the related discussions in [42–44]). However, it is still subdominant compared with the overall backreaction effect that depends on second derivatives of the metric.

In the next subsection we will discuss the results for the density parameters that are relevant for the interpretation of the energy budget of the Universe and its relation to the “dark components” in the standard model. Recall that we quantify fluctuations on Lagrangian domains in this paper; for a quantification of fluctuations of cosmic parameters on domains that correspond to actual observational geometries see [45].

2. Evolution of the density parameters

Let us consider the cosmological density parameters, Eq. (18), describing the energy content balance of the universe model. In the average scenario they are domain-dependent quantities. In Fig. 3 we show their evolution with cosmic time on an expanding typical (i.e. one-sigma) domain 200 Mpc. Figure 4 shows the situation again for an expanding domain of 200 Mpc, but here found with two-sigma probability. In Figs. 5 and 6 we give examples of smaller typical domains of 100 and 50 Mpc, but this time in the collapsing phase.

For the case of the EdS background with $\Omega_\Lambda = 0 = \Omega_\Lambda^D$ and $\Omega_k = 0$, the global matter density parameter is

$\Omega_m = 1$. Regionally, however, the plot shows that Ω_m^D is on 200 Mpc domains typically lower by 20%; i.e. this scale is dominated by underdense voids. In the relativistic framework the lower matter density is compensated by an emerging curvature parameter Ω_R^D .

It is interesting that the curvature deviation from the background curvature, \mathcal{W}_D , a second-order quantity, can be a lot bigger than the second-order quantity \mathcal{Q}_D . This is due to the fact that in the perturbative expansion the second-order terms both scale as a^{-1} , and in view of the integrability condition Eq. (12), the second-order contribution to \mathcal{W}_D is a factor of 5 larger than the second-order term of \mathcal{Q}_D . Therefore, even a small backreaction contribution of only 2% will already lead to a 10% modification of the averaged curvature.

For the Λ CDM background the matter density parameter is also reduced with respect to the background value and again we have curvature emerging from a flat background. Today, however, it is not as big as in the EdS case, since the cosmological constant dominates.

In both cases we see that, even though the backreaction contribution stays tiny (as was already found in [9]), we have a considerable amount of curvature. Comparing this curvature contribution with the flat geometry of the background on a given scale allows its interpretation in terms of dark energy: the standard interpretation is that the matter distribution evolves on a flat geometry; hence one would add a fundamental component to compensate the actually existing curvature that we model here. In other words, the matter distribution has to be seen on a curved space section and not on a flat background. Taking this point of view, the emerging curvature quantifies, on a given scale, the amount of dark energy that would be needed to compensate it in a quasi-Newtonian model. We find for the example of Fig. 3

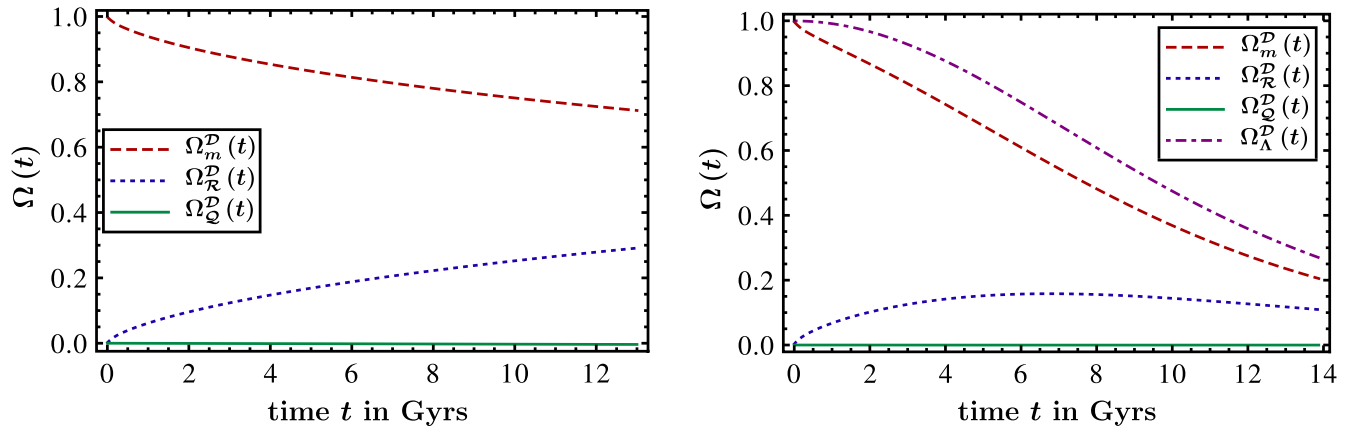


FIG. 4 (color online). This figure corresponds to Fig. 3 but shows the corresponding values for an expanding domain of two- σ fluctuations of the initial invariants of the perturbation one-form.

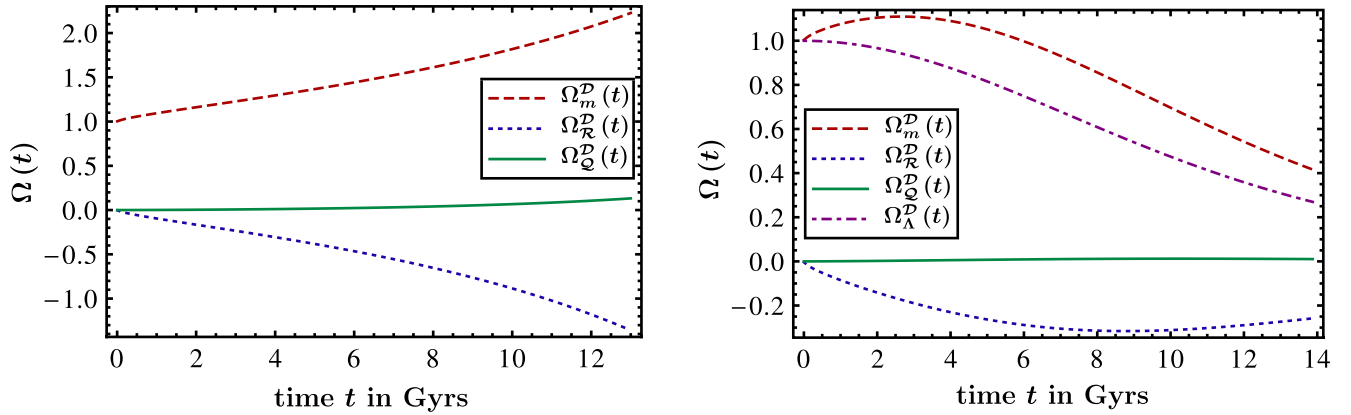


FIG. 5 (color online). Evolution of the domain-dependent cosmological parameters of Eq. (18) with cosmic time. One background is the EdS model with $\Omega_m = 1$ ($h = 0.5$, $\sigma_8 = 1$) (left), the other one the Λ CDM model with $\Omega_m = 0.27$ ($h = 0.7$, $\sigma_8 = 0.8$) (right: the background density parameter is plotted here as the lower curve). The figure shows values for a collapsing overdense domain of 100 Mpc effective diameter with one- σ fluctuations of the initial invariants of the perturbation one-form.

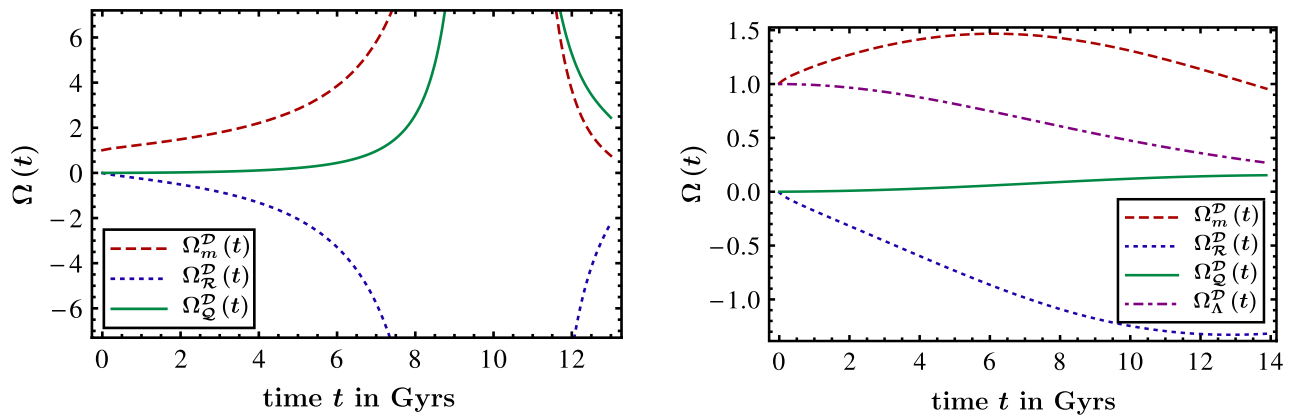


FIG. 6 (color online). This figure corresponds to Fig. 5 but shows the corresponding values for a collapsing domain of one- σ fluctuations on the scale of 50 Mpc. On this scale we appreciate a singular pancake collapse for the EdS model; Fig. 7 illustrates that the backreaction term now becomes not only qualitatively but also quantitatively significant.

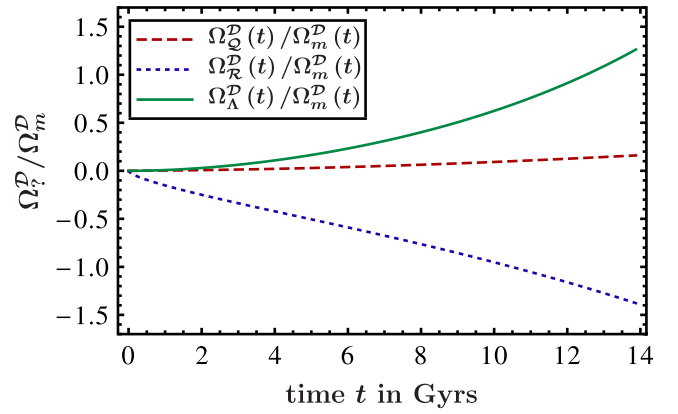
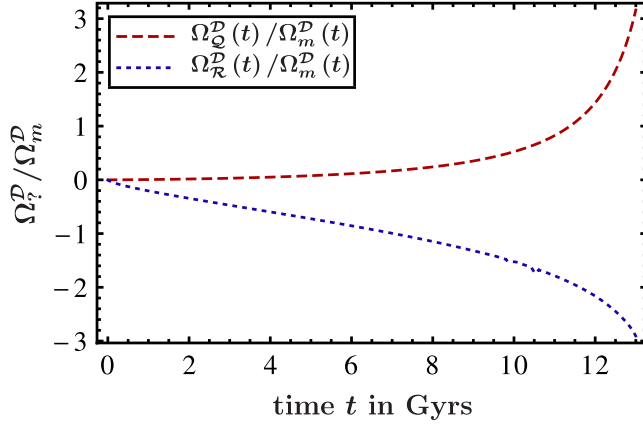


FIG. 7 (color online). Ratio of the cosmic parameters for backreaction and curvature to the matter density parameter for a scale of 50 Mpc as plotted in Fig. 6. As during the pancake collapse H_D becomes 0, the parameters themselves are no longer well defined. In the ratios plotted in this figure, however, $1/H_D^2$ cancels.

that our model predicts the existence of typical domains with a diameter of 200 Mpc of about 1/4 of the needed amount of dark energy. This amount can increase to 0, 4, if the 200 Mpc domain is slightly untypical (i.e. found with a two-sigma probability, Fig. 4).

Figures 5–7 show that the backreaction can have the opposite effect by looking at smaller scales. A typical, collapsing domain produces on average a positive curvature (corresponding to a negative curvature density parameter). In light of the interpretation above, the collapse produces a large amount of dark matter in the form of positive curvature. While the “cosmological parameters” start to lose their sense on this scale, we can clearly see, e.g., in Fig. 5, that the curvature contribution is of the order of the density contribution, since it compensates the produced overdensities. The physical interpretation is also clear: overdensities are hosted in positive-curvature environments. The order of magnitude of dark matter is also

comparable with that of dark energy in expanding domains; i.e. we can also here say that a substantial fraction of the density parameters is contained in the curvature parameter, while it is of comparable magnitude and not dominating.

Given the conservative assumptions of our model these values point to a highly significant effect of backreaction. Except in particular LTB models, where dark energy can be fully replaced, the only generic model that carries such a large effect has been investigated by Enqvist, Hotchkiss, and Rigopoulos [46] by employing gradient expansion techniques that allow one to go substantially beyond standard (Eulerian) perturbation methods [47,48]. These latter have to assume a universe model that stays close to the assumed background. For comparison, we wish to point the reader to a figure in [46]: Figure 8 corresponds to their Fig. 4 and shows the scale dependence of the X -matter component $\Omega_X^D := \Omega_Q^D + \Omega_R^D$ today: the X matter

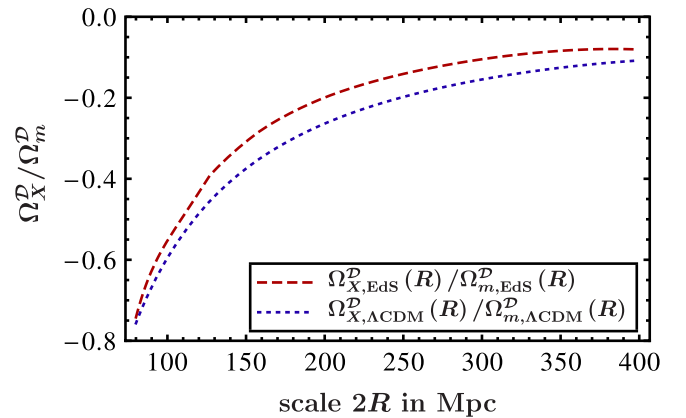
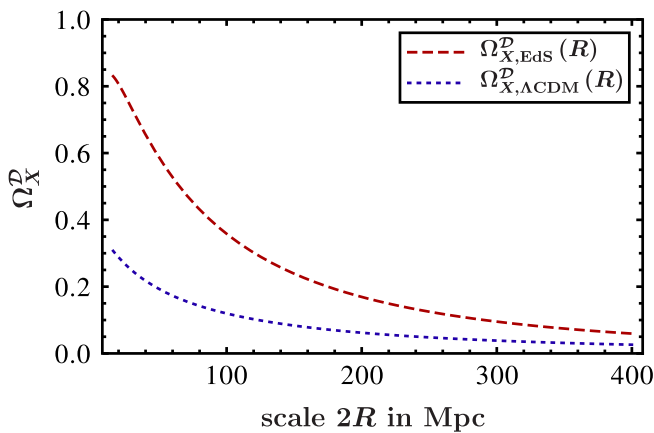


FIG. 8 (color online). Scale dependence of the X -matter component, $\Omega_X^D := \Omega_Q^D + \Omega_R^D$, today for typical (one-sigma) domains in the averaged RZA. Left: Underdense regions. Right: Overdense regions. As in this case the Hubble parameter goes through zero for small scales, we plot the ratio of the X matter to the matter density parameter. In this ratio H_D drops out.

produces dark energy on expanding, underdense domains (left panel), starts to compete with the matter density on scales below 100 Mpc and is still significant (6%) on domains of 400 Mpc; it produces dark matter on collapsing, overdense domains (right panel), starts to compete with the matter density again below the scales of 100 Mpc, and shows similar significance as dark energy on domains with diameter 400 Mpc. Note that the large-scale value is astonishingly big, since we are looking at a scale that is considered to comply with homogeneity.

VII. RESULTS, CONCLUSIONS AND OUTLOOK

Based on a nonlinear extrapolation of a first-order relativistic Lagrangian perturbation scheme investigated in [1], we modeled the fluctuations in extrinsic curvature in the form of the kinematical backreaction term $\mathcal{Q}_{\mathcal{D}}$ for the description of the average properties of irrotational dust models. We provided backreaction and intrinsic curvature expressions as nonlinear functionals of the first-order Lagrangian deformation and used them as an input for the general framework for the average dynamics. The large-scale behavior of the backreaction variables $\mathcal{Q}_{\mathcal{D}}$ and $\mathcal{W}_{\mathcal{D}}$ at second order is identical to the leading mode in the second-order perturbation theory (proportional to a^{-1} for an EdS background, while the leading curvature contribution is proportional to a^{-2} in conformity with the perturbative calculations of [22]). We showed that the backreaction model contains in limiting cases the Newtonian approximation investigated in [9], as well as special classes of exact GR solutions. We argued that this backreaction model is a powerful approximation due to the successes of the corresponding elements of this approximation in Newtonian cosmology (like the large-scale performance of the Newtonian Zel'dovich approximation in comparison with N -body simulations, e.g., [49,50]), and also due to the above-mentioned property of interpolating between exact GR solutions with orthogonal symmetries.

We discussed how substantial results of [9] can be translated to the relativistic context, the new feature being an emerging intrinsic averaged curvature. This translation is possible due to the fact that the Lagrangian description offered in [1] features a clear-cut Newtonian limit by sending the coefficients of the coframes to their Euclidean counterparts, $\eta^a_i \rightarrow f^a|_i$, where the existence of the vector field f^a in the Newtonian case describes the embedding into a global Euclidean vector space. The so-called *electric part of the Lagrange-Einstein system* [1] is formally identical to the Newtonian equations in Lagrangian form, where in the GR case the nonintegrability of the coframes is responsible for the emerging curvature.

We quantified the fluctuations of various cosmological parameters on different spatial domains for standard model backgrounds (CDM and Λ CDM power spectrum), and we now focus on these quantitative results.

A. Results

In order to discuss the quantitative results of our investigation, let us concentrate on the averaged scalar curvature $\langle \mathcal{R} \rangle_{\mathcal{D}}$. The principal idea is to compare the energy balance conditions, encoded in cosmological parameters (i) in the standard model and (ii) in the averaged model. The balance condition for (i) is entirely defined through the chosen background cosmology and furnished by the normalized Hamilton constraint. For the two chosen backgrounds we simply have

$$\Omega_m + \Omega_\Lambda = 1, \quad (101)$$

where initially Ω_m dominates and remains equal to 1 for the EdS background, while the importance of Λ increases in the late stages in the background of the ‘‘concordance model.’’ This balance has to be compared with the (scale-dependent) balance of the averaged cosmology furnished by the normalized averaged Hamilton constraint,

$$\Omega_m^{\mathcal{D}} + \Omega_X^{\mathcal{D}} = 1, \quad (102)$$

where $\Omega_X^{\mathcal{D}} = \Omega_Q^{\mathcal{D}} + \Omega_{\mathcal{R}}^{\mathcal{D}}$ comprises the emerging backreaction and curvature due to structure formation. This comparison allows us to interpret the X -matter energy density content as a candidate, on some scale, for the contribution by Ω_Λ that is added *ad hoc* in the standard model. One subtlety of this interpretation is related to the scale dependence of the averaged balance condition and also to its evaluation on a chosen background cosmology that is already supposed to contain a substantial amount of dark matter (for both backgrounds) and, additionally, dark energy (for the concordance model). We will now discuss the values obtained at the present day and denote the domain by \mathcal{D}_0 .

We found, in accord with previous analyses that were, e.g., summarized in the review papers [8,42,43,51–54], that on large underdense domains of the order of 100–400 Mpc fluctuations feature a negative averaged curvature due to the fact that on these scales the universe model is void dominated. Note that the curvature does not individually obey a conservation law like the density; only a combination of averaged curvature and backreaction is conserved [55].

Let us look, for example, at a scale of 200 Mpc. While the contribution by kinematical backreaction $\Omega_Q^{\mathcal{D}_0}$ remains small in this situation, the curvature density parameter (that is positive for negative curvature) $\Omega_{\mathcal{R}}^{\mathcal{D}_0}$ is, for one-sigma initial fluctuations, of the order of (e.g., on the EdS background) 20% and compensates the lowered matter density parameter on this scale, $\Omega_m^{\mathcal{D}_0} \cong 80\%$. This result, being physically plausible, nevertheless paints a completely different picture from the one advocated in standard cosmology, e.g., [56] (for a thorough treatment of the quality of quasi-Newtonian approximations see [57]),

claiming that intrinsic curvature is irrelevant down to the scales of neutron stars (see discussions and estimates in [26,42,58–61]). Since this latter prejudice led to the inclusion of a large amount of dark energy into the background, as modeled by a cosmological constant in the Λ CDM model, we are entitled to take the EdS background without dark energy and interpret the negative averaged curvature contribution as a replacement of the dark energy in the standard interpretation, where curvature is strictly zero on all scales. We have quantified this contribution as less than 20% for one-sigma fluctuations on 200 Mpc, which is—in this model—not enough to compensate the missing dark energy of the order of 73%. (Note that when speaking about dark energy we do not imply that the X -matter component always produces an accelerating universe model.) For two-sigma fluctuations on this scale we find 35% of the needed amount, but due to the architecture of our model, as discussed below, this contribution falls off rapidly on very large scales. We may compare this regional behavior in the sense of so-called “void models” that assume we are living in a large underdense region; see, e.g., [62–64], and references therein. In the spirit of these models our generic model “explains away” 1/2 of the dark energy for a slightly untypical region (two-sigma) on the scale of 200 Mpc.

Before we explain why we cannot expect this behavior on larger scales in our model, we briefly sketch the opposite behavior of the backreaction on smaller scales, where a strongly anisotropic collapse into sheetlike and filamentarylike structures features a negative backreaction term $Q_{\mathcal{D}}$ and a positive curvature term $\langle \mathcal{R} \rangle_{\mathcal{D}}$. This situation occurs, e.g., on the scale of 50 Mpc, below which the X -matter component starts to compete with the matter density and mimics the presence of dark matter. This result points to the high relevance of the backreaction effect for the explanation of dark matter in overdensities, while this effect decays on large scales leaving only a few percent remnant on an assumed homogeneity scale of about 400 Mpc. This latter is comparable with the dark energy remnant on such scales. Both effects thus are small but still alter the large-scale cosmological parameters at the percent level; see Fig. 8.

B. Conclusions

There are several serious shortcomings of our model that has a limited architecture as compared with the general situation, below discussed as *the background problem*, and also our model is, due to the choice of CDM and Λ CDM initial conditions, exploited in a regime where the backreaction effect is not yet effective on large scales, below discussed as *the amplitude problem*.

1. The background problem

In our model the background is “fictitious”: while our backreaction models have a generic structure, our

quantitative interpretations largely depend on the choice of background [65]. The reason is that both of our backgrounds contain a large amount of dark matter, while the effect we study produces kinematical contributions that act in a similar way. The same argument holds for the background containing dark energy. The implementation of a physical background, defined through the spatial average of the considered fluctuations, is the next step to be envisaged (compare the first results obtained in [14]). Such a physical background interacts with the backreaction models and thus changes their interpretation. While in the corresponding Newtonian model [9] the background forms the average distribution due to its torus architecture [10], i.e. the scale factor $a_{\mathcal{D}} \equiv a$ on the periodicity scale, the relativistic model studied here must be considered a hybrid construction featuring a volume-scale factor $a_{\mathcal{D}}$ and a background scale factor a that only approximately matches the volume-scale factor on the largest scales as a result of large-scale remnants of backreaction. An indication for this shortcoming is also that the backreaction model, seen on the FLRW background, only covers very restricted subcases of exact GR solutions, unlike the situation in Newtonian cosmology, where the plane and spherical collapse is in general covered by the backreaction model.

2. The amplitude problem

Closely related to the first problem of a background containing a dominating dark matter component, we here point out a serious mismatch between the time of onset of the actual large-scale backreaction effect and the age of the Universe as it is determined by the background model. We can take as an indicator the onset of an accelerating period that will set in as a result of dominance of backreaction effects over the density. Such an acceleration may not be needed to explain observational data; also, it is a strong requirement in view of the smallness of the kinematical backreaction term, which is the only backreaction component entering the volume acceleration equation, while the impact of X matter is considerably stronger. From our model we infer that this acceleration phase is seen in the future on scales roughly above 40 Mpc. This result confirms the amplitude problem already reported in [66], if we suppose that backreaction were to replace dark energy completely. Also, if we would consider the analyses presented in this paper in the period of onset of the effect, our quantitative conclusions on the amount of X matter would substantially change. Thus, all of our results remain conservative—although visible in the fluctuation properties on intermediate and small scales—since the actual effect is postponed to the future due to our standard initial data setting.

We can understand this by a simple consideration: the averaged density decays as $a_{\mathcal{D}}^{-3}$, the constant curvature (contained in the full averaged Ricci curvature) decays as

$a_{\mathcal{D}}^{-2}$, the leading large-scale mode of the backreaction model (the kinematical backreaction $\mathcal{Q}_{\mathcal{D}}$) decays as $a_{\mathcal{D}}^{-1}$ and, finally, the nondecaying cosmological constant. Thus, we expect several periods of onset of dominance of (i) the constant-curvature part and later (ii) the backreaction contribution, assuming that the cosmological constant is set to zero. As we saw, this latter period lies in the future for our initial data setting.

There are furthermore limitations due to the model's input of a first-order deformation. Obviously, this limits the range of applicability for the full backreaction regime that here lies in the future but would anyway not be covered by our extrapolation of a first-order solution. Generally, our model is applicable only to large scales, well beyond the scales of virialized objects. The matter distribution is assumed to be fairly smooth and does not take into account the small-scale discretization. The importance of the modeling of small-scale structure has to be emphasized. For example, the determination of the actual volume fraction in devoid regions is needed to precisely quantify the contribution of a negative curvature (and in turn of the contribution to what would be interpreted as dark energy in the standard model) [66–68], and this sensibly depends on the high-resolution modeling of small-scale structure. Collapse models, as the one based on [9,69], have to be refined by, e.g., including velocity dispersion. A corresponding argument, making use of the evolution of the fraction of virialized objects together with an attempt to also model the light-cone effect, may give an indication of its quantitative importance [70].

3. Outlook

Clearly, in view of the remarks above, further major efforts are needed, and the present investigation can only be considered as a further step towards quantifying backreaction effects. We expect, however, that the elements we provided in this work will prove useful for further considerations and the construction of improved models. At any rate, a model based on the RZA is expected to realistically model the large-scale skeleton and so lies at the heart of any model for structure formation including fully relativistic numerical simulations. Furthermore, the above remark has to be seen in junction with another major effort to reinterpret observational data in the relativistic models: the subtleties and complexity of the interpretation of curvature energies as dark energy and dark matter, seen on intermediate scales in our investigation, are not only a challenge in view of the scale dependence of the effect, but imply important changes of light propagation in these models [44,71,72]. Here again the precise modeling of small-scale structure is crucial.

The present investigation reveals, in a quantified way, how we have to proceed in order to master inhomogeneous

universe models. We identify four key questions to be addressed: first, how can we deal with the “kinematical dark matter” produced in the relativistic models on all scales, and how does this alter the initial conditions and their compatibility with cosmic microwave background constraints? Second, how can we improve the model to quantify the effect on smaller scales than those accessible here? Third, how can we implement a perturbation scheme on the physical background, given by the average of the model? Fourth, how are observational data reinterpreted in the relativistic models?

The answer to the first three questions would alter the time scale of the background, of structure formation and of our interpretation of backreaction. Would this imply a shift of the backreaction regime into an epoch before today? Interpreting the onset of an acceleration period as coinciding with an apparent acceleration is highly problematic [73], we are tempted to require such a shift to solve the “coincidence problem,” but an acceleration of the model may after all not be needed to explain observational data. The answer to the third question will deliver backreaction models that interact with the background; they will contain, e.g., the volume scale factor (and not the FLRW scale factor), which itself depends on the backreaction terms. An iterative scheme results that would substantially modify the time scales of our results. In turn, the amplitude of large-scale backreaction could substantially exceed the 6% effect already present in our model on an assumed homogeneity scale of 400 Mpc.

Finally, let us emphasize again that we constructed our model by keeping basic cornerstones of the standard model in the early stages of the cosmic evolution. Insufficiency of late-time backreaction effects may indicate that a more drastic paradigmatic change is needed that takes into account backreaction effects in the early Universe: consequences of emerging average properties from inhomogeneous inflationary models [74], and cosmological models that are built on global principles like globally stationary cosmologies [17], which inherently explain a large backreaction component contained in the physical background, have thus far not been thoroughly analyzed.

ACKNOWLEDGMENTS

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APPENDIX A: DEMONSTRATION OF PROPOSITIONS 1 AND 2

Proposition 1.—

$${}^{\text{RZA}}\mathcal{Q}_{\mathcal{D}} = 0 \Leftrightarrow \begin{cases} \langle \text{II}_i \rangle_I = \frac{1}{3} \langle \text{I}_i \rangle_I^2, \\ \langle \text{III}_i \rangle_I = \frac{1}{27} \langle \text{I}_i \rangle_I^3. \end{cases} \quad (\text{A1})$$

Proof.—

- (i) With Eq. (50), the demonstration of the vanishing of the backreaction term, given the right-hand side, is straightforward.
- (ii) For given vanishing backreaction, i.e. supposing that $\forall t$, ${}^{\text{RZA}}\mathcal{Q}_{\mathcal{D}} = 0$, we proceed as follows. Excluding the case $\dot{\xi}(t) = 0 \forall t$, using Eq. (50), the vanishing of the backreaction term is equivalent to

$$\forall t, \quad \gamma_1 + \xi(t)\gamma_2 + \xi^2(t)\gamma_3 = 0. \quad (\text{A2})$$

As the monomials in ξ are linearly independent, already the coefficients have to vanish. Therefore, $\gamma_1 = \gamma_2 = \gamma_3 = 0$, which, by the definition of Eq. (50), is equivalent to the right-hand side of Eq. (A1). \square

Proposition 2.—

$${}^{\text{RZA}}\mathcal{Q}_{\mathcal{D}} = 0 \Leftrightarrow \begin{cases} \langle {}^{\text{RZA}}\text{II}(\Theta^i_j) \rangle_{\mathcal{D}} = \frac{1}{3} \langle {}^{\text{RZA}}\text{I}(\Theta^i_j) \rangle_{\mathcal{D}}^2, \\ \langle {}^{\text{RZA}}\text{III}(\Theta^i_j) \rangle_{\mathcal{D}} = \frac{1}{27} \langle {}^{\text{RZA}}\text{I}(\Theta^i_j) \rangle_{\mathcal{D}}^3. \end{cases} \quad (\text{A3})$$

Proof.—

- (i) Once again the vanishing of the backreaction term, given the right-hand side, is straightforward.
- (ii) For the other way we use (Proposition 1): if, $\forall t$, ${}^{\text{RZA}}\mathcal{Q}_{\mathcal{D}} = 0$, then one can write $\langle \text{II}_i \rangle_I$ and $\langle \text{III}_i \rangle_I$ as functions of $\langle \text{I}_i \rangle_I$; with Eqs. (45) and (48) we have

$$\begin{aligned} \langle \mathcal{J} \rangle_I &= \left(1 + \frac{\xi}{3} \langle \text{I}_i \rangle_I \right)^3, \\ \langle {}^{\text{RZA}}\text{I}(\Theta^i_j) \rangle_{\mathcal{D}} &= 3 \frac{\dot{a}}{a} + \mathcal{X}, \\ \langle {}^{\text{RZA}}\text{II}(\Theta^i_j) \rangle_{\mathcal{D}} &= 3 \left(\frac{\dot{a}}{a} \right)^2 + 2 \frac{\dot{a}}{a} \mathcal{X} + \frac{1}{3} \mathcal{X}^2, \\ \langle {}^{\text{RZA}}\text{III}(\Theta^i_j) \rangle_{\mathcal{D}} &= \left(\frac{\dot{a}}{a} \right)^3 + \left(\frac{\dot{a}}{a} \right)^2 \mathcal{X} + \frac{1}{3} \frac{\dot{a}}{a} \mathcal{X}^2 + \frac{1}{27} \mathcal{X}^3, \end{aligned} \quad (\text{A4})$$

where $\mathcal{X} = \dot{\xi} \langle \text{I}_i \rangle_I / (1 + \frac{\xi}{3} \langle \text{I}_i \rangle_I)$. This finally leads to the equalities of the right-hand side of Eq. (A3). \square

APPENDIX B: ORTHONORMAL BASIS REPRESENTATION

We will, in this Appendix, give all the relevant expressions for the case where we employ the standard assumption of orthonormal frames (option 1 in the text; see Sec. III A 2). The coframes are in this case given by

$${}^{\text{RZA}}\eta^a_i(t, X^k) := a(t)(\delta^a_i + P^a_i + \xi(t)\dot{P}^a_i), \quad (\text{B1})$$

which combine, via δ_{ab} , to the complete metric coefficients

$$g_{ij} := \delta_{ab} \eta^a_i \eta^b_j. \quad (\text{B2})$$

We evaluate for this case some relevant fields furnishing the RZA and its average properties.

- (i) the coefficients of the metric tensor:

$$\begin{aligned} {}^{\text{RZA}}g_{ij}(t, X^k) &= a^2(t) \{ \delta_{ij} + 2P_{(ij)} + P_{ki}P^k_j \\ &\quad + 2\xi(t)(\dot{P}_{(ij)} + P_{k(i}\dot{P}^k_{j)}) \\ &\quad + \xi^2(t)\dot{P}_{ki}\dot{P}^k_j \}. \end{aligned} \quad (\text{B3})$$

- (ii) the local volume deformation:

$${}^{\text{RZA}}\mathcal{J} := \frac{1}{6} \epsilon_{abc} \epsilon^{ijk} \eta^a_i \eta^b_j \eta^c_k = a^3(t) \mathcal{J}. \quad (\text{B4})$$

- (iii) the first scalar invariant of the expansion tensor:

$${}^{\text{RZA}}\text{I}(\Theta^i_j) := \frac{1}{2\mathcal{J}} \epsilon_{abc} \epsilon^{ijk} \dot{\eta}^a_i \dot{\eta}^b_j \dot{\eta}^c_k = 3 \frac{\dot{a}(t)}{a(t)} + \frac{\dot{\mathcal{J}}}{\mathcal{J}}. \quad (\text{B5})$$

- (iv) the second scalar invariant of the expansion tensor:

$$\begin{aligned} {}^{\text{RZA}}\text{II}(\Theta^i_j) &:= \frac{1}{2\mathcal{J}} \epsilon_{abc} \epsilon^{ijk} \dot{\eta}^a_i \dot{\eta}^b_j \dot{\eta}^c_k \\ &= 3 \left(\frac{\dot{a}(t)}{a(t)} \right)^2 + 2 \frac{\dot{a}(t)}{a(t)} \frac{\dot{\mathcal{J}}}{\mathcal{J}} + \frac{1}{2} \left(\frac{\ddot{\mathcal{J}}}{\mathcal{J}} - \frac{\dot{\xi}(t)\dot{\mathcal{J}}}{\xi(t)\mathcal{J}} \right). \end{aligned} \quad (\text{B6})$$

- (v) the third scalar invariant of the expansion tensor:

$$\begin{aligned} {}^{\text{RZA}}\text{III}(\Theta^i_j) &:= \frac{1}{6\mathcal{J}} \epsilon_{abc} \epsilon^{ijk} \dot{\eta}^a_i \dot{\eta}^b_j \dot{\eta}^c_k \\ &= \left(\frac{\dot{a}(t)}{a(t)} \right)^3 + \left(\frac{\dot{a}(t)}{a(t)} \right)^2 \frac{\dot{\mathcal{J}}}{\mathcal{J}} \\ &\quad + \frac{1}{2} \frac{\dot{a}(t)}{a(t)} \left(\frac{\ddot{\mathcal{J}}}{\mathcal{J}} - \frac{\dot{\xi}(t)\dot{\mathcal{J}}}{\xi(t)\mathcal{J}} \right) + \frac{\dot{\xi}^3(t)D_{\text{ddd}}}{\mathcal{J}}. \end{aligned} \quad (\text{B7})$$

We have introduced the functions

$$\begin{aligned}\mathcal{J}(\xi(t), X^i) &:= S_i + \xi D_d + \xi^2 D_{dd} + \xi^3 D_{ddd}, \\ S_i &:= S(t_i, X^k) := 1 + I_i + II_i + III_i, \\ D_d &:= D_d(t_i, X^k) := I_d(1 + I_i + II_i) + P^i_j P^j_k \dot{P}^k_i \\ &\quad - (1 + I_i) P^a_b \dot{P}^b_a, \\ D_{dd} &:= D_{dd}(t_i, X^k) := II_d(1 + I_i) + P^i_j \dot{P}^j_k \dot{P}^k_i \\ &\quad - I_d P^a_b \dot{P}^b_a, \\ D_{ddd} &:= D_{ddd}(t_i, X^k) := III_d, \\ I_i &:= I(P^a_i) := \frac{1}{2} \epsilon_{abc} \epsilon^{ijk} P^a_i \delta^b_j \delta^c_k, \\ II_i &:= II(P^a_i) := \frac{1}{2} \epsilon_{abc} \epsilon^{ijk} P^a_i P^b_j \delta^c_k, \\ III_i &:= III(P^a_i) := \frac{1}{6} \epsilon_{abc} \epsilon^{ijk} P^a_i P^b_j P^c_k, \\ I_d &:= I(\dot{P}^a_i) := \frac{1}{2} \epsilon_{abc} \epsilon^{ijk} \dot{P}^a_i \delta^b_j \delta^c_k, \\ II_d &:= II(\dot{P}^a_i) := \frac{1}{2} \epsilon_{abc} \epsilon^{ijk} \dot{P}^a_i \dot{P}^b_j \delta^c_k, \\ III_d &:= III(\dot{P}^a_i) := \frac{1}{6} \epsilon_{abc} \epsilon^{ijk} \dot{P}^a_i \dot{P}^b_j \dot{P}^c_k.\end{aligned}$$

Using Eq. (45) one can express the scalar invariants of the peculiar-expansion tensor as a function of \mathcal{J} and ξ :

$${}^{\text{RZA}}I(\theta^i_j) = \frac{\dot{\mathcal{J}}}{\mathcal{J}}, \quad {}^{\text{RZA}}II(\theta^i_j) = \frac{1}{2} \left(\frac{\dot{\mathcal{J}}}{\mathcal{J}} - \frac{\ddot{\xi}(t)}{\xi(t)} \frac{\dot{\mathcal{J}}}{\mathcal{J}} \right), \quad (\text{B8})$$

$${}^{\text{RZA}}III(\theta^i_j) = \frac{1}{6} \left(\frac{\ddot{\mathcal{J}}}{\mathcal{J}} - \frac{\ddot{\xi}(t)}{\xi(t)} \frac{\dot{\mathcal{J}}}{\mathcal{J}} \right) - \frac{1}{2} \frac{\ddot{\xi}(t)}{\xi(t)} \left(\frac{\dot{\mathcal{J}}}{\mathcal{J}} - \frac{\ddot{\xi}(t)}{\xi(t)} \frac{\dot{\mathcal{J}}}{\mathcal{J}} \right). \quad (\text{B9})$$

Inserting Eqs. (B8) into Eq. (47), the backreaction term in the relativistic Zel'dovich approximation reads

$${}^{\text{RZA}}\mathcal{Q}_D = \frac{\langle \dot{\mathcal{J}} \rangle_I}{\langle \mathcal{J} \rangle_I} - \frac{\ddot{\xi}}{\xi} \frac{\langle \dot{\mathcal{J}} \rangle_I}{\langle \mathcal{J} \rangle_I} - \frac{2}{3} \left(\frac{\langle \dot{\mathcal{J}} \rangle_I}{\langle \mathcal{J} \rangle_I} \right)^2, \quad (\text{B10})$$

or finally, using the first equation of Eqs. (B8),

$${}^{\text{RZA}}\mathcal{Q}_D = \frac{\xi^2(\gamma_1 + \xi\gamma_2 + \xi^2\gamma_3)}{(\langle S_i \rangle_I + \xi \langle D_d \rangle_I + \xi^2 \langle D_{dd} \rangle_I + \xi^3 \langle D_{ddd} \rangle_I)^2},$$

with

$$\begin{aligned}\gamma_1 &:= 2 \langle S_i \rangle_I \langle D_{dd} \rangle_I - \frac{2}{3} \langle D_d \rangle_I^2, \\ \gamma_2 &:= 6 \langle S_i \rangle_I \langle D_{ddd} \rangle_I - \frac{2}{3} \langle D_d \rangle_I \langle D_{dd} \rangle_I, \\ \gamma_3 &:= 2 \langle D_d \rangle_I \langle D_{ddd} \rangle_I - \frac{2}{3} \langle D_{dd} \rangle_I^2.\end{aligned} \quad (\text{B11})$$

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